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REPRESENTING ELEMENTS OF STABLE HOMOTOPY GROUPS BY SYMMETRIC MAPS

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0. Introduction

Let S^m be the unit m -sphere. Let p be a prime and π the cyclic group of order p . Denote by $B\pi^{(r)}$ the r -skeleton of the classifying space $B\pi$. Recall that $B\pi$ is the infinite real projective space for $p=2$ and the infinite lens space for $p>2$. Let X be a space. Let m be a positive integer for the case $p=2$ and m an odd integer for the case $p>2$. Then a map $f: S^m \rightarrow X$ is called *symmetric* if there exists a map $\tilde{f}: B\pi^{(m)} \rightarrow X$ such that the following diagram is commutative:

$$(1) \quad \begin{array}{ccc} S^m & \xrightarrow{f} & X \\ \omega \searrow & & \nearrow \tilde{f} \\ & B\pi^{(m)} & \end{array}$$

, where $\omega: S^m \rightarrow B\pi^{(m)}$ is the canonical projection.

An element of the homotopy group $\pi_m(X)$ is called *symmetric* if it is represented by a symmetric map. For $p=2$, the definition of a symmetric map is due to J. H. C. Whitehead [14], in which he showed that if an essential element of $\pi_m(S^{m-1})$ is symmetric, then $m \equiv 3 \pmod{4}$. Some results about the symmetry of the elements of $\pi_m(X)$ are found in [4], [8], [10], [21] and [13].

Let X be an $(l-1)$ -connected, finite CW -complex. Then our purpose is to show the following

Theorem 1. *Every element of $\pi_m(X)$ is symmetric for any m satisfying $2 \dim X - l < m < 2l - 2$ and*

- i) $m \equiv -1 \pmod{2^{\phi(k+1)}} \quad \text{for } p=2,$
- ii) $m \equiv -1 \pmod{2p^{\lfloor (k+1)/2(p-1) \rfloor}} \quad \text{for } p>2,$

where $k=m-l$, $\phi(s)$ is the number of integers i such that $0 < i \leq s$ and $i \equiv 0, 1, 2$ or $4 \pmod{8}$ and $\lfloor s \rfloor$ indicates the integer part of a rational s .

Corollary 2. *For an arbitrary $k>0$, every element of the k -stem of the stable*

homotopy groups of spheres is symmetric.

To prove the above theorem we use the S -duality [11] and the Kahn-Priddy theorem [6] which is stated as follows for our use. Denote by ${}^p\{X, Y\}$ the p -primary component of $\{X, Y\} = \lim_{n \rightarrow \infty} [S^n X, S^n Y]$.

Theorem 3. [Kahn-Priddy]. *Let N be a sufficiently large integer and $h: S^N B\pi^{(s)} \rightarrow S^N$ a map such that the functional $\mathfrak{P}^1(Sq^2)$ -operation is non-trivial (respectively). Then for a connected, finite CW-complex X of dimension $< s$, $h_*: \{X, B\pi^{(s)}\} \rightarrow {}^p\{X, S^0\}$ is an epimorphism. Furthermore, assume that the functional $\mathfrak{P}^{[(s+1)/2(p-1)]}(Sq^{s+1})$ -operation of h is non-trivial for odd s (respectively), then h_* is an epimorphism for X of dimension $\leq s$.*

We express our thanks to H. Toda who suggested us to use the S -duality.

1. A proof of the Kahn-Priddy theorem

First we shall prove Theorem 3 for $p=2$. The notations of [6] are carried over to the present section unless otherwise stated.

Roughly speaking, the proof of Theorem 3 is to replace the infinite dimensional real projective space P^∞ with the s -dimensional one P^s and the map $\phi: P^\infty \rightarrow (QS^0)_0$ with a map $\text{adj}(h): P^s \rightarrow (QS^0)_0$ (cf. p. 985 of [6] and Theorem 7.3 of [9]) in the proof of Theorem 3.1 of [6].

$$\begin{aligned} \text{Let } t: B\mathfrak{E}_{2^k}^{(s)} &\rightarrow \hat{Q}_m(B\mathfrak{E}_{2^k}(2))^{(s)} \\ &= \hat{Q}_m(\underbrace{\hat{Q}_2 \cdots \hat{Q}_2}_{k-1} B\mathfrak{E}_2)^{(s)} \subset \hat{Q}_m(\underbrace{\hat{Q}_2 \cdots \hat{Q}_2}_{k-1} P^s) \end{aligned}$$

be a restriction of the pretransfer $T: B\mathfrak{E}_{2^k} \rightarrow \hat{Q}_m(B\mathfrak{E}_{2^k}(2))$ (Definition 3.1 of [6]) on the s -skeleton $B\mathfrak{E}_{2^k}^{(s)}$. Let $g_2': \hat{Q}_m(\underbrace{\hat{Q}_2 \cdots \hat{Q}_2}_{k-1} P^s) \rightarrow \hat{Q}_{m2^{k-1}}(P^s)$ be induced by the

wreath product and $g_3': \hat{Q}_{m2^{k-1}}(P^s) \rightarrow Q(P^s)$ a Dyer-Lashof map. Then we obtain a commutative diagram

$$\begin{array}{ccc} \sum^\infty B\mathfrak{E}_{2^k}^{(s)} & \xrightarrow{b} & \sum^\infty (QS^0)_0 \\ \downarrow a & & \downarrow r' \\ \sum^\infty P^s & \xrightarrow{h} & \sum^\infty S^0 \end{array}$$

, where $a = \text{adj}(g_2' g_3' t)$, b is a restriction of G_ϕ (p. 985 of [6]) on $\sum^\infty \mathfrak{E}_{2^k}^{(s)}$ and r' is defined by $r'(x \wedge f) = f(x)$ for $x \in \sum^\infty S^0$ and $f \in (QS^0)_0$. Remark that b is a restriction of $\sum^\infty \bar{\phi} \circ g_3 g_2 f_1$ on $\sum^\infty B\mathfrak{E}_{2^k}^{(s)}$.

For large k , $b_*: H_i(B\mathfrak{E}_{2^k}^{(s)}; Z_2) \rightarrow H_i(Q(S^0)_0; Z_2)$ is an isomorphism if $i < s$ (p. 985 of [6]). So, by the Whitehead-Serre theorem, $b_*: {}^2\{X, B\mathfrak{E}_{2^k}^{(s)}\} \rightarrow$

$\{X, (QS^0)_0\}$ is an isomorphism for a finite CW -complex X of dimension $< s-1$ and an epimorphism for X of dimension $< s$. It is clear that $r_*': \{X, (QS^0)_0\} \rightarrow \{X, S^0\}$ is an epimorphism if X is connected. Thus $(r'b)_*$ is an epimorphism on the 2-component and hence so is h_* . This proves the first part of Theorem 3 for $p=2$.

Under the first assumption of Theorem 3, the functional $\mathfrak{P}^i(Sq^{2^i})$ - and $\beta\mathfrak{P}^i(Sq^{2^i+1})$ - operations are non-trivial for $2i(p-1) \leq s$ ($2i \leq s$, respectively). This is easily seen by use of the cohomology structure of $B\pi^{(s)}$ and the Adem relation. So, by adding the second assumption, $b_*: H_i(B\mathcal{O}_{2^k}; \mathbb{Z}_2) \rightarrow H_i(Q(S^0)_0; \mathbb{Z}_2)$ is an isomorphism for $i < s$ and an epimorphism for $i \leq s$. This completes the proof of Theorem 3 for $p=2$.

For $p > 2$, the argument is quite parallel (cf. Remark 3.5 of [6] and Theorem 7.5 of [9]) and we omit it.

2. The S-duality

From now on we shall devote ourselves to the proof of Theorem 1. Denote by $B\pi_s^* = B\pi^{(s)}/B\pi^{(s-1)}$, where $B\pi_0^*$ means $B\pi^{(0)} \cup (\text{one point})$. Let X be an $(l-1)$ -connected, finite CW -complex of dimension j . Then $f: S^m \rightarrow X$ is symmetric if and only if there is a map $\tilde{f}: B\pi_n^m \rightarrow X$ for $1 \leq n \leq l$ such that the following diagram is commutative:

$$(2) \quad \begin{array}{ccc} S^m & \xrightarrow{f} & X \\ \omega' \searrow & & \nearrow \tilde{f}' \\ & B\pi_n^m & \end{array}$$

, where ω' is the map ω of (1) followed by the collapsing map from $B\pi^{(m)}$ to $B\pi_n^m$.

Let N be so large that $N \geq \max(2j+1, 2m+1)$ and take N -duals of everything in (2):

$$(2') \quad \begin{array}{ccc} D_N S^m & \xleftarrow{\Delta_N f} & D_N X \\ \Delta_N \omega' \nwarrow & & \nearrow \Delta_N(\tilde{f}') \\ & D_N(B\pi_n^m) & \end{array}$$

, where $D_N Y$ and $\Delta_N g$ are N -duals of a finite CW -complex Y and a map g [11].

If $m \leq 2n-2$, then we work in the stable range. So, we obtain the following

Proposition 4. *Let X be an $(l-1)$ -connected, finite CW -complex, $N \geq \max(2j+1, 2m+1)$ and $m \leq 2n-2$. Then a map $f: S^m \rightarrow X$ represents a symmetric element if and only if there is a map $\tilde{f}: D_N X \rightarrow D_N(B\pi_n^m)$ for $1 \leq n \leq l$ such that the following diagram is homotopy commutative:*

$$(3) \quad \begin{array}{ccc} & D_N f & \\ S^{N-m-1} & \xleftarrow{\quad} & D_N X \\ D_N \omega' & \searrow & \downarrow \tilde{f} \\ & D_N(B\pi_n^m) & \end{array}$$

3. The S-dual of $B\pi_m^n$

Take $N=N(a,s)=a2^{\phi(s)}$ for $p=2$ and $2ap^{\lceil s/2(p-1) \rceil}$ for $p>2$, where a is a sufficiently large integer.

Put $s=m-n$. Let $\varepsilon=\varepsilon(s)=0$ if $s \equiv -1 \pmod{2(p-1)}$ and $\varepsilon=1$ if $s \not\equiv -1 \pmod{2(p-1)}$ for $p>2$ and $\varepsilon=0$ for $p=2$. Then we have the following

Proposition 5. $D_N(B\pi_n^m)$ has the same homotopy type as $B\pi_{N-m-1}^{N-n-1}$ for $N=N(a, s+\varepsilon)$ with $s=m-n$.

Proof. For $p>2$, recall from Theorem 1 of [7] that the stunted lens space $B\pi_{2n}^{2m+1}=L^m(p)/L^{n-1}(p)$ is the Thom complex $(L^s(p))^{\pi_1^*r(\xi)}$, where $L^r(p)=B\pi^{(2r+1)}$ is the $(2r+1)$ -dimensional lens space, ξ is the canonical line bundle over the complex projective s -space CP^s , $r(\xi)$ is the real restriction of ξ and $\pi_1^*r(\xi)=r\pi_1^*(\xi)$ is the bundle induced by the natural projection $\pi_1:L^s(p) \rightarrow CP^s$.

First we shall show

$$(4) \quad \begin{aligned} D_N(B\pi_{2n}^{2m+1}) &\simeq (L^s(p))^{(N/2-(m+1))\pi_1^*r(\xi)} \\ &\simeq B\pi_{N-2m-2}^{N-2n-1} \quad \text{for } N=N(a, 2s+1). \end{aligned}$$

According to Theorem 3.3 of [3], the S-dual of $B\pi_{2n}^{2m+1}=(L^s(p))^{\pi_1^*r(\xi)}$ is $(L^s(p))^{-\pi_1^*r(\xi)-\tau}$, where τ is the tangent bundle over $L^s(p)$. As is well known, $\tau+1=(s+1)\pi_1^*r(\xi)$. So $(L^s(p))^{-\pi_1^*r(\xi)-\tau} \simeq (L^s(p))^{1-(m+1)\pi_1^*r(\xi)}$. By Theorem 2 of [7] the J -order of $\pi_1^*r(\xi)-2$ is $p^{\lceil s/2(p-1) \rceil}$. Obviously $[s/p-1]=[2s+1/2(p-1)]$ holds. So by Theorem 3 of [7], $(L^s(p))^{-(m+1)\pi_1^*r(\xi)}$ and $(L^s(p))^{(N/2-(m+1))\pi_1^*r(\xi)}$ have the same stable homotopy type. Therefore we have obtained (4).

Observe that $D_N(B\pi_{2n}^{2m})=D_N(B\pi_{2n}^{2m+1})/S^{N-2m-2}$ for $N=N(a, 2s)$ and also that $D_N(B\pi_{2n+1}^{2m+1})$ is obtained from $D_N(B\pi_{2n}^{2m+1})$ by deleting the top dimensional cell for $N=N(a, 2s)$. By the same way as above we obtain $D_N(B\pi_{2n+1}^{2m})$ from $D_N(B\pi_{2n}^{2m})$ for $N=N(a, 2s)$.

Similarly and more simply we have the assertion for $p=2$ (Theorem 6.1 of [3]). We note that the J -order of $\xi-1$ is $2^{\phi(s)}$, where ξ is the canonical line bundle over the s -dimensional real projective space P^s ([1] and [2]).

4. On the Kahn-Priddy map

Consider the cofibring sequence

$$(5) \quad \cdots \rightarrow S^m \xrightarrow{\omega'} B\pi_n^m \xrightarrow{i'} B\pi_n^{m+1} \xrightarrow{q'} S^{m+1} \rightarrow \cdots,$$

where i' and q' are the canonical inclusion and projection respectively.

Put $s=m-n$ and take $N(a, s+2)$ -duals of everything in (5) and use Theorem 6.2 of [11] and Proposition 5, then we have the following

Proposition 6. *There is a cofibring sequence*

$$\dots \leftarrow S^{N-m-1} \xleftarrow{h} B\pi_{N-m-1}^{N-n-1} \xleftarrow{q} B\pi_{N-m-2}^{N-n-1} \xleftarrow{i} S^{N-m-2} \leftarrow \dots$$

for $N=N(a, s+2)$, where $h: B\pi_{N-m-1}^{N-n-1} \simeq D_N(B\pi_n^m) \xrightarrow{D_N\omega'} S^{N-m-1}$, $q=D_N i'$ and $i=D_N q'$.

We note that the cofibre of h is $SB\pi_{N-m-2}^{N-n-1}$.

Proposition 7. *Let $m+1 \equiv 0 \pmod{2^{\phi(s+1)}}$ for $p=2$ and $m+1 \equiv 0 \pmod{2p^{[(s+1)/2(\phi-1)]}}$ for $p>2$. Then*

- i) $B\pi_{N-m-1}^{N-n-1} \simeq S^{N-m-1} B\pi_0^s \simeq S^{N-m-1} B\pi^{(s)} \vee S^{N-m-1}$.
- ii) $h|S^{N-m-1}$ is of degree p and $h|S^{N-m-1} B\pi^{(s)}$ has non-trivial functional $\mathfrak{P}^i(Sq^i)$ -operations for $2i(p-1) \leq s+1$ ($2 \leq i \leq s+1$), respectively).

Proof. Recall that $N=N(a, s+2)$ with sufficiently large a . Put $m+1=N(b, s+1)$ for any b with $0 < b < a-1$. Then $N-m-1=N(c, s+1)$ and $N-n-1=N(c, s+1)+s$ for some integer c . So by the James periodicity for $p=2$ ([5]) and by Theorem 4 of [7] for $p>2$, we have $B\pi_{N-m-1}^{N-n-1} \simeq S^{N-m-1} B\pi_0^s \simeq S^{N-m-1} B\pi^{(s)} \vee S^{N-m-1}$. This leads us to i).

Since $N-m-1$ is even, $h|S^{N-m-1}$ is of degree p . For $i > N-m-2$, there is the natural isomorphism $H^i(SB\pi_{N-m-2}^{N-n-1}; Z_p) \cong H^{i-1}(B\pi^{(N-n-1)}; Z_p)$. Let u and v be generators of $H^{i-1}(B\pi^{(N-n-1)}; Z_p)$ for $i=2$ and 3 respectively. Then the non-triviality of the functional \mathfrak{P}^i -operation follows directly from the following relation:

$$\mathfrak{P}^i(uv^{(N-m-3)/2}) = u\mathfrak{P}^i(v^{(N-m-3)/2}) = \binom{cp^{[(s+1)/2(\phi-1)]-1}}{i} uv^{(N-m-3)/2+i(\phi-1)} \neq 0$$

for $2i(p-1) \leq s+1$.

Similarly the functional Sq^i -operation is non-trivial for $2 \leq i \leq s+1$.

This completes the proof.

5. A proof of the main theorem

Obviously we have the following

Lemma 8. *If Y is an $(r-1)$ -connected CW-complex of dimension $r+s$ with $r>s$, then there exists an $(s-1)$ -connected CW-complex W of dimension $2s$ such that $Y \simeq S^{r-s}W$.*

Now we are ready to prove Theorem 1 supposing Theorem 3. If X is

$(l-1)$ -connected and $\dim X=j$, then $D_N X$ is $(N-j-2)$ -connected and $\dim D_N X=N-l-1$. Therefore, by the above lemma, there exists a $(j-l-1)$ -connected and $2(j-l)$ -dimensional CW -complex W such that $D_N X \simeq S^{N+l-2j-1}W$. If $m+l>2j$, then $S^{N+l-2j-1}W=S^{N-m-1}(S^{m+l-2j}W)$ and $\dim S^{m+l-2j}W=m-l=k$. Hence Propositions 4 and 7 for $n=l$ and Theorem 3 complete the proof of Theorem 1.

6. An example

Theorem 1 does not hold without the assumption $2 \dim X < m+l$. This is shown as follows.

Let $\iota \in \{S^0, S^0\}$, $\eta \in \{S^1, S^0\}$ and $\nu \in \{S^3, S^0\}$ be generators. Put $\alpha = \nu \vee 2\iota$ and $X = (S^{m-5} \vee S^{m-2}) \cup_{\alpha} e^{m-1}$. Then it is clear that $\pi_m(X) = \{\tilde{\eta}\} \cong Z_4$ for $m > 11$, where $\tilde{\eta}$ is a co-extension of η . It is shown as follows that $\tilde{\eta}$ is not symmetric for any $m > 10$.

If $\tilde{\eta}$ is represented by a symmetric map $f: S^m \rightarrow X$, then f is decomposed as (2) for $n=m-5$. It is easily seen that m is odd and $(f')^*: H^{m-1}(X; Z_2) \rightarrow H^{m-1}(P_{m-5}^m; Z_2)$ is an isomorphism. Put $m \equiv k \pmod{8}$, where $k=1, 3, 5$ or 7 . Since Sq^4 is non-trivial in $H^*(X; Z_2)$, we have $k=1$ or 3 and $(f')^*: H^{m-5}(X; Z_2) \rightarrow H^{m-5}(P_{m-5}^m; Z_2)$ is an isomorphism. The operation $Sq^2: H^{m-1}(P_{m-5}^{m+1}; Z_2) \rightarrow H^{m+1}(P_{m-5}^{m+1}; Z_2)$ is non-trivial and so we have $k=3$.

Consider the diagram (2)' for $n=m-5$. Then we have

$$\begin{array}{ccc} S^{N-m-1} & \xleftarrow{\Delta_N f} & D_N X \\ \Delta_N \omega' \swarrow & & \searrow \Delta_N(f') \\ & P_{N-m-1}^{N-m+4} & \end{array}$$

, where $N = a2^{\phi(5)} = 8a$ for sufficiently large a and $D_N X = S^{N-m} \cup e^{N-m+1} \cup e^{N-m+4}$. Put $N-m=8t+5$ and let $q: P_{8t+4}^{8t+9} \rightarrow S^{8t+9}$ be the collapsing map. Then it is clear that $q\Delta_N(f'): D_N X \rightarrow S^{8t+9}$ is also the collapsing map and $(q\Delta_N(f'))^*: \widetilde{KO}(S^{8t+9}) \rightarrow \widetilde{KO}(D_N X) \cong Z_2$ is an isomorphism. On the other hand, we have $\widetilde{KO}(P_{8t+4}^{8t+9}) \cong Z + Z_4$ by Theorem 7.4 of [1]. This is a contradiction. Hence $\tilde{\eta}$ is not symmetric for $m > 10$.

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