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REPRESENTING ELEMENTS OF STABLE HOMOTOPY GROUPS BY SYMMETRIC MAPS

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0. Introduction

Let S^m be the unit m-sphere. Let p be a prime and π the cyclic group of order p. Denote by $B\pi^{(r)}$ the r-skeleton of the classifying space $B\pi$. Recall that $B\pi$ is the infinite real projective space for $p=2$ and the infinite lens space for $p>2$. Let X be a space. Let m be a positive integer for the case $p=2$ and m an odd integer for the case $p>2$. Then a map $f: S^m \to X$ is called *symmetric* if there exists a map $\bar{f}: B\pi^{(m)} \to X$ such that the following diagram is commutative:

(1)
$$\begin{tikzcd} S^m \arrow[rr, "f"] \arrow[dr, "\omega"'] & & X \\ & B\pi^{(m)} \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

, where $\omega: S^m \to B\pi^{(m)}$ is the canonical projection.

An element of the homotopy group $\pi_m(X)$ is called *symmetric* if it is represented by a symmetric map. For $p=2$, the definition of a symmetric map is due to J. H. C. Whitehead [14], in which he showed that if an essential element of $\pi_m(S^{m-1})$ is symmetirc, then $m\equiv 3 \bmod 4$. Some results about the symmetricity of the elements of $\pi_m(X)$ are found in [4], [8], [10], [21] and [13].

Let X be an $(l-1)$-connected, finite CW-complex. Then our purpose is to show the following

Theorem 1. *Every element of* $\pi_m(X)$ *is symmetric for any* m *satisfying* $2 \dim X-l<m<2l-2$ *and*

i) $m\equiv -1 \bmod 2^{\phi(k+1)}$ *for* $p=2$,

ii) $m\equiv -1 \bmod 2p^{[(k+1)/2(p-1)]}$ *for* $p>2$,

where $k=m-l$, $\phi(s)$ *is the number of integers* i *such that* $0<i\leqq s$ *and* $i\equiv 0$, 1, 2 *or* $4 \bmod 8$ *and* $[s]$ *indicates the integer part of a rational* s.

Corollary 2. *For an arbitrary* $k>0$, *every element of the* k*-stem of the stable* *homotopy groups of spheres is symmetric.*

To prove the above theorem we use the S-duality [11] and the Kahn-Priddy theorem [6] which is stated as follows for our use. Denote by ${}^p\{X, Y\}$ the p-primary component of $\{X, Y\}=\lim_{n\to\infty}[S^nX, S^nY]$.

Theorem 3. [Kahn-Priddy]. *Let N be a sufficiently large integer and $h: S^N B\pi^{(s)} \to S^N$ a map such that the functional $\mathfrak{P}^1(Sq^2)$-operation is non-trivial (respectively). Then for a connected, finite CW-complex X of dimension $<s$, h_*: $\{X, B\pi^{(s)}\} \to {}^p\{X, S^0\}$ is an epimorphism. Furthermore, assume that the functional $\mathfrak{P}^{[(s+1)/2(p-1)]}(Sq^{s+1})$-operation of h is non-trivial for odd s (respectively), then h_* is an epimorphism for X of dimension $\leqq s$.*

We express our thanks to H. Toda who suggested us to use the S-duality.

1. A proof of the Kahn-Priddy theorem

First we shall prove Theorem 3 for $p=2$. The notations of [6] are carried over to the present section unless otherwise stated.

Roughly speaking, the proof of Theorem 3 is to replace the infinite dimensional real projective space P^∞ with the s-dimensional one P^s and the map $\phi: P^\infty \to (QS^0)_0$ with a map $\mathrm{adj}(h): P^s \to (QS^0)_0$ (cf. p. 985 of [6] and Theorem 7.3 of [9]) in the proof of Theorem 3.1 of [6].

Let
$$t: B\mathfrak{S}_{2^k}^{(s)} \to \hat{Q}_m(B\mathfrak{S}_{2^k}(2))^{(s)} = \hat{Q}_m(\underbrace{\hat{Q}_2\cdots\hat{Q}_2}_{k-1}B\mathfrak{S}_2)^{(s)} \subset \hat{Q}_m(\underbrace{\hat{Q}_2\cdots\hat{Q}_2}_{k-1}P^s)$$

be a restriction of the pretransfer $T: B\mathfrak{S}_{2^k} \to \hat{Q}_m(B\mathfrak{S}_{2^k}(2))$ (Definition 3.1 of [6]) on the s-skeleton $B\mathfrak{S}_{2^k}^{(s)}$. Let $g_2': \hat{Q}_m(\underbrace{\hat{Q}_2\cdots\hat{Q}_2}_{k-1}P^s) \to \hat{Q}_{m2^{k-1}}(P^s)$ be induced by the wreath product and $g_3': \hat{Q}_{m2^{k-1}}(P^s) \to Q(P^s)$ a Dyer-Lashof map. Then we obtain a commutative diagram

$$\begin{tikzcd} \Sigma^\infty B\mathfrak{S}_{2^k}^{(s)} \arrow[r, "b"] \arrow[d, "a"] & \Sigma^\infty (QS^0)_0 \arrow[d, "r'"] \\ \Sigma^\infty P^s \arrow[r, "h"] & \Sigma^\infty S^0 \end{tikzcd}$$

, where $a=\mathrm{adj}(g_2'g_3't)$, b is a restriction of G_ϕ (p. 985 of [6]) on $\Sigma^\infty \mathfrak{S}_{2^k}^{(s)}$ and r' is defined by $r'(x\wedge f)=f(x)$ for $x\in\Sigma^\infty S^0$ and $f\in(QS^0)_0$. Remark that b is a restriction of $\Sigma^\infty \bar{\phi}\circ g_3g_2f_1$ on $\Sigma^\infty B\mathfrak{S}_{2^k}^{(s)}$.

For large k, $b_*: H_i(B\mathfrak{S}_{2^k}^{(s)}; Z_2) \to H_i(Q(S^0)_0; Z_2)$ is an isomorphism if $i<s$ (p. 985 of [6]). So, by the Whitehead-Serre theorem, $b_*: {}^2\{X, B\mathfrak{S}_{2^k}^{(s)}\} \to$

${}^2\{X, (QS^0)_0\}$ is an isomorphism for a finite CW-complex X of dimension $<s-1$ and an epimorphism for X of dimension $<s$. It is clear that $r_*': \{X, (QS^0)_0\} \rightarrow \{X, S^0\}$ is an epimorphism if X is connected. Thus $(r'b)_*$ is an epimorphism on the 2-component and hence so is h_*. This proves the first part of Theorem 3 for $p=2$.

Under the first assumption of Theorem 3, the functional $\mathfrak{P}^i(Sq^{2i})$- and $\beta\mathfrak{P}^i(Sq^{2i+1})$- operations are non-trivial for $2i(p-1)\leqq s$ ($2i\leqq s$, respectively). This is easily seen by use of the cohomology structure of $B\pi^{(s)}$ and the Adem relation. So, by adding the second assumption, $b_*: H_i(B\mathfrak{S}_{2^k}^{(2)}; Z_2) \rightarrow H_i(Q(S^0)_0; Z_2)$ is an isomorphism for $i<s$ and an epimorphism for $i\leqq s$. This completes the proof of Theorem 3 for $p=2$.

For $p>2$, the argument is quite parallel (cf. Remark 3.5 of [6] and Theorem 7.5 of [9]) and we omit it.

2. The S-duality

From now on we shall devote ourselves to the proof of Theorem 1. Denote by $B\pi_s^r = B\pi^{(r)}/B\pi^{(s-1)}$, where $B\pi_0^r$ means $B\pi^{(r)} \cup$ (one point). Let X be an $(l-1)$-connected, finite CW-complex of dimension j. Then $f: S^m \rightarrow X$ is symmetric if and only if there is a map $\tilde{f}': B\pi_n^m \rightarrow X$ for $1\leqq n\leqq l$ such that the following diagram is commutative:

$$(2)\qquad \begin{tikzcd} S^m \arrow[rr, "f"] \arrow[dr, "\omega'"'] & & X \\ & B\pi_n^m \arrow[ur, dashed, "\tilde{f}'"'] & \end{tikzcd}$$

, where ω' is the map ω of (1) followed by the collapsing map from $B\pi^{(m)}$ to $B\pi_n^m$.

Let N be so large that $N\geqq \max(2j+1, 2m+1)$ and take N-duals of everything in (2):

$$(2')\qquad \begin{tikzcd} D_N S^m & & D_N X \arrow[ll, "\Delta_N f"'] \arrow[dl, dashed, "\Delta_N(\tilde{f}')"] \\ & D_N(B\pi_n^m) \arrow[ul, "\Delta_N \omega'"] & \end{tikzcd}$$

, where $D_N Y$ and $\Delta_N g$ are N-duals of a finite CW-complex Y and a map g [11].

If $m\leqq 2n-2$, then we work in the stable range. So, we obtain the following

Proposition 4. *Let X be an $(l-1)$-connected, finite CW-complex, $N\geqq \max(2j+1, 2m+1)$ and $m\leqq 2n-2$. Then a map $f: S^m \rightarrow X$ represents a symmetric element if and only if there is a map $\tilde{f}: D_N X \rightarrow D_N(B\pi_n^m)$ for $1\leqq n\leqq l$ such that the following diagram is homotopy commutative:*

$$
(3)\qquad
\begin{tikzcd}
S^{N-m-1} & & D_NX \arrow[ll, "D_Nf"'] \arrow[dl, dashed, "\tilde{f}"] \\
& D_N(B\pi_n^m) \arrow[ul, "D_N\omega'"] &
\end{tikzcd}
$$

3. The S-dual of $B\pi_m^n$

Take $N=N(a,s)=a2^{\phi(s)}$ for $p=2$ and $2ap^{[s/2(p-1)]}$ for $p>2$, where a is a sufficiently large integer.

Put $s=m-n$. Let $\varepsilon=\varepsilon(s)=0$ if $s\equiv-1 \bmod 2(p-1)$ and $\varepsilon=1$ if $s\not\equiv-1 \bmod 2(p-1)$ for $p>2$ and $\varepsilon=0$ for $p=2$. Then we have the following

Proposition 5. *$D_N(B\pi_n^m)$ has the same homotopy type as $B\pi_{N-m-1}^{N-n-1}$ for $N=N(a, s+\varepsilon)$ with $s=m-n$.*

Proof. For $p>2$, recall from Theorem 1 of [7] that the stunted lens space $B\pi_{2n}^{2m+1}=L^m(p)/L^{n-1}(p)$ is the Thom complex $(L^s(p))^{n\pi_1{}^*r(\xi)}$, where $L^r(p)=B\pi^{(2r+1)}$ is the $(2r+1)$-dimensional lens space, ξ is the canonical line bundle over the complex projective s-space CP^s, $r(\xi)$ is the real restriction of ξ and $\pi_1^*r(\xi)=r\pi_1^*(\xi)$ is the bundle induced by the natural projection $\pi_1: L^s(p)\to CP^s$.

First we shall show

$$
(4)\qquad
\begin{aligned}
D_N(B\pi_{2n}^{2m+1}) &\simeq (L^s(p))^{(N/2-(m+1))\pi_1{}^*r(\xi)}\\
&\simeq B\pi_{N-2m-2}^{N-2n-1} \qquad \text{for} \quad N=N(a, 2s+1)\,.
\end{aligned}
$$

According to Theorem 3.3 of [3], the S-dual of $B\pi_{2n}^{2m+1}\simeq(L^s(p))^{n\pi_1{}^*r(\xi)}$ is $(L^s(p))^{-n\pi_1{}^*r(\xi)-\tau}$, where τ is the tangent bundle over $L^s(p)$. As is well known, $\tau+1=(s+1)\pi_1^*r(\xi)$. So $(L^s(p))^{-n\pi_1{}^*r(\xi)-\tau}\simeq(L^s(p))^{1-(m+1)\pi_1{}^*r(\xi)}$. By Theorem 2 of [7] the J-order of $\pi_1^*r(\xi)-2$ is $p^{[s/(p-1)]}$. Obviously $[s/p-1]=[2s+1/2(p-1)]$ holds. So by Theorem 3 of [7], $(L^s(p))^{-(m+1)\pi_1{}^*r(\xi)}$ and $(L^s(p))^{(N/2-(m+1))\pi_1{}^*r(\xi)}$ have the same stable homotopy type. Therefore we have obtained (4).

Observe that $D_N(B\pi_{2n}^{2m})=D_N(B\pi_{2n}^{2m+1})/S^{N-2m-2}$ for $N=N(a, 2s)$ and also that $D_N(B\pi_{2n+1}^{2m+1})$ is obtained from $D_N(B\pi_{2n}^{2m+1})$ by deleting the top dimensional cell for $N=N(a, 2s)$. By the same way as above we obtain $D_N(B\pi_{2n+1}^{2m})$ from $D_N(B\pi_{2n}^{2m})$ for $N=N(a, 2s)$.

Similarly and more simply we have the assertion for $p=2$ (Theorem 6.1 of [3]). We note that the J-order of $\xi-1$ is $2^{\phi(s)}$, where ξ is the cannonical line bundle over the s-dimensional real projective space P^s ([1] and [2]).

4. On the Kahn-Priddy map

Consider the cofibring sequence

$$
(5)\qquad \cdots \to S^m \xrightarrow{\omega'} B\pi_n^m \xrightarrow{i'} B\pi_n^{m+1} \xrightarrow{q'} S^{m+1} \to \cdots,
$$

where i' and q' are the canoncial inclusion and projection respectively.

Put $s=m-n$ and take $N(a, s+2)$-duals of everything in (5) and use Theorem 6.2 of [11] and Proposition 5, then we have the following

Proposition 6. *There is a cofibring sequence*

$$\cdots \leftarrow S^{N-m-1} \xleftarrow{h} B\pi_{N-m-1}^{N-n-1} \xleftarrow{q} B\pi_{N-m-2}^{N-n-1} \xleftarrow{i} S^{N-m-2} \leftarrow \cdots$$

for $N=N(a, s+2)$, *where* $h\colon B\pi_{N-m-1}^{N-n-1} \simeq D_N(B\pi_n^m) \xrightarrow{D_N\omega'} S^{N-m-1}$, $q=D_N i'$ *and* $i=D_N q'$.

We note that the cofibre of h is $SB\pi_{N-m-2}^{N-n-1}$.

Proposition 7. *Let* $m+1\equiv 0 \bmod 2^{\phi(s+1)}$ *for* $p=2$ *and* $m+1\equiv 0 \bmod 2p^{[(s+1)/2(p-1)]}$ *for* $p>2$. *Then*

i) $B\pi_{N-m-1}^{N-n-1} \simeq S^{N-m-1}B\pi_0^s \simeq S^{N-m-1}B\pi^{(s)} \vee S^{N-m-1}$.

ii) $h\,|\,S^{N-m-1}$ *is of degree* p *and* $h\,|\,S^{N-m-1}B\pi^{(s)}$ *has non-trivial functional* $\mathfrak{P}^i(Sq^i)$*-operations for* $2i(p-1)\leqq s+1$ $(2\leqq i\leqq s+1)$, *respectively*).

Proof. Recall that $N=N(a, s+2)$ with sufficiently large a. Put $m+1=N(b, s+1)$ for any b with $0<b<a-1$. Then $N-m-1=N(c, s+1)$ and $N-n-1=N(c, s+1)+s$ for some integer c. So by the James periodicity for $p=2$ ([5]) and by Theorem 4 of [7] for $p>2$, we have $B\pi_{N-m-1}^{N-n-1} \simeq S^{N-m-1}B\pi_0^s = S^{N-m-1}B\pi^{(s)} \vee S^{N-m-1}$. This leads us to i).

Since $N-m-1$ is even, $h\,|\,S^{N-m-1}$ is of degree p. For $i>N-m-2$, there is the natural isomorphism $H^i(SB\pi_{N-m-2}^{N-n-1}; Z_p) \cong H^{i-1}(B\pi^{(N-n-1)}; Z_p)$. Let u and v be generators of $H^{i-1}(B\pi^{(N-n-1)}; Z_p)$ for $i=2$ and 3 respectively. Then the non-triviality of the functional $\mathfrak{P}^i$-operation follows directly from the following relation:

$$\mathfrak{P}^i(uv^{(N-m-3)/2}) = u\mathfrak{P}^i(v^{(N-m-3)/2}) = \binom{cp^{[(s+1)/2(p-1)]-1}}{i} uv^{(N-m-3)/2+i(p-1)} \neq 0$$

for $2i(p-1)\leq s+1$.

Similarly the functional Sq^i-operation is non-trivial for $2\leq i\leq s+1$.

This completes the proof.

5. A proof of the main theorem

Obviously we have the following

Lemma 8. *If* Y *is an* $(r-1)$*-connected CW-complex of dimension* $r+s$ *with* $r>s$, *then there exists an* $(s-1)$*-connected CW-complex* W *of dimension* $2s$ *such that* $Y\simeq S^{r-s}W$.

Now we are ready to prove Theorem 1 supposing Theorem 3. If X is

$(l-1)$-connected and dim $X=j$, then $D_N X$ is $(N-j-2)$-connected and dim $D_N X=N-l-1$. Therefore, by the above lemma, there exists a $(j-l-1)$-connected and $2(j-l)$-dimensional CW-complex W such that $D_N X \simeq S^{N+l-2j-1}W$. If $m+l>2j$, then $S^{N+l-2j-1}W=S^{N-m-1}(S^{m+l-2j}W)$ and dim $S^{m+l-2j}W=m-l=k$. Hence Propositions 4 and 7 for $n=l$ and Theorem 3 complete the proof of Theorem 1.

6. An example

Theorem 1 does not hold without the assumption 2 dim $X<m+l$. This is shown as follows.

Let $\iota \in \{S^0, S^0\}$, $\eta \in \{S^1, S^0\}$ and $\nu \in \{S^3, S^0\}$ be generators. Put $\alpha=\nu \vee 2\iota$ and $X=(S^{m-5} \vee S^{m-2}) \cup_\alpha e^{m-1}$. Then it is clear that $\pi_m(X)=\{\tilde{\eta}\} \cong Z_4$ for $m>11$, where $\tilde{\eta}$ is a co-extension of η. It is shown as follows that $\tilde{\eta}$ is not symmetric for any $m>10$.

If $\tilde{\eta}$ is represented by a symmetric map $f: S^m \to X$, then f is decomposed as (2) for $n=m-5$. It is easily seen that m is odd and $(\bar{f}')^*: H^{m-1}(X; Z_2) \to H^{m-1}(P^m_{m-5}; Z_2)$ is an isomorphism. Put $m \equiv k$ mod 8, where $k=1$, 3, 5 or 7. Since Sq^4 is non-trivial in $H^*(X; Z_2)$, we have $k=1$ or 3 and $(\bar{f}')^*: H^{m-5}(X; Z_2) \to H^{m-5}(P^m_{m-5}; Z_2)$ is an isomorphism. The operation $Sq^2: H^{m-1}(P^{m+1}_{m-5}; Z_2) \to H^{m+1}(P^{m+1}_{m-5}; Z_2)$ is non-trivial and so we have $k=3$.

Consider the diagram (2)′ for $n=m-5$. Then we have

$$
\begin{tikzcd}
S^{N-m-1} & & D_N X \arrow[ll, "\Delta_N f"'] \arrow[dl, dashed, "\Delta_N(\bar{f}')"] \\
& P^{N-m+4}_{N-m-1} \arrow[ul, "\Delta_N \omega'"] &
\end{tikzcd}
$$

, where $N=a2^{\phi(5)}=8a$ for sufficiently large a and $D_N X=S^{N-m} \cup e^{N-m+1} \cup e^{N-m+4}$. Put $N-m=8t+5$ and let $q: P^{8t+9}_{8t+4} \to S^{8t+9}$ be the collapsing map. Then it is clear that $q\Delta_N(\bar{f}'): D_N X \to S^{8t+9}$ is also the collapsing map and $(q\Delta_N(\bar{f}'))^*: \widetilde{KO}(S^{8t+9}) \to \widetilde{KO}(D_N X) \cong Z_2$ is an isomorphism. On the other hand, we have $\widetilde{KO}(P^{8t+9}_{8t+4}) \cong Z+Z_4$ by Theorem 7.4 of [1]. This is a contradition. Hence $\tilde{\eta}$ is not symmetric for $m>10$.

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