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Osaka University
0. Introduction

Let $S^m$ be the unit $m$-sphere. Let $p$ be a prime and $\pi$ the cyclic group of order $p$. Denote by $B\pi^{(r)}$ the $r$-skeleton of the classifying space $B\pi$. Recall that $B\pi$ is the infinite real projective space for $p=2$ and the infinite lens space for $p>2$. Let $X$ be a space. Let $m$ be a positive integer for the case $p=2$ and $m$ an odd integer for the case $p>2$. Then a map $f: S^m \to X$ is called symmetric if there exists a map $f: B\pi^{(m)} \to X$ such that the following diagram is commutative:

\[
\begin{array}{c}
S^m \quad \ x \quad \ X \\
\downarrow \omega \quad \downarrow f \\
B\pi^{(m)} \quad \ f
\end{array}
\]

where $\omega: S^m \to B\pi^{(m)}$ is the canonical projection.

An element of the homotopy group $\pi_m(X)$ is called symmetric if it is represented by a symmetric map. For $p=2$, the definition of a symmetric map is due to J. H. C. Whitehead [14], in which he showed that if an essential element of $\pi_m(S^{m-1})$ is symmetric, then $m \equiv 3 \mod 4$. Some results about the symmetricity of the elements of $\pi_m(X)$ are found in [4], [8], [10], [21] and [13].

Let $X$ be an $(l-1)$-connected, finite CW-complex. Then our purpose is to show the following

Theorem 1. Every element of $\pi_m(X)$ is symmetric for any $m$ satisfying $2 \dim X - l < m < 2l - 2$ and

i) $m \equiv -1 \mod 2^{k(k+1)}$ for $p=2$, 
ii) $m \equiv -1 \mod 2p^{[k+\frac{1}{2}(p-1)]}$ for $p>2$,

where $k=m-l$, $\phi(s)$ is the number of integers $i$ such that $0 < i \leq s$ and $i \equiv 0, 1, 2$ or $4 \mod 8$ and $[s]$ indicates the integer part of a rational $s$.

Corollary 2. For an arbitrary $k>0$, every element of the $k$-stem of the stable
homotopy groups of spheres is symmetric.

To prove the above theorem we use the $S$-duality [11] and the Kahn-Priddy theorem [6] which is stated as follows for our use. Denote by $^p\{X, Y\}$ the $p$-primary component of $\{X, Y\} = \lim_{n \to \infty} [S^n X, S^n Y]$.

**Theorem 3.** [Kahn-Priddy]. Let $N$ be a sufficiently large integer and $h: S^N B_{\pi^{(2)}} \to S^N$ a map such that the functional $\Psi(Sq^s)$-operation is non-trivial (respectively). Then for a connected, finite CW-complex $X$ of dimension $\leq s$, $h_*: \{X, B_{\pi^{(2)}}\} \to ^p\{X, S^s\}$ is an epimorphism. Furthermore, assume that the functional $\Psi^{(s+1)/(p-1)}(Sq^{s+1})$-operation of $h$ is non-trivial for odd $s$ (respectively), then $h_*$ is an epimorphism for $X$ of dimension $\leq s$.

We express our thanks to H. Toda who suggested us to use the $S$-duality.

1. **A proof of the Kahn-Priddy theorem**

First we shall prove Theorem 3 for $p=2$. The notations of [6] are carried over to the present section unless otherwise stated.

Roughly speaking, the proof of Theorem 3 is to replace the infinite dimensional real projective space $P^\infty$ with the $s$-dimensional one $P^s$ and the map $\phi: P^m \to (Q S^s)_b$ with a map $\text{adj}(h): P^s \to (Q S^s)_b$ (cf. p. 985 of [6] and Theorem 7.3 of [9]) in the proof of Theorem 3.1 of [6].

Let $t: B \mathcal{S}^s_2 \to \hat{Q}_m(B \mathcal{S}^s_2(2))^{(s)} = \hat{Q}_m(\hat{Q}_2 \cdots \hat{Q}_s B \mathcal{S}^s_2)^{(s)}$ be a restriction of the pretransfer $T: B \mathcal{S}^s_2 \to \hat{Q}_m(B \mathcal{S}^s_2(2))$ (Definition 3.1 of [6]) on the $s$-skeleton $B \mathcal{S}^s_2$. Let $g'_2: \hat{Q}_m(\hat{Q}_2 \cdots \hat{Q}_s P^s) \to \hat{Q}_{m2}^{-1}(P^s)$ be induced by the wreath product and $g'_2: \hat{Q}_{m2}^{-1}(P^s) \to Q(P^s)$ a Dyer-Lashof map. Then we obtain a commutative diagram

\[
\begin{array}{ccc}
\sum^\infty B \mathcal{S}^s_2^{(s)} & \xrightarrow{b} & \sum^\infty (Q S^s)_b \\
\downarrow a & & \downarrow r' \\
\sum^\infty P^s & \xrightarrow{h} & \sum^\infty S^s
\end{array}
\]

where $a = \text{adj}(g'_2 g'_2 t)$, $b$ is a restriction of $G_\phi$ (p. 985 of [6]) on $\sum^\infty \mathcal{S}^s_2^{(s)}$ and $r'$ is defined by $r'(x \wedge f) = f(x)$ for $x \in \sum^\infty S^s$ and $f \in (Q S^s)_b$. Remark that $b$ is a restriction of $\sum^\infty \phi g g f$, on $\sum^\infty B \mathcal{S}^s_2^{(s)}$.

For large $k$, $b_*: H_\bullet(B \mathcal{S}^s_2; Z_2) \to H_\bullet(Q(S^s)_b; Z_2)$ is an isomorphism if $i < s$ (p. 985 of [6]). So, by the Whitehead-Serre theorem, $b_*: \{X, B \mathcal{S}^s_2\} \to$
2\{X, (QS^n)\} is an isomorphism for a finite CW-complex X of dimension <s−1 and an epimorphism for X of dimension <s. It is clear that \(r_*': \{X, (QS^n)\} \to \{X, S^n\}\) is an epimorphism if X is connected. Thus \((r'b_*)_\#\) is an epimorphism on the 2-component and hence so is \(h_*\). This proves the first part of Theorem 3 for \(p=2\).

Under the first assumption of Theorem 3, the functional \(\Psi(S_q^{2i})\) and \(\beta\Psi(S_q^{2i+1})\) operations are non-trivial for \(2i(p−1)\leq s\) \((2i\leq s\), respectively). This is easily seen by use of the cohomology structure of \(B\pi^{(r)}\) and the Adem relation. So, by adding the second assumption, \(b_*: H_i(B\Omega Q; Z)\to H_i(Q(S^n); Z)\) is an isomorphism for \(i<s\) and an epimorphism for \(i\leq s\). This completes the proof of Theorem 3 for \(p=2\).

For \(p>2\), the argument is quite parallel (cf. Remark 3.5 of [6] and Theorem 7.5 of [9]) and we omit it.

2. The S-duality

From now on we shall devote ourselves to the proof of Theorem 1. Denote by \(B\pi^{(r)}=B\pi^{(s)}/B\pi^{(s−1)}\), where \(B\pi_0\) means \(B\pi^{(r)}\cup\{\text{one point}\}\). Let \(X\) be an \((l−1)\)-connected, finite CW-complex of dimension \(j\). Then \(f:S^m\to X\) is symmetric if and only if there is a map \(f':B\pi^{(r)}\to X\) for \(l<n\leq l\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
S^m & \xrightarrow{f} & X \\
\downarrow{\omega'} & & \downarrow{f'} \\
B\pi^{(r)} & \xrightarrow{f} & X
\end{array}
\]

where \(\omega'\) is the map \(\omega\) of (1) followed by the collapsing map from \(B\pi^{(r)}\) to \(B\pi^{(m)}\).

Let \(N\) be so large that \(N\geq \max (2j+1, 2m+1)\) and take \(N\)-duals of everything in (2):

\[
\begin{array}{ccc}
D_N S^m & \xrightarrow{\Delta_N f} & D_N X \\
\downarrow{\Delta_N \omega'} & & \downarrow{\Delta_N (f')} \\
D_N(B\pi^{(m)}) & \xrightarrow{\Delta_N g} & D_N(B\pi^{(m)})
\end{array}
\]

where \(D_N Y\) and \(\Delta_N g\) are \(N\)-duals of a finite CW-complex \(Y\) and a map \(g\) [11].

If \(m\leq 2n−2\), then we work in the stable range. So, we obtain the following

**Proposition 4.** Let \(X\) be an \((l−1)\)-connected, finite CW-complex, \(N\geq \max (2j+1, 2m+1)\) and \(m\leq 2n−2\). Then a map \(f:S^m\to X\) represents a symmetric element if and only if there is a map \(\tilde{f}:D_N X\to D_N(B\pi^{(m)})\) for \(1\leq n\leq l\) such that the following diagram is homotopy commutative:
3. The S-dual of $B\pi^m_\infty$

Take $N = N(a, s) = a2^s(e)$ for $p = 2$ and $2ap^{s/2}(p-1)$ for $p > 2$, where $a$ is a sufficiently large integer.

Put $s = m - n$. Let $\varepsilon = \varepsilon(s) = 0$ if $s \equiv -1 \mod 2(p-1)$ and $\varepsilon = 1$ if $s \equiv -1 \mod 2(p-1)$ for $p > 2$ and $\varepsilon = 0$ for $p = 2$. Then we have the following

**Proposition 5.** $D_N(B\pi^m_\infty)$ has the same homotopy type as $B\pi^m_\infty$ for $N = N(a, s + \varepsilon)$ with $s = m - n$.

Proof. For $p > 2$, recall from Theorem 1 of [7] that the stunted lens space $B\pi^{m+1}_{2n} = L^m(p)/L^{n-1}(p)$ is the Thom complex $(L^r(p))^{m, r, r(\xi)}$, where $L^r(p) = B\pi^{(r+1)}$ is the $(2r+1)$-dimensional lens space, $\xi$ is the canonical line bundle over the complex projective $r$-space $CP^r$, $r(\xi)$ is the real restriction of $\xi$ and $\pi^r_{\tau r}(\xi) = \pi^r_{\tau r}(\xi)$ is the bundle induced by the natural projection $\pi_r : L^r(p) \to CP^r$.

First we shall show

\[
D_N(B\pi^{2m+1}_{2n}) = (L^r(p))^{(N/2 - (m+1)\varepsilon, r(\xi))} \cong B\pi^{2m-2}_{2n-2} \quad \text{for} \quad N = N(a, 2s+1). 
\]

According to Theorem 3.3 of [3], the S-dual of $B\pi^{2m+1}_{2n} = (L^r(p))^{m, r, r(\xi)}$ is $(L^r(p))^{m, r, r(\xi)-\tau}$, where $\tau$ is the tangent bundle over $L^r(p)$. As is well known, $\tau + 1 = (s+1)\pi^r_{\tau r}(\xi)$. So $(L^r(p))^{m, r, r(\xi)} - \tau = (L^r(p))^{(m+1)\varepsilon, r(\xi)}$. By Theorem 2 of [7] the $f$-order of $\pi^r_{\tau r}(\xi) - 2 = p^{s/(p-1)}$. Obviously $[s/p-1] = [2s+1/2(p-1)]$ holds. So by Theorem 3 of [7], $(L^r(p))^{(m+1)\varepsilon, r(\xi)}$ and $(L^r(p))^{N/2 - (m+1)\varepsilon, r(\xi)}$ have the same stable homotopy type. Therefore we have obtained (4).

Observe that $D_N(B\pi^{2m+1}_{2n}) = D_N(B\pi^{2m+1}_{2n})/SN-2m-2$ for $N = N(a, 2s)$ and also that $D_N(B\pi^{2m+1}_{2n})$ is obtained from $D_N(B\pi^{2m+1}_{2n})$ by deleting the top dimensional cell for $N = N(a, 2s)$. By the same way as above we obtain $D_N(B\pi^{2m+1}_{2n})$ from $D_N(B\pi^{2m+1}_{2n})$ for $N = N(a, 2s)$.

Similarly and more simply we have the assertion for $p = 2$ (Theorem 6.1 of [3]). We note that the $f$-order of $\xi - 1$ is $2^{s(e)}$, where $\xi$ is the canonical line bundle over the $s$-dimensional real projective space $P^s$ ([1] and [2]).

4. On the Kahn-Priddy map

Consider the cofibring sequence

\[
\cdots \to S^m \xrightarrow{\omega'} B\pi^m_n \xrightarrow{i'} B\pi^{m+1}_n \xrightarrow{q'} S^{m+1} \to \cdots,
\]
where $i'$ and $q'$ are the canonical inclusion and projection respectively.

Put $s = m - n$ and take $N(a, s + 2)$-duals of everything in (5) and use Theorem 6.2 of [11] and Proposition 5, then we have the following

**Proposition 6.** There is a cofibring sequence

$$
\cdots \leftarrow S^{N-m-1} \leftarrow B\pi_{N-m-1}^{\pi, n} \leftarrow B\pi_{N-m-2}^{\pi, n} \leftarrow \cdots
$$

for $N = N(a, s + 2)$, where $h: B\pi_{N-m-1}^{\pi, n} \to D_N(B\pi_{n}^{\pi})$, $q = D_N i'$ and $i = D_N q'$.

We note that the cofibre of $h$ is $SB\pi_{N-m-1}^{\pi, n}$. 

**Proposition 7.** Let $m + 1 \equiv 0 \mod 2^{i(x+1)}$ for $p = 2$ and $m + 1 \equiv 0 \mod 2^{i(x+1)/2(p-1)}$ for $p > 2$. Then

i) \quad $B\pi_{N-m-1}^{\pi, n} = S^{N-m-1}B\pi_b = S^{N-m-1}B\pi_1^{(s)} \cup S^{N-m-1}$.

ii) \quad $h|S^{N-m-1}$ is of degree $p$ and $h|S^{N-m-1}B\pi^{(s)}$ has non-trivial functional $\Psi^i(Sq^i)$-operations for $2i(p-1) \leq s+1$ (2 $\leq i \leq s+1$, respectively).

Proof. Recall that $N = N(a, s+2)$ with sufficiently large $a$. Put $m + 1 = N(b, s+1)$ for any $b$ with $0 < b < a - 1$. Then $N - m - 1 = N(c, s+1)$ and $N - m - 2 = N(c, s+1) + s$ for some integer $c$. So by the James periodicity for $p = 2$ ([5]) and by Theorem 4 of [7] for $p > 2$, we have $B\pi_{N-m-1}^{\pi, n} = S^{N-m-1}B\pi_b = S^{N-m-1}B\pi_1^{(s)} \cup S^{N-m-1}$. This leads us to i).

Since $N - m - 1$ is even, $h|S^{N-m-1}$ is of degree $p$. For $i > N - m - 2$, there is the natural isomorphism $H^i(SB\pi_{N-m-1}^{\pi, n}; Z_p) \cong H^{i+1}(B\pi_{N-m-1}^{(s)}; Z_p)$. Let $u$ and $v$ be generators of $H^{i+1}(B\pi_{N-m-1}^{(s)}; Z_p)$ for $i = 2$ and 3 respectively. Then the non-triviality of the functional $\Psi^i$-operation follows directly from the following relation:

$$
\Psi^i(\sigma_{N-m-2}^{(s)}) = u\Psi^i(\sigma_{N-m-3}^{(s)}) = \left(\frac{cp^{(x+1)/2(p-1)}-1}{i}\right)\sigma_{N-m-2}^{(s)} = 0
$$

for $2i(p-1) \leq s+1$.

Similarly the functional $Sq^i$-operation is non-trivial for $2 \leq i \leq s+1$.

This completes the proof.

5. A proof of the main theorem

Obviously we have the following

**Lemma 8.** If $Y$ is an $(r-1)$-connected CW-complex of dimension $r+s$ with $r > s$, then there exists an $(s-1)$-connected CW-complex $W$ of dimension $2s$ such that $Y \simeq S^{r-s}W$.

Now we are ready to prove Theorem 1 supposing Theorem 3. If $X$ is
(l−1)-connected and \( \dim X = j \), then \( D_N X \) is \((N−j−2)\)-connected and \( \dim D_N X = N−l−1 \). Therefore, by the above lemma, there exists a \((j−l−1)\)-connected and \(2(j−l)\)-dimensional \(CW\)-complex \( W \) such that \( D_N X \cong S^{N+l−2j−1}W \). If \( m+l > 2j \), then \( S^{N+l−2j−1}W \cong S^{N−m−l}((S^{m+l−2j}W) \cong \dim S^{m+l−2j}W \cong m−l = k \). Hence Propositions 4 and 7 for \( n = l \) and Theorem 3 complete the proof of Theorem 1.

6. An example

Theorem 1 does not hold without the assumption \( \dim X < m+l \). This is shown as follows.

Let \( e \in \{S^0, S^1\} \), \( f \in \{S^0, S^1\} \) and \( \nu \in \{S^0, S^1\} \) be generators. Put \( \alpha = \nu \vee 2e \) and \( X = (S^{m−5} \vee S^{m−3}) \cup e^{m−1} \). Then it is clear that \( \pi_m(X) = \{\eta\} \cong \mathbb{Z} \) for \( m > 11 \), where \( \tilde{\eta} \) is a co-extension of \( \eta \). It is shown as follows that \( \tilde{\eta} \) is not symmetric for any \( m > 10 \).

If \( \tilde{\eta} \) is represented by a symmetric map \( f: S^m \to X \), then \( f \) is decomposed as (2) for \( n = m−5 \). It is easily seen that \( m \) is odd and \( (f')^*: H^{m−1}(X; \mathbb{Z}) \to H^{m−1}(P^m_{m−5}; \mathbb{Z}) \) is an isomorphism. Put \( m \equiv k \mod 8 \), where \( k = 1, 3, 5 \) or 7. Since \( Sq^4 \) is non-trivial in \( H^*(X; \mathbb{Z}) \), we have \( k = 1 \) or 3 and \( (f')^*: H^{m−5}(X; \mathbb{Z}) \to H^{m−5}(P^m_{m−5}; \mathbb{Z}) \) is an isomorphism. The operation \( Sq^2: H^{m−1}(P^m_{m−5}; \mathbb{Z}) \to H^{m+1}(P^m_{m−5}; \mathbb{Z}) \) is non-trivial and so we have \( k = 3 \).

Consider the diagram (2)' for \( n = m−5 \). Then we have

\[
\begin{array}{ccc}
S^{N−m−1} & \xrightarrow{\Delta_N f} & D_N X \\
\Delta_N g' & \searrow & \Delta_N (f') \\
P^N_{N−m−1} & \xrightarrow{p^N_{N−m−1}} &
\end{array}
\]

where \( N = a2^{k(3)} = 8a \) for sufficiently large \( a \) and \( D_N X = S^{N−m} \cup e^{N−m+1} \cup e^{N−m+4} \).

Put \( N = 8t + 5 \) and let \( q: P^{8t+9}_{8t+9} \to S^{8t+9} \) be the collapsing map. Then it is clear that \( q \Delta_N (f'): D_N X \to S^{8t+9} \) is also the collapsing map and \((q \Delta_N (f'))^*: \widetilde{KO}(S^{8t+9}) \to \widetilde{KO}(D_N X) \cong \mathbb{Z} \) is an isomorphism. On the other hand, we have \( \widetilde{KO}(P^{8t+9}_{8t+4}) \cong \mathbb{Z} \oplus \mathbb{Z}_4 \) by Theorem 7.4 of [1]. This is a contradiction. Hence \( \tilde{\eta} \) is not symmetric for \( m > 10 \).
References


