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Osaka University
GENUS ONE 1-BRIDGE KNOTS AS VIEWED FROM THE CURVE COMPLEX

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1. Introduction

W.J. Harvey [4] associated to a surface $S$ a finite-dimensional simplicial complex $C(S)$, called the curve complex, which we recall below.

For a connected orientable surface $F = F_{g,n}$ of genus $g$ with $n$ punctures, the curve complex $C(F)$ of $F$ is the complex whose $k$-simplexes are the isotopy classes of $k + 1$ collections of mutually non-isotopic essential loops in $F$ which can be realized disjointly. It is proved in [16] that the curve complex is connected if $F$ is not sporadic (where $F$ is sporadic if $g = 0$, $n \leq 4$ or $g = 1$, $n \leq 1$). For $[x]$ and $[y]$, vertices of $C(F)$, the distance $d([x],[y])$ between $[x]$ and $[y]$ is defined by the minimal number of 1-simplexes in a simplicial path joining $[x]$ to $[y]$. It is known that if $S$ is not sporadic, then $C(F)$ has infinite diameter with respect to the distance defined above (cf. [11], [16]). $C(F)$ is not locally finite in the sense that there are infinite edges around each vertex, and the dimension of $C(F)$ is $3g - 4 + n$.

Recently, J. Hempel [11] studied Heegaard splittings of closed 3-manifolds by using the curve complex of Heegaard surfaces. Let $M$ be a closed orientable 3-manifold and $(V_1,V_2;S)$ a genus $g \geq 2$ Heegaard splitting, that is, $V_i$ ($i = 1$ and 2) is a genus $g$ handlebody with $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = S$. By using the curve complex, Hempel defined the distance of the Heegaard splitting, denoted by $d(V_1,V_2)$, and proved the following results.

**Theorem 1.1** (J. Hempel). (1) Let $M$ be a closed, orientable, irreducible 3-manifold which is Seifert fibered or which contains essential tori. Then $d(V_1,V_2) \leq 2$ for any Heegaard splitting $(V_1,V_2;S)$ of $M$.

(2) There are Heegaard splittings of closed orientable 3-manifolds with distance $> n$ for any integer $n$.

In particular, the theorem above implies that a Haken manifold is hyperbolic if a Heegaard splitting of the manifold has distance $\geq 3$. Results along these lines were also obtained by A. Thompson [20]. Moreover, H. Goda, C. Hayashi and N. Yoshida [2] made detailed study of tunnel number one knots and C. Hayashi ([6], [7]) studied $(1,1)$-knots from similar points of view.
In this paper, we apply this idea to genus one 1-bridge knots. A knot \( K \) in an orientable closed 3-manifold \( M \) is called a genus one 1-bridge knot, a (1, 1)-knot briefly, if \( (M, K) = (V_i, t_i) \cup \rho(V_2, t_2) \), where \((V_1, V_2; P)\) is a genus one Heegaard splitting and \( t_i \) is a trivial arc in \( V_i \) \((i = 1 \text{ and } 2)\). (An arc \( t \) properly embedded in a solid torus \( V \) is said to be trivial if there is a disk \( D \) in \( V \) with \( t \subset \partial D \) and \( \partial D - t \subset \partial V \).) Set \( W_i = (V_i, t_i) \) \((i = 1 \text{ and } 2)\). We call the triple \((W_1, W_2; P)\) a (1, 1)-splitting of \((M, K)\).

In this paper, we study (1, 1)-splittings by using the distance of the curve complex. To define the distance of a (1, 1)-splitting, we use the twice punctured torus \( \Sigma = P - K \).

For \( i = 1 \text{ or } 2 \), let \( \mathcal{K}(W_i) \) be the maximal subcomplex of \( C(\Sigma) \) consisting of simplexes \( \langle [c_0], [c_1], \ldots, [c_k] \rangle \) such that an essential loop representing \([c_j]\) \( j = 0, 1, \ldots, k \) bounds a disk in \( V_i - t_i \).

**Definition 1.2.** We define the distance of a (1, 1)-splitting \((W_1, W_2; P)\) by

\[
d(W_1, W_2) = d(\mathcal{K}(W_1), \mathcal{K}(W_2)) = \min \{ d([x], [y]) \mid [x] \text{ a vertex in } \mathcal{K}(W_1), [y] \text{ a vertex in } \mathcal{K}(W_2) \}.
\]

In this paper, we give topological characterizations of the knots admitting (1, 1)-splittings of distance \( \leq 2 \) (Theorem 2.2, 2.3 and 2.5). As a corollary, we see that a (1, 1)-knot is hyperbolic if and only if it has a (1, 1)-splitting of distance \( \geq 3 \), except for certain knots (Corollary 2.6). Further we will prove that there are (1, 1)-splittings with arbitrarily high distance (Theorem 2.7).

### 2. Statement of results

Let \( K \) be a knot in a closed 3-manifold \( M \). By \( E(K) \), we mean the exterior of \( K \) in \( M \), i.e., \( E(K) = \text{cl}(M - N(K)) \), where \( N(K) \) is a regular neighborhood of \( K \) in \( M \).

**Definition 2.1.**
1. \( K \) is a trivial knot if \( K \) bounds a disk in \( M \).
2. \( K \) is a core knot if \( K \) is non-trivial and \( M \) admits a genus one Heegaard splitting \((V_i, V_2; P)\) such that \( K \) is isotopic to the core of \( V_i \) for \( i = 1 \text{ or } 2 \).
3. \( K \) is a torus knot if \( K \) is isotopic to a simple loop on a genus one Heegaard surface of \( M \) and is not a core knot.
4. \( K \) is a 2-bridge knot if there is a genus zero Heegaard splitting \((B_1, B_2; P_0)\) of \( S^3 \) such that \((B_1, B_2 \cap K) \) \((i = 1, 2)\) is a 2-string trivial tangle. (Note that a trivial knot in \( S^3 \) is also regarded as a 2-bridge knot.)
5. For a pair \( \alpha \geq 4 \) and \( \beta \) of coprime integers and an element \( r \in \mathbb{Q} \cup \{1/0\} \), \( K(\alpha, \beta; r) \) denotes the knot \( K_2 \) in \( K_1(r) \), where \( K_1 \cup K_2 \) is the 2-bridge link of type \( (\alpha, \beta) \) (cf. Chapter 10 of [22]) and \( K_1(r) \) is the manifold obtained by \( r \)-surgery on \( K_1 \).

By an argument similar to that in Section 1 of [18], we can see that \( K(\alpha, \beta; r) \) is a (1, 1)-knot. These knots form an important family of (1, 1)-knots (see [1], [3]...
and [8]).

For the definition of other standard terms in three-dimensional topology and knot theory, we refer to [10], [12] and [22].

In this paper, we prove the following theorems.

**Theorem 2.2.** Let $K$ be a $(1,1)$-knot in $M$ and $(W_1, W_2; P)$ a $(1,1)$-splitting of $(M, K)$. Then $d(W_1, W_2) = 0$ if and only if $K$ is a trivial knot.

Note that Theorem 1.1 of [9] essentially implies Theorem 2.2.

**Theorem 2.3.** Let $K$ be a $(1,1)$-knot in $M$ and $(W_1, W_2; P)$ a $(1,1)$-splitting of $(M, K)$. Then $d(W_1, W_2) = 1$ if and only if $M$ is $S^2 \times S^1$ and $K$ is a core knot.

**Theorem 2.4.** Let $K$ be a $(1,1)$-knot in $M$ and $(W_1, W_2; P)$ a $(1,1)$-splitting of $(M, K)$. If $d(W_1, W_2) = 2$, then one of the following holds.

1. $M$ is $S^3$ and $K$ is a non-trivial 2-bridge knot.
2. $M$ is a lens space and $K$ is a core knot.
3. $K$ is a non-trivial torus knot.
4. $E(K)$ contains an essential torus.
5. $K$ is non-trivial and $K = K(\alpha, \beta; r)$ for some $\alpha, \beta$ and $r$.

Conversely, if $(M, K)$ satisfies one of (1)–(4), then any $(1,1)$-splitting of $(M, K)$ has distance $= 2$.

In the above theorem, by a *lens space*, we mean a closed 3-manifold which admits a Heegaard splitting of genus one and is homeomorphic to neither $S^3$ nor $S^2 \times S^1$. To prove Theorem 2.4, we need the following results.

- The classification of $(1,1)$-splittings of 2-bridge knots in $S^3$ by T. Kobayashi and O. Saeki [15].
- The classification of $(1,1)$-splittings of core knots in lens paces by C. Hayashi [6].
- The classification of $(1,1)$-splittings of torus knots by K. Morimoto [17].
- A characterization of $(1,1)$-splittings of $(1,1)$-knots whose exteriors contain an essential torus (Proposition 6.1), which generalizes results of C. Hayashi [7] (cf. [18]).

Moreover, we prove the following characterization of $(1,1)$-knots whose exteriors contain an essential torus. A torus properly embedded in a compact orientable 3-manifold is called an *essential torus* if it is incompressible and not $\partial$-parallel in the 3-manifold.

**Theorem 2.5.** The exterior of a $(1,1)$-knot $K$ in $M$ contains an essential torus if and only if $K$ belongs to $K_1, K_2, K_3$ or $K_4$.

In the above theorem, $K_i$ ($i = 1, 2, 3, 4$) denote the families of $(1,1)$-knots defined
as follows.

1. \( K \in \mathcal{K}_1 \) if \( K \) is a knot in lens spaces which is the connected sum of a core knot in a lens space and a non-trivial 2-bridge knot.

2. \( K \in \mathcal{K}_2 \) if \( K \) is constructed as follows. Let \( K_0 \) be a non-trivial torus knot in a closed 3-manifold \( M \), and let \( L = K_1 \cup K_2 \) be a 2-bridge link of type \((\alpha, \beta)\) with \( \alpha \geq 4 \). Let \( \varphi : E(K_1) \rightarrow N(K_0) \) be an orientation-preserving homeomorphism which takes a meridian \( m_2 \subset \partial E(K_2) \) of \( K_2 \) to a regular fiber \( f \subset (\partial N(K_0) \cap P) \) of \( E(K_0) \). Then \( K = \varphi(K_1) \subset N(K_0) \subset M \).

3. \( K \in \mathcal{K}_3 \) if \( K \) is constructed as follows. Let \( K_0 \cup K_1 \cup K_2 \) be the connected sum of two Hopf links illustrated in Fig. 1, and let \( K'_1 \cup K'_2 \) be a non-trivial 2-bridge link. Set \( M = E(K_1 \cup K_2) \cup (\varphi_1, \varphi_2) E(K'_1 \cup K'_2) \), where \( \varphi_i : \partial E(K_i) \rightarrow \partial E(K'_i) \) is an orientation-reversing homeomorphism which takes a preferred longitude \( l_i \subset \partial E(K_i) \) of \( K_i \) to a meridian \( m_i \subset \partial E(K'_i) \) of \( K'_i \) \((i = 1 \text{ and } 2)\). Then \( K = K_0 \subset E(K_1 \cup K_2) \subset M \). It should be noted that \( M \cong S^2 \times S^1 \). This can be seen as follows. For \((i,j) = (1,2) \text{ and } (2,1)\), let \( D_i \) be a disk in \( E(K_j) \) bounded by \( l_i \). Then each of \( \text{cl}(E(K_1 \cup K_2) - N(D_1 \cup D_2)) \) and \( E(K'_1 \cup K'_2) \cup N(D_1 \cup D_2) \) is homeomorphic to \( S^2 \times [0,1] \).

4. \( K \in \mathcal{K}_4 \) if \( K \) is constructed as follows. Let \( K_0 \) be \( K(4,1;0) \) and \( K_1 \) a meridian of \( K_0 \) (see Fig. 2). Let \( l_1 \subset \partial E(K) \) be a longitude of \( K_1 \) which bounds a disk in \( E(K_1) \) intersecting \( K_0 \) transversely in a single point. Let \( K_2 \) be a non-trivial 2-bridge knot and \( \varphi : \partial E(K_1) \rightarrow \partial E(K_2) \) an orientation-reversing homeomorphism which takes \( l_1 \) to a meridian of \( K_2 \). Set \( M = E(K_1) \cup \varphi E(K_2) \). Then \( K = K_0 \subset E(K_1) \subset M \). It should be noted that \( M \cong S^2 \times S^1 \). This can be seen by using the fact that the union of \( E(K_2) \) and a regular neighbourhood of a disk in \( E(K_1) \) bounded by \( l_1 \) is a 3-ball.

By using Thurston's hyperbolization theorem of Haken manifolds (see for example [13]), we can obtain the following corollary.

**Corollary 2.6.** Let \( K \) be a \((1,1)\)-knot in \( M \). Suppose that \((M, K)\) is not equivalent to \( K(\alpha, \beta; r) \) for any \( \alpha, \beta \) and \( r \), and that the bridge index of \( K \) is at least three if \( M \cong S^3 \). Then \( K \) is a hyperbolic knot if and only if it has a \((1,1)\)-splitting.
with distance \( \geq 3 \).

In the last section, we construct \((1,1)\)-splittings with arbitrarily high distance.

**Theorem 2.7.** Let \( M \) be a closed 3-manifold which admits a genus one Heegaard splitting. Then for any positive integer \( n \), there is a \((1,1)\)-knot in \( M \) which has a \((1,1)\)-splitting with distance \( > n \).

3. The structure of \( \mathcal{K}(W) \)

In this section, we describe the structure of the simplicial complex \( \mathcal{K}(W) \). Throughout this section, \( W = (V,t) \) denotes a pair of a solid torus \( V \) and a trivial arc \( t \) properly embedded in \( V \), and \( \Sigma \) denotes the twice punctured torus \( \partial V - t \). The two punctures of \( \Sigma \) are denoted by \( p_1 \) and \( p_2 \). Two subspaces \( X \) and \( Y \) in \( W \) are said to be **pairwise isotopic**, if there is an ambient isotopy \( \{h_s\}_{0 \leq s \leq 1} \) of \( V \) such that \( h_0 = \text{id} \), \( h_s(t) = t \) and \( h_1(X) = Y \).

**Definition 3.1.** An essential loop in \( \Sigma \) is called an \( \varepsilon \)-loop (an \( \iota \)-loop resp.) if it is essential (inessential resp.) in \( \partial V \).

**Definition 3.2.** Let \( D \) be a properly embedded disk in \( V \).

1. \( D \) is called an \( \iota \)-disk in \( W \) if \( D \cap t = \emptyset \) and \( \partial D \) is an \( \iota \)-loop on \( \Sigma \).
2. \( D \) is called an \( \varepsilon_0 \)-disk in \( W \) if \( D \cap t = \emptyset \) and \( \partial D \) is an \( \varepsilon \)-loop on \( \Sigma \).
3. \( D \) is called an \( \varepsilon_1 \)-disk in \( W \) if \( D \cap t = \{1 \text{ point}\} \) and \( \partial D \) is an \( \varepsilon \)-loop on \( \Sigma \).

**Lemma 3.3.** Let \( D_0 \) be an \( \varepsilon_0 \)-disk in \( W \) with \( \alpha = \partial D_0 \), and let \( \beta \) be an essential loop in \( \Sigma \) disjoint from \( \alpha \). Then precisely one of the following conditions holds.

1. \( \beta \) is isotopic to \( \alpha \) in \( \Sigma \).
2. \( \beta \) bounds an \( \iota \)-disk in \( W \).
3. \( \beta \) bounds an \( \varepsilon_1 \)-disk in \( W \).

Proof. Let \( B \) be the 3-ball obtained by cutting \( V \) along \( D_0 \), and let \( D_0' \) and \( D_0'' \) be the copies of \( D_0 \) in \( \partial B \).

**Case 1.** Suppose that \( \beta \) does not separate \( D_0' \) and \( D_0'' \) in \( \partial B \).

Then \( \beta \) does not separate \( p_1 \) and \( p_2 \) in \( \partial B \), because \( \beta \) is essential in \( \Sigma \). Let \( t' \) be a properly embedded arc in \( B \) with \( \partial t' = \{p_1, p_2\} \) which is parallel to an arc in \( \partial B - \beta \) joining \( p_1 \) to \( p_2 \). Then \( \beta \) bounds a separating disk \( D_\beta \) in \( B \) disjoint from \( t' \). Since \( t' \) is isotopic to \( t \) in \( B \) relative \( D_0' \cup D_0'' \), the arc \( t' \) in \( V \) is isotopic to \( t \) in \( V \) relative \( \{p_1, p_2\} \). Moreover by the hypothesis of Case 1, \( D_\beta \) cuts \( (V_1,t) \) into \( (V_1',t) \) and \( (V_2,\emptyset) \), where \( V_1 \) is a 3-ball and \( V_2 \) is a solid torus. Hence the condition (2) holds.

**Case 2.** Suppose that \( \beta \) separates \( D_0' \) and \( D_0'' \) in \( \partial B \).

Then we can see, by an argument similar to the above, that the condition (3)
or (1) holds according as \( \beta \) separates \( \{ p_1, p_2 \} \) in \( \partial B \) or not.

This completes the proof of Lemma 3.3.

Lemma 3.4. Any two \( \varepsilon_0 \)-disks in \( W \) are pairwise isotopic.

Proof. Let \( D \) and \( D' \) be \( \varepsilon_0 \)-disks in \( W \). If \( D \cap D' = \emptyset \), then we can see that \( D \cup D' \) bounds a product region disjoint from \( r \) by an argument similar to that of Lemma 3.3. Hence we may assume that \( D \) and \( D' \) intersect transversely, \( |D \cap D'| \) is minimized up to pairwise isotopy in \( W \) and that \( |D \cap D'| > 0 \), where \( |\cdot| \) is the number of connected components. By a standard innermost disk argument, we can see that \( D \cap D' \) has no loop components. Let \( \gamma \) be a component of \( D \cap D' \) which is outermost in \( D' \) and \( \delta'_1 \) the outermost disk in \( D' \) with \( \gamma \subset \partial \delta'_1 \), that is, the interior of \( \delta'_1 \) is disjoint from \( D \). The arc \( \gamma \) also cuts \( D \) into two disks \( \delta_1 \) and \( \delta_2 \). Then each of \( \delta_1 \cup \delta'_1 \) and \( \delta_2 \cup \delta'_1 \) is a properly embedded disk in \( V \) disjoint from \( r \). If either \( \partial(\delta_1 \cup \delta'_1) \) or \( \partial(\delta_2 \cup \delta'_1) \) is inessential in \( \partial(V - t) \), then we can decrease \( |D \cap D'| \) by a pairwise isotopy of \( D \) in \( W \), a contradiction. So we may assume that \( \delta_1 \cup \delta'_1 \) and \( \delta_2 \cup \delta'_1 \) are \( \varepsilon_0 \)-disks or \( \iota \)-disks in \( W \).

Claim. At least one of \( \delta_1 \cup \delta'_1 \) and \( \delta_2 \cup \delta'_1 \) is an \( \iota \)-disk in \( W \).

Proof. Suppose that \( \delta_1 \cup \delta'_1 \) is a \( \varepsilon_0 \)-disk in \( W \) to show that \( \delta_2 \cup \delta'_1 \) is an \( \iota \)-disk.

Let \( B \) be the 3-ball obtained from \( V \) by cutting along \( D \), and let \( D_+ \) and \( D_- \) be the copies of \( D \) in \( \partial B \). We denote the image of \( \delta'_1 \) in \( B \) by the same symbol. Then we may assume \( \delta'_1 \cap D_+ = \emptyset \) and \( \delta'_1 \cap D_- = \gamma \). By cutting \( B \) along \( \delta'_1 \), we obtain 3-balls \( B_1 \) and \( B_2 \) with \( D_+ \subset \partial B_1 \), \( (\delta_1 \cup \delta'_1) \subset \partial B_1 \) and \( (\delta_2 \cup \delta'_1) \subset \partial B_2 \). Since \( D \) and \( \delta'_1 \) are disjoint from \( r \) in \( V \), precisely one of \( B_1 \) and \( B_2 \) contains \( r \). If \( r \subset B_1 \), then \( \partial(\delta_2 \cup \delta'_1) \) is inessential in \( \partial(V - t) \), a contradiction. Hence \( r \subset B_2 \), and \( \delta_2 \cup \delta'_1 \) is an \( \iota \)-disk in \( W \).

Let \( B_1, D_+, D_-, B_1 \) and \( B_2 \) be as above. Put \( \delta'_2 = \text{cl}(D' - \delta'_1) \), and let \( A \) be the annulus defined by \( A = \partial B_1 \cap (\partial B - \text{int}(D_+ \cup D_-)) \). Put \( \alpha = \partial D' \cap \partial \delta'_2 \), and let \( \partial \gamma \ni p_1, p_2, \ldots, p_n \subset \partial A \) be the components of \( \partial D' \cap \alpha \) sitting on \( \alpha \) in this order. Then by the minimality of \( |D \cap D'| \), we may assume that \( \alpha \cap \partial \delta'_2 \) consists of essential arcs in the annulus \( A \). Let \( \alpha_i \) be the subarc of \( \alpha \) joining \( p_i \) to \( p_{i+1} \) in \( \alpha \), and let \( p^+_i, p^-_i \), respectively the copies of \( p_i \) in \( \partial D_+ \) and \( \partial D_- \) \( (i = 1, 2, \ldots, n - 1) \). Then \( \alpha_1 \cap D_+ = p^+_1 \) and \( \alpha_1 \cap D_- = p^-_1 \), because \( \alpha_1 \) is essential in \( A \). Inductively, we obtain \( \alpha_i \cap D_+ = p^+_i \) and \( \alpha_i \cap D_- = p^-_{i+1} \) \( (i = 1, 2, \ldots, n - 1) \). In particular, \( \alpha_{n-1} \cap D_+ = p^+_n \) and \( \alpha_{n-1} \cap D_- = p^-_n \). This means that \( D' \) does not intersect \( D \) transversely in \( p_n \), a contradiction. Hence the interior of \( A \) is disjoint from \( \partial \delta'_2 \), and there is an \( \varepsilon_0 \)-disk obtained by moving \( D_+ \) so that it is disjoint from \( D' \). This means \( D' \) is isotopic to \( D \).

\( \square \)
Lemma 3.5. Let $[\alpha]$ be the vertex of $\mathcal{K}(W)$ represented by the boundary of an $\varepsilon_0$-disk, and let $[\beta]$ be an arbitrary vertex of $\mathcal{K}(W)$ different from $[\alpha]$. Then $[\beta]$ is represented by an $\iota$-loop disjoint from an $\varepsilon$-loop representing $[\alpha]$.

Proof. If $[\beta]$ is represented by an $\varepsilon$-loop, then we have $[\alpha] = [\beta]$ by Lemma 3.4, a contradiction. So $[\beta]$ is represented by an $\iota$-loop, say $\beta$. Let $D_\beta$ be a disk in $V - t$ bounded by $\beta$. Since $\beta$ is inessential in $V$, there is an essential disk $D$ in $V$ disjoint from $D_\beta$ (and hence disjoint from $t$). By Lemma 3.4, $\partial D$ represents $[\alpha]$ and hence we obtain the desired result.

Lemma 3.6. Any two mutually disjoint $\iota$-disks in $W$ are pairwise isotopic.

Proof. Let $D$ and $D'$ be mutually disjoint $\iota$-disks in $W$ and put $\beta = \partial D$ and $\beta' = \partial D'$. Then $D$ cuts $(V_1,t)$ into $(V_1,t)$ and $(V_2,0)$, where $V_1$ is a 3-ball and $V_2$ is a solid torus. If necessary, by exchanging the names $D$ and $D'$ of disks, we may assume that $D'$ is contained in $V_1$ and $\beta'$ is an inessential loop in $\partial V_1 - t$, because $D'$ is an $\iota$-disk and is disjoint from $D$. If $\beta'$ bounds a disk in $\partial V_1$ disjoint from the copy of $D$ in $\partial V_1$, then $\beta'$ is inessential in $\partial V - t$, a contradiction. Hence $\beta'$ separates the copy of $D$ from $\partial t$ in $\partial V_1$, and this implies $D$ and $D'$ are pairwise isotopic.

Let $\alpha$ be an $\varepsilon$-loop which bounds an $\varepsilon_0$-disk, say $D_\alpha$. We fix a properly embedded arc, say $t_0$, in $\partial V$ such that $\partial t_0 = \partial t$, $t_0 \cap \alpha = \emptyset$, and $t \cup t_0$ bounds a disk in $V$. Let $B$ be the 3-ball obtained by cutting $V$ along $D_\alpha$, and let $D'_\alpha$ and $D''_\alpha$ be the copies of $D_\alpha$ in $\partial B$. Set $\mathcal{P} = \partial t \cup \{\text{the centers of } D'_\alpha \text{ and } D''_\alpha\}$. Then $(\partial B, \mathcal{P})$ is identified with $(\mathbb{R}^2, \mathbb{Z}^2)/\Gamma$, where $\Gamma$ is the group of isometries of $\mathbb{R}^2$ generated by $\pi$-rotations about the points of the integral lattice $\mathbb{Z}^2$. Here $t_0$ is identified with a line in $\mathbb{R}^2$ of slope $1/0$, i.e., a lift of $t_0$ joins $(0,0)$ to $(0,1)$ in $\mathbb{R}^2$.

Let $\mathcal{A}$ be the set of the vertices of $\mathcal{K}(W)$ different from $[\alpha]$, where $[\alpha]$ is the vertex of $\mathcal{K}(W)$ represented by $\alpha$. In the following, we define a map $\varphi: \mathcal{A} \to \mathbb{Q} \cup \{1/0\}$. Let $[\beta]$ be an element of $\mathcal{A}$. Then by Lemma 3.5, $[\beta]$ is represented by an $\iota$-loop, say $\beta$, which is disjoint from $\alpha$. Let $t_\beta$ be an arc in $\partial V - \beta$ joining distinct components of $\partial t$. Note that $t_\beta$ is unique up to isotopy relative to the endpoints. Let $\hat{t}_\beta: [0,1] \to \mathbb{R}^2$ be a lift of $t_\beta: [0,1] \to (\partial B, \mathcal{P})$. Then $\hat{t}_\beta(1) - \hat{t}_\beta(0)$ is an integral vector, say $(p,q)$, in $\mathbb{R}^2$.

Lemma 3.7. Let $[\beta]$ and $(p,q)$ be as above. Then the rational number $q/p$ does not depend on the choice of a representative of $[\beta]$, and hence the correspondence $\beta \mapsto q/p$ induces a well-defined map $\varphi: \mathcal{A} \to \mathbb{Q} \cup \{1/0\}$. Moreover $\varphi$ is injective and the image is equal to $\{q/p \in \mathbb{Q} \cup \{1/0\} \mid (p,q) \equiv (0,1) \pmod{2}\}$. 
Proof. Let \( \beta' \) be another representative disjoint from \( \alpha \) of \([\beta]\). Then there is a homotopy in \( \Sigma \) between \( \beta \) and \( \beta' \). Since \( \alpha \) is an essential loop in \( \Sigma \) and is homotopic to neither \( \beta \) nor \( \beta' \), we can modify the homotopy so that it is disjoint from \( \alpha \). Hence \( \beta \) and \( \beta' \) are homotopic in \( \Sigma - \alpha \) and therefore in the four times punctured 2-sphere \( \partial B - \mathcal{P} \). This implies that \( \varphi \) is well-defined and injective, because it is well known that the correspondence \( \beta \mapsto q/p \) induces a well-defined injective map from the set of the isotopy classes of essential loops in \( \partial B - \mathcal{P} \) to \( \mathbb{Q} \cup \{1/0\} \) (cf. Section 2 of [5]). Moreover, since an \( \iota \)-loop representing \([\beta]\) does not separate \( \partial \iota \) in \( \partial \mathcal{V} \), we see \((p,q) \equiv (0,1) \pmod{2}\). On the other hand, it is easy to see that for any \( q/p \in \mathbb{Q} \cup \{1/0\} \) with \((p,q) \equiv (0,1) \pmod{2}\), there is a vertex \([\beta]\in \mathcal{A}\) with \( \varphi([\beta]) = q/p \). Hence we obtain the desired result. \(\square\)

**Proposition 3.8.** Let \([\alpha]\) be the vertex of \(\mathcal{K}(\mathcal{W})\) represented by the boundary of an \(\varepsilon_0\)-disk of \(\mathcal{W}\), and let \(\mathcal{A}\) be the countably infinite set as above. Then \(\mathcal{K}(\mathcal{W})\) is isomorphic to the join \(\{[\alpha]\} \ast \mathcal{A}\).

Proof. By Lemma 3.4, we see that \([\alpha]\) is unique. Lemma 3.5 indicates that for any vertex \([\beta]\) of \(\mathcal{A}\), there is an edge joining \([\beta]\) to \([\alpha]\). On the other hand, by Lemma 3.6, there are no edges of \(C(\Sigma)\) joining distinct vertices of \(\mathcal{A}\). \(\square\)

4. \((1,1)\)-splittings of distance = 0

**Lemma 4.1.** Let \(K\) be a \((1,1)\)-knot in \(M\) and \((W_1, W_2; P)\) a \((1,1)\)-splitting of \((M, K)\). Then \(K\) is a trivial knot if and only if there are an \(\iota\)-disk \(D_i\) in \(W_i\) with \(\partial D_1 = \partial D_2\) (\(i = 1\) and \(2\)).

Proof. We first prove the “only if part”. Suppose that \(K\) is trivial. Let \(D\) be a disk in \(M\) with \(\partial D = K\). Then by Theorem 1.1 of [9], we can isotope \(D\) so that \(D \cap P\) separates \(D\) into two disks. Set \(D_i = \partial N(D) \cap V_i\) (\(i = 1\) and \(2\)). Then we see that \(D_i\) is an \(\iota\)-disk and \(\partial D_1 = \partial D_2\) (\(i = 1\) and \(2\)).

We next prove the “if part”. Suppose that there are an \(\iota\)-disk \(D_i\) in \(W_i\) (\(i = 1\) and \(2\)). Then \(D_1 \cup D_2\) forms a 2-sphere which cuts \((M, K)\) into \((M - \text{int} B^3, \emptyset)\) and \((B^3, 1\text{-bridge knot})\) and hence \(K\) is a trivial knot. \(\square\)

**Lemma 4.2.** Let \(K\) be a \((1,1)\)-knot in \(S^2 \times S^1\) and \((W_1, W_2; P)\) a \((1,1)\)-splitting of \((S^2 \times S^1, K)\). Then \(K\) is a trivial knot if and only if there are an \(\varepsilon_0\)-disk \(D_1\) in \(W_1\) and an \(\varepsilon_0\)-disk \(D_2\) in \(W_2\) with \(\partial D_1 = \partial D_2\).

Proof. We first prove the “if part”. Suppose that the latter condition in Lemma 4.2 holds. Then there are \(\iota\)-disks \(D_i^1\) and \(D_i^2\) in \(W_i\) and \(W_2\), respectively, with \(\partial D_i^1 \cap \partial D_i^2 = \emptyset\) (\(i = 1, 2\)) and \(\partial D_i^1 = \partial D_i^2\). Hence by Lemma 4.1, \(K\) is a trivial knot.

Suppose conversely that \(K\) is a trivial knot in \(S^2 \times S^1\). By Lemma 4.1, there are
an $\iota$-disk $\delta_i$ in $W_i$ with $\partial \delta_1 = \partial \delta_2$ ($i = 1$ and 2). Then there are $\epsilon_0$-disks in each of $W_1$ and $W_2$ such that they are disjoint from $\delta_1 \cup \delta_2$ and they share their boundaries since the manifold is $S^2 \times S^1$. Hence we see that the latter condition holds.

Proof of Theorem 2.2. Suppose that $K$ is a trivial knot in $M$. Then by Lemma 4.1, we have $d(W_1, W_2) = 0$.

Conversely, let $K$ be a $(1, 1)$-knot in $M$ and $(W_1, W_2; P)$ a $(1, 1)$-splitting of $(M, K)$ with $d(W_1, W_2) = 0$. Then there is an essential loop $x$ in $\Sigma = P - K$ which bounds a disk in $V_i - t_i$ for each $i = 1$ and 2.

If $x$ is an $\epsilon_0$-loop, then $(W_1, W_2; P)$ satisfies the condition of Lemma 4.2. Hence $M$ is $S^2 \times S^1$ and $K$ is a trivial knot.

If $x$ is an $\iota$-loop, then $(W_1, W_2; P)$ satisfies the condition of Lemma 4.1, that is, $K$ is a trivial knot in $M$.

We have completed the proof of Theorem 2.2.

5. $(1, 1)$-splittings of distance $= 1$

Proposition 5.1. Let $K$ be a $(1, 1)$-knot in $S^2 \times S^1$ and $(W_1, W_2; P)$ a $(1, 1)$-splitting of $(S^2 \times S^1, K)$. Then $K$ is a core knot if and only if there are an $\epsilon_0$-disk $D_i$ in $W_i$ and an $\epsilon_1$-disk $D_j$ in $W_j$ with $\partial D_i = \partial D_j$ for $(i, j) = (1, 2)$ or $(2, 1)$.

Proof. The “if part” follows from the light bulb theorem (cf. Chapter 9, Section E, 4 Exercise of [22]).

To prove the “only if part”, suppose that $K$ is a core knot in $S^2 \times S^1$. Then there is an essential 2-sphere $S$ which intersects $K$ in one point. Put $S_i = S \cap V_i$ ($i = 1$ and 2). We may assume that each component of $S_1$ is either an $\epsilon_0$-disk, an $\epsilon_1$-disk or an $\iota$-disk in $W_i = (V_i, t_i)$. Note that $|S_1| > 0$ and that $S_1$ contains at most one $\epsilon_1$-disk component. Let $D$ be an $\epsilon_0$-disk in $W_2$ such that $D$ intersects $S_2$ transversely. We choose $S$ and $D$ so that each component of $S_1$ is either an $\epsilon_0$-disk, an $\epsilon_1$-disk or an $\iota$-disk in $W_1$, and the pair $(|S_1|, |S_2 \cap D|)$ is minimized with respect to the lexicographic order.

If $|S_1| = 1$, then $S \cap P$ is an $\iota$-loop because $S$ is an essential 2-sphere in $S^2 \times S^1$. Hence the assertion obviously holds. So we may assume $|S_1| > 1$.

Claim 1. $S_2 \cap D \neq \emptyset$.

Proof. Suppose that $S_2$ is disjoint from $D$. Let $B$ be the 3-ball obtained by cutting $V_2$ along $D$. Then there is a disk $E$ on $\partial B$ with $E \cap S_2 = \partial E$ and $|E \cap K| \leq 1$. Let $E'$ be the disk obtained from $E$ by pushing the interior of $E$ into the interior of $B$. Then $\partial E'$ cuts $S$ into two disks $Q_1$ and $Q_2$. Precisely one of them, say $Q_1$, is a component of $S_1$.

Suppose that $|E' \cap K| = 0$. If $|Q_1 \cap K| = 1$, then $Q_1 \cup E'$ is a 2-sphere which inter-
sects $K$ in one point. Hence the disks $Q_1$ and $E'$ satisfy the desired condition. So we may assume that $|Q_1 \cap K| = 0$ and hence $|Q_2 \cap K| = 1$. Let $S'$ be the 2-sphere obtained from $Q_2 \cup E'$ by pushing $\partial E'$ into the interior of $V_2$ slightly. Then each component of $S'_1 := S' \cap V_1$ is either an $\epsilon_0$-disk, an $\epsilon_1$-disk or an $\iota$-disk in $W_1$, and $|S'_1| < |S_1|$, a contradiction.

Suppose that $|E' \cap K| = 1$. If $|Q_1 \cap K| = 0$, then $Q_1 \cup E'$ is a 2-sphere which intersects $K$ in one point, and hence the disks $Q_1$ and $E'$ satisfy the desired condition. So we may assume that $|Q_1 \cap K| = 1$ and hence $|Q_2 \cap K| = 0$. Let $S'$ be the 2-sphere obtained from $Q_2 \cup E'$ by pushing $\partial E'$ into $V_2$ slightly. Then each component of $S'_1 := S' \cap V_1$ is either an $\epsilon_0$-disk, an $\epsilon_1$-disk or an $\iota$-disk in $W_1$, and $|S'_1| < |S_1|$, a contradiction.

\textbf{Claim 2.} $S_2 \cap D$ has no loop components.

Proof. Suppose that $S_2 \cap D$ has a loop component. Let $\sigma$ be a loop component of $S_2 \cap D$ which is innermost in $D$ and $D_\sigma$ the innermost disk with $\sigma = \partial D_\sigma$, that is, the interior of $D_\sigma$ is disjoint from $S_2$. Then $\sigma$ cuts $S$ into two disks $E_1$ and $E_2$. We can assume that $|E_1 \cap K| = 1$. Since $D_\sigma$ is disjoint from $K$, $S' = E_1 \cup D_\sigma$ is a 2-sphere which intersects $K$ in one point. Put $S'_i = S' \cap V_i$ ($i = 1$ and 2). Note that $S'_1$ is either an $\epsilon_0$-disk, an $\epsilon_1$-disk or an $\iota$-disk in $W_1$. If $\sigma$ is essential in $S_2$, then $|S'_1| < |S_1|$, a contradiction. If $\sigma$ is inessential in $S_2$, then $|S'_1| = |S_1|$. In this case, by isotoping $S'$ so that $D_\sigma$ is disjoint from $D$, we see that $|S'_1 \cap D| < |S_2 \cap D|$, a contradiction.

By Claim 1 and Claim 2, there is an arc component $\gamma$ of $S_2 \cap D$ which is outermost in $D$. Let $D_\gamma \subset D$ be the outermost disk with $\gamma \subset \partial D_\gamma$. Put $\gamma' = \text{cl}(\partial D_\gamma - \gamma)$. Let $F$ be the component of the surface obtained by cutting $\partial V_1$ along $\partial S_1$ such that $\gamma' \subset F$. Let $S^{(1)}_1$ be a 2-sphere obtained by isotoping $S$ along $D_\gamma$ near the arc $\gamma$, and put $S^{(1)}_i = S^{(1)} \cap V_i$ ($i = 1$ and 2).

\textbf{Claim 3.} The arc $\gamma'$ is essential in $F$.

Proof. Suppose that $\gamma'$ is inessential in $F$. Then we obtain an annulus component $A$ in $S^{(1)}_1$ such that one of the components of $\partial A$ bounds a disk $E$ in $\partial V_1$. Note that $|E \cap K| \leq 2$ and $\partial E$ cuts $S$ into two disks $R_1$ and $R_2$. Since $S$ intersects $K$ transversely in one point, we may assume that $|R_1 \cap K| = 1$ and $|R_2 \cap K| = 0$.

Suppose that $|E \cap K| = 0$. If $A \subset R_1$, let $S'$ be a 2-sphere obtained from $R_1 \cup E$ by pushing $E$ into the interior of $V_1$; otherwise, let $S'$ be a 2-sphere obtained from $R_1 \cup E$ by pushing the interior of $E$ into the interior of $V_2$. Then we see that each component of $S'_1$ is either an $\epsilon_0$-disk, an $\epsilon_1$-disk or an $\iota$-disk in $W_1$, and that $(|S'_1|, |S_2 \cap D|) < (|S_1|, |S_2 \cap D|)$, a contradiction.

Suppose that $|E \cap K| = 1$. If $A \subset R_2$, let $S'$ be a 2-sphere obtained from $R_2 \cup E$ by
pushing $E$ into the interior of $V_1$; otherwise, let $S'$ be a 2-sphere obtained from $R_2 \cup E$ by pushing the interior of $E$ into the interior of $V_2$. Then we see that each component of $S'_1$ is either an $\varepsilon_0$-disk, an $\varepsilon_1$-disk or an $\iota$-disk in $W_1$, and that $(|S'_1|, |S'_2 \cap D|) < (|S_1|, |S_2 \cap D|)$, a contradiction.

Suppose that $|E \cap K| = 2$. If $\gamma'$ joins an $\iota$-disk to itself, then $E' := \text{cl}(F - E)$ is a disk bounded by a component of $\partial A$. Since $E'$ is disjoint from $K$, by an argument similar to the case of $|E \cap K| = 0$, we obtain a contradiction by using the disk $E'$ instead of $E$. So we may assume that $\gamma'$ joins an $\varepsilon_0$-disk to itself. Then there is an $\varepsilon_0$-disk disjoint from $\partial E$. By Lemma 3.3, $\partial E$ bounds an $\iota$-disk. Hence by an argument similar to the case of $|E \cap K| = 0$, we obtain a contradiction by using the $\iota$-disk instead of $E$.

**CLAIM 4.** $S_1$ has no $\varepsilon_1$-disk components.

**Proof.** Suppose that $S_1$ has an $\varepsilon_1$-disk component. Then $S_1$ has no $\iota$-disk components. Thus $S_1$ has $\varepsilon_0$-disk components, because $|S_1| > 1$. Hence by Claim 3, $\gamma'$ joins distinct components of $S_1$.

**CASE 1.** The arc $\gamma'$ joins distinct $\varepsilon_0$-disks.

Let $\delta$ be the disk component of $S'_1$ obtained from these disks. Then we can push $\delta$ out of $V_1$ fixing $t_1$. After this operation, we see that each component of $S'_1$ is either an $\varepsilon_0$-disk or an $\varepsilon_1$-disk in $W_1$, and that $|S'_1| < |S_1|$, a contradiction.

**CASE 2.** The arc $\gamma'$ joins an $\varepsilon_0$-disk to an $\varepsilon_1$-disk.

Then $S'_1$ has the disk component $\delta'$ from these disks. Note that $\delta'$ cuts $(V_1, t_1)$ into $(V'_1, t'_1)$ and $(V''_1, t''_1)$. where $V'_1$ is a 3-ball, $t'_1$ is a trivial arc in $V'_1$. $V''_1$ is a solid torus and $t''_1$ is a trivial arc in $V''_1$. So we can push $\delta'$ out of $V_1$ through $(V'_1, t'_1)$. After this operation, each component of $S'_1$ is either an $\varepsilon_0$-disk or an $\varepsilon_1$-disk in $W_1$, and we have $|S'_1| < |S_1|$, a contradiction.

**CLAIM 5.** $S_1$ has no $\varepsilon_0$-disk components.

**Proof.** Suppose that $S_1$ has an $\varepsilon_0$-disk component. Note that $S_1$ may have $\iota$-disk components, because $S_1$ has no $\varepsilon_1$-disk components by Claim 4. Since $\gamma'$ is essential in $F$ by Claim 3, we have the following cases.

**CASE 1.** The arc $\gamma'$ joins distinct $\varepsilon_0$-disks, or joins distinct $\iota$-disks.

By an argument similar to Case 1 in the proof of Claim 4, we obtain a contradiction.

**CASE 2.** The arc $\gamma'$ joins an $\varepsilon_0$-disk to an $\iota$-disk.

Then $S'_1$ is either an $\varepsilon_0$-disk, an $\varepsilon_1$-disk or an $\iota$-disk in $W_1$, and $|S'_1| < |S_1|$, a contradiction.

**CASE 3.** The arc $\gamma'$ joins an $\varepsilon_0$-disk to itself.

By Claim 3, $\gamma'$ must be essential in $F$. Hence $S_1$ must consist of an $\varepsilon_0$-disk and
ι-disks, and we obtain a Möbius band in \( S_1^{(1)} \), a contradiction.

**CASE 4.** The arc \( \gamma' \) joins an ι-disk to itself.

Let \( \delta \) be the ι-disk component of \( S_1 \) with \( \partial \gamma' \subset \partial \delta \), and let \( \gamma_1 \) and \( \gamma_2 \) be arcs such that \( \partial \delta = \gamma_1 \cup \gamma_2 \) and \( \partial \gamma_1 = \partial \gamma_2 = \partial \gamma' \). Since \( S_1 \) has \( \varepsilon_0 \)-disk components, by Claim 3, \( \gamma' \cup \gamma_1 \) bounds an \( \varepsilon_0 \)-disk, say \( E' \), whose interior is disjoint from \( S \). Hence by an argument similar to Claim 3, we have a contradiction by using the disk \( E' \).

By Claim 4 and Claim 5, \( S_1 \) consists of ι-disks, because \( |S_1| > 1 \). But this implies that \( S \) is inessential in \( S^2 \times S^1 \), a contradiction.

This completes the proof of Proposition 5.1.

**Proof of Theorem 2.3.** Suppose that \( K \) is a core knot in \( S^2 \times S^1 \). By Proposition 5.1, we may assume that there are an \( \varepsilon_0 \)-disk \( D_1 \) in \( W_1 \) and an \( \varepsilon_1 \)-disk \( D_2 \) in \( W_2 \) with \( \partial D_1 = \partial D_2 \). Then there is an \( \varepsilon_0 \)-disk \( D'_2 \) in \( W_2 \) which is disjoint from \( D_2 \). Hence we have \( d(W_1, W_2) = 1 \) since Theorem 2.2 implies \( d(W_1, W_2) \neq 0 \) for \((1,1)\)-splittings of the core knot in \( S^2 \times S^1 \).

Conversely, we suppose \( d(W_1, W_2) = 1 \), that is, there are mutually disjoint essential loops \( x \) and \( y \) in \( \Sigma = P - K \) which bound disks in \( V_1 - t_1 \) and \( V_2 - t_2 \), respectively. Suppose that either \( x \) or \( y \), say \( y \), is an ι-loop. If \( x \) bounds an \( \varepsilon_0 \)-disk, then \( y \) bounds an ι-disk in \( W_1 \) by Lemma 3.3. (Otherwise, \( y \) is pairwise isotopic to \( x \).) Hence \( K \) is a trivial knot, a contradiction. So we may suppose that \( x \) (resp. \( y \)) bounds an \( \varepsilon_0 \)-disk in \( W_1 \) (resp. \( W_2 \)). Then \( x \) bounds an \( \varepsilon_1 \)-disk in \( W_2 \) by Lemma 3.3. Hence \( K \) is a core knot in \( S^2 \times S^1 \) by Proposition 5.1.

We have completed the proof of Theorem 2.3.

**6. (1, 1)-knots whose exteriors contain essential tori**

In this section, we study (1, 1)-knots whose exteriors contain an essential torus and prove Theorem 2.5 and the following Proposition 6.1.

**Proposition 6.1.** Let \( K \) be a (1, 1)-knot in \( M \) whose exterior contains an essential torus. Then every (1, 1)-splitting \((W_1, W_2; P)\) of \((M, K)\) satisfies one of the following conditions.

\((\#_a)\) There are \( \varepsilon_0 \)-disk \( D_i \) in \( W_i \) and an \( \varepsilon_1 \)-disk \( D_j \) in \( W_j \) such that \( \partial D_i \cap \partial D_j = \emptyset \) for \((i, j) = (1, 2) \) or \((2, 1)\).

\((\#_b)\) There is an annulus \( Z \subset P \) which is incompressible in both \( V_1 \) and \( V_2 \), and there is an ι-disk \( D_i \) in \( W_i \) with \( \partial D_i \subset Z \) for each \( i = 1 \) and \( 2 \).

\((\#_c)\) There are an \( \varepsilon_1 \)-disk \( D_1 \) in \( W_1 \) and an \( \varepsilon_1 \)-disk \( D_2 \) in \( W_2 \) with \( \partial D_1 = \partial D_2 \).

Before proving Theorem 2.5 and Proposition 6.1, we present lemmas which describe topological consequences of the conclusions in Proposition 6.1.
Lemma 6.2 ([7] Lemma 2.1). Let $K$ be a non-trivial $(1,1)$-knot in $M$ with a $(1,1)$-splitting $(W_1, W_2; P)$ satisfying the condition $(\#_a)$ of Proposition 6.1. Then one of the following holds.

1. $K$ is a 2-bridge knot.
2. $K$ is a core knot in a lens space.
3. $K$ belongs to $K_1$.

Remark 6.3. Though this lemma is proved under the assumption that $M \not\cong S^2 \times S^1$ in [7], we can easily see that the same conclusion holds even if $M \cong S^2 \times S^1$. In fact, we can show by using the light bulb theorem that $K$ is a core knot in this case.

Lemma 6.4. Let $K$ be a non-trivial $(1,1)$-knot in $M$ with a $(1,1)$-splitting $(W_1, W_2; P)$ satisfying the condition $(\#_b)$ of Proposition 6.1. Then one of the following holds.

1. $K$ is a core knot or a torus knot.
2. $K = K(\alpha, \beta; r)$ for some $\alpha$, $\beta$ and $r$.
3. $K$ belongs to $K_2$.

Proof. Let $Z$ be an annulus which satisfies the condition $(\#_b)$ of Proposition 6.1. For each $i = 1$ and 2, since $Z$ is incompressible in $V_i$, $\partial D_i$ bounds a disk $D_i^j$ in $Z$. Let $A_i$ be an annulus in $V_i$ obtained from $Z_i := \text{cl}(Z - D_i^j \cup D_i)$ by pushing the interior of $Z_i$ into the interior of $V_i$. For each $i = 1$ and 2, let $(V_{i1}, t_i)$ and $(V_{i2}, t_i)$ be the pair obtained from $(V_i, t_i)$ by cutting along $A_i$, where each of $V_{i1}$ and $V_{i2}$ is a solid torus and $t_i$ is a trivial arc in $V_{i2}$. Then we see that $V_{11} \cup V_{12}$ is either a solid torus or the exterior of a torus knot. On the other hand, $(V_{i2}, t_i)$ is identified with $(\text{cl}(B^3 - \tau_1), \tau_2)$, where $(B^3, \tau_1 \cup \tau_2)$ is a 2-string trivial tangle, in such a way that the copy of $A_i$ corresponds to the boundary of the regular neighbourhood of $\tau_1$. Since $V_{11} \cap V_{21}$ is a 2-sphere with two holes which contains the two points $P \cap K$, we see that $(V_{11} \cup V_{21}, K)$ is identified with $(E(K_2), K_1)$, where $K_1 \cup K_2 = L$ is a 2-bridge link.

Suppose that $L$ is a trivial link. Then $K_1$ bounds a disk in $E(K_2)$ and hence $K$ is a trivial knot, a contradiction.

Suppose that $L$ is a Hopf link. Then $K_1$ is isotopic to $K_2$. So we can put $K$ on $P$. Hence $K$ is a core knot or a torus knot.

Suppose that $V_{11} \cup V_{12}$ is a solid torus. Then we see that $K = K(\alpha, \beta; r)$ for some $\alpha$, $\beta$ and $r$.

In other cases, we see that $A_1 \cup A_2$ is an essential torus. Hence $K$ belongs to $K_2$.

\qed
Lemma 6.5. Let $K$ be a non-trivial $(1, 1)$-knot in $M$ and $(W_1, W_2; P)$ a $(1, 1)$-splitting of $(M, K)$. Suppose that $(W_1, W_2; P)$ satisfies the condition $(\#_e)$ of Proposition 6.1. Then $M \cong S^2 \times S^1$ and either

1. $K = K(4, 1; 0)$, or
2. $K$ belongs to $K_3$ or $K_4$.

Proof. Let $D_1$ and $D_2$ be a pair of disks which give the condition $(\#_e)$ of Proposition 6.1, and put $V_i^- = \text{cl}(V_i - N(t_i))$ ($i = 1$ and 2). Let $\alpha_{ij}$ ($j = 1$ and 2) be the components of $\partial(V_i^- \cap N(t_i))$, and let $A_{ij}$ ($j = 1$ and 2) be annuli properly embedded in $V_i^-$ satisfying the following conditions (see Fig. 3).

1. $A_{ij}$ is parallel to $D_i \cap V_i^-$ in $V_i$.
2. $A_{ij} \cap N(t_i) = \emptyset$.
3. $\alpha_{ij}$ is parallel to a component of $\partial A_{ij}$ in $\text{cl}(\partial V_i^- - N(t_i))$.
4. $\partial(A_{11} \cup A_{12}) = \partial(A_{21} \cup A_{22})$.

For each $i = 1$ and 2, let $(V_{i1}, \emptyset)$ and $(V_{i2}, t_i)$ be the pairs obtained from $(V_i, t_i)$ by cutting along $A_{i1} \cup A_{i2}'$, where $V_{i1}$ is a genus two handlebody, $V_{i2}$ is a 3-ball and $t_i$ is a trivial arc in $V_{i2}$. Then $V_{i1}$ is identified with the exterior of a 2-string trivial tangle $(B^3, \tau)$ in such a way that the copy of $A_{i1} \cup A_{i2}$ corresponds to the boundary of the regular neighbourhood of $\tau$.

**Case 1.** $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$ composes two tori.

Suppose that one of the tori, say $T_0$, is inessential in $E(K)$. Then since $T_0$ is not parallel to $\partial N(K)$, $T_0$ is compressible in $E(K)$. So we can obtain the 2-sphere $S$ by compressing $T_0$. Note that $S$ is essential, because $T_0$ is non-separating in $E(K)$. Hence $S$ is an essential 2-sphere in $E(K)$. This implies that $K$ is a trivial knot by Proposition 2.9 of [2], a contradiction. Hence $T_0$ is an essential torus in $E(K)$. In the following, we show that $K$ belongs to $K_3$. Since $V_{11} \cap V_{21}$ is a 2-sphere with four holes, we see that $V_{11} \cup V_{21}$ is the exterior of a non-trivial 2-bridge link, say $L$. On the other hand, we can recognize $(M_0, k_0) := (V_{12}, t_1) \cup (V_{22}, t_2)$ as follows. We first note that $(V_{i2}, t_i)$ is identified with $(B^3, \tau)$, where $\tau$ is a trivial arc in $B^3$, in such a way that
the copy of $A_{11} \cup A_{21}$ corresponds to a regular neighborhood on $\partial B^3$ of two homotopically non-trivial simple loops in $\partial B^3 - \tau$. Moreover, $(V_{12}, t_1) \cap (V_{22}, t_2)$ consists of an annulus and two copies of $(D^2, o)$, where $o$ is the center of the disk. By using this fact, we can see that $E(k_0)$ is identified with $B \times S^1$, an orientable $S^1$-bundle over a two-holed disk $B$, and that a meridian of $E(k_0)$ is isotopic to a fiber. Here the $S^1$-bundle structure is obtained by gluing the $S^1$-bundle structure of $E(t_1)$ and $E(t_2)$. Now let $K_0 \cup K_1 \cup K_2$ be as in the definition of $K_3$. Since $E(K_0 \cup K_1 \cup K_2)$ is identified with $B \times S^1$, where fibers are $1/2$. Hence is a torus knot in $2 \times 1$ which intersects $2 \times \{1\text{point}\}$ in two points. This implies $K = \langle 4, 1, 0 \rangle$.

**Case 2.** $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$ composes a torus $T$.

Since $V_{11} \cap V_{21}$ is a 2-sphere with four holes, we see that $V_{11} \cup V_{21}$ is the exterior of a 2-bridge knot, say $K_2$. On the other hand, we can recognize $(M_0, k_0) := (V_{12}, t_1) \cup (V_{22}, t_2)$ as follows. We first note that $(V_{12}, t_1)$ is identified with $(B^3, \tau)$, where $\tau$ is a trivial arc in $B^3$ in such a way that the copy of $A_{12}$ corresponds to a regular neighborhood on $\partial B^3$ of two homotopically non-trivial simple loops in $\partial B^3 - \tau$. Moreover, $(V_{12}, t_1) \cap (V_{22}, t_2)$ consists of an annulus and two copies of $(D^2, \emptyset)$. By using this fact, we can see that $E(k_0)$ is identified with $B \times S^1$, an orientable twisted $S^1$-bundle over a one-holed Möbius band $B$, and that a meridian of $E(k_0)$ is isotopic to a fiber. Here the $S^1$-bundle structure is obtained by gluing the $S^1$-bundle structure of $E(t_1)$ and $E(t_2)$. Now let $K_0 \cup K_1 \subset S^2 \times S^1$ and $l_1 \subset \partial E(K_1)$ be as in the definition of $K_4$. Then $(V_{12}, t_1) \cup (V_{22}, t_2) = E(k_0), \emptyset) \cup (N(k_0), k_0)$ is identified with $(E(K_1), K_0)$, where $l_1$ corresponds to a fiber (with respect to the bundle structure $B \times S^1$ on $E(k_0)$). This can be seen as follows. Since $K_0 = K(4, 1; 0)$, $K_0$ intersects each fiber $S^2$ in two points. So $E(K_0)$ is a twisted annulus bundle over $S^2$, and hence it is a twisted $S^1$-bundle over a Möbius band. Moreover, the meridian $K_1$ of $K_0$ corresponds to a regular fiber. This implies that $E(K_0 \cup K_1)$ is identified with $B \times S^1$, where $l_1$ corresponds to a fiber of $B \times S^1$. Hence $(E(K_0), K_1) = E(K_0 \cup K_1), \emptyset) \cup (N(K_0), K_0)$ is identified with $(E(k_0), \emptyset) \cup (N(k_0), k_0)$. Thus we have $(M, K) = (V_{11}, \emptyset) \cup (V_{21}, \emptyset) \cup (V_{21}, t_1) \cup (V_{22}, t_2) = E(K_2), \emptyset) \cup (E(K_1), K_0)$.

Suppose that $T$ is essential in $E(K)$. Then $K_2$ is non-trivial. Hence $K$ belongs to $K_4$.

Suppose that $T$ is inessential in $E(K)$. Then we see that $K_2$ is trivial. Hence $E(K)$ is homeomorphic to $B \times S^1$, where $B$ is a Möbius band. Hence $E(K)$ is a Seifert fibered space whose base space is a disk with two singular points, and the Seifert invariant of the singular fibers are $1/2$. Hence $K$ is a torus knot in $S^2 \times S^1$ which intersects $S^2 \times \{1\text{point}\}$ in two points. This implies $K = K(4, 1, 0)$. \qed
To prove Proposition 6.1, we prepare some lemmas which are obtained by an argument similar to those in Section 3 of [14]. An annulus properly embedded in an orientable 3-manifold is called essential if it is incompressible and not $\partial$-parallel. For a solid torus $V$ and a trivial arc $t$ in $V$, an annulus properly embedded in $V - t$ is called essential in $(V, t)$ if it is essential in $V - t$.

**Lemma 6.6.** Let $V$ be a solid torus and $t$ a trivial arc in $V$, and let $A$ be an essential annulus in $(V, t)$. Then one of the following holds (see Fig. 4).

1. $A$ cuts $(V, t)$ into $(V_1, \emptyset)$ and $(V_2, t)$, where $V_1$ is a genus two handlebody, $V_2$ is a 3-ball and $t$ is a trivial arc in $V_1$.
2. $A$ cuts $(V, t)$ into $(V_1, \emptyset)$ and $(V_2, t)$, where $V_1$ is a solid torus, $V_2$ is a genus two handlebody and $t$ is a trivial arc in $V_2$.
3. $A$ is a non-separating annulus in $V - t$ and there are an $\varepsilon_0$-disk $D$ and an $\varepsilon_1$-disk $D'$ in $(V, t)$ with $D \cap D' = \emptyset$ and $A \cap (D \cup D') = \emptyset$.

**Proof.** Let $D$ be a disjoint union of an $\varepsilon_0$-disk and an $\varepsilon$-disk in $(V, t)$. Since $A$ is incompressible in $V - t$, $A$ intersects $D$. By a standard innermost/outermost disk argument, we can find a disk $\delta$ in $V$ such that $\delta \cap t = \emptyset$, $\delta \cap A = a$ is an essential arc in $A$ and $\delta \cap \partial V = b$ is an arc with $\partial a = \partial b$ and $a \cup b = \partial \delta$. By performing a $\partial$-compression of $A$ along $\delta$, we obtain a disk $D$ properly embedded in $V - t$. Since $A$ is essential in $V - t$, $D$ is essential in $V - t$. 

![Fig. 4.](image-url)
**Case 1.** $D$ is an $\varepsilon$-disk.

Then $D$ cuts $(V, t)$ into $(V', t)$ and $(V'', \emptyset)$, where $V'$ is a 3-ball, $t$ is a trivial arc in $V'$ and $V''$ is a solid torus. If $A - D \subset V'$, then we obtain the conclusion (1). Otherwise, we obtain the conclusion (2).

**Case 2.** $D$ is an $\varepsilon_0$-disk.

Then $D$ cuts $(V, t)$ into $(B, t)$, where $B$ is a 3-ball and $t$ is a trivial arc in $B$. By a pairwise isotopy of $(B, t)$, we may assume $A \subset \partial B$. Then since $A$ is essential in $V - t$, the core $\alpha$ of $A$ separates the two punctures of $\partial B - t$. Hence by Lemma 3.3, $\alpha$ bounds an $\varepsilon_1$-disk $D'$ in $(V, t)$. By moving $D$ and $D'$ so that $(D \cup D') \cap A = \emptyset$, we obtain the conclusion (3).

**Lemma 6.7.** Let $V$ be a solid torus and $t$ a trivial arc in $V$, and let $A = A_1 \cup A_2$ be a disjoint union of non-parallel essential annuli in $(V, t)$. Then one of the following holds (see Fig. 5).

1. $A$ cuts $(V, t)$ into $(V_1, \emptyset)$ and $(V_2, t)$, where $V_1$ is a genus two handlebody, $V_2$ is a 3-ball and $t$ is a trivial arc in $V_2$, which satisfy $A \subset \partial V_j$ ($j = 1$ and 2). Moreover, there are an $\varepsilon_0$-disk $D$ and an $\varepsilon_1$-disk $D'$ in $(V, t)$ with $D \cap D' = \emptyset$ and $A \cap (D \cup D') = \emptyset$.

2. $A$ cuts $(V, t)$ into $(V_1, \emptyset)$, $(V_2, \emptyset)$ and $(V_3, t)$, where $V_1$ is a solid torus, $V_2$ is...
a genus two handlebody, $V_3$ is a 3-ball and $t$ is a trivial arc in $V_3$, which satisfy $\mathcal{A} \cap \partial V_1 = A_1$, $\mathcal{A} \subset \partial V_2$ and $\mathcal{A} \cap \partial V_3 = A_2$ after changing the subscripts. Moreover, there is an $\iota$-disk in $(V, t)$ disjoint from $\mathcal{A}$.

(3) $\mathcal{A}$ cuts $(V, t)$ into $(V_1, \emptyset)$ and $(V_2, t)$, where $V_1$ is a genus two handlebody, $V_2$ is a 3-ball and $t$ is a trivial arc in $V_2$, which satisfy $\mathcal{A} \subset \partial V_1$ and $\mathcal{A} \cap \partial V_2 = A_2$ after changing the subscripts.

**Proof.** By performing $\partial$-compressions of $A_1$ and $A_2$, we obtain mutually disjoint disks $D_1$ and $D_2$ properly embedded in $V - t$. Since $A_1$ and $A_2$ are essential in $V - t$, $D_1$ and $D_2$ are essential in $V - t$. Suppose that both $D_1$ and $D_2$ are $\varepsilon_0$-disks. Then we obtain the conclusion (1). Suppose next that both $D_1$ and $D_2$ are $\iota$-disks. Then we obtain the conclusion (2). Suppose finally that precisely one of $D_1$ and $D_2$, say $D_1$, is an $\varepsilon_0$-disk and $D_2$ is an $\iota$-disk. Note that $A_2$ is disjoint from $D_2$. This implies that $A_2$ is parallel to $\partial \bar{N}(K)$. Hence we obtain the condition (3). \hfill \Box

The following lemma is obtained by using Lemma 3.3 of [14].

**Lemma 6.8.** Let $V$ be a solid torus and $t$ a trivial arc in $V$, and let $\mathcal{A} = A_1 \cup A_2 \cup A_3$ be a disjoint union of non-parallel essential annuli in $(V, t)$. Then $\mathcal{A}$ cuts $(V, t)$ into $(V_1, \emptyset)$, $(V_2, \emptyset)$ and $(V_3, t)$, where $V_1$ is a genus two handlebody, $V_2$ is a solid torus and $V_3$ is a 3-ball and $t$ is a trivial arc in $V_3$, which satisfy $\mathcal{A} \cap \partial V_1 = A_1 \cup A_2$, $\mathcal{A} \subset \partial V_2$ and $\mathcal{A} \cap \partial V_3 = A_3$ after changing the subscripts (see Fig. 6).

**Proof.** Note that $A_1 \cup A_2$ satisfies one of the conclusions of Lemma 6.7. Suppose that $A_1 \cup A_2$ satisfies the conclusion (2) of Lemma 6.7. Then $A_1 \cup A_2$ cuts $(V, t)$ into $(V_1, \emptyset)$, $(V_2, \emptyset)$ and $(V_3, t)$, where $V_1$ is a solid torus, $V_2$ is a genus two handlebody, $V_3$ is a 3-ball and $t$ is a trivial arc in $V_3$. If $A_3 \subset V_1$ or $V_3$, then $A_3$ is parallel to $A_1$ or $A_2$. If $A_3 \subset V_2$, then by Lemma 3.3 of [14], $A_3$ is parallel to $A_1$ or $A_2$. Hence we may assume that $A_1 \cup A_2$ satisfies the conclusion (1) or (3) of Lemma 6.8.

Suppose $A_1 \cup A_2$ satisfies the conclusion (1) of Lemma 6.7. Then $A_1 \cup A_2$ cuts $(V, t)$ into $(V_1, \emptyset)$ and $(V_2, t)$, where $V_1$ is a genus two handlebody, $V_2$ is a 3-ball and
$t$ is a trivial arc in $V_2$. By Lemma 3.3 of [14], $A_3$ must be contained in $V_2$. Hence $A_3$ is parallel to $\partial N(t)$.

Suppose $A_1 \cup A_2$ satisfies the conclusion (3) of Lemma 3.3. Then $A_1 \cup A_2$ cuts $(V, t)$ into $(V_1, \emptyset)$ and $(V_2, t)$, where $V_1$ is a genus two handlebody, $V_2$ is a 3-ball and $t$ is a trivial arc in $V_2$. By Lemma 3.3 of [14], $A_3$ is parallel to an annulus, say $A'$, in $\partial V_2$. Since $A_3$ is essential in $V - t$ and is not parallel to $A_i$ ($i = 1$ and 2), $A'$ contains $\partial A_1 \cup \partial A_2$. This implies $A_3$ satisfies the condition (3) of Lemma 6.6. Then by changing the subscripts, we can see that $A$ satisfies the condition of Lemma 6.8.

Proof of Proposition 6.1. Let $(W_1, W_2; P)$ be a $(1, 1)$-splitting of $(M, K)$ and $T$ an essential torus in $E(K)$. We put $T_i = T \cap V_i$.

Claim. We may assume that $T_i$ consists of essential annuli in $W_i$ ($i = 1$ and 2).

Proof. Since $\chi(T) = 0$, we have only to show that $T_i$ has no disks.

We may assume that after an isotopy, each disk of $T_i$ is essential in $V_i - t_i$ ($i = 1$ and 2). Suppose that both $T_1$ and $T_2$ have disk components. Then this implies $d(W_1, W_2) \leq 1$ because $\partial T_1 = \partial T_2$. Hence we see that $K$ is a trivial knot or a core knot in $S^2 \times S^1$ by Theorem 2.2 and Theorem 2.3, a contradiction. Hence we may assume that either $T_1$ or $T_2$, say $T_2$, has no disk components. Further we assume that the number of disk components of $T_1$ is minimal among all essential tori satisfying the condition as above. Let $\Delta$ be the union of the disk components of $T_1$. Choose a disjoint union $D$ of an $\varepsilon_0$-disk and an $\iota$-disk in $W_2$ which intersect $T_2$ transversely.

Note that $E(K)$ is irreducible, i.e., $E(K)$ contains no essential 2-spheres. Otherwise, $K$ is a trivial knot by Proposition 2.9 of [2], a contradiction. Hence by a standard argument, we can eliminate all loop components of $T_2 \cap D$ by an ambient isotopy on $E(K)$.

Suppose that $\Delta \cap D = \emptyset$. Then each component of $\partial \Delta$ is isotopic to one of the components of $\partial D$ because each component of $\partial \Delta$ is either an $\varepsilon$-loop or an $\iota$-loop. This implies that $\partial \Delta$ bounds a disk in $V_2 - t_2$, and hence $d(W_1, W_2) = 0$. By Theorem 2.2, $K$ is a trivial knot, a contradiction. So $\Delta \cap D \neq \emptyset$.

Let $\Gamma$ be the union of the arc components of $T_2 \cap D$ incident to $\partial \Delta \cap D$. Let $\gamma$ be a component of $\Gamma$ such that $\gamma$ clips a disk, say $\delta_{\gamma}$, from $D$ with $\delta_{\gamma} \cap \Gamma = \gamma$. Suppose that $\delta_{\gamma} \cap T_2 \neq \gamma$. Then there is a component $\gamma'$ of $\delta \cap T_2$ which clips a disk $\delta_{\gamma'}$ with $\delta_{\gamma'} \cap T_2 = \gamma'$. We can isotope $T$ along $\delta_{\gamma'}$ near $\gamma'$ without increasing the number of disks of $T_1$. By repeating this operation, if necessary, we may suppose that $\delta_{\gamma} \cap T_2 = \gamma$. By isotoping $T$ along $\delta_{\gamma}$, we can reduce the number of disk components of $T_1$ at least by one, a contradiction.

This completes the proof of the claim.
Let $A_i$ be a union of mutually disjoint, non-parallel, essential annuli in $W_i = (V_i, t_i)$ of which $T_i$ consists of parallel copies ($i = 1$ and 2). Note that $|A_1| \leq 3$ by Lemmas 6.6–6.8. By changing the subscripts, if necessary, we may assume that $|A_1| \geq |A_2|$.  

**Case 1.** $|A_1| = 3$.  

Note that one of the following holds.  

- $A_2$ consists of an annulus satisfying one of the conditions in Lemma 6.6.  
- $A_2$ consists of two annuli satisfying one of the conditions in Lemma 6.7.  
- $A_2$ consists of three annuli satisfying the condition in Lemma 6.6.  

Suppose that $A_2$ satisfies the condition (1) of Lemma 6.6, the condition (2) of Lemma 6.7, the condition (3) of Lemma 6.7, or the condition of Lemma 6.8. Here, the sentence “$A_2$ satisfies the condition (1) of Lemma 6.6” means that $A_2$ consists of an annulus satisfying the condition (1) in Lemma 6.6. Then $T_1 \cup T_2$ contains a torus which is parallel to $\partial N(K)$, a contradiction.

Suppose that $A_2$ satisfies the condition (2) of Lemma 6.6 or the condition (3) of Lemma 6.6. Let $\{p_1, p_2\}$ be points of $P \cap K$. Note that $A_1$ has a component which is isotopic to $\partial N(p_i; P)$ for each $i = 1$ and 2. On the other hand, for $i = 1$ or 2, $A_2$ does not have a component which is isotopic to $\partial N(p_i; P)$. This implies that $\partial T_1 \neq \partial T_2$, a contradiction.

Suppose that $A_2$ satisfies the condition (1) of Lemma 6.7. Put $A_1 = A_{11} \cup A_{12} \cup A_{13}$ and $A_2 = A_{21} \cup A_{22}$. We may assume that $A_{13}$ is isotopic to $\partial N(K) \cap V_1$. Suppose that $T_1$ consists of $m_1$ parallel copies of $A_{11}$, $m_2$ parallel copies of $A_{12}$, and $m_3$ parallel copies of $A_{13}$, and $T_2$ consists of $n_1$ parallel copies of $A_{21}$ and $n_2$ parallel copies of $A_{22}$. Then since $\partial T_1 = \partial T_2$, we have $m_1 + m_2 = n_1 + n_2$, $m_1 + m_3 = n_1$, and $m_2 + m_3 = n_2$. This implies that $m_3 = 0$, a contradiction. Hence Case 1 does not occur.

**Case 2.** $|A_1| = 2$.  

Set $A_1 = A_{11} \cup A_{12}$. We have the following three subcases by Lemma 6.7.

**Case 2.1.** $A_1$ satisfies the condition (1) of Lemma 6.7.  

By an argument similar to Case 1, we see that $A_2$ satisfies the condition (1) or (2) of Lemma 6.7. Set $A_2 = A_{21} \cup A_{22}$.  

Suppose that $A_2$ satisfies the condition (1) of Lemma 6.7. Then we see $|T_1| = |T_2| = 2$. (Otherwise $T_1 \cup T_2$ has plural components.) So we may assume $T_i = A_{i1} \cup A_{i2}$ ($i = 1$ and 2) (cf. Fig. 3). Since $M \cong S^2 \times S^1$, we can find an $\varepsilon_1$-disk $D_i$ in $W_i$ ($i = 1$ and 2) with $\partial D_1 = \partial D_2$. Hence $(W_1, W_2; P)$ satisfies the condition $(\#_c)$ of Proposition 6.1.

Suppose that $A_2 = A_{21} \cup A_{22}$ satisfies the condition (2) of Lemma 6.7. Then we can find an $\varepsilon_1$-disk $D_1$ in $W_1$ and an $\varepsilon$-disk $D_2$ in $W_2$ which satisfy the condition $(\#_a)$ of Proposition 6.1 (see Fig. 7). Hence by the remark below Lemma 6.2, $K$ is a core knot, a contradiction.

**Case 2.2.** $A_1$ satisfies the condition (2) of Lemma 6.7.

Then by an argument similar to Case 1, we see that $A_2$ satisfies the condition (1)
of Lemma 6.7. Hence by changing the subscripts, Case 2.2 is equivalent to the latter case of Case 2.1.

**Case 2.3.** \(A_1\) satisfies the condition (3) of Lemma 6.7.

Then by an argument similar to Case 1, we see that Case 2.3 is impossible.

**Case 3.** \(|A_1| = 1\).

By Lemma 6.6, we have the following three subcases.

**Case 3.1.** \(A_1\) satisfies the condition (1) of Lemma 6.6.

By an argument similar to Case 1, we see that \(A_2\) satisfies the condition (1) of Lemma 6.6. Hence \(T_1 \cup T_2\) contains a torus which is parallel to \(\partial N(K)\), a contradiction.

**Case 3.2.** \(A_1\) satisfies the condition (2) of Lemma 6.6.

By an argument similar to Case 1, we see that \(A_2\) satisfies the condition (2) of Lemma 6.6. Moreover \(T_j\) consists of an annulus \((i = 1\) and 2\). (Otherwise, \(T_1 \cup T_2\) consists of plural components.) Let \(z\) be one of the components of \(\partial A_1 = \partial A_2\). For each \(i = 1\) and 2, let \(\Delta_i\) be a disk in \(V_i\) such that \(t_i \subset \partial \Delta_i\), and \(\Delta_i \cap \partial V_i = \text{cl}(\partial \Delta_i - t_i) =: t'_i\) is disjoint from \(z\). Then there are \(u\)-disks \(D_i\) in \(W_i\) with \(\partial D_i = \partial N(t'_i; P)\) for each \(i = 1\) and 2. Hence \(Z := \text{cl}(P - N(z; P))\) gives the condition (\#b) of Proposition 6.1.

**Case 3.3.** \(A_1\) satisfies the condition (3) of Lemma 6.6.

By an argument similar to Case 1, we see that \(A_2\) satisfies the condition (3) of Lemma 6.6. Then there are an \(\varepsilon_1\)-disk \(D_i\) in \(W_i\) \((i = 1\) and 2\) with \(\partial D_1 = \partial D_2\). Hence \((W_1, W_2; P)\) satisfies the condition (\#c) of Proposition 6.1.

This completes the proof of Proposition 6.1.

Proof of Theorem 2.5. Let \(K\) be a \((1, 1)\)-knot in \(M\) and \((W_1, W_2; P)\) a \((1, 1)\)-splitting of \((M, K)\). By Proposition 6.1, \((W_1, W_2; P)\) satisfies one of the conditions in Proposition 6.1.

Suppose that \((W_1, W_2; P)\) satisfies the condition (\#a) of Proposition 6.1. Then by Lemma 6.2, \(K\) belongs to \(K_1\), because the exteriors of 2-bridge knots and core knots do not contain essential tori (see [5]).

Suppose that \((W_1, W_2; P)\) satisfies the condition (\#b) of Proposition 6.1. Then by
arguments in the proof of Lemma 6.4 and the proof of Proposition 6.1, \( K \) belongs to \( K_2 \), because \( E(K) \) contains an essential torus.

Suppose that \((W_1, W_2; P)\) satisfies the condition \((\#_c)\) of Proposition 6.1. Then by Lemma 6.5, \( K \) belongs to \( K_3 \) or \( K_4 \).

We have thus proved Theorem 2.5.

7. \((1, 1)\)-splittings of distance = 2

In this section, we give the proof of Theorem 2.4.

Proof of Theorem 2.4. We first assume \( d(W_1, W_2) = 2 \), that is, there is an essential loop \( x \) (y resp.) in \( \Sigma := P - K \) which bounds a disk in \( V_1 - t_1 \) (\( V_2 - t_2 \) resp.) such that \( x \) and \( y \) intersect each other, and there is an essential loop \( z \) in \( \Sigma \) with \( z \cap (x \cup y) = \emptyset \).

**Case 1.** Both \( x \) and \( y \) are \( \epsilon \)-loops.

If \( z \) is an \( \iota \)-loop, then \( z \) bounds an \( \iota \)-disk in each of \( W_1 \) and \( W_2 \) by Lemma 3.3. This implies that \((W_1, W_2; P)\) is of distance = 0, a contradiction. Hence by Lemma 3.3, \( z \) must be an \( \epsilon \)-loop and \( z \) bounds an \( \epsilon_0 \)-disk or an \( \epsilon_1 \)-disk in each of \( W_1 \) and \( W_2 \).

Suppose that \( z \) bounds an \( \epsilon_0 \)-disk in each of \( W_1 \) and \( W_2 \). Then this means that \( d(W_1, W_2) \leq 1 \), a contradiction.

Suppose that \( z \) bounds an \( \epsilon_1 \)-disk in each of \( W_1 \) and \( W_2 \). Then \((W_1, W_2; P)\) satisfies the condition \((\#_c)\) of Proposition 6.1. By Lemma 6.5, \( K = K(4, 1, 0) \) or \( E(K) \) contains an essential torus.

**Case 2.** Precisely one of \( x \) and \( y \), say \( x \), is an \( \epsilon \)-loop.

We see that \( z \) is an \( \epsilon \)-loop by an argument similar to Case 1. Then by Lemma 3.3, \( z \) bounds an \( \epsilon_1 \)-disk in \( W_1 \). So \((W_1, W_2; P)\) satisfies the condition \((\#_b)\) of Proposition 6.1, and hence \((M, K)\) satisfies one of the conditions (1)–(3) of Lemma 6.2. Note that if \( K \) satisfies the condition (3), we can find an essential torus in \( E(K) \) by making an appropriate “swallow-follow torus”.

**Case 3.** Both \( x \) and \( y \) are \( \iota \)-loops.

Then \( z \) must be an \( \epsilon \)-loop by the same argument as above. In particular, \( z \) must be contained in the surface \( T_0 \) obtained from the torus \( P \) by removing the interior of the disk bounded by \( x \). So all components of \( y \cap T_0 \) (\( \neq \emptyset \)) are parallel in \( T_0 \). Note that we can regard \( y \) as \( \partial N(t'_i; P) \), where \( t'_i \) is an arc in \( P \) such that \( t_2 \cup t'_i \) bounds a disk in \( V_2 \). By an isotopy on \( \Sigma \), we may assume that \( |x \cap y| \) is minimal.

**Case 3.1.** \(|y \cap T_0| = 2\).

Then \( K \) is isotopic to a knot in \( P \), and hence \( K \) satisfies the condition (2) or (3) of Theorem 2.4.

**Case 3.2.** \(|y \cap T_0| > 2\).

Let \( A_1 \) in \( V_1^- \) (\( A_2 \) in \( V_2^- \) resp.) be an annulus obtained by pushing the interior of \( N(z; P) \) into the interior of \( V_i \) (\( V_2^- \) resp.), where \( V_i^- = \text{cl}(V_i - \overline{N(t_i)}) \) (\( i = 1, 2 \)). So \( T := A_1 \cup A_2 \) is a torus in \( E(K) \) (see Fig. 8).

\( A_1 \) (\( A_2 \) resp.) cuts \( V_1^- \) (\( V_2^- \) resp.) into a solid torus \( V_1^{11} \) (\( V_2^{21} \) resp.) and a genus
two handlebody $V_{12}^-$ ($V_{22}^-$ resp.). $M_1 = V_{11}^- \cup V_{21}^-$ is the exterior of a trivial knot, a core knot or a torus knot. $M_2 = V_{21}^- \cup V_{22}^-$ is the exterior of a 2-bridge link, and $M_2 \cap N(K)$ should be a solid torus. If $M_1$ is a solid torus, then $(M, K)$ is equivalent to $K(\alpha, \beta; r)$ for some $\alpha$, $\beta$ and $\gamma$. If not, by the hypothesis of Case 3.2, we can see that $T$ is not parallel to $\partial N(K)$. Hence $T$ is an essential torus in $E(K)$.

This completes the proof of the first part of Theorem 2.4.

Next, we prove the second part of Theorem 2.4.

CASE (1). $K$ is a non-trivial 2-bridge knot in $S^3$.

By Theorem 8.2 of [15], every $(1, 1)$-splitting of a non-trivial 2-bridge knot is isotopic to that constructed as follows. For a non-trivial 2-bridge knot $K$, let $(B_1, a_1 \cup a_2) \cup S (B_2, b_1 \cup b_2)$ be a 2-bridge decomposition. Put $V_1 = B_1 \cup N(b_2; B_2)$, $V_2 = \text{cl}(B_2 - N(b_2; B_2))$, $t_1 = a_1 \cup a_2 \cup b_2$ and $t_2 = b_1$. Then $W_i := (V_i, t_i)$ is a pair of a solid torus $V_i$ and a trivial arc $t_i$ in $V_i$ ($i = 1, 2$), and $(W_1, W_2; P)$ gives a $(1, 1)$-splitting of $(S^3, K)$. In the following, we show that this $(1, 1)$-splitting has distance $= 2$.

Let $D_i$ be a properly embedded disk in $B_i$ such that $D_i$ separates two trivial arcs in $B_i$ ($i = 1, 2$). Then $D_1$ determines an $\varepsilon_0$-disk in $W_1$, and $D_2$ determines an $\iota$-disk in $W_2$. Further, $\partial D_1$ and $\partial D_2$ are disjoint from an essential loop $z$ in $\Sigma := P - K$, where $z$ is one of the boundary components of the meridian disks $B_1 \cap N(b_2; B_2)$. Hence $d(W_1, W_2) \leq 2$. By Theorem 2.2 and Theorem 2.3, we have $d(W_1, W_2) = 2$. 

Fig. 8.
CASE (2) and (3). $K$ is a core knot in a lens space or a torus knot in $M$.

By Theorem C of [6] and Theorem 3 of [17], every $(1, 1)$-splitting of $(M, K)$ is isotopic to that constructed as follows. Let $(V_1, V_2; P)$ be a genus one Heegaard splitting of $M$ such that $K \subset P$. Let $p_1$ and $p_2$ be distinct points in $K$. Then $p_1 \cup p_2$ cuts $K$ into two arcs $I_1$ and $I_2$. Let $I_i$ be the properly embedded arc by slightly pushing the interior of $I_i$ into the interior of $V_i$, and put $W_i = (V_i, I_i)$ ($i = 1$ and 2). Then $(W_1, W_2; P)$ is a $(1, 1)$-splitting of $(M, K)$.

Let $z$ be a core of the annulus cl$(P - N(K; P))$. Then $\partial N(I_i; P)$ bounds an $\epsilon$-disk in $W_i$ ($i = 1, 2$), and $\partial N(I_1; P)$ and $\partial N(I_2; P)$ are disjoint from the essential loop $z$ in $P$. So we have $d(W_1, W_2) \leq 2$. By Theorem 2.2 and Theorem 2.3, we obtain $d(W_1, W_2) = 2$.

CASE (4). $E(K)$ contains an essential torus.

Let $(W_1, W_2; P)$ be a $(1, 1)$-splitting of $(M, K)$. By Proposition 6.1, $(W_1, W_2; P)$ satisfies one of the conditions $(\#_a)$, $(\#_b)$ and $(\#_c)$.

Suppose that $(W_1, W_2; P)$ satisfies the condition $(\#_a)$. Let $D_1$ (resp.) be an $\epsilon$-disk (an $\epsilon_1$-disk resp.) in $W_1$ (resp.) such that $\partial D_1 \cap \partial D_2 = \emptyset$. By cutting $W_2 = (V_2, I_2)$ along $D_2$, we obtain a 2-string trivial tangle $(B, \tau)$. Let $D_2^+ \cap D_2^-$ be the copy of $D_2$ in $\partial B$. Let $D_2'$ be a disk properly embedded in $B$ such that $D_2' \cap (D_2^+ \cup D_2^-) = \emptyset$ and $D_2'$ separates a component of $\tau$ from the other. Then $D_2'$ determines an $\epsilon_1$-disk $W_2$, and $D_2'$ is disjoint from $D_2$. Hence $\partial D_1$ and $\partial D_2'$ give $d(W_1, W_2) = 2$.

We can easily see that the condition $(\#_b)$ directly gives $d(W_1, W_2) \leq 2$.

Finally, if the condition $(\#_c)$ is satisfied, then we can also obtain $d(W_1, W_2) \leq 2$ by using an argument similar to that in case of the condition $(\#_a)$. By Theorem 2.2 and Theorem 2.3, we obtain $d(W_1, W_2) = 2$.

We have completed the proof of Theorem 2.4.

Proof of Corollary 2.6. By Thurston’s hyperbolization theorem of Haken manifolds (see, for example, [13]), a knot $K$ is hyperbolic if and only if $E(K)$ is irreducible, $E(K)$ contains no essential torus, and $E(K)$ is not a Seifert fibered space.

CASE 1. $E(K)$ is reducible.

By Proposition 2.9 of [2], $E(K)$ is reducible if and only if $K$ is a trivial knot. Hence $d(W_1, W_2) = 0$ by Theorem 2.2.

CASE 2. $E(K)$ contains an essential torus.

Then by Theorem 2.6, $d(W_1, W_2) = 2$.

CASE 3. $E(K)$ is a Seifert fibered space whose regular fiber is not a meridian of $K$.

Then by Lemma 5.2 of [14], if $E(K)$ is a Seifert fibered space whose regular fiber is not a meridian of $K$ and $\partial E(K)$ is incompressible in $E(K)$, then one of the following holds: (1) the base space is a disk with two singular points, where the regular fiber in $\partial E(K)$ intersects the meridian in one point, (2) the base space is a Möbius
band with one singular point, where the regular fiber in $\partial E(K)$ intersects the meridian in one point. (3) $E(K)$ is a twisted $S^1$-bundle over a Möbius band. If $E(K)$ satisfies the condition (1) or (3), then $K$ is a torus knot. If $E(K)$ satisfies the condition (2), then there is an essential torus in $E(K)$. Hence by Theorem 2.4, $d(W_1, W_2) = 2$.

Suppose that $\partial E(K)$ is compressible in $E(K)$. Then we obtain a 2-sphere $S$ in $E(K)$ by compressing $\partial E(K)$. If $S$ bounds a 3-ball in $E(K)$, then $E(K)$ is a solid torus and hence $K$ is a trivial knot or a core knot. Otherwise, since $S$ is essential in $E(K)$, $K$ is a trivial knot by Proposition 2.9 of [2]. Hence by Theorems 2.2 and 2.3, we have $d(W_1, W_2) = 0$ or 1.

**CASE 4.** $E(K)$ is a Seifert fibered space whose regular fiber is a meridian of $K$.
Let $B$ be the base orbifold of $E(K)$. Then $\pi_1(M) = \pi_1(E(K))/\langle f \rangle$, where $f$ is the element of $\pi_1(E(K))$ represented by a regular fiber, is isomorphic to the orbifold fundamental group $\pi_1(B)$. Since $M$ is a lens space, $\pi_1(B)$ is cyclic. It is known that such an orbifold is isomorphic to a disk with only one singular point (see, for example, Section 3 of [19]). Therefore $E(K)$ is a solid torus, and hence $K$ is a core knot. Hence by Theorem 2.3, we have $d(W_1, W_2) = 1$.

Hence by Theorems 2.2–2.4 and the hypothesis of Proposition 2.6, $d(W_1, W_2) \leq 2$ if and only if $E(K)$ is a Seifert fibered space or contains an essential 2-sphere or torus. By Thurston’s hyperbolization theorem, we obtain the desired result. 

**8. (1, 1)-splittings of distance $\geq 3$**

Theorem 2.7 can be proved by the arguments of J. Hempel in Section 2 of [11]. To this end, we first recall the covering distance introduced in [11].

Let $S$ be a connected, compact, orientable surface. We say that a covering space $p: \tilde{S} \to S$ separates essential loops $x$ and $y$ in $S$ if there are components $\tilde{x}$ of $p^{-1}(x)$ and $\tilde{y}$ of $p^{-1}(y)$ with $\tilde{x} \cap \tilde{y} = \emptyset$. A finite covering $p: \tilde{S} \to S$ is sub-solvable if $p$ can be factored as a composition of cyclic coverings.

**DEFINITION 8.1** ([11] Section 2). Let $[x]$ and $[y]$ be distinct vertices of $C(S)$, and let $x$ (y resp.) be a representative of $[x]$ ([y] resp.). Then we define the covering distance between $[x]$ and $[y]$ as follows.

$$cd([x], [y]) = 1 + \min \left\{ \eta \mid \text{there is a degree } 2^\eta \text{ sub-solvable covering of } S \text{ which separates } x \text{ and } y \right\}.$$ 

As an analogy of Lemma 2.3 in [11], we obtain the following.

**Lemma 8.2.** Let $[x]$ and $[y]$ be distinct vertices of $C(S)$. Then
(1) $d([x], [y]) = 2$ if and only if $cd([x], [y]) = 2$ and
(2) $cd([x], [y]) \leq d([x], [y])$. 

Proof. Let \(x\) (resp.) be a representative of \([x]\) (resp.) \(\).

(1) Suppose that \((x, y) = 2\), that is, \(x \cap y \neq \emptyset\) and there is an essential loop \(z\) in \(S\) with \(z \cap (x \cup y) = \emptyset\).

CASE 1. \(z\) is an \(\varepsilon\)-loop.

Since an \(\varepsilon\)-loop is a non-separating loop in \(S\), \(S' := \text{cl}(S - N(z))\) is connected. We can construct a double cover \(\tilde{S}\) of \(S\) by gluing two copies \(S'_1\) and \(S'_2\) of \(S'\) along \(z\). Hence \(\tilde{x}\) in \(S'_1\) and \(\tilde{y}\) on \(S'_2\) can give \(cd([x], [y]) = 2\).

CASE 2. \(z\) is an \(\iota\)-loop.

Let \(\gamma\) be an essential arc which joins two punctures of \(S\) such that \(\gamma\) is disjoint from \(z\). Then we can construct a double cover \(\tilde{S}\) of \(S\) by gluing two copies of \(\text{cl}(S - N(\gamma))\). Therefore we can also get \(cd([x], [y]) = 2\).

The converse follows from the proof of Lemma 2.3 in [11].

(2) The second assertion can also be proved by the same argument as that in the proof of Lemma 2.3 of [11].

This completes the proof of Lemma 8.2. \(\square\)

By Lemma 8.2, we can get a lower estimation of the distance between distinct vertices on \(C(S)\). For the covering distance, the following lemma is proved in [11].

**Lemma 8.3** ([11] Theorem 2.5). If \([x]\) and \([y]\) are vertices of \(C(S)\) and \(h : S \to S\) is a pseudo-Anosov homeomorphism, then \(\lim_{n \to \infty} cd([x], [h^n(y)]) = \infty\).

Proof of Theorem 2.7. We first construct a pseudo-Anosov map \(f\) of \(\Sigma := P - K\) whose extension to \(P\) is isotopic to \(id\). To this end, let \(a\) and \(b\) be essential loops on \(\Sigma\) illustrated in Fig. 9, and put \(f = \tau_a^{-1} \circ \tau_b\), where \(\tau_a\) (resp.) a right-hand Dehn twist along \(a\) (resp.). Then \(f\) is pseudo-Anosov by Theorem 3.1 of [21], because \(a \cup b\) fills \(\Sigma\). Since \(a\) and \(b\) are isotopic in \(P\), the extension \(\hat{f}\) of \(f\) to \(P\) is isotopic to the identity.

Now let \(M\) be a 3-manifold with a genus one Heegaard splitting. Pick a
(1,1)-knot $K$ in $M$ and its (1,1)-splitting $(W_1, W_2; P)$. Let $x$ (y resp.) be an $\varepsilon$-loop in $\Sigma$ which bounds an $\varepsilon_0$-disk in $W_1$ (resp. $W_2$). By Lemma 8.2 and Lemma 8.3, for any positive integer $n$, there is an integer $N$ such that $d([x], [f^N(y)]) > n + 2$, where $[x]$ ($[f^N(y)]$ resp.) is represented by $x$ ($f^N(y)$ resp.). Since $\hat{f} \simeq id$, the manifold obtained from $M$ by cutting along $P$ and regluing it after composing $\hat{f}$ is homeomorphic to $M$. Let $(W'_1, W'_2; P)$ be a (1,1)-splitting obtained in the above way. Then by Proposition 3.8, we have $d(W'_1, W'_2) \geq d([x], [f^N(y)]) - 2 > n$.

We have completed the proof of Theorem 2.7.

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