



Title	Genus one 1-bridge knots as viewed from the curve complex
Author(s)	Saito, Toshio
Citation	Osaka Journal of Mathematics. 2004, 41(2), p. 427-454
Version Type	VoR
URL	<a href="https://doi.org/10.18910/11881">https://doi.org/10.18910/11881</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

# GENUS ONE 1-BRIDGE KNOTS AS VIEWED FROM THE CURVE COMPLEX

TOSHIO SAITO

(Received September 13, 2002)

## 1. Introduction

W.J. Harvey [4] associated to a surface  $S$  a finite-dimensional simplicial complex  $C(S)$ , called the curve complex, which we recall below.

For a connected orientable surface  $F = F_{g,n}$  of genus  $g$  with  $n$  punctures, the *curve complex*  $C(F)$  of  $F$  is the complex whose  $k$ -simplexes are the isotopy classes of  $k + 1$  collections of mutually non-isotopic essential loops in  $F$  which can be realized disjointly. It is proved in [16] that the curve complex is connected if  $F$  is not sporadic (where  $F$  is *sporadic* if  $g = 0$ ,  $n \leq 4$  or  $g = 1$ ,  $n \leq 1$ ). For  $[x]$  and  $[y]$ , vertices of  $C(F)$ , the *distance*  $d([x], [y])$  between  $[x]$  and  $[y]$  is defined by the minimal number of 1-simplexes in a simplicial path joining  $[x]$  to  $[y]$ . It is known that if  $S$  is not sporadic, then  $C(F)$  has infinite diameter with respect to the distance defined above (cf. [11], [16]),  $C(F)$  is not locally finite in the sense that there are infinite edges around each vertex, and the dimension of  $C(F)$  is  $3g - 4 + n$ .

Recently, J. Hempel [11] studied Heegaard splittings of closed 3-manifolds by using the curve complex of Heegaard surfaces. Let  $M$  be a closed orientable 3-manifold and  $(V_1, V_2; S)$  a genus  $g \geq 2$  Heegaard splitting, that is,  $V_i$  ( $i = 1$  and  $2$ ) is a genus  $g$  handlebody with  $M = V_1 \cup V_2$  and  $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = S$ . By using the curve complex, Hempel defined the distance of the Heegaard splitting, denoted by  $d(V_1, V_2)$ , and proved the following results.

**Theorem 1.1** (J. Hempel). (1) *Let  $M$  be a closed, orientable, irreducible 3-manifold which is Seifert fibered or which contains essential tori. Then  $d(V_1, V_2) \leq 2$  for any Heegaard splitting  $(V_1, V_2; S)$  of  $M$ .*  
 (2) *There are Heegaard splittings of closed orientable 3-manifolds with distance  $> n$  for any integer  $n$ .*

In particular, the theorem above implies that a Haken manifold is hyperbolic if a Heegaard splitting of the manifold has distance  $\geq 3$ . Results along these lines were also obtained by A. Thompson [20]. Moreover, H. Goda, C. Hayashi and N. Yoshida [2] made detailed study of tunnel number one knots and C. Hayashi ([6], [7]) studied  $(1, 1)$ -knots from similar points of view.

In this paper, we apply this idea to genus one 1-bridge knots. A knot  $K$  in an orientable closed 3-manifold  $M$  is called a *genus one 1-bridge knot*, a  $(1, 1)$ -knot briefly, if  $(M, K) = (V_1, t_1) \cup_P (V_2, t_2)$ , where  $(V_1, V_2; P)$  is a genus one Heegaard splitting and  $t_i$  is a trivial arc in  $V_i$  ( $i = 1$  and  $2$ ). (An arc  $t$  properly embedded in a solid torus  $V$  is said to be *trivial* if there is a disk  $D$  in  $V$  with  $t \subset \partial D$  and  $\partial D - t \subset \partial V$ .) Set  $W_i = (V_i, t_i)$  ( $i = 1$  and  $2$ ). We call the triple  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$ . In this paper, we study  $(1, 1)$ -splittings by using the distance of the curve complex. To define the distance of a  $(1, 1)$ -splitting, we use the twice punctured torus  $\Sigma = P - K$ .

For  $i = 1$  or  $2$ , let  $\mathcal{K}(W_i)$  be the maximal subcomplex of  $C(\Sigma)$  consisting of simplexes  $\langle [c_0], [c_1], \dots, [c_k] \rangle$  such that an essential loop representing  $[c_j]$  ( $j = 0, 1, \dots, k$ ) bounds a disk in  $V_i - t_i$ .

DEFINITION 1.2. We define the *distance* of a  $(1, 1)$ -splitting  $(W_1, W_2; P)$  by

$$\begin{aligned} d(W_1, W_2) &= d(\mathcal{K}(W_1), \mathcal{K}(W_2)) \\ &= \min\{d([x], [y]) \mid [x]: \text{a vertex in } \mathcal{K}(W_1), [y]: \text{a vertex in } \mathcal{K}(W_2)\}. \end{aligned}$$

In this paper, we give topological characterizations of the knots admitting  $(1, 1)$ -splittings of distance  $\leq 2$  (Theorem 2.2, 2.3 and 2.5). As a corollary, we see that a  $(1, 1)$ -knot is hyperbolic if and only if it has a  $(1, 1)$ -splitting of distance  $\geq 3$ , except for certain knots (Corollary 2.6). Further we will prove that there are  $(1, 1)$ -splittings with arbitrarily high distance (Theorem 2.7).

## 2. Statement of results

Let  $K$  be a knot in a closed 3-manifold  $M$ . By  $E(K)$ , we mean the *exterior* of  $K$  in  $M$ , i.e.,  $E(K) = \text{cl}(M - N(K))$ , where  $N(K)$  is a regular neighborhood of  $K$  in  $M$ .

DEFINITION 2.1. (1)  $K$  is a *trivial knot* if  $K$  bounds a disk in  $M$ .

(2)  $K$  is a *core knot* if  $K$  is non-trivial and  $M$  admits a genus one Heegaard splitting  $(V_1, V_2; P)$  such that  $K$  is isotopic to the core of  $V_i$  for  $i = 1$  or  $2$ .

(3)  $K$  is a *torus knot* if  $K$  is isotopic to a simple loop on a genus one Heegaard surface of  $M$  and is not a core knot.

(4)  $K$  is a *2-bridge knot* if there is a genus zero Heegaard splitting  $(B_1, B_2; P_0)$  of  $S^3$  such that  $(B_i, B_i \cap K)$  ( $i = 1, 2$ ) is a 2-string trivial tangle. (Note that a trivial knot in  $S^3$  is also regarded as a 2-bridge knot.)

(5) For a pair  $\alpha$  ( $\geq 4$ ) and  $\beta$  of coprime integers and an element  $r \in \mathbb{Q} \cup \{1/0\}$ ,  $K(\alpha, \beta; r)$  denotes the knot  $K_2$  in  $K_1(r)$ , where  $K_1 \cup K_2$  is the 2-bridge link of type  $(\alpha, \beta)$  (cf. Chapter 10 of [22]) and  $K_1(r)$  is the manifold obtained by  $r$ -surgery on  $K_1$ .

By an argument similar to that in Section 1 of [18], we can see that  $K(\alpha, \beta; r)$  is a  $(1, 1)$ -knot. These knots form an important family of  $(1, 1)$ -knots (see [1], [3])

and [8]).

For the definition of other standard terms in three-dimensional topology and knot theory, we refer to [10], [12] and [22].

In this paper, we prove the following theorems.

**Theorem 2.2.** *Let  $K$  be a  $(1, 1)$ -knot in  $M$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$ . Then  $d(W_1, W_2) = 0$  if and only if  $K$  is a trivial knot.*

Note that Theorem 1.1 of [9] essentially implies Theorem 2.2.

**Theorem 2.3.** *Let  $K$  be a  $(1, 1)$ -knot in  $M$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$ . Then  $d(W_1, W_2) = 1$  if and only if  $M$  is  $S^2 \times S^1$  and  $K$  is a core knot.*

**Theorem 2.4.** *Let  $K$  be a  $(1, 1)$ -knot in  $M$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$ . If  $d(W_1, W_2) = 2$ , then one of the following holds.*

- (1)  $M$  is  $S^3$  and  $K$  is a non-trivial 2-bridge knot.
- (2)  $M$  is a lens space and  $K$  is a core knot.
- (3)  $K$  is a non-trivial torus knot.
- (4)  $E(K)$  contains an essential torus.
- (5)  $K$  is non-trivial and  $K = K(\alpha, \beta; r)$  for some  $\alpha, \beta$  and  $r$ .

*Conversely, if  $(M, K)$  satisfies one of (1)–(4), then any  $(1, 1)$ -splitting of  $(M, K)$  has distance  $= 2$ .*

In the above theorem, by a *lens space*, we mean a closed 3-manifold which admits a Heegaard splitting of genus one and is homeomorphic to neither  $S^3$  nor  $S^2 \times S^1$ . To prove Theorem 2.4, we need the following results.

- The classification of  $(1, 1)$ -splittings of 2-bridge knots in  $S^3$  by T. Kobayashi and O. Saeki [15].
- The classification of  $(1, 1)$ -splittings of core knots in lens spaces by C. Hayashi [6].
- The classification of  $(1, 1)$ -splittings of torus knots by K. Morimoto [17].
- A characterization of  $(1, 1)$ -splittings of  $(1, 1)$ -knots whose exteriors contain an essential torus (Proposition 6.1), which generalizes results of C. Hayashi [7] (cf. [18]).

Moreover, we prove the following characterization of  $(1, 1)$ -knots whose exteriors contain an essential torus. A torus properly embedded in a compact orientable 3-manifold is called an *essential torus* if it is incompressible and not  $\partial$ -parallel in the 3-manifold.

**Theorem 2.5.** *The exterior of a  $(1, 1)$ -knot  $K$  in  $M$  contains an essential torus if and only if  $K$  belongs to  $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$  or  $\mathcal{K}_4$ .*

In the above theorem,  $\mathcal{K}_i$  ( $i = 1, 2, 3, 4$ ) denote the families of  $(1, 1)$ -knots defined

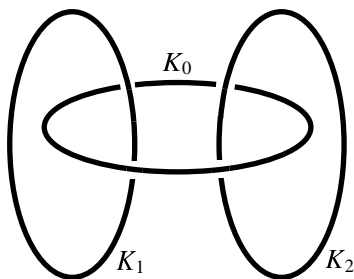


Fig. 1.

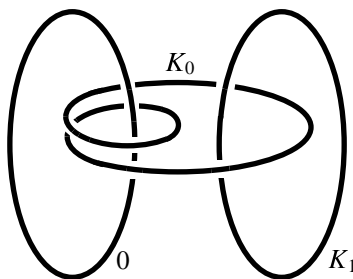


Fig. 2.

as follows.

(1)  $K \in \mathcal{K}_1$  if  $K$  is a knot in lens spaces which is the connected sum of a core knot in a lens space and a non-trivial 2-bridge knot.

(2)  $K \in \mathcal{K}_2$  if  $K$  is constructed as follows. Let  $K_0$  be a non-trivial torus knot in a closed 3-manifold  $M$ , and let  $L = K_1 \cup K_2$  be a 2-bridge link of type  $(\alpha, \beta)$  with  $\alpha \geq 4$ . Let  $\varphi: E(K_2) \rightarrow N(K_0)$  be an orientation-preserving homeomorphism which takes a meridian  $m_2 \subset \partial E(K_2)$  of  $K_2$  to a regular fiber  $f \subset (\partial N(K_0) \cap P)$  of  $E(K_0)$ . Then  $K = \varphi(K_1) \subset N(K_0) \subset M$ .

(3)  $K \in \mathcal{K}_3$  if  $K$  is constructed as follows. Let  $K_0 \cup K_1 \cup K_2$  be the connected sum of two Hopf links illustrated in Fig. 1, and let  $K'_1 \cup K'_2$  be a non-trivial 2-bridge link. Set  $M = E(K_1 \cup K_2) \cup_{(\varphi_1, \varphi_2)} E(K'_1 \cup K'_2)$ , where  $\varphi_i: \partial E(K_i) \rightarrow \partial E(K'_i)$  is an orientation-reversing homeomorphism which takes a preferred longitude  $l_i \subset \partial E(K_i)$  of  $K_i$  to a meridian  $m_i \subset \partial E(K'_i)$  of  $K'_i$  ( $i = 1$  and  $2$ ). Then  $K = K_0 \subset E(K_1 \cup K_2) \subset M$ . It should be noted that  $M \cong S^2 \times S^1$ . This can be seen as follows. For  $(i, j) = (1, 2)$  and  $(2, 1)$ , let  $D_i$  be a disk in  $E(K_j)$  bounded by  $l_i$ . Then each of  $\text{cl}(E(K_1 \cup K_2) - N(D_1 \cup D_2))$  and  $E(K'_1 \cup K'_2) \cup N(D_1 \cup D_2)$  is homeomorphic to  $S^2 \times [0, 1]$ .

(4)  $K \in \mathcal{K}_4$  if  $K$  is constructed as follows. Let  $K_0$  be  $K(4, 1; 0)$  and  $K_1$  a meridian of  $K_0$  (see Fig. 2). Let  $l_1 \subset \partial E(K)$  be a longitude of  $K_1$  which bounds a disk in  $E(K_1)$  intersecting  $K_0$  transversely in a single point. Let  $K_2$  be a non-trivial 2-bridge knot and  $\varphi: \partial E(K_1) \rightarrow \partial E(K_2)$  an orientation-reversing homeomorphism which takes  $l_1$  to a meridian of  $K_2$ . Set  $M = E(K_1) \cup_{\varphi} E(K_2)$ . Then  $K = K_0 \subset E(K_1) \subset M$ . It should be noted that  $M \cong S^2 \times S^1$ . This can be seen by using the fact that the union of  $E(K_2)$  and a regular neighbourhood of a disk in  $E(K_1)$  bounded by  $l_1$  is a 3-ball.

By using Thurston's hyperbolization theorem of Haken manifolds (see for example [13]), we can obtain the following corollary.

**Corollary 2.6.** *Let  $K$  be a  $(1, 1)$ -knot in  $M$ . Suppose that  $(M, K)$  is not equivalent to  $K(\alpha, \beta; r)$  for any  $\alpha, \beta$  and  $r$ , and that the bridge index of  $K$  is at least three if  $M \cong S^3$ . Then  $K$  is a hyperbolic knot if and only if it has a  $(1, 1)$ -splitting*

with distance  $\geq 3$ .

In the last section, we construct  $(1, 1)$ -splittings with arbitrarily high distance.

**Theorem 2.7.** *Let  $M$  be a closed 3-manifold which admits a genus one Heegaard splitting. Then for any positive integer  $n$ , there is a  $(1, 1)$ -knot in  $M$  which has a  $(1, 1)$ -splitting with distance  $> n$ .*

### 3. The structure of $\mathcal{K}(W_i)$

In this section, we describe the structure of the simplicial complex  $\mathcal{K}(W_i)$ . Throughout this section,  $W = (V, t)$  denotes a pair of a solid torus  $V$  and a trivial arc  $t$  properly embedded in  $V$ , and  $\Sigma$  denotes the twice punctured torus  $\partial V - t$ . The two punctures of  $\Sigma$  are denoted by  $p_1$  and  $p_2$ . Two subspaces  $X$  and  $Y$  in  $W$  are said to be *pairwise isotopic*, if there is an ambient isotopy  $\{h_s\}_{0 \leq s \leq 1}$  of  $V$  such that  $h_0 = id$ ,  $h_s(t) = t$  and  $h_1(X) = Y$ .

**DEFINITION 3.1.** An essential loop in  $\Sigma$  is called an  $\varepsilon$ -loop (an  $\iota$ -loop resp.) if it is essential (inessential resp.) in  $\partial V$ .

**DEFINITION 3.2.** Let  $D$  be a properly embedded disk in  $V$ .

- (1)  $D$  is called an  $\iota$ -disk in  $W$  if  $D \cap t = \emptyset$  and  $\partial D$  is an  $\iota$ -loop on  $\Sigma$ .
- (2)  $D$  is called an  $\varepsilon_0$ -disk in  $W$  if  $D \cap t = \emptyset$  and  $\partial D$  is an  $\varepsilon$ -loop on  $\Sigma$ .
- (3)  $D$  is called an  $\varepsilon_1$ -disk in  $W$  if  $D \cap t = \{1 \text{ point}\}$  and  $\partial D$  is an  $\varepsilon$ -loop on  $\Sigma$ .

**Lemma 3.3.** *Let  $D_0$  be an  $\varepsilon_0$ -disk in  $W$  with  $\alpha = \partial D_0$ , and let  $\beta$  be an essential loop in  $\Sigma$  disjoint from  $\alpha$ . Then precisely one of the following conditions holds.*

- (1)  $\beta$  is isotopic to  $\alpha$  in  $\Sigma$ .
- (2)  $\beta$  bounds an  $\iota$ -disk in  $W$ .
- (3)  $\beta$  bounds an  $\varepsilon_1$ -disk in  $W$ .

**Proof.** Let  $B$  be the 3-ball obtained by cutting  $V$  along  $D_0$ , and let  $D'_0$  and  $D''_0$  be the copies of  $D_0$  in  $\partial B$ .

**CASE 1.** Suppose that  $\beta$  does not separate  $D'_0$  and  $D''_0$  in  $\partial B$ .

Then  $\beta$  does not separate  $p_1$  and  $p_2$  in  $\partial B$ , because  $\beta$  is essential in  $\Sigma$ . Let  $t'$  be a properly embedded arc in  $B$  with  $\partial t' = \{p_1, p_2\}$  which is parallel to an arc in  $\partial B - \beta$  joining  $p_1$  to  $p_2$ . Then  $\beta$  bounds a separating disk  $D_\beta$  in  $B$  disjoint from  $t'$ . Since  $t'$  is isotopic to  $t$  in  $B$  relative  $D'_0 \cup D''_0$ , the arc  $t'$  in  $V$  is isotopic to  $t$  in  $V$  relative  $\{p_1, p_2\}$ . Moreover by the hypothesis of Case 1,  $D_\beta$  cuts  $(V, t)$  into  $(V_1, t)$  and  $(V_2, \emptyset)$ , where  $V_1$  is a 3-ball and  $V_2$  is a solid torus. Hence the condition (2) holds.

**CASE 2.** Suppose that  $\beta$  separates  $D'_0$  and  $D''_0$  in  $\partial B$ .

Then we can see, by an argument similar to the above, that the condition (3)

or (1) holds according as  $\beta$  separates  $\{p_1, p_2\}$  in  $\partial B$  or not.

This completes the proof of Lemma 3.3.  $\square$

**Lemma 3.4.** *Any two  $\varepsilon_0$ -disks in  $W$  are pairwise isotopic.*

*Proof.* Let  $D$  and  $D'$  be  $\varepsilon_0$ -disks in  $W$ . If  $D \cap D' = \emptyset$ , then we can see that  $D \cup D'$  bounds a product region disjoint from  $t$  by an argument similar to that of Lemma 3.3. Hence we may assume that  $D$  and  $D'$  intersect transversely,  $|D \cap D'|$  is minimized up to pairwise isotopy in  $W$  and that  $|D \cap D'| > 0$ , where  $|\cdot|$  is the number of connected components. By a standard innermost disk argument, we can see that  $D \cap D'$  has no loop components. Let  $\gamma$  be a component of  $D \cap D'$  which is outermost in  $D'$  and  $\delta'_1$  the outermost disk in  $D'$  with  $\gamma \subset \partial\delta'_1$ , that is, the interior of  $\delta'_1$  is disjoint from  $D$ . The arc  $\gamma$  also cuts  $D$  into two disks  $\delta_1$  and  $\delta_2$ . Then each of  $\delta_1 \cup \delta'_1$  and  $\delta_2 \cup \delta'_1$  is a properly embedded disk in  $V$  disjoint from  $t$ . If either  $\partial(\delta_1 \cup \delta'_1)$  or  $\partial(\delta_2 \cup \delta'_1)$  is inessential in  $\partial(V - t)$ , then we can decrease  $|D \cap D'|$  by a pairwise isotopy of  $D$  in  $W$ , a contradiction. So we may assume that  $\delta_1 \cup \delta'_1$  and  $\delta_2 \cup \delta'_1$  are  $\varepsilon_0$ -disks or  $\iota$ -disks in  $W$ .

CLAIM. At least one of  $\delta_1 \cup \delta'_1$  and  $\delta_2 \cup \delta'_1$  is an  $\iota$ -disk in  $W$ .

*Proof.* Suppose that  $\delta_1 \cup \delta'_1$  is a  $\varepsilon_0$ -disk in  $W$  to show that  $\delta_2 \cup \delta'_1$  is an  $\iota$ -disk. Let  $B$  be the 3-ball obtained from  $V$  by cutting along  $D$ , and let  $D_+$  and  $D_-$  be the copies of  $D$  in  $\partial B$ . We denote the image of  $\delta'_1$  in  $B$  by the same symbol. Then we may assume  $\delta'_1 \cap D_+ = \emptyset$  and  $\delta'_1 \cap D_- = \gamma$ . By cutting  $B$  along  $\delta'_1$ , we obtain 3-balls  $B_1$  and  $B_2$  with  $D_+ \subset \partial B_1$ ,  $(\delta_1 \cup \delta'_1) \subset \partial B_1$  and  $(\delta_2 \cup \delta'_1) \subset \partial B_2$ . Since  $D$  and  $\delta'_1$  are disjoint from  $t$  in  $V$ , precisely one of  $B_1$  and  $B_2$  contains  $t$ . If  $t \subset B_1$ , then  $\partial(\delta_2 \cup \delta'_1)$  is inessential in  $\partial(V - t)$ , a contradiction. Hence  $t \subset B_2$ , and  $\delta_2 \cup \delta'_1$  is an  $\iota$ -disk in  $W$ .  $\square$

Let  $B$ ,  $D_+$ ,  $D_-$ ,  $B_1$  and  $B_2$  be as above. Put  $\delta'_2 = \text{cl}(D' - \delta'_1)$ , and let  $A$  be the annulus defined by  $A = \partial B_1 \cap (\partial B - \text{int}(D_+ \cup D_-))$ . Put  $\alpha = \partial D' \cap \partial\delta'_2$ , and let  $\partial\gamma \ni p_1, p_2, \dots, p_n \in \partial\gamma$  be the components of  $\partial D \cap \alpha$  sitting on  $\alpha$  in this order. Then by the minimality of  $|D \cap D'|$ , we may assume that  $A \cap \partial\delta'_2$  consists of essential arcs in the annulus  $A$ . Let  $\alpha_i$  be the subarc of  $\alpha$  joining  $p_i$  to  $p_{i+1}$  in  $\alpha$ , and let  $p_i^+$ ,  $p_i^-$ , respectively the copies of  $p_i$  in  $\partial D_+$  and  $\partial D_-$  ( $i = 1, 2, \dots, n-1$ ). Then  $\alpha_1 \cap D_+ = p_1^+$  and  $\alpha_1 \cap D_- = p_2^-$ , because  $\alpha_1$  is essential in  $A$ . Inductively, we obtain  $\alpha_i \cap D_+ = p_i^+$  and  $\alpha_i \cap D_- = p_{i+1}^-$  ( $i = 1, 2, \dots, n-1$ ). In particular,  $\alpha_{n-1} \cap D_+ = p_{n-1}^+$  and  $\alpha_{n-1} \cap D_- = p_n^-$ . This means that  $D'$  does not intersect  $D$  transversely in  $p_n$ , a contradiction. Hence the interior of  $A$  is disjoint from  $\partial\delta'_2$ , and there is an  $\varepsilon_0$ -disk obtained by moving  $D_+$  so that it is disjoint from  $D'$ . This means  $D'$  is isotopic to  $D$ .  $\square$

**Lemma 3.5.** *Let  $[\alpha]$  be the vertex of  $\mathcal{K}(W)$  represented by the boundary of an  $\varepsilon_0$ -disk, and let  $[\beta]$  be an arbitrary vertex of  $\mathcal{K}(W)$  different from  $[\alpha]$ . Then  $[\beta]$  is represented by an  $\iota$ -loop disjoint from an  $\varepsilon$ -loop representing  $[\alpha]$ .*

*Proof.* If  $[\beta]$  is represented by an  $\varepsilon$ -loop, then we have  $[\alpha] = [\beta]$  by Lemma 3.4, a contradiction. So  $[\beta]$  is represented by an  $\iota$ -loop, say  $\beta$ . Let  $D_\beta$  be a disk in  $V - t$  bounded by  $\beta$ . Since  $\beta$  is inessential in  $V$ , there is an essential disk  $D$  in  $V$  disjoint from  $D_\beta$  (and hence disjoint from  $t$ ). By Lemma 3.4,  $\partial D$  represents  $[\alpha]$  and hence we obtain the desired result.  $\square$

**Lemma 3.6.** *Any two mutually disjoint  $\iota$ -disks in  $W$  are pairwise isotopic.*

*Proof.* Let  $D$  and  $D'$  be mutually disjoint  $\iota$ -disks in  $W$  and put  $\beta = \partial D$  and  $\beta' = \partial D'$ . Then  $D$  cuts  $(V, t)$  into  $(V_1, t)$  and  $(V_2, \emptyset)$ , where  $V_1$  is a 3-ball and  $V_2$  is a solid torus. If necessary, by exchanging the names  $D$  and  $D'$  of disks, we may assume that  $D'$  is contained in  $V_1$  and  $\beta'$  is an inessential loop in  $\partial V_1 - t$ , because  $D'$  is an  $\iota$ -disk and is disjoint from  $D$ . If  $\beta'$  bounds a disk in  $\partial V_1$  disjoint from the copy of  $D$  in  $\partial V_1$ , then  $\beta'$  is inessential in  $\partial V - t$ , a contradiction. Hence  $\beta'$  separates the copy of  $D$  from  $\partial t$  in  $\partial V_1$ , and this implies  $D$  and  $D'$  are pairwise isotopic.  $\square$

Let  $\alpha$  be an  $\varepsilon$ -loop which bounds an  $\varepsilon_0$ -disk, say  $D_\alpha$ . We fix a properly embedded arc, say  $t_0$ , in  $\partial V$  such that  $\partial t_0 = \partial t$ ,  $t_0 \cap \alpha = \emptyset$  and  $t \cup t_0$  bounds a disk in  $V$ . Let  $B$  be the 3-ball obtained by cutting  $V$  along  $D_\alpha$ , and let  $D'_\alpha$  and  $D''_\alpha$  be the copies of  $D_\alpha$  in  $\partial B$ . Set  $\mathcal{P} = \partial t \cup \{\text{the centers of } D'_\alpha \text{ and } D''_\alpha\}$ . Then  $(\partial B, \mathcal{P})$  is identified with  $(\mathbb{R}^2, \mathbb{Z}^2)/\Gamma$ , where  $\Gamma$  is the group of isometries of  $\mathbb{R}^2$  generated by  $\pi$ -rotations about the points of the integral lattice  $\mathbb{Z}^2$ . Here  $t_0$  is identified with a line in  $\mathbb{R}^2$  of slope  $1/0$ , i.e., a lift of  $t_0$  joins  $(0, 0)$  to  $(0, 1)$  in  $\mathbb{R}^2$ .

Let  $\mathcal{A}$  be the set of the vertices of  $\mathcal{K}(W)$  different from  $[\alpha]$ , where  $[\alpha]$  is the vertex of  $\mathcal{K}(W)$  represented by  $\alpha$ . In the following, we define a map  $\varphi: \mathcal{A} \rightarrow \mathbb{Q} \cup \{1/0\}$ . Let  $[\beta]$  be an element of  $\mathcal{A}$ . Then by Lemma 3.5,  $[\beta]$  is represented by an  $\iota$ -loop, say  $\beta$ , which is disjoint from  $\alpha$ . Let  $t_\beta$  be an arc in  $\partial V - \beta$  joining distinct components of  $\partial t$ . Note that  $t_\beta$  is unique up to isotopy relative to the endpoints. Let  $\tilde{t}_\beta: [0, 1] \rightarrow \mathbb{R}^2$  be a lift of  $t_\beta: [0, 1] \rightarrow (\partial B, \mathcal{P})$ . Then  $\tilde{t}_\beta(1) - \tilde{t}_\beta(0)$  is an integral vector, say  $(p, q)$ , in  $\mathbb{R}^2$ .

**Lemma 3.7.** *Let  $[\beta]$  and  $(p, q)$  be as above. Then the rational number  $q/p$  does not depend on the choice of a representative of  $[\beta]$ , and hence the correspondence  $\beta \mapsto q/p$  induces a well-defined map  $\varphi: \mathcal{A} \rightarrow \mathbb{Q} \cup \{1/0\}$ . Moreover  $\varphi$  is injective and the image is equal to  $\{q/p \in \mathbb{Q} \cup \{1/0\} \mid (p, q) \equiv (0, 1) \pmod{2}\}$ .*

Proof. Let  $\beta'$  be another representative disjoint from  $\alpha$  of  $[\beta]$ . Then there is a homotopy in  $\Sigma$  between  $\beta$  and  $\beta'$ . Since  $\alpha$  is an essential loop in  $\Sigma$  and is homotopic to neither  $\beta$  nor  $\beta'$ , we can modify the homotopy so that it is disjoint from  $\alpha$ . Hence  $\beta$  and  $\beta'$  are homotopic in  $\Sigma - \alpha$  and therefore in the four times punctured 2-sphere  $\partial B - \mathcal{P}$ . This implies that  $\varphi$  is well-defined and injective, because it is well known that the correspondence  $\beta \mapsto q/p$  induces a well-defined injective map from the set of the isotopy classes of essential loops in  $\partial B - \mathcal{P}$  to  $\mathbb{Q} \cup \{1/0\}$  (cf. Section 2 of [5]). Moreover, since an  $\iota$ -loop representing  $[\beta]$  does not separate  $\partial t$  in  $\partial V$ , we see  $(p, q) \equiv (0, 1) \pmod{2}$ . On the other hand, it is easy to see that for any  $q/p \in \mathbb{Q} \cup \{1/0\}$  with  $(p, q) \equiv (0, 1) \pmod{2}$ , there is a vertex  $[\beta] \in \mathcal{A}$  with  $\varphi([\beta]) = q/p$ . Hence we obtain the desired result.  $\square$

**Proposition 3.8.** *Let  $[\alpha]$  be the vertex of  $\mathcal{K}(W)$  represented by the boundary of an  $\varepsilon_0$ -disk of  $W$ , and let  $\mathcal{A}$  be the countably infinite set as above. Then  $\mathcal{K}(W)$  is isomorphic to the join  $\{[\alpha]\} * \mathcal{A}$ .*

Proof. By Lemma 3.4, we see that  $[\alpha]$  is unique. Lemma 3.5 indicates that for any vertex  $[\beta]$  of  $\mathcal{A}$ , there is an edge joining  $[\beta]$  to  $[\alpha]$ . On the other hand, by Lemma 3.6, there are no edges of  $C(\Sigma)$  joining distinct vertices of  $\mathcal{A}$ .  $\square$

#### 4. (1, 1)-splittings of distance = 0

**Lemma 4.1.** *Let  $K$  be a (1, 1)-knot in  $M$  and  $(W_1, W_2; P)$  a (1, 1)-splitting of  $(M, K)$ . Then  $K$  is a trivial knot if and only if there are an  $\iota$ -disk  $D_i$  in  $W_i$  with  $\partial D_1 = \partial D_2$  ( $i = 1$  and  $2$ ).*

Proof. We first prove the “only if part”. Suppose that  $K$  is trivial. Let  $D$  be a disk in  $M$  with  $\partial D = K$ . Then by Theorem 1.1 of [9], we can isotope  $D$  so that  $D \cap P$  separates  $D$  into two disks. Set  $D_i = \partial N(D) \cap V_i$  ( $i = 1$  and  $2$ ). Then we see that  $D_i$  is an  $\iota$ -disk and  $\partial D_1 = \partial D_2$  ( $i = 1$  and  $2$ ).

We next prove the “if part”. Suppose that there are an  $\iota$ -disk  $D_i$  in  $W_i$  ( $i = 1$  and  $2$ ). Then  $D_1 \cup D_2$  forms a 2-sphere which cuts  $(M, K)$  into  $(M - \text{int } B^3, \emptyset)$  and  $(B^3, 1\text{-bridge knot})$  and hence  $K$  is a trivial knot.  $\square$

**Lemma 4.2.** *Let  $K$  be a (1, 1)-knot in  $S^2 \times S^1$  and  $(W_1, W_2; P)$  a (1, 1)-splitting of  $(S^2 \times S^1, K)$ . Then  $K$  is a trivial knot if and only if there are an  $\varepsilon_0$ -disk  $D_1$  in  $W_1$  and an  $\varepsilon_0$ -disk  $D_2$  in  $W_2$  with  $\partial D_1 = \partial D_2$ .*

Proof. We first prove the “if part”. Suppose that the latter condition in Lemma 4.2 holds. Then there are  $\iota$ -disks  $D'_1$  and  $D'_2$  in  $W_1$  and  $W_2$ , respectively, with  $\partial D'_i \cap \partial D_i = \emptyset$  ( $i = 1, 2$ ) and  $\partial D'_1 = \partial D'_2$ . Hence by Lemma 4.1,  $K$  is a trivial knot.

Suppose conversely that  $K$  is a trivial knot in  $S^2 \times S^1$ . By Lemma 4.1, there are

an  $\iota$ -disk  $\delta_i$  in  $W_i$  with  $\partial\delta_1 = \partial\delta_2$  ( $i = 1$  and  $2$ ). Then there are  $\varepsilon_0$ -disks in each of  $W_1$  and  $W_2$  such that they are disjoint from  $\delta_1 \cup \delta_2$  and they share their boundaries since the manifold is  $S^2 \times S^1$ . Hence we see that the latter condition holds.  $\square$

**Proof of Theorem 2.2.** Suppose that  $K$  is a trivial knot in  $M$ . Then by Lemma 4.1, we have  $d(W_1, W_2) = 0$ .

Conversely, let  $K$  be a  $(1, 1)$ -knot in  $M$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$  with  $d(W_1, W_2) = 0$ . Then there is an essential loop  $x$  in  $\Sigma = P - K$  which bounds a disk in  $V_i - t_i$  for each  $i = 1$  and  $2$ .

If  $x$  is an  $\varepsilon_0$ -loop, then  $(W_1, W_2; P)$  satisfies the condition of Lemma 4.2. Hence  $M$  is  $S^2 \times S^1$  and  $K$  is a trivial knot.

If  $x$  is an  $\iota$ -loop, then  $(W_1, W_2; P)$  satisfies the condition of Lemma 4.1, that is,  $K$  is a trivial knot in  $M$ .

We have completed the proof of Theorem 2.2.  $\square$

## 5. $(1, 1)$ -splittings of distance = 1

**Proposition 5.1.** *Let  $K$  be a  $(1, 1)$ -knot in  $S^2 \times S^1$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(S^2 \times S^1, K)$ . Then  $K$  is a core knot if and only if there are an  $\varepsilon_0$ -disk  $D_i$  in  $W_i$  and an  $\varepsilon_1$ -disk  $D_j$  in  $W_j$  with  $\partial D_i = \partial D_j$  for  $(i, j) = (1, 2)$  or  $(2, 1)$ .*

**Proof.** The “if part” follows from the light bulb theorem (cf. Chapter 9, Section E, 4 Exercise of [22]).

To prove the “only if part”, suppose that  $K$  is a core knot in  $S^2 \times S^1$ . Then there is an essential 2-sphere  $S$  which intersects  $K$  in one point. Put  $S_i = S \cap V_i$  ( $i = 1$  and  $2$ ). We may assume that each component of  $S_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_i = (V_i, t_i)$ . Note that  $|S_1| > 0$  and that  $S_1$  contains at most one  $\varepsilon_1$ -disk component. Let  $D$  be an  $\varepsilon_0$ -disk in  $W_2$  such that  $D$  intersects  $S_2$  transversely. We choose  $S$  and  $D$  so that each component of  $S_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ , and the pair  $(|S_1|, |S_2 \cap D|)$  is minimized with respect to the lexicographic order.

If  $|S_1| = 1$ , then  $S \cap P$  is an  $\varepsilon$ -loop because  $S$  is an essential 2-sphere in  $S^2 \times S^1$ . Hence the assertion obviously holds. So we may assume  $|S_1| > 1$ .

**CLAIM 1.**  $S_2 \cap D \neq \emptyset$ .

**Proof.** Suppose that  $S_2$  is disjoint from  $D$ . Let  $B$  be the 3-ball obtained by cutting  $V_2$  along  $D$ . Then there is a disk  $E$  on  $\partial B$  with  $E \cap S_2 = \partial E$  and  $|E \cap K| \leq 1$ . Let  $E'$  be the disk obtained from  $E$  by pushing the interior of  $E$  into the interior of  $B$ . Then  $\partial E'$  cuts  $S$  into two disks  $Q_1$  and  $Q_2$ . Precisely one of them, say  $Q_1$ , is a component of  $S_1$ .

Suppose that  $|E' \cap K| = 0$ . If  $|Q_1 \cap K| = 1$ , then  $Q_1 \cup E'$  is a 2-sphere which inter-

sects  $K$  in one point. Hence the disks  $Q_1$  and  $E'$  satisfy the desired condition. So we may assume that  $|Q_1 \cap K| = 0$  and hence  $|Q_2 \cap K| = 1$ . Let  $S'$  be the 2-sphere obtained from  $Q_2 \cup E'$  by pushing  $\partial E'$  into the interior of  $V_2$  slightly. Then each component of  $S'_1 := S' \cap V_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ , and  $|S'_1| < |S_1|$ , a contradiction.

Suppose that  $|E' \cap K| = 1$ . If  $|Q_1 \cap K| = 0$ , then  $Q_1 \cup E'$  is a 2-sphere which intersects  $K$  in one point, and hence the disks  $Q_1$  and  $E'$  satisfy the desired condition. So we may assume that  $|Q_1 \cap K| = 1$  and hence  $|Q_2 \cap K| = 0$ . Let  $S'$  be the 2-sphere obtained from  $Q_2 \cup E'$  by pushing  $\partial E'$  into  $V_2$  slightly. Then each component of  $S'_1 := S' \cap V_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ , and  $|S'_1| < |S_1|$ , a contradiction.  $\square$

CLAIM 2.  $S_2 \cap D$  has no loop components.

Proof. Suppose that  $S_2 \cap D$  has a loop component. Let  $\sigma$  be a loop component of  $S_2 \cap D$  which is innermost in  $D$  and  $D_\sigma$  the innermost disk with  $\sigma = \partial D_\sigma$ , that is, the interior of  $D_\sigma$  is disjoint from  $S_2$ . Then  $\sigma$  cuts  $S$  into two disks  $E_1$  and  $E_2$ . We can assume that  $|E_1 \cap K| = 1$ . Since  $D_\sigma$  is disjoint from  $K$ ,  $S' = E_1 \cup D_\sigma$  is a 2-sphere which intersects  $K$  in one point. Put  $S'_i = S' \cap V_i$  ( $i = 1$  and  $2$ ). Note that  $S'_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ . If  $\sigma$  is essential in  $S_2$ , then  $|S'_1| < |S_1|$ , a contradiction. If  $\sigma$  is inessential in  $S_2$ , then  $|S'_1| = |S_1|$ . In this case, by isotoping  $S'$  so that  $D_\sigma$  is disjoint from  $D$ , we see that  $|S'_2 \cap D| < |S_2 \cap D|$ , a contradiction.  $\square$

By Claim 1 and Claim 2, there is an arc component  $\gamma$  of  $S_2 \cap D$  which is outermost in  $D$ . Let  $D_\gamma \subset D$  be the outermost disk with  $\gamma \subset \partial D_\gamma$ . Put  $\gamma' = \text{cl}(\partial D_\gamma - \gamma)$ . Let  $F$  be the component of the surface obtained by cutting  $\partial V_1$  along  $\partial S_1$  such that  $\gamma' \subset F$ . Let  $S^{(1)}$  be a 2-sphere obtained by isotoping  $S$  along  $D_\gamma$  near the arc  $\gamma$ , and put  $S_i^{(1)} = S^{(1)} \cap V_i$  ( $i = 1$  and  $2$ ).

CLAIM 3. The arc  $\gamma'$  is essential in  $F$ .

Proof. Suppose that  $\gamma'$  is inessential in  $F$ . Then we obtain an annulus component  $A$  in  $S_1^{(1)}$  such that one of the components of  $\partial A$  bounds a disk  $E$  in  $\partial V_1$ . Note that  $|E \cap K| \leq 2$  and  $\partial E$  cuts  $S$  into two disks  $R_1$  and  $R_2$ . Since  $S$  intersects  $K$  transversely in one point, we may assume that  $|R_1 \cap K| = 1$  and  $|R_2 \cap K| = 0$ .

Suppose that  $|E \cap K| = 0$ . If  $A \subset R_1$ , let  $S'$  be a 2-sphere obtained from  $R_1 \cup E$  by pushing  $E$  into the interior of  $V_1$ ; otherwise, let  $S'$  be a 2-sphere obtained from  $R_1 \cup E$  by pushing the interior of  $E$  into the interior of  $V_2$ . Then we see that each component of  $S'_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ , and that  $(|S'_1|, |S'_2 \cap D|) < (|S_1|, |S_2 \cap D|)$ , a contradiction.

Suppose that  $|E \cap K| = 1$ . If  $A \subset R_2$ , let  $S'$  be a 2-sphere obtained from  $R_2 \cup E$  by

pushing  $E$  into the interior of  $V_1$ ; otherwise, let  $S'$  be a 2-sphere obtained from  $R_2 \cup E$  by pushing the interior of  $E$  into the interior of  $V_2$ . Then we see that each component of  $S'_1$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ , and that  $(|S'_1|, |S'_2 \cap D|) < (|S_1|, |S_2 \cap D|)$ , a contradiction.

Suppose that  $|E \cap K| = 2$ . If  $\gamma'$  joins an  $\iota$ -disk to itself, then  $E' := \text{cl}(F - E)$  is a disk bounded by a component of  $\partial A$ . Since  $E'$  is disjoint from  $K$ , by an argument similar to the case of  $|E \cap K| = 0$ , we obtain a contradiction by using the disk  $E'$  instead of  $E$ . So we may assume that  $\gamma'$  joins an  $\varepsilon_0$ -disk to itself. Then there is an  $\varepsilon_0$ -disk disjoint from  $\partial E$ . By Lemma 3.3,  $\partial E$  bounds an  $\iota$ -disk. Hence by an argument similar to the case of  $|E \cap K| = 0$ , we obtain a contradiction by using the  $\iota$ -disk instead of  $E$ .  $\square$

CLAIM 4.  $S_1$  has no  $\varepsilon_1$ -disk components.

Proof. Suppose that  $S_1$  has an  $\varepsilon_1$ -disk component. Then  $S_1$  has no  $\iota$ -disk components. Thus  $S_1$  has  $\varepsilon_0$ -disk components, because  $|S_1| > 1$ . Hence by Claim 3,  $\gamma'$  joins distinct components of  $S_1$ .

CASE 1. The arc  $\gamma'$  joins distinct  $\varepsilon_0$ -disks.

Let  $\delta$  be the disk component of  $S_1^{(1)}$  obtained from these disks. Then we can push  $\delta$  out of  $V_1$  fixing  $t_1$ . After this operation, we see that each component of  $S_1^{(1)}$  is either an  $\varepsilon_0$ -disk or an  $\varepsilon_1$ -disk in  $W_1$ , and that  $|S_1^{(1)}| < |S_1|$ , a contradiction.

CASE 2. The arc  $\gamma'$  joins an  $\varepsilon_0$ -disk to an  $\varepsilon_1$ -disk.

Then  $S_1^{(1)}$  has the disk component  $\delta'$  from these disks. Note that  $\delta'$  cuts  $(V_1, t_1)$  into  $(V'_1, t'_1)$  and  $(V''_1, t''_1)$ , where  $V'_1$  is a 3-ball,  $t'_1$  is a trivial arc in  $V'_1$ ,  $V''_1$  is a solid torus and  $t''_1$  is a trivial arc in  $V''_1$ . So we can push  $\delta'$  out of  $V_1$  through  $(V'_1, t'_1)$ . After this operation, each component of  $S_1^{(1)}$  is either an  $\varepsilon_0$ -disk or an  $\varepsilon_1$ -disk in  $W_1$ , and we have  $|S_1^{(1)}| < |S_1|$ , a contradiction.  $\square$

CLAIM 5.  $S_1$  has no  $\varepsilon_0$ -disk components.

Proof. Suppose that  $S_1$  has an  $\varepsilon_0$ -disk component. Note that  $S_1$  may have  $\iota$ -disk components, because  $S_1$  has no  $\varepsilon_1$ -disk components by Claim 4. Since  $\gamma'$  is essential in  $F$  by Claim 3, we have the following cases.

CASE 1. The arc  $\gamma'$  joins distinct  $\varepsilon_0$ -disks, or joins distinct  $\iota$ -disks.

By an argument similar to Case 1 in the proof of Claim 4, we obtain a contradiction.

CASE 2. The arc  $\gamma'$  joins an  $\varepsilon_0$ -disk to an  $\iota$ -disk.

Then  $S_1^{(1)}$  is either an  $\varepsilon_0$ -disk, an  $\varepsilon_1$ -disk or an  $\iota$ -disk in  $W_1$ , and  $|S_1^{(1)}| < |S_1|$ , a contradiction.

CASE 3. The arc  $\gamma'$  joins an  $\varepsilon_0$ -disk to itself.

By Claim 3,  $\gamma'$  must be essential in  $F$ . Hence  $S_1$  must consist of an  $\varepsilon_0$ -disk and

$\iota$ -disks, and we obtain a Möbius band in  $S_1^{(1)}$ , a contradiction.

CASE 4. The arc  $\gamma'$  joins an  $\iota$ -disk to itself.

Let  $\delta$  be the  $\iota$ -disk component of  $S_1$  with  $\partial\gamma' \subset \partial\delta$ , and let  $\gamma_1$  and  $\gamma_2$  be arcs such that  $\partial\delta = \gamma_1 \cup \gamma_2$  and  $\partial\gamma_1 = \partial\gamma_2 = \partial\gamma'$ . Since  $S_1$  has  $\varepsilon_0$ -disk components, by Claim 3,  $\gamma' \cup \gamma_1$  bounds an  $\varepsilon_0$ -disk, say  $E'$ , whose interior is disjoint from  $S$ . Hence by an argument similar to Claim 3, we have a contradiction by using the disk  $E'$ .  $\square$

By Claim 4 and Claim 5,  $S_1$  consists of  $\iota$ -disks, because  $|S_1| > 1$ . But this implies that  $S$  is inessential in  $S^2 \times S^1$ , a contradiction.

This completes the proof of Proposition 5.1.  $\square$

Proof of Theorem 2.3. Suppose that  $K$  is a core knot in  $S^2 \times S^1$ . By Proposition 5.1, we may assume that there are an  $\varepsilon_0$ -disk  $D_1$  in  $W_1$  and an  $\varepsilon_1$ -disk  $D_2$  in  $W_2$  with  $\partial D_1 = \partial D_2$ . Then there is an  $\varepsilon_0$ -disk  $D'_2$  in  $W_2$  which is disjoint from  $D_2$ . Hence we have  $d(W_1, W_2) = 1$  since Theorem 2.2 implies  $d(W_1, W_2) \neq 0$  for  $(1, 1)$ -splittings of the core knot in  $S^2 \times S^1$ .

Conversely, we suppose  $d(W_1, W_2) = 1$ , that is, there are mutually disjoint essential loops  $x$  and  $y$  in  $\Sigma = P - K$  which bound disks in  $V_1 - t_1$  and  $V_2 - t_2$ , respectively. Suppose that either  $x$  or  $y$ , say  $y$ , is an  $\iota$ -loop. If  $x$  bounds an  $\varepsilon_0$ -disk, then  $y$  bounds an  $\iota$ -disk in  $W_1$  by Lemma 3.3. (Otherwise,  $y$  is pairwise isotopic to  $x$ .) Hence  $K$  is a trivial knot, a contradiction. So we may suppose that  $x$  ( $y$  resp.) bounds an  $\varepsilon_0$ -disk in  $W_1$  ( $W_2$  resp.). Then  $x$  bounds an  $\varepsilon_1$ -disk in  $W_2$  by Lemma 3.3. Hence  $K$  is a core knot in  $S^2 \times S^1$  by Proposition 5.1.

We have completed the proof of Theorem 2.3.  $\square$

## 6. $(1, 1)$ -knots whose exteriors contain essential tori

In this section, we study  $(1, 1)$ -knots whose exteriors contain an essential torus and prove Theorem 2.5 and the following Proposition 6.1.

**Proposition 6.1.** *Let  $K$  be a  $(1, 1)$ -knot in  $M$  whose exterior contains an essential torus. Then every  $(1, 1)$ -splitting  $(W_1, W_2; P)$  of  $(M, K)$  satisfies one of the following conditions.*

(#<sub>a</sub>) *There are an  $\iota$ -disk  $D_i$  in  $W_i$  and an  $\varepsilon_1$ -disk  $D_j$  in  $W_j$  such that  $\partial D_i \cap \partial D_j = \emptyset$  for  $(i, j) = (1, 2)$  or  $(2, 1)$ .*

(#<sub>b</sub>) *There is an annulus  $Z \subset P$  which is incompressible in both  $V_1$  and  $V_2$ , and there is an  $\iota$ -disk  $D_i$  in  $W_i$  with  $\partial D_i \subset Z$  for each  $i = 1$  and  $2$ .*

(#<sub>c</sub>) *There are an  $\varepsilon_1$ -disk  $D_1$  in  $W_1$  and an  $\varepsilon_1$ -disk  $D_2$  in  $W_2$  with  $\partial D_1 = \partial D_2$ .*

Before proving Theorem 2.5 and Proposition 6.1, we present lemmas which describe topological consequences of the conclusions in Proposition 6.1.

**Lemma 6.2** ([7] Lemma 2.1). *Let  $K$  be a non-trivial  $(1, 1)$ -knot in  $M$  with a  $(1, 1)$ -splitting  $(W_1, W_2; P)$  satisfying the condition  $(\#_a)$  of Proposition 6.1. Then one of the following holds.*

- (1)  $K$  is a 2-bridge knot.
- (2)  $K$  is a core knot in a lens space.
- (3)  $K$  belongs to  $\mathcal{K}_1$ .

**REMARK 6.3.** Though this lemma is proved under the assumption that  $M \not\cong S^2 \times S^1$  in [7], we can easily see that the same conclusion holds even if  $M \cong S^2 \times S^1$ . In fact, we can show by using the light bulb theorem that  $K$  is a core knot in this case.

**Lemma 6.4.** *Let  $K$  be a non-trivial  $(1, 1)$ -knot in  $M$  with a  $(1, 1)$ -splitting  $(W_1, W_2; P)$  satisfying the condition  $(\#_b)$  of Proposition 6.1. Then one of the following holds.*

- (1)  $K$  is a core knot or a torus knot.
- (2)  $K = K(\alpha, \beta; r)$  for some  $\alpha, \beta$  and  $r$ .
- (3)  $K$  belongs to  $\mathcal{K}_2$ .

**Proof.** Let  $Z$  be an annulus which satisfies the condition  $(\#_b)$  of Proposition 6.1. For each  $i = 1$  and  $2$ , since  $Z$  is incompressible in  $V_i$ ,  $\partial D_i$  bounds a disk  $D'_i$  in  $Z$ . Let  $A_i$  be an annulus in  $V_i$  obtained from  $Z_i := \text{cl}((Z - D'_i) \cup D_i)$  by pushing the interior of  $Z_i$  into the interior of  $V_i$ . For each  $i = 1$  and  $2$ , let  $(V_{i1}, \emptyset)$  and  $(V_{i2}, t_i)$  be the pair obtained from  $(V_i, t_i)$  by cutting along  $A_i$ , where each of  $V_{i1}$  and  $V_{i2}$  is a solid torus and  $t_i$  is a trivial arc in  $V_{i2}$ . Then we see that  $V_{11} \cup V_{12}$  is either a solid torus or the exterior of a torus knot. On the other hand,  $(V_{i2}, t_i)$  is identified with  $(\text{cl}(B^3 - \tau_1), \tau_2)$ , where  $(B^3, \tau_1 \cup \tau_2)$  is a 2-string trivial tangle, in such a way that the copy of  $A_i$  corresponds to the boundary of the regular neighbourhood of  $\tau_1$ . Since  $V_{11} \cap V_{21}$  is a 2-sphere with two holes which contains the two points  $P \cap K$ , we see that  $(V_{11} \cup V_{21}, K)$  is identified with  $(E(K_2), K_1)$ , where  $K_1 \cup K_2 = L$  is a 2-bridge link.

Suppose that  $L$  is a trivial link. Then  $K_1$  bounds a disk in  $E(K_2)$  and hence  $K$  is a trivial knot, a contradiction.

Suppose that  $L$  is a Hopf link. Then  $K_1$  is isotopic to  $K_2$ . So we can put  $K$  on  $P$ . Hence  $K$  is a core knot or a torus knot.

Suppose that  $V_{11} \cup V_{12}$  is a solid torus. Then we see that  $K = K(\alpha, \beta; r)$  for some  $\alpha, \beta$  and  $r$ .

In other cases, we see that  $A_1 \cup A_2$  is an essential torus. Hence  $K$  belongs to  $\mathcal{K}_2$ .  $\square$

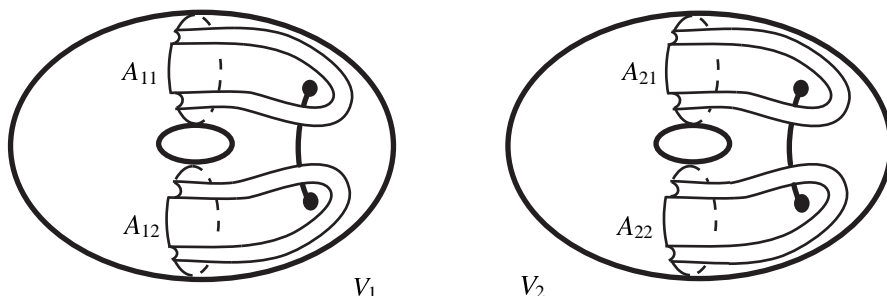


Fig. 3.

**Lemma 6.5.** *Let  $K$  be a non-trivial  $(1, 1)$ -knot in  $M$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$ . Suppose that  $(W_1, W_2; P)$  satisfies the condition  $(\#_c)$  of Proposition 6.1. Then  $M \cong S^2 \times S^1$  and either*

- (1)  $K = K(4, 1; 0)$ , or
- (2)  $K$  belongs to  $\mathcal{K}_3$  or  $\mathcal{K}_4$ .

*Proof.* Let  $D_1$  and  $D_2$  be a pair of disks which give the condition  $(\#_c)$  of Proposition 6.1, and put  $V_i^- = \text{cl}(V_i - N(t_i))$  ( $i = 1$  and  $2$ ). Let  $\alpha_{ij}$  ( $j = 1$  and  $2$ ) be the components of  $\partial(V_i^- \cap N(t_i))$ , and let  $A_{ij}$  ( $j = 1$  and  $2$ ) be annuli properly embedded in  $V_i^-$  satisfying the following conditions (see Fig. 3).

- (1)  $A_{ij}$  is parallel to  $D_i \cap V_i^-$  in  $V_i$ .
- (2)  $A_{ij} \cap N(t_i) = \emptyset$ .
- (3)  $\alpha_{ij}$  is parallel to a component of  $\partial A_{ij}$  in  $\text{cl}(\partial V_i^- - N(t_i))$ .
- (4)  $\partial(A_{11} \cup A_{12}) = \partial(A_{21} \cup A_{22})$ .

For each  $i = 1$  and  $2$ , let  $(V_{i1}, \emptyset)$  and  $(V_{i2}, t_i)$  be the pairs obtained from  $(V_i, t_i)$  by cutting along  $A_{i1} \cup A'_{i2}$ , where  $V_{i1}$  is a genus two handlebody,  $V_{i2}$  is a 3-ball and  $t_i$  is a trivial arc in  $V_{i2}$ . Then  $V_{i1}$  is identified with the exterior of a 2-string trivial tangle  $(B^3, \tau)$  in such a way that the copy of  $A_{i1} \cup A_{i2}$  corresponds to the boundary of the regular neighbourhood of  $\tau$ .

CASE 1.  $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$  composes two tori.

Suppose that one of the tori, say  $T_0$ , is inessential in  $E(K)$ . Then since  $T_0$  is not parallel to  $\partial N(K)$ ,  $T_0$  is compressible in  $E(K)$ . So we can obtain the 2-sphere  $S$  by compressing  $T_0$ . Note that  $S$  is essential, because  $T_0$  is non-separating in  $E(K)$ . Hence  $S$  is an essential 2-sphere in  $E(K)$ . This implies that  $K$  is a trivial knot by Proposition 2.9 of [2], a contradiction. Hence  $T_0$  is an essential torus in  $E(K)$ . In the following, we show that  $K$  belongs to  $\mathcal{K}_3$ . Since  $V_{11} \cap V_{21}$  is a 2-sphere with four holes, we see that  $V_{11} \cup V_{21}$  is the exterior of a non-trivial 2-bridge link, say  $L$ . On the other hand, we can recognize  $(M_0, k_0) := (V_{12}, t_1) \cup (V_{22}, t_2)$  as follows. We first note that  $(V_{12}, t_1)$  is identified with  $(B^3, \tau)$ , where  $\tau$  is a trivial arc in  $B^3$ , in such a way that

the copy of  $A_{i1} \cup A_{i2}$  corresponds to a regular neighborhood on  $\partial B^3$  of two homotopically non-trivial simple loops in  $\partial B^3 - \tau$ . Moreover,  $(V_{12}, t_1) \cap (V_{22}, t_2)$  consists of an annulus and two copies of  $(D^2, o)$ , where  $o$  is the center of the disk. By using this fact, we can see that  $E(k_0)$  is identified with  $B \times S^1$ , an orientable  $S^1$ -bundle over a two-holed disk  $B$ , and that a meridian of  $E(k_0)$  is isotopic to a fiber. Here the  $S^1$ -bundle structure is obtained by glueing the  $S^1$ -bundle structure of  $E(t_1)$  and  $E(t_2)$ . Now let  $K_0 \cup K_1 \cup K_2$  be as in the definition of  $\mathcal{K}_3$ . Since  $E(K_0 \cup K_1 \cup K_2)$  is identified with  $B \times S^1$ , where longitudes of  $K_1$  and  $K_2$  correspond to fibers of  $B \times S^1$ ,  $(V_{12}, t_1) \cup (V_{22}, t_2) = (E(k_0), \emptyset) \cup (N(k_0), k_0)$  is identified with  $(E(K_0 \cup K_1 \cup K_2), K_0)$ , where a longitude of  $k_0$  corresponds to a fiber (with respect to the bundle structure  $B \times S^1$  on  $E(k_0)$ ). Hence  $(E(K_1 \cup K_2), K_0) = (E(K_0 \cup K_1 \cup K_2), \emptyset) \cup (N(K_0), K_0)$  is identified with  $(E(k_0), \emptyset) \cup (N(k_0), k_0)$ . Thus we have  $(M, K) = (V_{11}, \emptyset) \cup (V_{21}, \emptyset) \cup (V_{21}, t_1) \cup (V_{22}, t_2) = (E(L), \emptyset) \cup (E(K_1 \cup K_2), K_0)$ . Hence  $K$  belongs to  $\mathcal{K}_3$ .

CASE 2.  $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$  composes a torus  $T$ .

Since  $V_{11} \cap V_{21}$  is a 2-sphere with four holes, we see that  $V_{11} \cup V_{21}$  is the exterior of a 2-bridge knot, say  $K_2$ . On the other hand, we can recognize  $(M_0, k_0) := (V_{12}, t_1) \cup (V_{22}, t_2)$  as follows. We first note that  $(V_{i2}, t_i)$  is identified with  $(B^3, \tau)$ , where  $\tau$  is a trivial arc in  $B^3$  in such a way that the copy of  $A_{i1} \cup A_{i2}$  corresponds to a regular neighborhood on  $\partial B^3$  of two homotopically non-trivial simple loops in  $\partial B^3 - \tau$ . Moreover,  $(V_{12}, t_1) \cap (V_{22}, t_2)$  consists of an annulus and two copies of  $(D^2, \emptyset)$ . By using this fact, we can see that  $E(k_0)$  is identified with  $B \widetilde{\times} S^1$ , an orientable twisted  $S^1$ -bundle over a one-holed Möbius band  $B$ , and that a meridian of  $E(k_0)$  is isotopic to a fiber. Here the  $S^1$ -bundle structure is obtained by glueing the  $S^1$ -bundle structure of  $E(t_1)$  and  $E(t_2)$ . Now let  $K_0 \cup K_1 \subset S^2 \times S^1$  and  $l_1 \subset \partial E(K_1)$  be as in the definition of  $\mathcal{K}_4$ . Then  $(V_{12}, t_1) \cup (V_{22}, t_2) = (E(k_0), \emptyset) \cup (N(k_0), k_0)$  is identified with  $(E(K_1), K_0)$ , where  $l_1$  corresponds to a fiber (with respect to the bundle structure  $B \widetilde{\times} S^1$  on  $E(k_0)$ ). This can be seen as follows. Since  $K_0 = K(4, 1; 0)$ ,  $K_0$  intersects each fiber  $S^2$  in two points. So  $E(K_0)$  is a twisted annulus bundle over  $S^1$ , and hence it is a twisted  $S^1$ -bundle over a Möbius band. Moreover, the meridian  $K_1$  of  $K_0$  corresponds to a regular fiber. This implies that  $E(K_0 \cup K_1)$  is identified with  $B \widetilde{\times} S^1$ , where  $l_1$  corresponds to a fiber of  $B \widetilde{\times} S^1$ . Hence  $(E(K_0), K_1) = (E(K_0 \cup K_1), \emptyset) \cup (N(K_0), K_0)$  is identified with  $(E(k_0), \emptyset) \cup (N(k_0), k_0)$ . Thus we have  $(M, K) = (V_{11}, \emptyset) \cup (V_{21}, \emptyset) \cup (V_{21}, t_1) \cup (V_{22}, t_2) = (E(K_2), \emptyset) \cup (E(K_1), K_0)$ .

Suppose that  $T$  is essential in  $E(K)$ . Then  $K_2$  is non-trivial. Hence  $K$  belongs to  $\mathcal{K}_4$ .

Suppose that  $T$  is inessential in  $E(K)$ . Then we see that  $K_2$  is trivial. Hence  $E(K)$  is homeomorphic to  $B \widetilde{\times} S^1$ , where  $B$  is a Möbius band. Hence  $E(K)$  is a Seifert fibered space whose base space is a disk with two singular points, and the Seifert invariant of the singular fibers are  $1/2$ . Hence  $K$  is a torus knot in  $S^2 \times S^1$  which intersects  $S^2 \times \{1\text{point}\}$  in two points. This implies  $K = K(4, 1, 0)$ .  $\square$

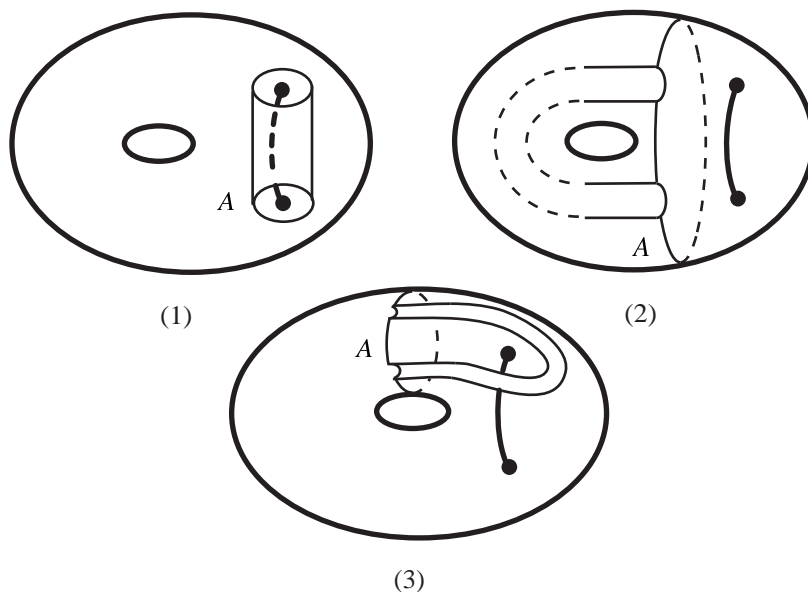


Fig. 4.

To prove Proposition 6.1, we prepare some lemmas which are obtained by an argument similar to those in Section 3 of [14]. An annulus properly embedded in an orientable 3-manifold is called *essential* if it is incompressible and not  $\partial$ -parallel. For a solid torus  $V$  and a trivial arc  $t$  in  $V$ , an annulus properly embedded in  $V - t$  is called *essential in  $(V, t)$*  if it is essential in  $V - t$ .

**Lemma 6.6.** *Let  $V$  be a solid torus and  $t$  a trivial arc in  $V$ , and let  $A$  be an essential annulus in  $(V, t)$ . Then one of the following holds (see Fig. 4).*

- (1)  *$A$  cuts  $(V, t)$  into  $(V_1, \emptyset)$  and  $(V_2, t)$ , where  $V_1$  is a genus two handlebody,  $V_2$  is a 3-ball and  $t$  is a trivial arc in  $V_1$ .*
- (2)  *$A$  cuts  $(V, t)$  into  $(V_1, \emptyset)$  and  $(V_2, t)$ , where  $V_1$  is a solid torus,  $V_2$  is a genus two handlebody and  $t$  is a trivial arc in  $V_2$ .*
- (3)  *$A$  is a non-separating annulus in  $V - t$  and there are an  $\varepsilon_0$ -disk  $D$  and an  $\varepsilon_1$ -disk  $D'$  in  $(V, t)$  with  $D \cap D' = \emptyset$  and  $A \cap (D \cup D') = \emptyset$ .*

**Proof.** Let  $\mathcal{D}$  be a disjoint union of an  $\varepsilon_0$ -disk and an  $\varepsilon_1$ -disk in  $(V, t)$ . Since  $A$  is incompressible in  $V - t$ ,  $A$  intersects  $\mathcal{D}$ . By a standard innermost/outermost disk argument, we can find a disk  $\delta$  in  $V$  such that  $\delta \cap t = \emptyset$ ,  $\delta \cap A = a$  is an essential arc in  $A$  and  $\delta \cap \partial V = b$  is an arc with  $\partial a = \partial b$  and  $a \cup b = \partial \delta$ . By performing a  $\partial$ -compression of  $A$  along  $\delta$ , we obtain a disk  $D$  properly embedded in  $V - t$ . Since  $A$  is essential in  $V - t$ ,  $D$  is essential in  $V - t$ .

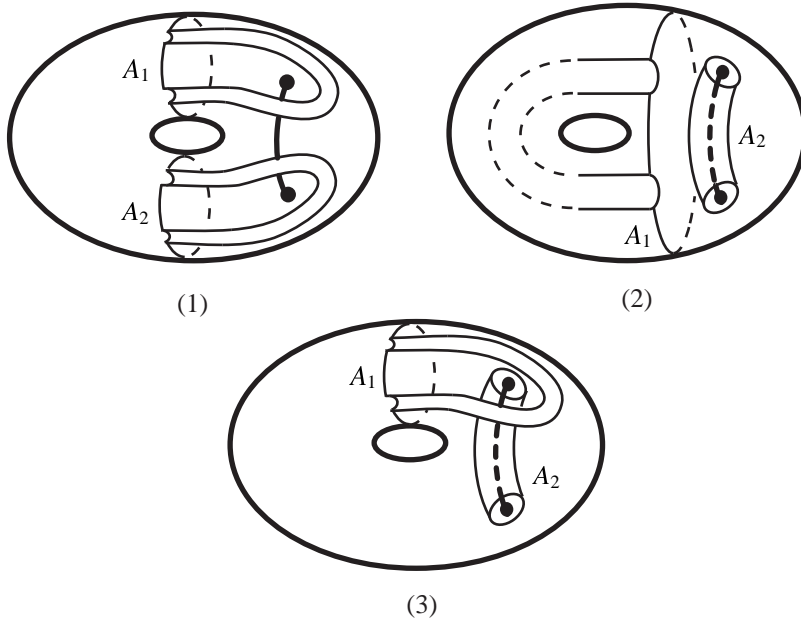


Fig. 5.

CASE 1.  $D$  is an  $\iota$ -disk.

Then  $D$  cuts  $(V, t)$  into  $(V', t)$  and  $(V'', \emptyset)$ , where  $V'$  is a 3-ball,  $t$  is a trivial arc in  $V'$  and  $V''$  is a solid torus. If  $A - D \subset V'$ , then we obtain the conclusion (1). Otherwise, we obtain the conclusion (2).

CASE 2.  $D$  is an  $\varepsilon_0$ -disk.

Then  $D$  cuts  $(V, t)$  into  $(B, t)$ , where  $B$  is a 3-ball and  $t$  is a trivial arc in  $B$ . By a pairwise isotopy of  $(B, t)$ , we may assume  $A \subset \partial B$ . Then since  $A$  is essential in  $V - t$ , the core  $\alpha$  of  $A$  separates the two punctures of  $\partial B - t$ . Hence by Lemma 3.3,  $\alpha$  bounds an  $\varepsilon_1$ -disk  $D'$  in  $(V, t)$ . By moving  $D$  and  $D'$  so that  $(D \cup D') \cap A = \emptyset$ , we obtain the conclusion (3).  $\square$

**Lemma 6.7.** *Let  $V$  be a solid torus and  $t$  a trivial arc in  $V$ , and let  $\mathcal{A} = A_1 \cup A_2$  be a disjoint union of non-parallel essential annuli in  $(V, t)$ . Then one of the following holds (see Fig. 5).*

- (1)  $\mathcal{A}$  cuts  $(V, t)$  into  $(V_1, \emptyset)$  and  $(V_2, t)$ , where  $V_1$  is a genus two handlebody,  $V_2$  is a 3-ball and  $t$  is a trivial arc in  $V_2$ , which satisfy  $\mathcal{A} \subset \partial V_j$  ( $j = 1$  and  $2$ ). Moreover, there are an  $\varepsilon_0$ -disk  $D$  and an  $\varepsilon_1$ -disk  $D'$  in  $(V, t)$  with  $D \cap D' = \emptyset$  and  $\mathcal{A} \cap (D \cup D') = \emptyset$ .
- (2)  $\mathcal{A}$  cuts  $(V, t)$  into  $(V_1, \emptyset)$ ,  $(V_2, \emptyset)$  and  $(V_3, t)$ , where  $V_1$  is a solid torus,  $V_2$  is

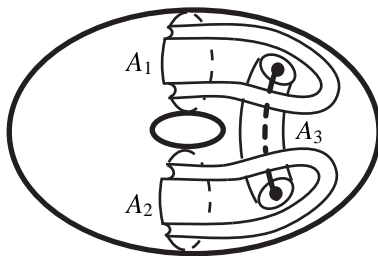


Fig. 6.

a genus two handlebody,  $V_3$  is a 3-ball and  $t$  is a trivial arc in  $V_3$ , which satisfy  $\mathcal{A} \cap \partial V_1 = A_1$ ,  $\mathcal{A} \subset \partial V_2$  and  $\mathcal{A} \cap \partial V_3 = A_2$  after changing the subscripts. Moreover, there is an  $\iota$ -disk in  $(V, t)$  disjoint from  $\mathcal{A}$ .

(3)  $\mathcal{A}$  cuts  $(V, t)$  into  $(V_1, \emptyset)$  and  $(V_2, t)$ , where  $V_1$  is a genus two handlebody,  $V_2$  is a 3-ball and  $t$  is a trivial arc in  $V_2$ , which satisfy  $\mathcal{A} \subset \partial V_1$  and  $\mathcal{A} \cap \partial V_2 = A_2$  after changing the subscripts.

**Proof.** By performing  $\partial$ -compressions of  $A_1$  and  $A_2$ , we obtain mutually disjoint disks  $D_1$  and  $D_2$  properly embedded in  $V - t$ . Since  $A_1$  and  $A_2$  are essential in  $V - t$ ,  $D_1$  and  $D_2$  are essential in  $V - t$ . Suppose that both  $D_1$  and  $D_2$  are  $\varepsilon_0$ -disks. Then we obtain the conclusion (1). Suppose next that both  $D_1$  and  $D_2$  are  $\iota$ -disks. Then we obtain the conclusion (2). Suppose finally that precisely one of  $D_1$  and  $D_2$ , say  $D_1$ , is an  $\varepsilon_0$ -disk and  $D_2$  is an  $\iota$ -disk. Note that  $A_2$  is disjoint from  $D_2$ . This implies that  $A_2$  is parallel to  $\partial N(K)$ . Hence we obtain the condition (3).  $\square$

The following lemma is obtained by using Lemma 3.3 of [14].

**Lemma 6.8.** *Let  $V$  be a solid torus and  $t$  a trivial arc in  $V$ , and let  $\mathcal{A} = A_1 \cup A_2 \cup A_3$  be a disjoint union of non-parallel essential annuli in  $(V, t)$ . Then  $\mathcal{A}$  cuts  $(V, t)$  into  $(V_1, \emptyset)$ ,  $(V_2, \emptyset)$  and  $(V_3, t)$ , where  $V_1$  is a genus two handlebody,  $V_2$  is a solid torus and  $V_3$  is a 3-ball and  $t$  is a trivial arc in  $V_3$ , which satisfy  $\mathcal{A} \cap \partial V_1 = A_1 \cup A_2$ ,  $\mathcal{A} \subset \partial V_2$  and  $\mathcal{A} \cap \partial V_3 = A_3$  after changing the subscripts (see Fig. 6).*

**Proof.** Note that  $A_1 \cup A_2$  satisfies one of the conclusions of Lemma 6.7. Suppose that  $A_1 \cup A_2$  satisfies the conclusion (2) of Lemma 6.7. Then  $A_1 \cup A_2$  cuts  $(V, t)$  into  $(V_1, \emptyset)$ ,  $(V_2, \emptyset)$  and  $(V_3, t)$ , where  $V_1$  is a solid torus,  $V_2$  is a genus two handlebody,  $V_3$  is a 3-ball and  $t$  is a trivial arc in  $V_3$ . If  $A_3 \subset V_1$  or  $V_3$ , then  $A_3$  is parallel to  $A_1$  or  $A_2$ . If  $A_3 \subset V_2$ , then by Lemma 3.3 of [14],  $A_3$  is parallel to  $A_1$  or  $A_2$ . Hence we may assume that  $A_1 \cup A_2$  satisfies the conclusion (1) or (3) of Lemma 6.8.

Suppose  $A_1 \cup A_2$  satisfies the conclusion (1) of Lemma 6.7. Then  $A_1 \cup A_2$  cuts  $(V, t)$  into  $(V_1, \emptyset)$  and  $(V_2, t)$ , where  $V_1$  is a genus two handlebody,  $V_2$  is a 3-ball and

$t$  is a trivial arc in  $V_2$ . By Lemma 3.3 of [14],  $A_3$  must be contained in  $V_2$ . Hence  $A_3$  is parallel to  $\partial N(t)$ .

Suppose  $A_1 \cup A_2$  satisfies the conclusion (3) of Lemma 3.3. Then  $A_1 \cup A_2$  cuts  $(V, t)$  into  $(V_1, \emptyset)$  and  $(V_2, t)$ , where  $V_1$  is a genus two handlebody,  $V_2$  is a 3-ball and  $t$  is a trivial arc in  $V_2$ . By Lemma 3.3 of [14],  $A_3$  is parallel to an annulus, say  $A'$ , in  $\partial V_2$ . Since  $A_3$  is essential in  $V - t$  and is not parallel to  $A_i$  ( $i = 1$  and  $2$ ),  $A'$  contains  $\partial A_1 \cup \partial A_2$ . This implies  $A_3$  satisfies the condition (3) of Lemma 6.6. Then by changing the subscripts, we can see that  $\mathcal{A}$  satisfies the condition of Lemma 6.8.  $\square$

**Proof of Proposition 6.1.** Let  $(W_1, W_2; P)$  be a  $(1, 1)$ -splitting of  $(M, K)$  and  $T$  an essential torus in  $E(K)$ . We put  $T_i = T \cap V_i$ .

**CLAIM.** We may assume that  $T_i$  consists of essential annuli in  $W_i$  ( $i = 1$  and  $2$ ).

**Proof.** Since  $\chi(T) = 0$ , we have only to show that  $T_i$  has no disks.

We may assume that after an isotopy, each disk of  $T_i$  is essential in  $V_i - t_i$  ( $i = 1$  and  $2$ ). Suppose that both  $T_1$  and  $T_2$  have disk components. Then this implies  $d(W_1, W_2) \leq 1$  because  $\partial T_1 = \partial T_2$ . Hence we see that  $K$  is a trivial knot or a core knot in  $S^2 \times S^1$  by Theorem 2.2 and Theorem 2.3, a contradiction. Hence we may assume that either  $T_1$  or  $T_2$ , say  $T_2$ , has no disk components. Further we assume that the number of disk components of  $T_1$  is minimal among all essential tori satisfying the condition as above. Let  $\Delta$  be the union of the disk components of  $T_1$ . Choose a disjoint union  $\mathcal{D}$  of an  $\varepsilon_0$ -disk and an  $\iota$ -disk in  $W_2$  which intersect  $T_2$  transversely.

Note that  $E(K)$  is irreducible, i.e.,  $E(K)$  contains no essential 2-spheres. Otherwise,  $K$  is a trivial knot by Proposition 2.9 of [2], a contradiction. Hence by a standard argument, we can eliminate all loop components of  $T_2 \cap \mathcal{D}$  by an ambient isotopy on  $E(K)$ .

Suppose that  $\Delta \cap \mathcal{D} = \emptyset$ . Then each component of  $\partial \Delta$  is isotopic to one of the components of  $\partial \mathcal{D}$  because each component of  $\partial \Delta$  is either an  $\varepsilon$ -loop or an  $\iota$ -loop. This implies that  $\partial \Delta$  bounds a disk in  $V_2 - t_2$ , and hence  $d(W_1, W_2) = 0$ . By Theorem 2.2,  $K$  is a trivial knot, a contradiction. So  $\Delta \cap \mathcal{D} \neq \emptyset$ .

Let  $\Gamma$  be the union of the arc components of  $T_2 \cap \mathcal{D}$  incident to  $\partial \Delta \cap \mathcal{D}$ . Let  $\gamma$  be a component of  $\Gamma$  such that  $\gamma$  clips a disk, say  $\delta_\gamma$ , from  $\mathcal{D}$  with  $\delta_\gamma \cap \Gamma = \gamma$ . Suppose that  $\delta_\gamma \cap T_2 \neq \gamma$ . Then there is a component  $\gamma'$  of  $\delta \cap T_2$  which clips a disk  $\delta_{\gamma'}$  with  $\delta_{\gamma'} \cap T_2 = \gamma'$ . We can isotope  $T$  along  $\delta_{\gamma'}$  near  $\gamma'$  without increasing the number of disks of  $T_1$ . By repeating this operation, if necessary, we may suppose that  $\delta_\gamma \cap T_2 = \gamma$ . By isotoping  $T$  along  $\delta_\gamma$ , we can reduce the number of disk components of  $T_1$  at least by one, a contradiction.

This completes the proof of the claim.  $\square$

Let  $\mathcal{A}_i$  be a union of mutually disjoint, non-parallel, essential annuli in  $W_i = (V_i, t_i)$  of which  $T_i$  consists of parallel copies ( $i = 1$  and  $2$ ). Note that  $|\mathcal{A}_1| \leq 3$  by Lemmas 6.6–6.8. By changing the subscripts, if necessary, we may assume that  $|\mathcal{A}_1| \geq |\mathcal{A}_2|$ .

CASE 1.  $|\mathcal{A}_1| = 3$ .

Note that one of the following holds.

- $\mathcal{A}_2$  consists of an annulus satisfying one of the conditions in Lemma 6.6.
- $\mathcal{A}_2$  consists of two annuli satisfying one of the conditions in Lemma 6.7.
- $\mathcal{A}_2$  consists of three annuli satisfying the condition in Lemma 6.6.

Suppose that  $\mathcal{A}_2$  satisfies the condition (1) of Lemma 6.6, the condition (2) of Lemma 6.7, the condition (3) of Lemma 6.7, or the condition of Lemma 6.8. Here, the sentence “ $\mathcal{A}_2$  satisfies the condition (1) of Lemma 6.6” means that  $\mathcal{A}_2$  consists of an annulus satisfying the condition (1) in Lemma 6.6. Then  $T_1 \cup T_2$  contains a torus which is parallel to  $\partial N(K)$ , a contradiction.

Suppose that  $\mathcal{A}_2$  satisfies the condition (2) of Lemma 6.6 or the condition (3) of Lemma 6.6. Let  $\{p_1, p_2\}$  be points of  $P \cap K$ . Note that  $\mathcal{A}_1$  has a component which is isotopic to  $\partial N(p_i; P)$  for each  $i = 1$  and  $2$ . On the other hand, for  $i = 1$  or  $2$ ,  $\mathcal{A}_2$  does not have a component which is isotopic to  $\partial N(p_i; P)$ . This implies that  $\partial T_1 \neq \partial T_2$ , a contradiction.

Suppose that  $\mathcal{A}_2$  satisfies the condition (1) of Lemma 6.7. Put  $\mathcal{A}_1 = A_{11} \cup A_{12} \cup A_{13}$  and  $\mathcal{A}_2 = A_{21} \cup A_{22}$ . We may assume that  $A_{13}$  is isotopic to  $\partial N(K) \cap V_1$ . Suppose that  $T_1$  consists of  $m_1$  parallel copies of  $A_{11}$ ,  $m_2$  parallel copies of  $A_{12}$  and  $m_3$  parallel copies of  $A_{13}$ , and  $T_2$  consists of  $n_1$  parallel copies of  $A_{21}$  and  $n_2$  parallel copies of  $A_{22}$ . Then since  $\partial T_1 = \partial T_2$ , we have  $m_1 + m_2 = n_1 + n_2$ ,  $m_1 + m_3 = n_1$  and  $m_2 + m_3 = n_2$ . This implies that  $m_3 = 0$ , a contradiction. Hence Case 1 does not occur.

CASE 2.  $|\mathcal{A}_1| = 2$ .

Set  $\mathcal{A}_1 = A_{11} \cup A_{12}$ . We have the following three subcases by Lemma 6.7.

CASE 2.1.  $\mathcal{A}_1$  satisfies the condition (1) of Lemma 6.7.

By an argument similar to Case 1, we see that  $\mathcal{A}_2$  satisfies the condition (1) or (2) of Lemma 6.7. Set  $\mathcal{A}_2 = A_{21} \cup A_{22}$ .

Suppose that  $\mathcal{A}_2$  satisfies the condition (1) of Lemma 6.7. Then we see  $|T_1| = |T_2| = 2$ . (Otherwise  $T_1 \cup T_2$  has plural components.) So we may assume  $T_i = A_{i1} \cup A_{i2}$  ( $i = 1$  and  $2$ ) (cf. Fig. 3). Since  $M \cong S^2 \times S^1$ , we can find an  $\varepsilon_1$ -disks  $D_i$  in  $W_i$  ( $i = 1$  and  $2$ ) with  $\partial D_1 = \partial D_2$ . Hence  $(W_1, W_2; P)$  satisfies the condition  $(\#_c)$  of Proposition 6.1.

Suppose that  $\mathcal{A}_2 = A_{21} \cup A_{22}$  satisfies the condition (2) of Lemma 6.7. Then we can find an  $\varepsilon_1$ -disk  $D_1$  in  $W_1$  and an  $\iota$ -disk  $D_2$  in  $W_2$  which satisfy the condition  $(\#_a)$  of Proposition 6.1 (see Fig. 7). Hence by the remark below Lemma 6.2,  $K$  is a core knot, a contradiction.

CASE 2.2.  $\mathcal{A}_1$  satisfies the condition (2) of Lemma 6.7.

Then by an argument similar to Case 1, we see that  $\mathcal{A}_2$  satisfies the condition (1)

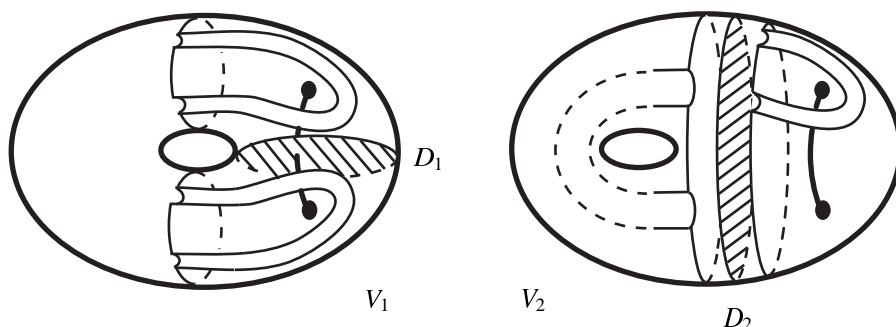


Fig. 7.

of Lemma 6.7. Hence by changing the subscripts, Case 2.2 is equivalent to the latter case of Case 2.1.

CASE 2.3.  $\mathcal{A}_1$  satisfies the condition (3) of Lemma 6.7.

Then by an argument similar to Case 1, we see that Case 2.3 is impossible.

CASE 3.  $|\mathcal{A}_1| = 1$ .

By Lemma 6.6, we have the following three subcases.

CASE 3.1.  $\mathcal{A}_1$  satisfies the condition (1) of Lemma 6.6.

By an argument similar to Case 1, we see that  $\mathcal{A}_2$  satisfies the condition (1) of Lemma 6.6. Hence  $T_1 \cup T_2$  contains a torus which is parallel to  $\partial N(K)$ , a contradiction.

CASE 3.2.  $\mathcal{A}_1$  satisfies the condition (2) of Lemma 6.6.

By an argument similar to Case 1, we see that  $\mathcal{A}_2$  satisfies the condition (2) of Lemma 6.6. Moreover  $T_i$  consists of an annulus ( $i = 1$  and  $2$ ). (Otherwise,  $T_1 \cup T_2$  consists of plural components.) Let  $z$  be one of the components of  $\partial \mathcal{A}_1 = \partial \mathcal{A}_2$ . For each  $i = 1$  and  $2$ , let  $\Delta_i$  be a disk in  $V_i$  such that  $t_i \subset \partial \Delta$ , and  $\Delta_i \cap \partial V_i = \text{cl}(\partial \Delta_i - t_i) =: t'_i$  is disjoint from  $z$ . Then there are  $\iota$ -disks  $D_i$  in  $W_i$  with  $\partial D_i = \partial N(t'_i; P)$  for each  $i = 1$  and  $2$ . Hence  $Z := \text{cl}(P - N(z; P))$  gives the condition  $(\#_b)$  of Proposition 6.1.

CASE 3.3.  $\mathcal{A}_1$  satisfies the condition (3) of Lemma 6.6.

By an argument similar to Case 1, we see that  $\mathcal{A}_2$  satisfies the condition (3) of Lemma 6.6. Then there are an  $\varepsilon_1$ -disk  $D_i$  in  $W_i$  ( $i = 1$  and  $2$ ) with  $\partial D_1 = \partial D_2$ . Hence  $(W_1, W_2; P)$  satisfies the condition  $(\#_c)$  of Proposition 6.1.

This completes the proof of Proposition 6.1.  $\square$

**Proof of Theorem 2.5.** Let  $K$  be a  $(1, 1)$ -knot in  $M$  and  $(W_1, W_2; P)$  a  $(1, 1)$ -splitting of  $(M, K)$ . By Proposition 6.1,  $(W_1, W_2; P)$  satisfies one of the conditions in Proposition 6.1.

Suppose that  $(W_1, W_2; P)$  satisfies the condition  $(\#_a)$  of Proposition 6.1. Then by Lemma 6.2,  $K$  belongs to  $\mathcal{K}_1$ , because the exteriors of 2-bridge knots and core knots do not contain essential tori (see [5]).

Suppose that  $(W_1, W_2; P)$  satisfies the condition  $(\#_b)$  of Proposition 6.1. Then by

arguments in the proof of Lemma 6.4 and the proof of Proposition 6.1,  $K$  belongs to  $\mathcal{K}_2$ , because  $E(K)$  contains an essential torus.

Suppose that  $(W_1, W_2; P)$  satisfies the condition  $(\#_c)$  of Proposition 6.1. Then by Lemma 6.5,  $K$  belongs to  $\mathcal{K}_3$  or  $\mathcal{K}_4$ .

We have thus proved Theorem 2.5.  $\square$

## 7. (1, 1)-splittings of distance = 2

In this section, we give the proof of Theorem 2.4.

**Proof of Theorem 2.4.** We first assume  $d(W_1, W_2) = 2$ , that is, there is an essential loop  $x$  ( $y$  resp.) in  $\Sigma := P - K$  which bounds a disk in  $V_1 - t_1$  ( $V_2 - t_2$  resp.) such that  $x$  and  $y$  intersect each other, and there is an essential loop  $z$  in  $\Sigma$  with  $z \cap (x \cup y) = \emptyset$ .

**CASE 1.** Both  $x$  and  $y$  are  $\varepsilon$ -loops.

If  $z$  is an  $\iota$ -loop, then  $z$  bounds an  $\iota$ -disk in each of  $W_1$  and  $W_2$  by Lemma 3.3. This implies that  $(W_1, W_2; P)$  is of distance = 0, a contradiction. Hence by Lemma 3.3,  $z$  must be an  $\varepsilon$ -loop and  $z$  bounds an  $\varepsilon_0$ -disk or an  $\varepsilon_1$ -disk in each of  $W_1$  and  $W_2$ .

Suppose that  $z$  bounds an  $\varepsilon_0$ -disk in each of  $W_1$  and  $W_2$ . Then this means that  $d(W_1, W_2) \leq 1$ , a contradiction.

Suppose that  $z$  bounds an  $\varepsilon_1$ -disk in each of  $W_1$  and  $W_2$ . Then  $(W_1, W_2; P)$  satisfies the condition  $(\#_c)$  of Proposition 6.1. By Lemma 6.5,  $K = K(4, 1, 0)$  or  $E(K)$  contains an essential torus.

**CASE 2.** Precisely one of  $x$  and  $y$ , say  $x$ , is an  $\varepsilon$ -loop.

We see that  $z$  is an  $\varepsilon$ -loop by an argument similar to Case 1. Then by Lemma 3.3,  $z$  bounds an  $\varepsilon_1$ -disk in  $W_1$ . So  $(W_1, W_2; P)$  satisfies the condition  $(\#_a)$  of Proposition 6.1, and hence  $(M, K)$  satisfies one of the conditions (1)–(3) of Lemma 6.2. Note that if  $K$  satisfies the condition (3), we can find an essential torus in  $E(K)$  by making an appropriate “swallow-follow torus”.

**CASE 3.** Both  $x$  and  $y$  are  $\iota$ -loops.

Then  $z$  must be an  $\varepsilon$ -loop by the same argument as above. In particular,  $z$  must be contained in the surface  $T_0$  obtained from the torus  $P$  by removing the interior of the disk bounded by  $x$ . So all components of  $y \cap T_0$  ( $\neq \emptyset$ ) are parallel in  $T_0$ . Note that we can regard  $y$  as  $\partial N(t'_2; P)$ , where  $t'_2$  is an arc in  $P$  such that  $t_2 \cup t'_2$  bounds a disk in  $V_2$ . By an isotopy on  $\Sigma$ , we may assume that  $|x \cap y|$  is minimal.

**CASE 3.1.**  $|y \cap T_0| = 2$ .

Then  $K$  is isotopic to a knot in  $P$ , and hence  $K$  satisfies the condition (2) or (3) of Theorem 2.4.

**CASE 3.2.**  $|y \cap T_0| > 2$ .

Let  $A_1$  in  $V_1^-$  ( $A_2$  in  $V_2^-$  resp.) be an annulus obtained by pushing the interior of  $N(z; P)$  into the interior of  $V_1$  ( $V_2$  resp.), where  $V_i^- = \text{cl}(V_i - N(t_i))$  ( $i = 1, 2$ ). So  $T := A_1 \cup A_2$  is a torus in  $E(K)$  (see Fig. 8).

$A_1$  ( $A_2$  resp.) cuts  $V_1^-$  ( $V_2^-$  resp.) into a solid torus  $V_{11}^-$  ( $V_{21}^-$  resp.) and a genus

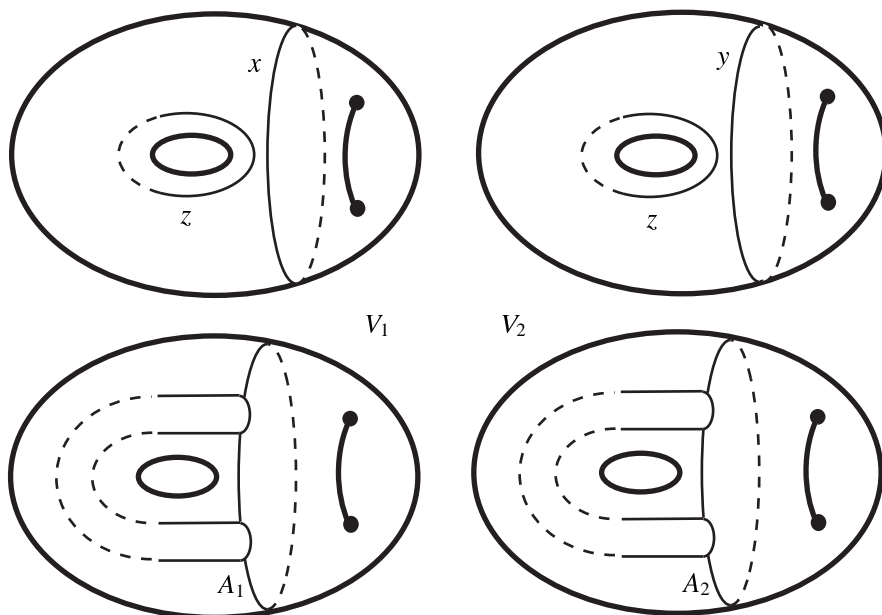


Fig. 8.

two handlebody  $V_{12}^-$  ( $V_{22}^-$  resp.).  $M_1 = V_{11}^- \cup V_{21}^-$  is the exterior of a trivial knot, a core knot or a torus knot.  $M_2 = V_{21}^- \cup V_{22}^-$  is the exterior of a 2-bridge link, and  $M_2 \cup N(K)$  should be a solid torus. If  $M_1$  is a solid torus, then  $(M, K)$  is equivalent to  $K(\alpha, \beta; r)$  for some  $\alpha, \beta$  and  $r$ . If not, by the hypothesis of Case 3.2, we can see that  $T$  is not parallel to  $\partial N(K)$ . Hence  $T$  is an essential torus in  $E(K)$ .

This completes the proof of the first part of Theorem 2.4.

Next, we prove the second part of Theorem 2.4.

CASE (1).  $K$  is a non-trivial 2-bridge knot in  $S^3$ .

By Theorem 8.2 of [15], every (1, 1)-splitting of a non-trivial 2-bridge knot is isotopic to that constructed as follows. For a non-trivial 2-bridge knot  $K$ , let  $(B_1, a_1 \cup a_2) \cup_S (B_2, b_1 \cup b_2)$  be a 2-bridge decomposition. Put  $V_1 = B_1 \cup N(b_2; B_2)$ ,  $V_2 = \text{cl}(B_2 - N(b_2; B_2))$ ,  $t_1 = a_1 \cup a_2 \cup b_2$  and  $t_2 = b_1$ . Then  $W_i := (V_i, t_i)$  is a pair of a solid torus  $V_i$  and a trivial arc  $t_i$  in  $V_i$  ( $i = 1, 2$ ), and  $(W_1, W_2; P)$  gives a (1, 1)-splitting of  $(S^3, K)$ . In the following, we show that this (1, 1)-splitting has distance = 2.

Let  $D_i$  be a properly embedded disk in  $B_i$  such that  $D_i$  separates two trivial arcs in  $B_i$  ( $i = 1, 2$ ). Then  $D_1$  determines an  $\varepsilon_0$ -disk in  $W_1$ , and  $D_2$  determines an  $\iota$ -disk in  $W_2$ . Further,  $\partial D_1$  and  $\partial D_2$  are disjoint from an essential loop  $z$  in  $\Sigma := P - K$ , where  $z$  is one of the boundary components of the meridian disks  $B_1 \cap N(b_2; B_2)$ . Hence  $d(W_1, W_2) \leq 2$ . By Theorem 2.2 and Theorem 2.3, we have  $d(W_1, W_2) = 2$ .

CASE (2) and (3).  $K$  is a core knot in a lens space or a torus knot in  $M$ .

By Theorem C of [6] and Theorem 3 of [17], every  $(1, 1)$ -splitting of  $(M, K)$  is isotopic to that constructed as follows. Let  $(V_1, V_2; P)$  be a genus one Heegaard splitting of  $M$  such that  $K \subset P$ . Let  $p_1$  and  $p_2$  be distinct points in  $K$ . Then  $p_1 \cup p_2$  cuts  $K$  into two arcs  $l_1$  and  $l_2$ . Let  $t_i$  be the properly embedded arc by slightly pushing the interior of  $l_i$  into the interior of  $V_i$ , and put  $W_i = (V_i, t_i)$  ( $i = 1$  and  $2$ ). Then  $(W_1, W_2; P)$  is a  $(1, 1)$ -splitting of  $(M, K)$ .

Let  $z$  be a core of the annulus  $\text{cl}(P - N(K; P))$ . Then  $\partial N(l_i; P)$  bounds an  $\iota$ -disk in  $W_i$  ( $i = 1, 2$ ), and  $\partial N(l_1; P)$  and  $\partial N(l_2; P)$  are disjoint from the essential loop  $z$  in  $P$ . So we have  $d(W_1, W_2) \leq 2$ . By Theorem 2.2 and Theorem 2.3, we obtain  $d(W_1, W_2) = 2$ .

CASE (4).  $E(K)$  contains an essential torus.

Let  $(W_1, W_2; P)$  be a  $(1, 1)$ -splitting of  $(M, K)$ . By Proposition 6.1,  $(W_1, W_2; P)$  satisfies one of the conditions  $(\#_a)$ ,  $(\#_b)$  and  $(\#_c)$ .

Suppose that  $(W_1, W_2; P)$  satisfies the condition  $(\#_a)$ . Let  $D_1$  ( $D_2$  resp.) be an  $\iota$ -disk (an  $\varepsilon_1$ -disk resp.) in  $W_1$  ( $W_2$  resp.) such that  $\partial D_1 \cap \partial D_2 = \emptyset$ . By cutting  $W_2 = (V_2, t_2)$  along  $D_2$ , we obtain a 2-string trivial tangle  $(B, \tau)$ . Let  $D_2^+$  and  $D_2^-$  be the copy of  $D_2$  in  $\partial B$ . Let  $D'_2$  be a disk properly embedded in  $B$  such that  $D'_2 \cap (D_2^+ \cup D_2^-) = \emptyset$  and  $D'_2$  separates a component of  $\tau$  from the other. Then  $D'_2$  determines an  $\varepsilon_1$ -disk  $W_2$ , and  $D'_2$  is disjoint from  $D_2$ . Hence  $\partial D_1$  and  $\partial D_2$  give  $d(W_1, W_2) \leq 2$ .

We can easily see that the condition  $(\#_b)$  directly gives  $d(W_1, W_2) \leq 2$ .

Finally, if the condition  $(\#_c)$  is satisfied, then we can also obtain  $d(W_1, W_2) \leq 2$  by using an argument similar to that in case of the condition  $(\#_a)$ . By Theorem 2.2 and Theorem 2.3, we obtain  $d(W_1, W_2) = 2$ .

We have completed the proof of Theorem 2.4. □

**Proof of Corollary 2.6.** By Thurston's hyperbolization theorem of Haken manifolds (see, for example, [13]), a knot  $K$  is hyperbolic if and only if  $E(K)$  is irreducible,  $E(K)$  contains no essential torus, and  $E(K)$  is not a Seifert fibered space.

CASE 1.  $E(K)$  is reducible.

By Proposition 2.9 of [2],  $E(K)$  is reducible if and only if  $K$  is a trivial knot. Hence  $d(W_1, W_2) = 0$  by Theorem 2.2.

CASE 2.  $E(K)$  contains an essential torus.

Then by Theorem 2.6,  $d(W_1, W_2) = 2$ .

CASE 3.  $E(K)$  is a Seifert fibered space whose regular fiber is not a meridian of  $K$ .

Then by Lemma 5.2 of [14], if  $E(K)$  is a Seifert fibered space whose regular fiber is not a meridian of  $K$  and  $\partial E(K)$  is incompressible in  $E(K)$ , then one of the following holds: (1) the base space is a disk with two singular points, where the regular fiber in  $\partial E(K)$  intersects the meridian in one point, (2) the base space is a Möbius

band with one singular point, where the regular fiber in  $\partial E(K)$  intersects the meridian in one point, (3)  $E(K)$  is a twisted  $S^1$ -bundle over a Möbius band. If  $E(K)$  satisfies the condition (1) or (3), then  $K$  is a torus knot. If  $E(K)$  satisfies the condition (2), then there is an essential torus in  $E(K)$ . Hence by Theorem 2.4,  $d(W_1, W_2) = 2$ .

Suppose that  $\partial E(K)$  is compressible in  $E(K)$ . Then we obtain a 2-sphere  $S$  in  $E(K)$  by compressing  $\partial E(K)$ . If  $S$  bounds a 3-ball in  $E(K)$ , then  $E(K)$  is a solid torus and hence  $K$  is a trivial knot or a core knot. Otherwise, since  $S$  is essential in  $E(K)$ ,  $K$  is a trivial knot by Proposition 2.9 of [2]. Hence by Theorems 2.2 and 2.3, we have  $d(W_1, W_2) = 0$  or 1.

CASE 4.  $E(K)$  is a Seifert fibered space whose regular fiber is a meridian of  $K$ .

Let  $B$  be the base orbifold of  $E(K)$ . Then  $\pi_1(M) = \pi_1(E(K))/\langle f \rangle$ , where  $f$  is the element of  $\pi_1(E(K))$  represented by a regular fiber, is isomorphic to the orbifold fundamental group  $\pi_1(B)$ . Since  $M$  is a lens space,  $\pi_1(B)$  is cyclic. It is known that such an orbifold is isomorphic to a disk with only one singular point (see, for example, Section 3 of [19]). Therefore  $E(K)$  is a solid torus, and hence  $K$  is a core knot. Hence by Theorem 2.3, we have  $d(W_1, W_2) = 1$ .

Hence by Theorems 2.2–2.4 and the hypothesis of Proposition 2.6,  $d(W_1, W_2) \leq 2$  if and only if  $E(K)$  is a Seifert fibered space or contains an essential 2-sphere or torus. By Thurston's hyperbolization theorem, we obtain the desired result.  $\square$

## 8. (1, 1)-splittings of distance $\geq 3$

Theorem 2.7 can be proved by the arguments of J. Hempel in Section 2 of [11]. To this end, we first recall the covering distance introduced in [11].

Let  $S$  be a connected, compact, orientable surface. We say that a covering space  $p: \tilde{S} \rightarrow S$  separates essential loops  $x$  and  $y$  in  $S$  if there are components  $\tilde{x}$  of  $p^{-1}(x)$  and  $\tilde{y}$  of  $p^{-1}(y)$  with  $\tilde{x} \cap \tilde{y} = \emptyset$ . A finite covering  $p: \tilde{S} \rightarrow S$  is *sub-solvable* if  $p$  can be factored as a composition of cyclic coverings.

DEFINITION 8.1 ([11] Section 2). Let  $[x]$  and  $[y]$  be distinct vertices of  $C(S)$ , and let  $x$  ( $y$  resp.) be a representative of  $[x]$  ( $[y]$  resp.). Then we define the *covering distance* between  $[x]$  and  $[y]$  as follows.

$$cd([x], [y]) = 1 + \min \left\{ n \mid \begin{array}{l} \text{there is a degree } 2^n \text{ sub-solvable covering of } S \\ \text{which separates } x \text{ and } y \end{array} \right\}.$$

As an analogy of Lemma 2.3 in [11], we obtain the following.

**Lemma 8.2.** *Let  $[x]$  and  $[y]$  be distinct vertices of  $C(S)$ . Then*

- (1)  $d([x], [y]) = 2$  if and only if  $cd([x], [y]) = 2$  and
- (2)  $cd([x], [y]) \leq d([x], [y])$ .

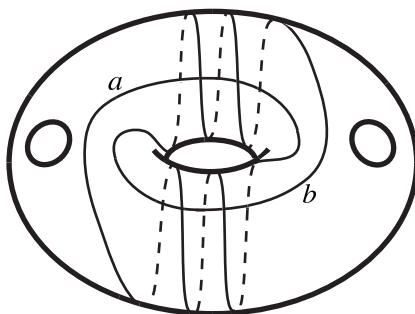


Fig. 9.

Proof. Let  $x$  ( $y$  resp.) be a representative of  $[x]$  ( $[y]$  resp.).

(1) Suppose that  $d([x], [y]) = 2$ , that is,  $x \cap y \neq \emptyset$  and there is an essential loop  $z$  in  $S$  with  $z \cap (x \cup y) = \emptyset$ .

CASE 1.  $z$  is an  $\varepsilon$ -loop.

Since an  $\varepsilon$ -loop is a non-separating loop in  $S$ ,  $S' := \text{cl}(S - N(z))$  is connected. We can construct a double cover  $\tilde{S}$  of  $S$  by gluing two copies  $S'_1$  and  $S'_2$  of  $S'$  along  $z$ . Hence  $\tilde{x}$  in  $S'_1$  and  $\tilde{y}$  on  $S'_2$  can give  $cd([x], [y]) = 2$ .

CASE 2.  $z$  is an  $\iota$ -loop.

Let  $\gamma$  be an essential arc which joins two punctures of  $S$  such that  $\gamma$  is disjoint from  $z$ . Then we can construct a double cover  $\tilde{S}$  of  $S$  by gluing two copies of  $\text{cl}(S - N(\gamma))$ . Therefore we can also get  $cd([x], [y]) = 2$ .

The converse follows from the proof of Lemma 2.3 in [11].

(2) The second assertion can also be proved by the same argument as that in the proof of Lemma 2.3 of [11].

This completes the proof of Lemma 8.2.  $\square$

By Lemma 8.2, we can get a lower estimation of the distance between distinct vertices on  $C(S)$ . For the covering distance, the following lemma is proved in [11].

**Lemma 8.3** ([11] Theorem 2.5). *If  $[x]$  and  $[y]$  are vertices of  $C(S)$  and  $h: S \rightarrow S$  is a pseudo-Anosov homeomorphism, then  $\lim_{n \rightarrow \infty} cd([x], [h^n(y)]) = \infty$ .*

Proof of Theorem 2.7. We first construct a pseudo-Anosov map  $f$  of  $\Sigma := P - K$  whose extension to  $P$  is isotopic to  $id$ . To this end, let  $a$  and  $b$  be essential loops on  $\Sigma$  illustrated in Fig. 9, and put  $f = \tau_a^{-1} \circ \tau_b$ , where  $\tau_a$  ( $\tau_b$  resp.) a right-hand Dehn twist along  $a$  ( $b$  resp.). Then  $f$  is pseudo-Anosov by Theorem 3.1 of [21], because  $a \cup b$  fills  $\Sigma$ . Since  $a$  and  $b$  are isotopic in  $P$ , the extension  $\hat{f}$  of  $f$  to  $P$  is isotopic to the identity.

Now let  $M$  be a 3-manifold with a genus one Heegaard splitting. Pick a

$(1, 1)$ -knot  $K$  in  $M$  and its  $(1, 1)$ -splitting  $(W_1, W_2; P)$ . Let  $x$  ( $y$  resp.) be an  $\varepsilon$ -loop in  $\Sigma$  which bounds an  $\varepsilon_0$ -disk in  $W_1$  ( $W_2$  resp.). By Lemma 8.2 and Lemma 8.3, for any positive integer  $n$ , there is an integer  $N$  such that  $d([x], [f^N(y)]) > n + 2$ , where  $[x]$  ( $[f^N(y)]$  resp.) is represented by  $x$  ( $f^N(y)$  resp.). Since  $\hat{f} \simeq id$ , the manifold obtained from  $M$  by cutting along  $P$  and regluing it after composing  $\hat{f}^N$  is homeomorphic to  $M$ . Let  $(W'_1, W'_2; P)$  be a  $(1, 1)$ -splitting obtained in the above way. Then by Proposition 3.8, we have  $d(W'_1, W'_2) \geq d([x], [f^N(y)]) - 2 > n$ .

We have completed the proof of Theorem 2.7.  $\square$

ACKNOWLEDGEMENTS. I would like to express my thanks to Prof. Makoto Sakuma for leading me to this subject, many instructive suggestions and conversations. And I would also like to thank Prof. Hiroshi Goda, Prof. Chuichiro Hayashi, Prof. Tsuyoshi Kobayashi and Prof. Kanji Morimoto for helpful conversations.

---

### References

- [1] D.H. Choi and K.H. Ko: *On 1-bridge torus knots*, preprint.
- [2] H. Goda, C. Hayashi and N. Yoshida: *Genus two Heegaard splittings of exteriors of knots and the disjoint curve property*, Kobe J. Math. **18** (2001), 79–114.
- [3] H. Goda and C. Hayashi: *Genus two Heegaard splittings for 1-genus 1-bridge knot exteriors*, preprint.
- [4] W.J. Harvey: *Boundary structure of the modular group, Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference*, Princeton Univ. Press, Princeton, N.J. 1981.
- [5] A. Hatcher and W. Thurston: *Incompressible surfaces in 2-bridge knot complements*, Invent. math. **79** (1981), 225–246.
- [6] C. Hayashi: *Genus one 1-bridge positions for the trivial knot and torus knots*, Math. Proc. Camb. Phil. Soc. **125** (1999), 53–65.
- [7] C. Hayashi: *Satellite knots in 1-genus 1-bridge positions*, Osaka J. Math. **36** (1999), 711–729.
- [8] C. Hayashi: *1-genus 1-bridge positions for knots in the 3-sphere and lens spaces*, preprint.
- [9] C. Hayashi and K. Shimokawa: *Heegaard splittings of the trivial knot*, J. Knot Theory Ramifications **7** (1998), 1073–1085.
- [10] J. Hempel: *3-manifolds*, Princeton Univ. Press, Princeton, N.J. (1976).
- [11] J. Hempel: *3-manifolds as viewed from the curve complex*, Topology, **40** (2001), 631–657.
- [12] W. Jaco: *Lectures on three manifold topology*, Conference board of Math. Amer. Math. Soc. (1980).
- [13] M. Kapovich: *Hyperbolic Manifolds and Discrete Groups*, Progress in Math. **183**, Birkhäuser (2001).
- [14] T. Kobayashi: *Structures of the Haken manifolds with Heegaard splittings of genus two*, Osaka J. Math. **21** (1984), 437–455.
- [15] T. Kobayashi and O. Saeki: *The Rubinstein-Scharlemann graphic of a 3-manifold as the discriminant set of a stable map*, Pacific J. Math. **195** (2000), 101–156.
- [16] H.A. Masur and Y.N. Minsky: *Geometry of the complex of curves I. hyperbolicity*, Invent. Math. **138** (1999), 103–149.
- [17] K. Morimoto: *On minimum Heegaard splitting of some orientable closed 3-manifolds*, Tokyo J. Math. **12** (1989), 321–355.
- [18] K. Morimoto and M. Sakuma: *On unknotting tunnels for knots*, Math. Ann. **289** (1991),

- 143–167.
- [19] P. Scott: *The geometries of 3-manifolds*, Bull. London. Math. Soc. **15** (1983), 401–487.
  - [20] A. Thompson: *The disjoint curve property and genus 2 manifolds*, Topology Appl. **97** (1999), 273–279.
  - [21] R.C. Penner: *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **310** (1988), 179–197.
  - [22] D. Rolfsen: *Knots and links*, Publish or Perish Inc. (1976).

Department of Mathematics  
Graduate School of Science  
Osaka University  
Toyonaka, Osaka 560-0043, Japan  
e-mail: [saito@gaia.math.wani.osaka-u.ac.jp](mailto:saito@gaia.math.wani.osaka-u.ac.jp)