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GENUS ONE 1-BRIDGE KNOTS AS VIEWED FROM THE CURVE COMPLEX

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1. Introduction

W.J. Harvey [4] associated to a surface S a finite-dimensional simplicial complex $C(S)$, called the curve complex, which we recall below.

For a connected orientable surface $F = F_{g,n}$ of genus g with n punctures, the *curve complex* $C(F)$ of F is the complex whose k-simplexes are the isotopy classes of $k+1$ collections of mutually non-isotopic essential loops in F which can be realized disjointly. It is proved in [16] that the curve complex is connected if F is not sporadic (where F is *sporadic* if $g = 0$, $n \leq 4$ or $g = 1$, $n \leq 1$). For $[x]$ and $[y]$, vertices of $C(F)$, the *distance* $d([x],[y])$ between $[x]$ and $[y]$ is defined by the minimal number of 1-simplexes in a simplicial path joining $[x]$ to $[y]$. It is known that if S is not sporadic, then $C(F)$ has infinite diameter with respect to the distance defined above (cf. [11], [16]), $C(F)$ is not locally finite in the sense that there are infinite edges around each vertex, and the dimension of $C(F)$ is $3g-4+n$.

Recently, J. Hempel [11] studied Heegaard splittings of closed 3-manifolds by using the curve complex of Heegaard surfaces. Let M be a closed orientable 3-manifold and $(V_1, V_2; S)$ a genus $g \geq 2$ Heegaard splitting, that is, V_i ($i = 1$ and 2) is a genus g handlebody with $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = S$. By using the curve complex, Hempel defined the distance of the Heegaard splitting, denoted by $d(V_1, V_2)$, and proved the following results.

Theorem 1.1 (J. Hempel). (1) *Let M be a closed, orientable, irreducible 3-manifold which is Seifert fibered or which contains essential tori. Then $d(V_1, V_2) \leq 2$ for any Heegaard splitting $(V_1, V_2; S)$ of M.*
(2) *There are Heegaard splittings of closed orientable 3-manifolds with distance $> n$ for any integer n.*

In particular, the theorem above implies that a Haken manifold is hyperbolic if a Heegaard splitting of the manifold has distance $\geq$ 3. Results along these lines were also obtained by A. Thompson [20]. Moreover, H. Goda, C. Hayashi and N. Yoshida [2] made detailed study of tunnel number one knots and C. Hayashi ([6], [7]) studied $(1,1)$-knots from similar points of view.

In this paper, we apply this idea to genus one 1-bridge knots. A knot K in an orientable closed 3-manifold M is called a *genus one* 1-*bridge knot*, a $(1,1)$-*knot* briefly, if $(M, K) = (V_1, t_1) \cup_P (V_2, t_2)$, where $(V_1, V_2;\ P)$ is a genus one Heegaard splitting and t_i is a trivial arc in V_i ($i = 1$ and 2). (An arc t properly embedded in a solid torus V is said to be *trivial* if there is a disk D in V with $t \subset \partial D$ and $\partial D - t \subset \partial V$.) Set $W_i = (V_i, t_i)$ ($i = 1$ and 2). We call the triple $(W_1, W_2;\ P)$ a $(1,1)$-*splitting* of (M, K). In this paper, we study $(1,1)$-splittings by using the distance of the curve complex. To define the distance of a $(1,1)$-splitting, we use the twice punctured torus $\Sigma = P - K$.

For $i = 1$ or 2, let $\mathcal{K}(W_i)$ be the maximal subcomplex of $C(\Sigma)$ consisting of simplexes $\langle [c_0], [c_1], \ldots, [c_k] \rangle$ such that an essential loop representing $[c_j]$ ($j = 0, 1, \ldots, k$) bounds a disk in $V_i - t_i$.

DEFINITION 1.2. We define the *distance* of a $(1,1)$-splitting $(W_1, W_2;\ P)$ by

$$\begin{aligned} d(W_1, W_2) &= d(\mathcal{K}(W_1), \mathcal{K}(W_2)) \\ &= \min\{d([x],[y]) \mid [x]\colon \text{a vertex in } \mathcal{K}(W_1),\ \ [y]\colon \text{a vertex in } \mathcal{K}(W_2)\}. \end{aligned}$$

In this paper, we give topological characterizations of the knots admitting $(1,1)$-splittings of distance ≤ 2 (Theorem 2.2, 2.3 and 2.5). As a corollary, we see that a $(1,1)$-knot is hyperbolic if and only if it has a $(1,1)$-splitting of distance ≥ 3, except for certain knots (Corollary 2.6). Further we will prove that there are $(1,1)$-splittings with arbitrarily high distance (Theorem 2.7).

2. Statement of results

Let K be a knot in a closed 3-manifold M. By $E(K)$, we mean the *exterior* of K in M, i.e., $E(K) = \mathrm{cl}(M - N(K))$, where $N(K)$ is a regular neighborhood of K in M.

DEFINITION 2.1. (1) K is a *trivial knot* if K bounds a disk in M.
(2) K is a *core knot* if K is non-trivial and M admits a genus one Heegaard splitting $(V_1, V_2;\ P)$ such that K is isotopic to the core of V_i for $i = 1$ or 2.
(3) K is a *torus knot* if K is isotopic to a simple loop on a genus one Heegaard surface of M and is not a core knot.
(4) K is a 2-*bridge knot* if there is a genus zero Heegaard splitting $(B_1, B_2;\ P_0)$ of S^3 such that $(B_i, B_i \cap K)$ ($i = 1, 2$) is a 2-string trivial tangle. (Note that a trivial knot in S^3 is also regarded as a 2-bridge knot.)
(5) For a pair α (≥ 4) and β of coprime integers and an element $r \in \mathbb{Q} \cup \{1/0\}$, $K(\alpha, \beta;\ r)$ denotes the knot K_2 in $K_1(r)$, where $K_1 \cup K_2$ is the 2-bridge link of type (α, β) (cf. Chapter 10 of [22]) and $K_1(r)$ is the manifold obtained by r-surgery on K_1.

By an argument similar to that in Section 1 of [18], we can see that $K(\alpha, \beta;\ r)$ is a $(1,1)$-knot. These knots form an important family of $(1,1)$-knots (see [1], [3] and [8]).

For the definition of other standard terms in three-dimensional topology and knot theory, we refer to [10], [12] and [22].

In this paper, we prove the following theorems.

Theorem 2.2. *Let K be a $(1, 1)$-knot in M and $(W_1, W_2; P)$ a $(1, 1)$-splitting of (M, K). Then $d(W_1, W_2) = 0$ if and only if K is a trivial knot.*

Note that Theorem 1.1 of [9] essentially implies Theorem 2.2.

Theorem 2.3. *Let K be a $(1, 1)$-knot in M and $(W_1, W_2; P)$ a $(1, 1)$-splitting of (M, K). Then $d(W_1, W_2) = 1$ if and only if M is $S^2 \times S^1$ and K is a core knot.*

Theorem 2.4. *Let K be a $(1, 1)$-knot in M and $(W_1, W_2; P)$ a $(1, 1)$-splitting of (M, K). If $d(W_1, W_2) = 2$, then one of the following holds.*
(1) *M is S^3 and K is a non-trivial 2-bridge knot.*
(2) *M is a lens space and K is a core knot.*
(3) *K is a non-trivial torus knot.*
(4) *$E(K)$ contains an essential torus.*
(5) *K is non-trivial and $K = K(\alpha, \beta; r)$ for some α, β and r.*

Conversely, if (M, K) satisfies one of (1)–(4), then any $(1, 1)$-splitting of (M, K) has distance $= 2$.

In the above theorem, by a *lens space*, we mean a closed 3-manifold which admits a Heegaard splitting of genus one and is homeomorphic to neither S^3 nor $S^2 \times S^1$. To prove Theorem 2.4, we need the following results.

- The classification of $(1, 1)$-splittings of 2-bridge knots in S^3 by T. Kobayashi and O. Saeki [15].
- The classification of $(1, 1)$-splittings of core knots in lens paces by C. Hayashi [6].
- The classification of $(1, 1)$-splittings of torus knots by K. Morimoto [17].
- A characterization of $(1, 1)$-splittings of $(1, 1)$-knots whose exteriors contain an essential torus (Proposition 6.1), which generalizes results of C. Hayashi [7] (cf. [18]).

Moreover, we prove the following characterization of $(1, 1)$-knots whose exteriors contain an essential torus. A torus properly embedded in a compact orientable 3-manifold is called an *essential torus* if it is incompressible and not ∂-parallel in the 3-manifold.

Theorem 2.5. *The exterior of a $(1, 1)$-knot K in M contains an essential torus if and only if K belongs to $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ or $\mathcal{K}_4$.*

In the above theorem, $\mathcal{K}_i$ $(i = 1, 2, 3, 4)$ denote the families of $(1, 1)$-knots defined

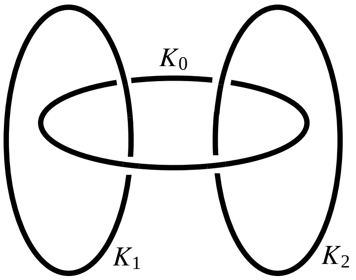

Fig. 1.

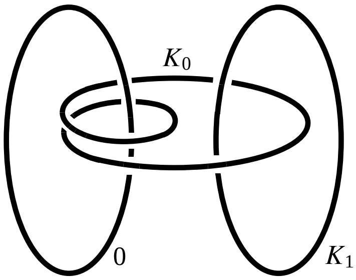

Fig. 2.

as follows.
(1) $K \in \mathcal{K}_1$ if K is a knot in lens spaces which is the connected sum of a core knot in a lens space and a non-trivial 2-bridge knot.
(2) $K \in \mathcal{K}_2$ if K is constructed as follows. Let K_0 be a non-trivial torus knot in a closed 3-manifold M, and let $L = K_1 \cup K_2$ be a 2-bridge link of type (α, β) with $\alpha \geq 4$. Let $\varphi\colon E(K_2) \to N(K_0)$ be an orientation-preserving homeomorphism which takes a meridian $m_2 \subset \partial E(K_2)$ of K_2 to a regular fiber $f \subset (\partial N(K_0) \cap P)$ of $E(K_0)$. Then $K = \varphi(K_1) \subset N(K_0) \subset M$.
(3) $K \in \mathcal{K}_3$ if K is constructed as follows. Let $K_0 \cup K_1 \cup K_2$ be the connected sum of two Hopf links illustrated in Fig. 1, and let $K'_1 \cup K'_2$ be a non-trivial 2-bridge link. Set $M = E(K_1 \cup K_2) \cup_{(\varphi_1, \varphi_2)} E(K'_1 \cup K'_2)$, where $\varphi_i\colon \partial E(K_i) \to \partial E(K'_i)$ is an orientation-reversing homeomorphism which takes a preferred longitude $l_i \subset \partial E(K_i)$ of K_i to a meridian $m_i \subset \partial E(K'_i)$ of K'_i ($i = 1$ and 2). Then $K = K_0 \subset E(K_1 \cup K_2) \subset M$. It should be noted that $M \cong S^2 \times S^1$. This can be seen as follows. For $(i, j) = (1, 2)$ and $(2, 1)$, let D_i be a disk in $E(K_j)$ bounded by l_i. Then each of $\mathrm{cl}(E(K_1 \cup K_2) - N(D_1 \cup D_2))$ and $E(K'_1 \cup K'_2) \cup N(D_1 \cup D_2)$ is homeomorphic to $S^2 \times [0, 1]$.
(4) $K \in \mathcal{K}_4$ if K is constructed as follows. Let K_0 be $K(4, 1; 0)$ and K_1 a meridian of K_0 (see Fig. 2). Let $l_1 \subset \partial E(K)$ be a longitude of K_1 which bounds a disk in $E(K_1)$ intersecting K_0 transversely in a single point. Let K_2 be a non-trivial 2-bridge knot and $\varphi\colon \partial E(K_1) \to \partial E(K_2)$ an orientation-reversing homeomorphism which takes l_1 to a meridian of K_2. Set $M = E(K_1) \cup_\varphi E(K_2)$. Then $K = K_0 \subset E(K_1) \subset M$. It should be noted that $M \cong S^2 \times S^1$. This can be seen by using the fact that the union of $E(K_2)$ and a regular neighbourhood of a disk in $E(K_1)$ bounded by l_1 is a 3-ball.

By using Thurston's hyperbolization theorem of Haken manifolds (see for example [13]), we can obtain the following corollary.

Corollary 2.6. *Let K be a $(1, 1)$-knot in M. Suppose that (M, K) is not equivalent to $K(\alpha, \beta; r)$ for any α, β and r, and that the bridge index of K is at least three if $M \cong S^3$. Then K is a hyperbolic knot if and only if it has a $(1, 1)$-splitting* *with distance* ≥ 3.

In the last section, we construct $(1, 1)$-splittings with arbitrarily high distance.

Theorem 2.7. *Let M be a closed 3-manifold which admits a genus one Heegaard splitting. Then for any positive integer n, there is a $(1, 1)$-knot in M which has a $(1, 1)$-splitting with distance $> n$.*

3. The structure of $\mathcal{K}(W_i)$

In this section, we describe the structure of the simplicial complex $\mathcal{K}(W_i)$. Throughout this section, $W = (V, t)$ denotes a pair of a solid torus V and a trivial arc t properly embedded in V, and Σ denotes the twice punctured torus $\partial V - t$. The two punctures of Σ are denoted by p_1 and p_2. Two subspaces X and Y in W are said to be *pairwise isotopic*, if there is an ambient isotopy $\{h_s\}_{0\leq s\leq 1}$ of V such that $h_0 = id$, $h_s(t) = t$ and $h_1(X) = Y$.

DEFINITION 3.1. An essential loop in Σ is called an *ε-loop* (an *ι-loop* resp.) if it is essential (inessential resp.) in ∂V.

DEFINITION 3.2. Let D be a properly embedded disk in V.
(1) D is called an *ι-disk* in W if $D \cap t = \emptyset$ and ∂D is an ι-loop on Σ.
(2) D is called an *ε_0-disk* in W if $D \cap t = \emptyset$ and ∂D is an ε-loop on Σ.
(3) D is called an *ε_1-disk* in W if $D \cap t = \{1 \text{ point}\}$ and ∂D is an ε-loop on Σ.

Lemma 3.3. *Let D_0 be an ε_0-disk in W with $\alpha = \partial D_0$, and let β be an essential loop in Σ disjoint from α. Then precisely one of the following conditions holds.*
(1) *β is isotopic to α in Σ.*
(2) *β bounds an ι-disk in W.*
(3) *β bounds an ε_1-disk in W.*

Proof. Let B be the 3-ball obtained by cutting V along D_0, and let D_0' and D_0'' be the copies of D_0 in ∂B.

CASE 1. Suppose that β does not separate D_0' and D_0'' in ∂B.

Then β does not separate p_1 and p_2 in ∂B, because β is essential in Σ. Let t' be a properly embedded arc in B with $\partial t' = \{p_1, p_2\}$ which is parallel to an arc in $\partial B - \beta$ joining p_1 to p_2. Then β bounds a separating disk D_β in B disjoint from t'. Since t' is isotopic to t in B relative $D_0' \cup D_0''$, the arc t' in V is isotopic to t in V relative $\{p_1, p_2\}$. Moreover by the hypothesis of Case 1, D_β cuts (V, t) into (V_1, t) and $(V_2, \emptyset)$, where V_1 is a 3-ball and V_2 is a solid torus. Hence the condition (2) holds.

CASE 2. Suppose that β separates D_0' and D_0'' in ∂B.

Then we can see, by an argument similar to the above, that the condition (3) or (1) holds according as β separates $\{p_1, p_2\}$ in ∂B or not.

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Any two ε_0-disks in W are pairwise isotopic.*

Proof. Let D and D' be ε_0-disks in W. If $D \cap D' = \emptyset$, then we can see that $D \cup D'$ bounds a product region disjoint from t by an argument similar to that of Lemma 3.3. Hence we may assume that D and D' intersect transversely, $|D \cap D'|$ is minimized up to pairwise isotopy in W and that $|D \cap D'| > 0$, where $|\cdot|$ is the number of connected components. By a standard innermost disk argument, we can see that $D \cap D'$ has no loop components. Let γ be a component of $D \cap D'$ which is outermost in D' and δ'_1 the outermost disk in D' with $\gamma \subset \partial\delta'$, that is, the interior of δ'_1 is disjoint from D. The arc γ also cuts D into two disks δ_1 and δ_2. Then each of $\delta_1 \cup \delta'_1$ and $\delta_2 \cup \delta'_1$ is a properly embedded disk in V disjoint from t. If either $\partial(\delta_1 \cup \delta'_1)$ or $\partial(\delta_2 \cup \delta'_1)$ is inessential in $\partial(V - t)$, then we can decrease $|D \cap D'|$ by a pairwise isotopy of D in W, a contradiction. So we may assume that $\delta_1 \cup \delta'_1$ and $\delta_2 \cup \delta'_1$ are ε_0-disks or ι-disks in W.

CLAIM. At least one of $\delta_1 \cup \delta'_1$ and $\delta_2 \cup \delta'_1$ is an ι-disk in W.

Proof. Suppose that $\delta_1 \cup \delta'_1$ is a ε_0-disk in W to show that $\delta_2 \cup \delta'_1$ is an ι-disk. Let B be the 3-ball obtained from V by cutting along D, and let D_+ and D_- be the copies of D in ∂B. We denote the image of δ'_1 in B by the same symbol. Then we may assume $\delta'_1 \cap D_+ = \emptyset$ and $\delta'_1 \cap D_- = \gamma$. By cutting B along δ'_1, we obtain 3-balls B_1 and B_2 with $D_+ \subset \partial B_1$, $(\delta_1 \cup \delta'_1) \subset \partial B_1$ and $(\delta_2 \cup \delta'_1) \subset \partial B_2$. Since D and δ'_1 are disjoint from t in V, precisely one of B_1 and B_2 contains t. If $t \subset B_1$, then $\partial(\delta_2 \cup \delta'_1)$ is inessential in $\partial(V - t)$, a contradiction. Hence $t \subset B_2$, and $\delta_2 \cup \delta'_1$ is an ι-disk in W. □

Let B, D_+, D_-, B_1 and B_2 be as above. Put $\delta'_2 = \mathrm{cl}(D' - \delta'_1)$, and let A be the annulus defined by $A = \partial B_1 \cap (\partial B - \mathrm{int}(D_+ \cup D_-))$. Put $\alpha = \partial D' \cap \partial\delta'_2$, and let $\partial\gamma \ni p_1, p_2, \dots, p_n \in \partial\gamma$ be the components of $\partial D \cap \alpha$ sitting on α in this order. Then by the minimality of $|D \cap D'|$, we may assume that $A \cap \partial\delta'_2$ consists of essential arcs in the annulus A. Let α_i be the subarc of α joining p_i to p_{i+1} in α, and let p_i^+, p_i^-, respectively the copies of p_i in ∂D_+ and ∂D_- ($i = 1, 2, \dots, n-1$). Then $\alpha_1 \cap D_+ = p_1^+$ and $\alpha_1 \cap D_- = p_2^-$, because α_1 is essential in A. Inductively, we obtain $\alpha_i \cap D_+ = p_i^+$ and $\alpha_i \cap D_- = p_{i+1}^-$ ($i = 1, 2, \dots, n-1$). In particular, $\alpha_{n-1} \cap D_+ = p_{n-1}^+$ and $\alpha_{n-1} \cap D_- = p_n^-$. This means that D' does not intersect D transversely in p_n, a contradiction. Hence the interior of A is disjoint from $\partial\delta'_2$, and there is an ε_0-disk obtained by moving D_+ so that it is disjoint from D'. This means D' is isotopic to D. □

Lemma 3.5. *Let $[\alpha]$ be the vertex of $\mathcal{K}(W)$ represented by the boundary of an ε_0-disk, and let $[\beta]$ be an arbitrary vertex of $\mathcal{K}(W)$ different from $[\alpha]$. Then $[\beta]$ is represented by an ι-loop disjoint from an ε-loop representing $[\alpha]$.*

Proof. If $[\beta]$ is represented by an ε-loop, then we have $[\alpha] = [\beta]$ by Lemma 3.4, a contradiction. So $[\beta]$ is represented by an ι-loop, say β. Let D_β be a disk in $V - t$ bounded by β. Since β is inessential in V, there is an essential disk D in V disjoint from D_β (and hence disjoint from t). By Lemma 3.4, ∂D represents $[\alpha]$ and hence we obtain the desired result. □

Lemma 3.6. *Any two mutually disjoint ι-disks in W are pairwise isotopic.*

Proof. Let D and D' be mutually disjoint ι-disks in W and put $\beta = \partial D$ and $\beta' = \partial D'$. Then D cuts (V, t) into (V_1, t) and $(V_2, \emptyset)$, where V_1 is a 3-ball and V_2 is a solid torus. If necessary, by exchanging the names D and D' of disks, we may assume that D' is contained in V_1 and β' is an inessential loop in $\partial V_1 - t$, because D' is an ι-disk and is disjoint from D. If β' bounds a disk in ∂V_1 disjoint from the copy of D in ∂V_1, then β' is inessential in $\partial V - t$, a contradiction. Hence β' separates the copy of D from ∂t in ∂V_1, and this implies D and D' are pairwise isotopic. □

Let α be an ε-loop which bounds an ε_0-disk, say D_α. We fix a properly embedded arc, say t_0, in ∂V such that $\partial t_0 = \partial t$, $t_0 \cap \alpha = \emptyset$ and $t \cup t_0$ bounds a disk in V. Let B be the 3-ball obtained by cutting V along D_α, and let D'_α and D''_α be the copies of D_α in ∂B. Set $\mathcal{P} = \partial t \cup \{$the centers of D'_α and $D''_\alpha\}$. Then $(\partial B, \mathcal{P})$ is identified with $(\mathbb{R}^2, \mathbb{Z}^2)/\Gamma$, where Γ is the group of isometries of $\mathbb{R}^2$ generated by π-rotations about the points of the integral lattice $\mathbb{Z}^2$. Here t_0 is identified with a line in $\mathbb{R}^2$ of slope $1/0$, i.e., a lift of t_0 joins $(0, 0)$ to $(0, 1)$ in $\mathbb{R}^2$.

Let $\mathcal{A}$ be the set of the vertices of $\mathcal{K}(W)$ different from $[\alpha]$, where $[\alpha]$ is the vertex of $\mathcal{K}(W)$ represented by α. In the following, we define a map $\varphi\colon \mathcal{A} \to \mathbb{Q} \cup \{1/0\}$. Let $[\beta]$ be an element of $\mathcal{A}$. Then by Lemma 3.5, $[\beta]$ is represented by an ι-loop, say β, which is disjoint from α. Let t_β be an arc in $\partial V - \beta$ joining distinct components of ∂t. Note that t_β is unique up to isotopy relative to the endpoints. Let $\tilde{t}_\beta\colon [0, 1] \to \mathbb{R}^2$ be a lift of $t_\beta\colon [0, 1] \to (\partial B, \mathcal{P})$. Then $\tilde{t}_\beta(1) - \tilde{t}_\beta(0)$ is an integral vector, say (p, q), in $\mathbb{R}^2$.

Lemma 3.7. *Let $[\beta]$ and (p, q) be as above. Then the rational number q/p does not depend on the choice of a representative of $[\beta]$, and hence the correspondence $\beta \mapsto q/p$ induces a well-defined map $\varphi\colon \mathcal{A} \to \mathbb{Q} \cup \{1/0\}$. Moreover φ is injective and the image is equal to $\{q/p \in \mathbb{Q} \cup \{1/0\} \mid (p, q) \equiv (0, 1) \pmod 2\}$.*

Proof. Let β' be another representative disjoint from α of $[\beta]$. Then there is a homotopy in Σ between β and β'. Since α is an essential loop in Σ and is homotopic to neither β nor β', we can modify the homotopy so that it is disjoint from α. Hence β and β' are homotopic in $\Sigma - \alpha$ and therefore in the four times punctured 2-sphere $\partial B - \mathcal{P}$. This implies that φ is well-defined and injective, because it is well known that the correspondence $\beta \mapsto q/p$ induces a well-defined injective map from the set of the isotopy classes of essential loops in $\partial B - \mathcal{P}$ to $\mathbb{Q} \cup \{1/0\}$ (cf. Section 2 of [5]). Moreover, since an ι-loop representing $[\beta]$ does not separate ∂t in ∂V, we see $(p, q) \equiv (0, 1) \pmod 2$. On the other hand, it is easy to see that for any $q/p \in \mathbb{Q} \cup \{1/0\}$ with $(p, q) \equiv (0, 1) \pmod 2$, there is a vertex $[\beta] \in \mathcal{A}$ with $\varphi([\beta]) = q/p$. Hence we obtain the desired result. $\square$

Proposition 3.8. *Let $[\alpha]$ be the vertex of $\mathcal{K}(W)$ represented by the boundary of an ε_0-disk of W, and let $\mathcal{A}$ be the countably infinite set as above. Then $\mathcal{K}(W)$ is isomorphic to the join $\{[\alpha]\} * \mathcal{A}$.*

Proof. By Lemma 3.4, we see that $[\alpha]$ is unique. Lemma 3.5 indicates that for any vertex $[\beta]$ of $\mathcal{A}$, there is an edge joining $[\beta]$ to $[\alpha]$. On the other hand, by Lemma 3.6, there are no edges of $C(\Sigma)$ joining distinct vertices of $\mathcal{A}$. $\square$

4. (1, 1)-splittings of distance = 0

Lemma 4.1. *Let K be a $(1, 1)$-knot in M and $(W_1, W_2; P)$ a $(1, 1)$-splitting of (M, K). Then K is a trivial knot if and only if there are an ι-disk D_i in W_i with $\partial D_1 = \partial D_2$ ($i = 1$ and 2).*

Proof. We first prove the "only if part". Suppose that K is trivial. Let D be a disk in M with $\partial D = K$. Then by Theorem 1.1 of [9], we can isotope D so that $D \cap P$ separates D into two disks. Set $D_i = \partial N(D) \cap V_i$ ($i = 1$ and 2). Then we see that D_i is an ι-disk and $\partial D_1 = \partial D_2$ ($i = 1$ and 2).

We next prove the "if part". Suppose that there are an ι-disk D_i in W_i ($i = 1$ and 2). Then $D_1 \cup D_2$ forms a 2-sphere which cuts (M, K) into $(M - \operatorname{int} B^3, \emptyset)$ and $(B^3, \text{1-bridge knot})$ and hence K is a trivial knot. $\square$

Lemma 4.2. *Let K be a $(1, 1)$-knot in $S^2 \times S^1$ and $(W_1, W_2; P)$ a $(1, 1)$-splitting of $(S^2 \times S^1, K)$. Then K is a trivial knot if and only if there are an ε_0-disk D_1 in W_1 and an ε_0-disk D_2 in W_2 with $\partial D_1 = \partial D_2$.*

Proof. We first prove the "if part". Suppose that the latter condition in Lemma 4.2 holds. Then there are ι-disks D'_1 and D'_2 in W_1 and W_2, respectively, with $\partial D'_i \cap \partial D_i = \emptyset$ ($i = 1, 2$) and $\partial D'_1 = \partial D'_2$. Hence by Lemma 4.1, K is a trivial knot.

Suppose conversely that K is a trivial knot in $S^2 \times S^1$. By Lemma 4.1, there are an ι-disk δ_i in W_i with $\partial\delta_1 = \partial\delta_2$ ($i = 1$ and 2). Then there are ε_0-disks in each of W_1 and W_2 such that they are disjoint from $\delta_1 \cup \delta_2$ and they share their boundaries since the manifold is $S^2 \times S^1$. Hence we see that the latter condition holds. □

Proof of Theorem 2.2. Suppose that K is a trivial knot in M. Then by Lemma 4.1, we have $d(W_1, W_2) = 0$.

Conversely, let K be a $(1,1)$-knot in M and $(W_1, W_2; P)$ a $(1,1)$-splitting of (M, K) with $d(W_1, W_2) = 0$. Then there is an essential loop x in $\Sigma = P - K$ which bounds a disk in $V_i - t_i$ for each $i = 1$ and 2.

If x is an ε_0-loop, then $(W_1, W_2; P)$ satisfies the condition of Lemma 4.2. Hence M is $S^2 \times S^1$ and K is a trivial knot.

If x is an ι-loop, then $(W_1, W_2; P)$ satisfies the condition of Lemma 4.1, that is, K is a trivial knot in M.

We have completed the proof of Theorem 2.2. □

5. (1, 1)-splittings of distance = 1

Proposition 5.1. *Let K be a $(1,1)$-knot in $S^2 \times S^1$ and $(W_1, W_2; P)$ a $(1,1)$-splitting of $(S^2 \times S^1, K)$. Then K is a core knot if and only if there are an ε_0-disk D_i in W_i and an ε_1-disk D_j in W_j with $\partial D_i = \partial D_j$ for $(i, j) = (1, 2)$ or $(2, 1)$.*

Proof. The "if part" follows from the light bulb theorem (cf. Chapter 9, Section E, 4 Exercise of [22]).

To prove the "only if part", suppose that K is a core knot in $S^2 \times S^1$. Then there is an essential 2-sphere S which intersects K in one point. Put $S_i = S \cap V_i$ ($i = 1$ and 2). We may assume that each component of S_1 is either an ε_0-disk, an ε_1-disk or an ι-disk in $W_i = (V_i, t_i)$. Note that $|S_1| > 0$ and that S_1 contains at most one ε_1-disk component. Let D be an ε_0-disk in W_2 such that D intersects S_2 transversely. We choose S and D so that each component of S_1 is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1, and the pair $(|S_1|, |S_2 \cap D|)$ is minimized with respect to the lexicographic order.

If $|S_1| = 1$, then $S \cap P$ is an ε-loop because S is an essential 2-sphere in $S^2 \times S^1$. Hence the assertion obviously holds. So we may assume $|S_1| > 1$.

CLAIM 1. $S_2 \cap D \neq \emptyset$.

Proof. Suppose that S_2 is disjoint from D. Let B be the 3-ball obtained by cutting V_2 along D. Then there is a disk E on ∂B with $E \cap S_2 = \partial E$ and $|E \cap K| \leq 1$. Let E' be the disk obtained from E by pushing the interior of E into the interior of B. Then $\partial E'$ cuts S into two disks Q_1 and Q_2. Precisely one of them, say Q_1, is a component of S_1.

Suppose that $|E' \cap K| = 0$. If $|Q_1 \cap K| = 1$, then $Q_1 \cup E'$ is a 2-sphere which intersects K in one point. Hence the disks Q_1 and E' satisfy the desired condition. So we may assume that $|Q_1 \cap K| = 0$ and hence $|Q_2 \cap K| = 1$. Let S' be the 2-sphere obtained from $Q_2 \cup E'$ by pushing $\partial E'$ into the interior of V_2 slightly. Then each component of $S'_1 := S' \cap V_1$ is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1, and $|S'_1| < |S_1|$, a contradiction.

Suppose that $|E' \cap K| = 1$. If $|Q_1 \cap K| = 0$, then $Q_1 \cup E'$ is a 2-sphere which intersects K in one point, and hence the disks Q_1 and E' satisfy the desired condition. So we may assume that $|Q_1 \cap K| = 1$ and hence $|Q_2 \cap K| = 0$. Let S' be the 2-sphere obtained from $Q_2 \cup E'$ by pushing $\partial E'$ into V_2 slightly. Then each component of $S'_1 := S' \cap V_1$ is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1, and $|S'_1| < |S_1|$, a contradiction. □

Claim 2. $S_2 \cap D$ has no loop components.

Proof. Suppose that $S_2 \cap D$ has a loop component. Let σ be a loop component of $S_2 \cap D$ which is innermost in D and D_σ the innermost disk with $\sigma = \partial D_\sigma$, that is, the interior of D_σ is disjoint from S_2. Then σ cuts S into two disks E_1 and E_2. We can assume that $|E_1 \cap K| = 1$. Since D_σ is disjoint from K, $S' = E_1 \cup D_\sigma$ is a 2-sphere which intersects K in one point. Put $S'_i = S' \cap V_i$ ($i = 1$ and 2). Note that S'_1 is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1. If σ is essential in S_2, then $|S'_1| < |S_1|$, a contradiction. If σ is inessential in S_2, then $|S'_1| = |S_1|$. In this case, by isotoping S' so that D_σ is disjoint from D, we see that $|S'_2 \cap D| < |S_2 \cap D|$, a contradiction. □

By Claim 1 and Claim 2, there is an arc component γ of $S_2 \cap D$ which is outermost in D. Let $D_\gamma \subset D$ be the outermost disk with $\gamma \subset \partial D_\gamma$. Put $\gamma' = \mathrm{cl}(\partial D_\gamma - \gamma)$. Let F be the component of the surface obtained by cutting ∂V_1 along ∂S_1 such that $\gamma' \subset F$. Let $S^{(1)}$ be a 2-sphere obtained by isotoping S along D_γ near the arc γ, and put $S^{(1)}_i = S^{(1)} \cap V_i$ ($i = 1$ and 2).

Claim 3. The arc γ' is essential in F.

Proof. Suppose that γ' is inessential in F. Then we obtain an annulus component A in $S^{(1)}_1$ such that one of the components of ∂A bounds a disk E in ∂V_1. Note that $|E \cap K| \leq 2$ and ∂E cuts S into two disks R_1 and R_2. Since S intersects K transversely in one point, we may assume that $|R_1 \cap K| = 1$ and $|R_2 \cap K| = 0$.

Suppose that $|E \cap K| = 0$. If $A \subset R_1$, let S' be a 2-sphere obtained from $R_1 \cup E$ by pushing E into the interior of V_1; otherwise, let S' be a 2-sphere obtained from $R_1 \cup E$ by pushing the interior of E into the interior of V_2. Then we see that each component of S'_1 is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1, and that $(|S'_1|, |S'_2 \cap D|) < (|S_1|, |S_2 \cap D|)$, a contradiction.

Suppose that $|E \cap K| = 1$. If $A \subset R_2$, let S' be a 2-sphere obtained from $R_2 \cup E$ by pushing E into the interior of V_1; otherwise, let S' be a 2-sphere obtained from $R_2 \cup E$ by pushing the interior of E into the interior of V_2. Then we see that each component of S'_1 is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1, and that $(|S'_1|, |S'_2 \cap D|) < (|S_1|, |S_2 \cap D|)$, a contradiction.

Suppose that $|E \cap K| = 2$. If γ' joins an ι-disk to itself, then $E' := \mathrm{cl}(F - E)$ is a disk bounded by a component of ∂A. Since E' is disjoint from K, by an argument similar to the case of $|E \cap K| = 0$, we obtain a contradiction by using the disk E' instead of E. So we may assume that γ' joins an ε_0-disk to itself. Then there is an ε_0-disk disjoint from ∂E. By Lemma 3.3, ∂E bounds an ι-disk. Hence by an argument similar to the case of $|E \cap K| = 0$, we obtain a contradiction by using the ι-disk instead of E. $\square$

CLAIM 4. S_1 has no ε_1-disk components.

Proof. Suppose that S_1 has an ε_1-disk component. Then S_1 has no ι-disk components. Thus S_1 has ε_0-disk components, because $|S_1| > 1$. Hence by Claim 3, γ' joins distinct components of S_1.

CASE 1. The arc γ' joins distinct ε_0-disks.

Let δ be the disk component of $S_1^{(1)}$ obtained from these disks. Then we can push δ out of V_1 fixing t_1. After this operation, we see that each component of $S_1^{(1)}$ is either an ε_0-disk or an ε_1-disk in W_1, and that $|S_1^{(1)}| < |S_1|$, a contradiction.

CASE 2. The arc γ' joins an ε_0-disk to an ε_1-disk.

Then $S_1^{(1)}$ has the disk component δ' from these disks. Note that δ' cuts (V_1, t_1) into (V'_1, t'_1) and (V''_1, t''_1), where V'_1 is a 3-ball, t'_1 is a trivial arc in V'_1, V''_1 is a solid torus and t''_1 is a trivial arc in V''_1. So we can push δ' out of V_1 through (V'_1, t'_1). After this operation, each component of $S_1^{(1)}$ is either an ε_0-disk or an ε_1-disk in W_1, and we have $|S_1^{(1)}| < |S_1|$, a contradiction. $\square$

CLAIM 5. S_1 has no ε_0-disk components.

Proof. Suppose that S_1 has an ε_0-disk component. Note that S_1 may have ι-disk components, because S_1 has no ε_1-disk components by Claim 4. Since γ' is essential in F by Claim 3, we have the following cases.

CASE 1. The arc γ' joins distinct ε_0-disks, or joins distinct ι-disks.

By an argument similar to Case 1 in the proof of Claim 4, we obtain a contradiction.

CASE 2. The arc γ' joins an ε_0-disk to an ι-disk.

Then $S_1^{(1)}$ is either an ε_0-disk, an ε_1-disk or an ι-disk in W_1, and $|S_1^{(1)}| < |S_1|$, a contradiction.

CASE 3. The arc γ' joins an ε_0-disk to itself.

By Claim 3, γ' must be essential in F. Hence S_1 must consist of an ε_0-disk and

ι-disks, and we obtain a Möbius band in $S_1^{(1)}$, a contradiction.

CASE 4. The arc γ' joins an ι-disk to itself.

Let δ be the ι-disk component of S_1 with $\partial\gamma' \subset \partial\delta$, and let γ_1 and γ_2 be arcs such that $\partial\delta = \gamma_1 \cup \gamma_2$ and $\partial\gamma_1 = \partial\gamma_2 = \partial\gamma'$. Since S_1 has ε_0-disk components, by Claim 3, $\gamma' \cup \gamma_1$ bounds an ε_0-disk, say E', whose interior is disjoint from S. Hence by an argument similar to Claim 3, we have a contradiction by using the disk E'. $\square$

By Claim 4 and Claim 5, S_1 consists of ι-disks, because $|S_1| > 1$. But this implies that S is inessential in $S^2 \times S^1$, a contradiction.

This completes the proof of Proposition 5.1. $\square$

Proof of Theorem 2.3. Suppose that K is a core knot in $S^2 \times S^1$. By Proposition 5.1, we may assume that there are an ε_0-disk D_1 in W_1 and an ε_1-disk D_2 in W_2 with $\partial D_1 = \partial D_2$. Then there is an ε_0-disk D_2' in W_2 which is disjoint from D_2. Hence we have $d(W_1, W_2) = 1$ since Theorem 2.2 implies $d(W_1, W_2) \neq 0$ for $(1, 1)$-splittings of the core knot in $S^2 \times S^1$.

Conversely, we suppose $d(W_1, W_2) = 1$, that is, there are mutually disjoint essential loops x and y in $\Sigma = P - K$ which bound disks in $V_1 - t_1$ and $V_2 - t_2$, respectively. Suppose that either x or y, say y, is an ι-loop. If x bounds an ε_0-disk, then y bounds an ι-disk in W_1 by Lemma 3.3. (Otherwise, y is pairwise isotopic to x.) Hence K is a trivial knot, a contradiction. So we may suppose that x (y resp.) bounds an ε_0-disk in W_1 (W_2 resp.). Then x bounds an ε_1-disk in W_2 by Lemma 3.3. Hence K is a core knot in $S^2 \times S^1$ by Proposition 5.1.

We have completed the proof of Theorem 2.3. $\square$

6. (1, 1)-knots whose exteriors contain essential tori

In this section, we study $(1, 1)$-knots whose exteriors contain an essential torus and prove Theorem 2.5 and the following Proposition 6.1.

Proposition 6.1. *Let K be a $(1, 1)$-knot in M whose exterior contains an essential torus. Then every $(1, 1)$-splitting $(W_1, W_2; P)$ of (M, K) satisfies one of the following conditions.*
($\#_a$) *There are an ι-disk D_i in W_i and an ε_1-disk D_j in W_j such that $\partial D_i \cap \partial D_j = \emptyset$ for $(i, j) = (1, 2)$ or $(2, 1)$.*
($\#_b$) *There is an annulus $Z \subset P$ which is incompressible in both V_1 and V_2, and there is an ι-disk D_i in W_i with $\partial D_i \subset Z$ for each $i = 1$ and 2.*
($\#_c$) *There are an ε_1-disk D_1 in W_1 and an ε_1-disk D_2 in W_2 with $\partial D_1 = \partial D_2$.*

Before proving Theorem 2.5 and Proposition 6.1, we present lemmas which describe topological consequences of the conclusions in Proposition 6.1.

Lemma 6.2 ([7] Lemma 2.1). *Let K be a non-trivial $(1, 1)$-knot in M with a $(1, 1)$-splitting $(W_1, W_2; P)$ satisfying the condition $(\#_a)$ of* Proposition 6.1. *Then one of the following holds*.
(1) *K is a 2-bridge knot.*
(2) *K is a core knot in a lens space.*
(3) *K belongs to $\mathcal{K}_1$.*

REMARK 6.3. Though this lemma is proved under the assumption that $M \not\cong S^2 \times S^1$ in [7], we can easily see that the same conclusion holds even if $M \cong S^2 \times S^1$. In fact, we can show by using the light bulb theorem that K is a core knot in this case.

Lemma 6.4. *Let K be a non-trivial $(1, 1)$-knot in M with a $(1, 1)$-splitting $(W_1, W_2; P)$ satisfying the condition $(\#_b)$ of* Proposition 6.1. *Then one of the following holds.*
(1) *K is a core knot or a torus knot.*
(2) *$K = K(\alpha, \beta; r)$ for some α, β and r.*
(3) *K belongs to $\mathcal{K}_2$.*

Proof. Let Z be an annulus which satisfies the condition $(\#_b)$ of Proposition 6.1. For each $i = 1$ and 2, since Z is incompressible in V_i, ∂D_i bounds a disk D_i' in Z. Let A_i be an annulus in V_i obtained from $Z_i := \mathrm{cl}((Z - D_i') \cup D_i)$ by pushing the interior of Z_i into the interior of V_i. For each $i = 1$ and 2, let $(V_{i1}, \emptyset)$ and (V_{i2}, t_i) be the pair obtained from (V_i, t_i) by cutting along A_i, where each of V_{i1} and V_{i2} is a solid torus and t_i is a trivial arc in V_{i2}. Then we see that $V_{11} \cup V_{12}$ is either a solid torus or the exterior of a torus knot. On the other hand, (V_{i2}, t_i) is identified with $(\mathrm{cl}(B^3 - \tau_1), \tau_2)$, where $(B^3, \tau_1 \cup \tau_2)$ is a 2-string trivial tangle, in such a way that the copy of A_i corresponds to the boundary of the regular neighbourhood of τ_1. Since $V_{11} \cap V_{21}$ is a 2-sphere with two holes which contains the two points $P \cap K$, we see that $(V_{11} \cup V_{21}, K)$ is identified with $(E(K_2), K_1)$, where $K_1 \cup K_2 = L$ is a 2-bridge link.

Suppose that L is a trivial link. Then K_1 bounds a disk in $E(K_2)$ and hence K is a trivial knot, a contradiction.

Suppose that L is a Hopf link. Then K_1 is isotopic to K_2. So we can put K on P. Hence K is a core knot or a torus knot.

Suppose that $V_{11} \cup V_{12}$ is a solid torus. Then we see that $K = K(\alpha, \beta; r)$ for some α, β and r.

In other cases, we see that $A_1 \cup A_2$ is an essential torus. Hence K belongs to $\mathcal{K}_2$. □

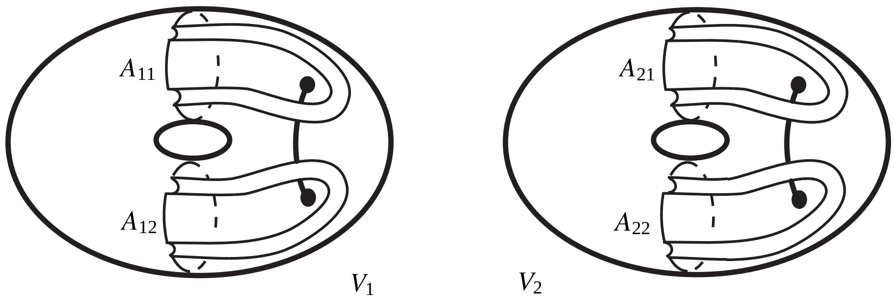

Fig. 3.

Lemma 6.5. *Let K be a non-trivial* $(1,1)$*-knot in M and* $(W_1, W_2; P)$ *a* $(1,1)$*-splitting of* (M, K)*. Suppose that* $(W_1, W_2; P)$ *satisfies the condition* $(\#_c)$ *of* Proposition 6.1. *Then $M \cong S^2 \times S^1$ and either*
(1) $K = K(4, 1; 0)$*, or*
(2) *K belongs to $\mathcal{K}_3$ or $\mathcal{K}_4$.*

Proof. Let D_1 and D_2 be a pair of disks which give the condition $(\#_c)$ of Proposition 6.1, and put $V_i^- = \mathrm{cl}(V_i - N(t_i))$ ($i = 1$ and 2). Let α_{ij} ($j = 1$ and 2) be the components of $\partial(V_i^- \cap N(t_i))$, and let A_{ij} ($j = 1$ and 2) be annuli properly embedded in V_i^- satisfying the following conditions (see Fig. 3).
(1) A_{ij} is parallel to $D_i \cap V_i^-$ in V_i.
(2) $A_{ij} \cap N(t_i) = \emptyset$.
(3) α_{ij} is parallel to a component of ∂A_{ij} in $\mathrm{cl}(\partial V_i^- - N(t_i))$.
(4) $\partial(A_{11} \cup A_{12}) = \partial(A_{21} \cup A_{22})$.

For each $i = 1$ and 2, let $(V_{i1}, \emptyset)$ and (V_{i2}, t_i) be the pairs obtained from (V_i, t_i) by cutting along $A_{i1} \cup A'_{i2}$, where V_{i1} is a genus two handlebody, V_{i2} is a 3-ball and t_i is a trivial arc in V_{i2}. Then V_{i1} is identified with the exterior of a 2-string trivial tangle (B^3, τ) in such a way that the copy of $A_{i1} \cup A_{i2}$ corresponds to the boundary of the regular neighbourhood of τ.

CASE 1. $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$ composes two tori.

Suppose that one of the tori, say T_0, is inessential in $E(K)$. Then since T_0 is not parallel to $\partial N(K)$, T_0 is compressible in $E(K)$. So we can obtain the 2-sphere S by compressing T_0. Note that S is essential, because T_0 is non-separating in $E(K)$. Hence S is an essential 2-sphere in $E(K)$. This implies that K is a trivial knot by Proposition 2.9 of [2], a contradiction. Hence T_0 is an essential torus in $E(K)$. In the following, we show that K belongs to $\mathcal{K}_3$. Since $V_{11} \cap V_{21}$ is a 2-sphere with four holes, we see that $V_{11} \cup V_{21}$ is the exterior of a non-trivial 2-bridge link, say L. On the other hand, we can recognize $(M_0, k_0) := (V_{12}, t_1) \cup (V_{22}, t_2)$ as follows. We first note that (V_{i2}, t_i) is identified with (B^3, τ), where τ is a trivial arc in B^3, in such a way that the copy of $A_{i1} \cup A_{i2}$ corresponds to a regular neighborhood on ∂B^3 of two homotopically non-trivial simple loops in $\partial B^3 - \tau$. Moreover, $(V_{12}, t_1) \cap (V_{22}, t_2)$ consists of an annulus and two copies of (D^2, o), where o is the center of the disk. By using this fact, we can see that $E(k_0)$ is identified with $B \times S^1$, an orientable S^1-bundle over a two-holed disk B, and that a meridian of $E(k_0)$ is isotopic to a fiber. Here the S^1-bundle structure is obtained by glueing the S^1-bundle structure of $E(t_1)$ and $E(t_2)$. Now let $K_0 \cup K_1 \cup K_2$ be as in the definition of $\mathcal{K}_3$. Since $E(K_0 \cup K_1 \cup K_2)$ is identified with $B \times S^1$, where longitudes of K_1 and K_2 correspond to fibers of $B \times S^1$, $(V_{12}, t_1) \cup (V_{22}, t_2) = (E(k_0), \emptyset) \cup (N(k_0), k_0)$ is identified with $(E(K_0 \cup K_1 \cup K_2), K_0)$, where a longitude of k_0 corresponds to a fiber (with respect to the bundle structure $B \times S^1$ on $E(k_0)$). Hence $(E(K_1 \cup K_2), K_0) = (E(K_0 \cup K_1 \cup K_2), \emptyset) \cup (N(K_0), K_0)$ is identified with $(E(k_0), \emptyset) \cup (N(k_0), k_0)$. Thus we have $(M, K) = (V_{11}, \emptyset) \cup (V_{21}, \emptyset) \cup (V_{21}, t_1) \cup (V_{22}, t_2) = (E(L), \emptyset) \cup (E(K_1 \cup K_2), K_0)$. Hence K belongs to $\mathcal{K}_3$.

CASE 2. $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$ composes a torus T.

Since $V_{11} \cap V_{21}$ is a 2-sphere with four holes, we see that $V_{11} \cup V_{21}$ is the exterior of a 2-bridge knot, say K_2. On the other hand, we can recognize $(M_0, k_0) := (V_{12}, t_1) \cup (V_{22}, t_2)$ as follows. We first note that (V_{i2}, t_i) is identified with (B^3, τ), where τ is a trivial arc in B^3 in such a way that the copy of $A_{i1} \cup A_{i2}$ corresponds to a regular neighborhood on ∂B^3 of two homotopically non-trivial simple loops in $\partial B^3 - \tau$. Moreover, $(V_{12}, t_1) \cap (V_{22}, t_2)$ consists of an annulus and two copies of $(D^2, \emptyset)$. By using this fact, we can see that $E(k_0)$ is identified with $B \tilde{\times} S^1$, an orientable twisted S^1-bundle over a one-holed Möbius band B, and that a meridian of $E(k_0)$ is isotopic to a fiber. Here the S^1-bundle structure is obtained by glueing the S^1-bundle structure of $E(t_1)$ and $E(t_2)$. Now let $K_0 \cup K_1 \subset S^2 \times S^1$ and $l_1 \subset \partial E(K_1)$ be as in the definition of $\mathcal{K}_4$. Then $(V_{12}, t_1) \cup (V_{22}, t_2) = (E(k_0), \emptyset) \cup (N(k_0), k_0)$ is identified with $(E(K_1), K_0)$, where l_1 corresponds to a fiber (with respect to the bundle structure $B \tilde{\times} S^1$ on $E(k_0)$). This can be seen as follows. Since $K_0 = K(4, 1; 0)$, K_0 intersects each fiber S^2 in two points. So $E(K_0)$ is a twisted annuls bundle over S^1, and hence it is a twisted S^1-bundle over a Möbius band. Moreover, the meridian K_1 of K_0 corresponds to a regular fiber. This implies that $E(K_0 \cup K_1)$ is identified with $B \tilde{\times} S^1$, where l_1 corresponds to a fiber of $B \tilde{\times} S^1$. Hence $(E(K_0), K_1) = (E(K_0 \cup K_1), \emptyset) \cup (N(K_0), K_0)$ is identified with $(E(k_0), \emptyset) \cup (N(k_0), k_0)$. Thus we have $(M, K) = (V_{11}, \emptyset) \cup (V_{21}, \emptyset) \cup (V_{21}, t_1) \cup (V_{22}, t_2) = (E(K_2), \emptyset) \cup (E(K_1), K_0)$.

Suppose that T is essential in $E(K)$. Then K_2 is non-trivial. Hence K belongs to $\mathcal{K}_4$.

Suppose that T is inessential in $E(K)$. Then we see that K_2 is trivial. Hence $E(K)$ is homeomorphic to $B \tilde{\times} S^1$, where B is a Möbius band. Hence $E(K)$ is a Seifert fibered space whose base space is a disk with two singular points, and the Seifert invariant of the singular fibers are $1/2$. Hence K is a torus knot in $S^2 \times S^1$ which intersects $S^2 \times \{1\text{point}\}$ in two points. This implies $K = K(4, 1, 0)$. $\square$

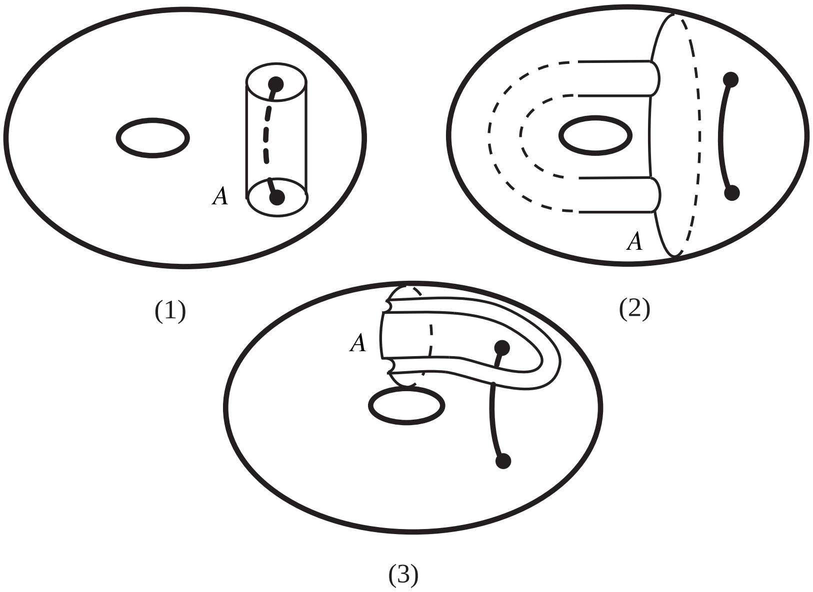

Fig. 4.

To prove Proposition 6.1, we prepare some lemmas which are obtained by an argument similar to those in Section 3 of [14]. An annulus properly embedded in an orientable 3-manifold is called *essential* if it is incompressible and not ∂-parallel. For a solid torus V and a trivial arc t in V, an annulus properly embedded in $V - t$ is called *essential in* (V, t) if it is essential in $V - t$.

Lemma 6.6. *Let V be a solid torus and t a trivial arc in V, and let A be an essential annulus in (V, t). Then one of the following holds* (*see* Fig. 4).
(1) *A cuts (V, t) into $(V_1, \emptyset)$ and (V_2, t), where V_1 is a genus two handlebody, V_2 is a* 3*-ball and t is a trivial arc in V_1.*
(2) *A cuts (V, t) into $(V_1, \emptyset)$ and (V_2, t), where V_1 is a solid torus, V_2 is a genus two handlebody and t is a trivial arc in V_2.*
(3) *A is a non-separating annulus in $V - t$ and there are an ε_0-disk D and an ε_1-disk D' in (V, t) with $D \cap D' = \emptyset$ and $A \cap (D \cup D') = \emptyset$.*

Proof. Let $\mathcal{D}$ be a disjoint union of an ε_0-disk and an ι-disk in (V, t). Since A is incompressible in $V - t$, A intersects $\mathcal{D}$. By a standard innermost/outermost disk argument, we can find a disk δ in V such that $\delta \cap t = \emptyset$, $\delta \cap A = a$ is an essential arc in A and $\delta \cap \partial V = b$ is an arc with $\partial a = \partial b$ and $a \cup b = \partial\delta$. By performing a ∂-compression of A along δ, we obtain a disk D properly embedded in $V - t$. Since A is essential in $V - t$, D is essential in $V - t$.

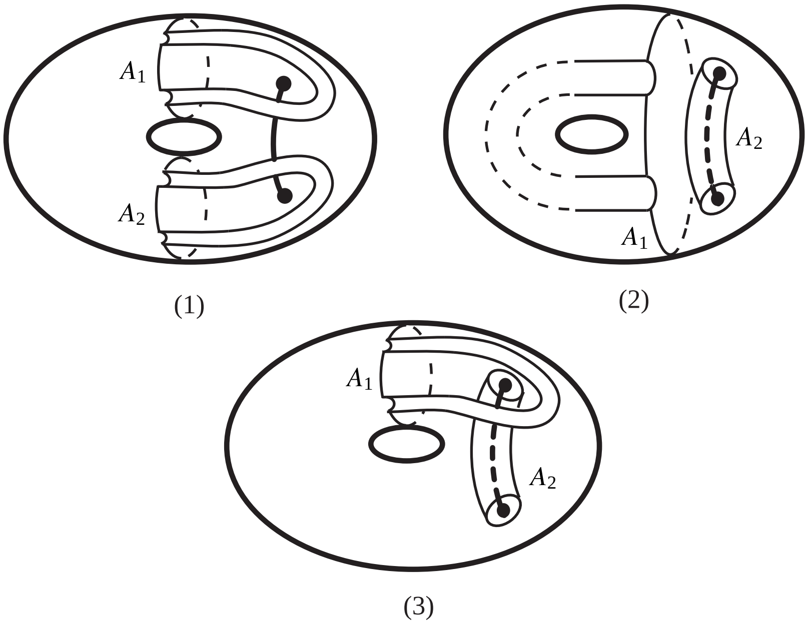

Fig. 5.

CASE 1. D is an ι-disk.

Then D cuts (V, t) into (V', t) and $(V'', \emptyset)$, where V' is a 3-ball, t is a trivial arc in V' and V'' is a solid torus. If $A - D \subset V'$, then we obtain the conclusion (1). Otherwise, we obtain the conclusion (2).

CASE 2. D is an ε_0-disk.

Then D cuts (V, t) into (B, t), where B is a 3-ball and t is a trivial arc in B. By a pairwise isotopy of (B, t), we may assume $A \subset \partial B$. Then since A is essential in $V - t$, the core α of A separates the two punctures of $\partial B - t$. Hence by Lemma 3.3, α bounds an ε_1-disk D' in (V, t). By moving D and D' so that $(D \cup D') \cap A = \emptyset$, we obtain the conclusion (3). □

Lemma 6.7. *Let V be a solid torus and t a trivial arc in V, and let $\mathcal{A} = A_1 \cup A_2$ be a disjoint union of non-parallel essential annuli in (V, t). Then one of the following holds (see* Fig. 5*).*

(1) *$\mathcal{A}$ cuts (V, t) into $(V_1, \emptyset)$ and (V_2, t), where V_1 is a genus two handlebody, V_2 is a* 3*-ball and t is a trivial arc in V_2, which satisfy $\mathcal{A} \subset \partial V_j$ (j = 1 and 2). Moreover, there are an ε_0-disk D and an ε_1-disk D' in (V, t) with $D \cap D' = \emptyset$ and $\mathcal{A} \cap (D \cup D') = \emptyset$.*

(2) *$\mathcal{A}$ cuts (V, t) into $(V_1, \emptyset)$, $(V_2, \emptyset)$ and (V_3, t), where V_1 is a solid torus, V_2 is*

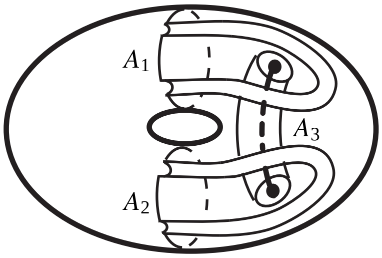

Fig. 6.

a genus two handlebody, V_3 *is a* 3*-ball and* t *is a trivial arc in* V_3*, which satisfy* $\mathcal{A} \cap \partial V_1 = A_1$*,* $\mathcal{A} \subset \partial V_2$ *and* $\mathcal{A} \cap \partial V_3 = A_2$ *after changing the subscripts. Moreover, there is an* ι*-disk in* (V, t) *disjoint from* $\mathcal{A}$*.*
(3) $\mathcal{A}$ *cuts* (V, t) *into* $(V_1, \emptyset)$ *and* (V_2, t)*, where* V_1 *is a genus two handlebody,* V_2 *is a* 3*-ball and* t *is a trivial arc in* V_2*, which satisfy* $\mathcal{A} \subset \partial V_1$ *and* $\mathcal{A} \cap \partial V_2 = A_2$ *after changing the subscripts.*

Proof. By performing ∂-compressions of A_1 and A_2, we obtain mutually disjoint disks D_1 and D_2 properly embedded in $V - t$. Since A_1 and A_2 are essential in $V - t$, D_1 and D_2 are essential in $V - t$. Suppose that both D_1 and D_2 are ε_0-disks. Then we obtain the conclusion (1). Suppose next that both D_1 and D_2 are ι-disks. Then we obtain the conclusion (2). Suppose finally that precisely one of D_1 and D_2, say D_1, is an ε_0-disk and D_2 is an ι-disk. Note that A_2 is disjoint from D_2. This implies that A_2 is parallel to $\partial N(K)$. Hence we obtain the condition (3). □

The following lemma is obtained by using Lemma 3.3 of [14].

Lemma 6.8. *Let* V *be a solid torus and* t *a trivial arc in* V*, and let* $\mathcal{A} = A_1 \cup A_2 \cup A_3$ *be a disjoint union of non-parallel essential annuli in* (V, t)*. Then* $\mathcal{A}$ *cuts* (V, t) *into* $(V_1, \emptyset)$*,* $(V_2, \emptyset)$ *and* (V_3, t)*, where* V_1 *is a genus two handlebody,* V_2 *is a solid torus and* V_3 *is a* 3*-ball and* t *is a trivial arc in* V_3*, which satisfy* $\mathcal{A} \cap \partial V_1 = A_1 \cup A_2$*,* $\mathcal{A} \subset \partial V_2$ *and* $\mathcal{A} \cap \partial V_3 = A_3$ *after changing the subscripts* (*see* Fig. 6).

Proof. Note that $A_1 \cup A_2$ satisfies one of the conclusions of Lemma 6.7. Suppose that $A_1 \cup A_2$ satisfies the conclusion (2) of Lemma 6.7. Then $A_1 \cup A_2$ cuts (V, t) into $(V_1, \emptyset)$, $(V_2, \emptyset)$ and (V_3, t), where V_1 is a solid torus, V_2 is a genus two handlebody, V_3 is a 3-ball and t is a trivial arc in V_3. If $A_3 \subset V_1$ or V_3, then A_3 is parallel to A_1 or A_2. If $A_3 \subset V_2$, then by Lemma 3.3 of [14], A_3 is parallel to A_1 or A_2. Hence we may assume that $A_1 \cup A_2$ satisfies the conclusion (1) or (3) of Lemma 6.8.

Suppose $A_1 \cup A_2$ satisfies the conclusion (1) of Lemma 6.7. Then $A_1 \cup A_2$ cuts (V, t) into $(V_1, \emptyset)$ and (V_2, t), where V_1 is a genus two handlebody, V_2 is a 3-ball and

t is a trivial arc in V_2. By Lemma 3.3 of [14], A_3 must be contained in V_2. Hence A_3 is parallel to $\partial N(t)$.

Suppose $A_1 \cup A_2$ satisfies the conclusion (3) of Lemma 3.3. Then $A_1 \cup A_2$ cuts (V, t) into $(V_1, \emptyset)$ and (V_2, t), where V_1 is a genus two handlebody, V_2 is a 3-ball and t is a trivial arc in V_2. By Lemma 3.3 of [14], A_3 is parallel to an annulus, say A', in ∂V_2. Since A_3 is essential in $V - t$ and is not parallel to A_i ($i = 1$ and 2), A' contains $\partial A_1 \cup \partial A_2$. This implies A_3 satisfies the condition (3) of Lemma 6.6. Then by changing the subscripts, we can see that $\mathcal{A}$ satisfies the condition of Lemma 6.8. □

Proof of Proposition 6.1. Let $(W_1, W_2; P)$ be a $(1, 1)$-splitting of (M, K) and T an essential torus in $E(K)$. We put $T_i = T \cap V_i$.

CLAIM. We may assume that T_i consists of essential annuli in W_i ($i = 1$ and 2).

Proof. Since $\chi(T) = 0$, we have only to show that T_i has no disks.

We may assume that after an isotopy, each disk of T_i is essential in $V_i - t_i$ ($i = 1$ and 2). Suppose that both T_1 and T_2 have disk components. Then this implies $d(W_1, W_2) \leq 1$ because $\partial T_1 = \partial T_2$. Hence we see that K is a trivial knot or a core knot in $S^2 \times S^1$ by Theorem 2.2 and Theorem 2.3, a contradiction. Hence we may assume that either T_1 or T_2, say T_2, has no disk components. Further we assume that the number of disk components of T_1 is minimal among all essential tori satisfying the condition as above. Let Δ be the union of the disk components of T_1. Choose a disjoint union $\mathcal{D}$ of an ε_0-disk and an ι-disk in W_2 which intersect T_2 transversely.

Note that $E(K)$ is irreducible, i.e., $E(K)$ contains no essential 2-spheres. Otherwise, K is a trivial knot by Proposition 2.9 of [2], a contradiction. Hence by a standard argument, we can eliminate all loop components of $T_2 \cap \mathcal{D}$ by an ambient isotopy on $E(K)$.

Suppose that $\Delta \cap \mathcal{D} = \emptyset$. Then each component of $\partial\Delta$ is isotopic to one of the components of $\partial\mathcal{D}$ because each component of $\partial\Delta$ is either an ε-loop or an ι-loop. This implies that $\partial\Delta$ bounds a disk in $V_2 - t_2$, and hence $d(W_1, W_2) = 0$. By Theorem 2.2, K is a trivial knot, a contradiction. So $\Delta \cap \mathcal{D} \neq \emptyset$.

Let Γ be the union of the arc components of $T_2 \cap \mathcal{D}$ incident to $\partial\Delta \cap \mathcal{D}$. Let γ be a component of Γ such that γ clips a disk, say δ_γ, from $\mathcal{D}$ with $\delta_\gamma \cap \Gamma = \gamma$. Suppose that $\delta_\gamma \cap T_2 \neq \gamma$. Then there is a component γ' of $\delta \cap T_2$ which clips a disk $\delta_{\gamma'}$ with $\delta_{\gamma'} \cap T_2 = \gamma'$. We can isotope T along $\delta_{\gamma'}$ near γ' without increasing the number of disks of T_1. By repeating this operation, if necessary, we may suppose that $\delta_\gamma \cap T_2 = \gamma$. By isotoping T along δ_γ, we can reduce the number of disk components of T_1 at least by one, a contradiction.

This completes the proof of the claim. □

Let $\mathcal{A}_i$ be a union of mutually disjoint, non-parallel, essential annuli in $W_i = (V_i, t_i)$ of which T_i consists of parallel copies ($i = 1$ and 2). Note that $|\mathcal{A}_1| \leq 3$ by Lemmas 6.6–6.8. By changing the subscripts, if necessary, we may assume that $|\mathcal{A}_1| \geq |\mathcal{A}_2|$.

CASE 1. $|\mathcal{A}_1| = 3$.

Note that one of the following holds.

- $\mathcal{A}_2$ consists of an annulus satisfying one of the conditions in Lemma 6.6.
- $\mathcal{A}_2$ consists of two annuli satisfying one of the conditions in Lemma 6.7.
- $\mathcal{A}_2$ consists of three annuli satisfying the condition in Lemma 6.6.

Suppose that $\mathcal{A}_2$ satisfies the condition (1) of Lemma 6.6, the condition (2) of Lemma 6.7, the condition (3) of Lemma 6.7, or the condition of Lemma 6.8. Here, the sentence "$\mathcal{A}_2$ satisfies the condition (1) of Lemma 6.6" means that $\mathcal{A}_2$ consists of an annulus satisfying the condition (1) in Lemma 6.6. Then $T_1 \cup T_2$ contains a torus which is parallel to $\partial N(K)$, a contradiction.

Suppose that $\mathcal{A}_2$ satisfies the condition (2) of Lemma 6.6 or the condition (3) of Lemma 6.6. Let $\{p_1, p_2\}$ be points of $P \cap K$. Note that $\mathcal{A}_1$ has a component which is isotopic to $\partial N(p_i;\, P)$ for each $i = 1$ and 2. On the other hand, for $i = 1$ or 2, $\mathcal{A}_2$ does not have a component which is isotopic to $\partial N(p_i;\, P)$. This implies that $\partial T_1 \neq \partial T_2$, a contradiction.

Suppose that $\mathcal{A}_2$ satisfies the condition (1) of Lemma 6.7. Put $\mathcal{A}_1 = A_{11} \cup A_{12} \cup A_{13}$ and $\mathcal{A}_2 = A_{21} \cup A_{22}$. We may assume that A_{13} is isotopic to $\partial N(K) \cap V_1$. Suppose that T_1 consists of m_1 parallel copies of A_{11}, m_2 parallel copies of A_{12} and m_3 parallel copies of A_{13}, and T_2 consists of n_1 parallel copies of A_{21} and n_2 parallel copies of A_{22}. Then since $\partial T_1 = \partial T_2$, we have $m_1 + m_2 = n_1 + n_2$, $m_1 + m_3 = n_1$ and $m_2 + m_3 = n_2$. This implies that $m_3 = 0$, a contradiction. Hence Case 1 does not occur.

CASE 2. $|\mathcal{A}_1| = 2$.

Set $\mathcal{A}_1 = A_{11} \cup A_{12}$. We have the following three subcases by Lemma 6.7.

CASE 2.1. $\mathcal{A}_1$ satisfies the condition (1) of Lemma 6.7.

By an argument similar to Case 1, we see that $\mathcal{A}_2$ satisfies the condition (1) or (2) of Lemma 6.7. Set $\mathcal{A}_2 = A_{21} \cup A_{22}$.

Suppose that $\mathcal{A}_2$ satisfies the condition (1) of Lemma 6.7. Then we see $|T_1| = |T_2| = 2$. (Otherwise $T_1 \cup T_2$ has plural components.) So we may assume $T_i = A_{i1} \cup A_{i2}$ ($i = 1$ and 2) (cf. Fig. 3). Since $M \cong S^2 \times S^1$, we can find an ε_1-disks D_i in W_i ($i = 1$ and 2) with $\partial D_1 = \partial D_2$. Hence $(W_1, W_2;\, P)$ satisfies the condition ($\#_c$) of Proposition 6.1.

Suppose that $\mathcal{A}_2 = A_{21} \cup A_{22}$ satisfies the condition (2) of Lemma 6.7. Then we can find an ε_1-disk D_1 in W_1 and an ι-disk D_2 in W_2 which satisfy the condition ($\#_a$) of Proposition 6.1 (see Fig. 7). Hence by the remark below Lemma 6.2, K is a core knot, a contradiction.

CASE 2.2. $\mathcal{A}_1$ satisfies the condition (2) of Lemma 6.7.

Then by an argument similar to Case 1, we see that $\mathcal{A}_2$ satisfies the condition (1)

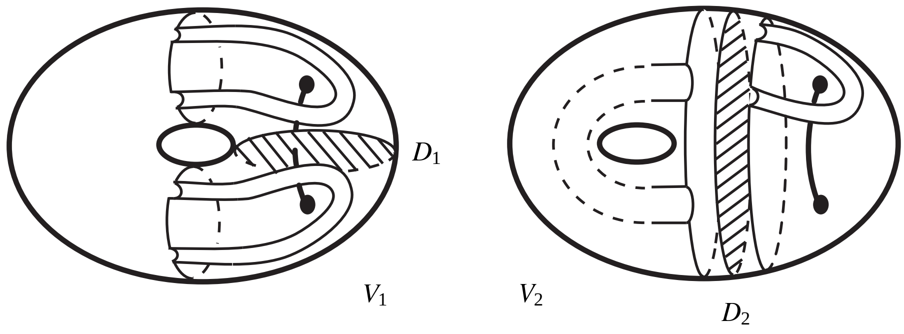

Fig. 7.

of Lemma 6.7. Hence by changing the subscripts, Case 2.2 is equivalent to the latter case of Case 2.1.

CASE 2.3. $\mathcal{A}_1$ satisfies the condition (3) of Lemma 6.7.

Then by an argument similar to Case 1, we see that Case 2.3 is impossible.

CASE 3. $|\mathcal{A}_1| = 1$.

By Lemma 6.6, we have the following three subcases.

CASE 3.1. $\mathcal{A}_1$ satisfies the condition (1) of Lemma 6.6.

By an argument similar to Case 1, we see that $\mathcal{A}_2$ satisfies the condition (1) of Lemma 6.6. Hence $T_1 \cup T_2$ contains a torus which is parallel to $\partial N(K)$, a contradiction.

CASE 3.2. $\mathcal{A}_1$ satisfies the condition (2) of Lemma 6.6.

By an argument similar to Case 1, we see that $\mathcal{A}_2$ satisfies the condition (2) of Lemma 6.6. Moreover T_i consists of an annulus ($i = 1$ and 2). (Otherwise, $T_1 \cup T_2$ consists of plural components.) Let z be one of the components of $\partial\mathcal{A}_1 = \partial\mathcal{A}_2$. For each $i = 1$ and 2, let Δ_i be a disk in V_i such that $t_i \subset \partial\Delta$, and $\Delta_i \cap \partial V_i = \mathrm{cl}(\partial\Delta_i - t_i) =: t_i'$ is disjoint from z. Then there are ι-disks D_i in W_i with $\partial D_i = \partial N(t_i'; P)$ for each $i = 1$ and 2. Hence $Z := \mathrm{cl}(P - N(z; P))$ gives the condition $(\#_b)$ of Proposition 6.1.

CASE 3.3. $\mathcal{A}_1$ satisfies the condition (3) of Lemma 6.6.

By an argument similar to Case 1, wee see that $\mathcal{A}_2$ satisfies the condition (3) of Lemma 6.6. Then there are an ε_1-disk D_i in W_i ($i = 1$ and 2) with $\partial D_1 = \partial D_2$. Hence $(W_1, W_2; P)$ satisfies the condition $(\#_c)$ of Proposition 6.1.

This completes the proof of Proposition 6.1. □

Proof of Theorem 2.5. Let K be a $(1, 1)$-knot in M and $(W_1, W_2; P)$ a $(1, 1)$-splitting of (M, K). By Proposition 6.1, $(W_1, W_2; P)$ satisfies one of the conditions in Proposition 6.1.

Suppose that $(W_1, W_2; P)$ satisfies the condition $(\#_a)$ of Proposition 6.1. Then by Lemma 6.2, K belongs to $\mathcal{K}_1$, because the exteriors of 2-bridge knots and core knots do not contain essential tori (see [5]).

Suppose that $(W_1, W_2; P)$ satisfies the condition $(\#_b)$ of Proposition 6.1. Then by arguments in the proof of Lemma 6.4 and the proof of Proposition 6.1, K belongs to $\mathcal{K}_2$, because $E(K)$ contains an essential torus.

Suppose that $(W_1, W_2; P)$ satisfies the condition $(\#_c)$ of Proposition 6.1. Then by Lemma 6.5, K belongs to $\mathcal{K}_3$ or $\mathcal{K}_4$.

We have thus proved Theorem 2.5. □

7. (1, 1)-splittings of distance = 2

In this section, we give the proof of Theorem 2.4.

Proof of Theorem 2.4. We first assume $d(W_1, W_2) = 2$, that is, there is an essential loop x (y resp.) in $\Sigma := P - K$ which bounds a disk in $V_1 - t_1$ ($V_2 - t_2$ resp.) such that x and y intersect each other, and there is an essential loop z in Σ with $z \cap (x \cup y) = \emptyset$.

CASE 1. Both x and y are ε-loops.

If z is an ι-loop, then z bounds an ι-disk in each of W_1 and W_2 by Lemma 3.3. This implies that $(W_1, W_2; P)$ is of distance = 0, a contradiction. Hence by Lemma 3.3, z must be an ε-loop and z bounds an ε_0-disk or an ε_1-disk in each of W_1 and W_2.

Suppose that z bounds an ε_0-disk in each of W_1 and W_2. Then this means that $d(W_1, W_2) \leq 1$, a contradiction.

Suppose that z bounds an ε_1-disk in each of W_1 and W_2. Then $(W_1, W_2; P)$ satisfies the condition $(\#_c)$ of Proposition 6.1. By Lemma 6.5, $K = K(4, 1, 0)$ or $E(K)$ contains an essential torus.

CASE 2. Precisely one of x and y, say x, is an ε-loop.

We see that z is an ε-loop by an argument similar to Case 1. Then by Lemma 3.3, z bounds an ε_1-disk in W_1. So $(W_1, W_2; P)$ satisfies the condition $(\#_a)$ of Proposition 6.1, and hence (M, K) satisfies one of the conditions (1)–(3) of Lemma 6.2. Note that if K satisfies the condition (3), we can find an essential torus in $E(K)$ by making an appropriate "swallow-follow torus".

CASE 3. Both x and y are ι-loops.

Then z must be an ε-loop by the same argument as above. In particular, z must be contained in the surface T_0 obtained from the torus P by removing the interior of the disk bounded by x. So all components of $y \cap T_0$ $(\neq \emptyset)$ are parallel in T_0. Note that we can regard y as $\partial N(t_2'; P)$, where t_2' is an arc in P such that $t_2 \cup t_2'$ bounds a disk in V_2. By an isotopy on Σ, we may assume that $|x \cap y|$ is minimal.

CASE 3.1. $|y \cap T_0| = 2$.

Then K is isotopic to a knot in P, and hence K satisfies the condition (2) or (3) of Theorem 2.4.

CASE 3.2. $|y \cap T_0| > 2$.

Let A_1 in V_1^- (A_2 in V_2^- resp.) be an annulus obtained by pushing the interior of $N(z; P)$ into the interior of V_1 (V_2 resp.), where $V_i^- = \mathrm{cl}(V_i - N(t_i))$ $(i = 1, 2)$. So $T := A_1 \cup A_2$ is a torus in $E(K)$ (see Fig. 8).

A_1 (A_2 resp.) cuts V_1^- (V_2^- resp.) into a solid torus V_{11}^- (V_{21}^- resp.) and a genus

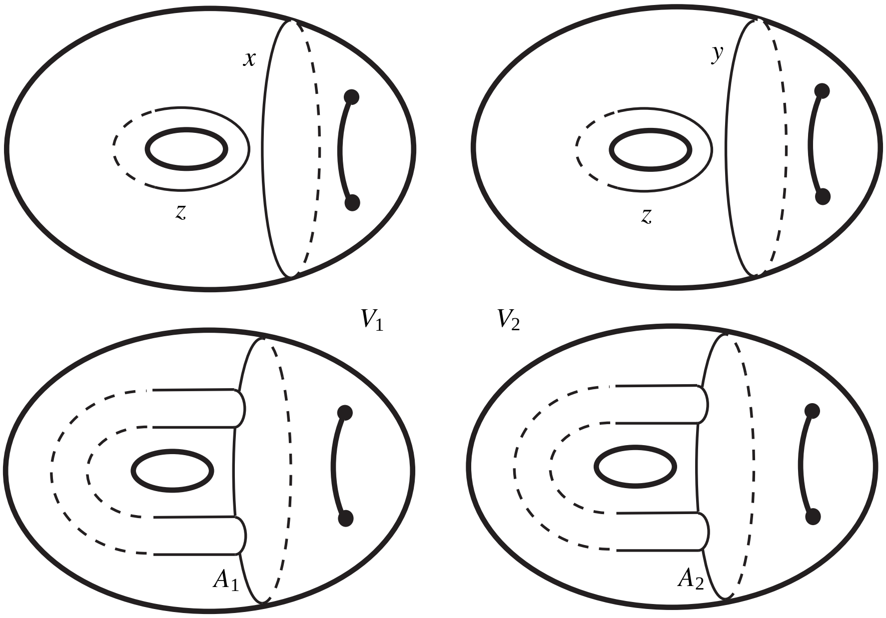

Fig. 8.

two handlebody V_{12}^- (V_{22}^- resp.). $M_1 = V_{11}^- \cup V_{21}^-$ is the exterior of a trivial knot, a core knot or a torus knot. $M_2 = V_{21}^- \cup V_{22}^-$ is the exterior of a 2-bridge link, and $M_2 \cup N(K)$ should be a solid torus. If M_1 is a solid torus, then (M, K) is equivalent to $K(\alpha, \beta; r)$ for some α, β and γ. If not, by the hypothesis of Case 3.2, we can see that T is not parallel to $\partial N(K)$. Hence T is an essential torus in $E(K)$.

This completes the proof of the first part of Theorem 2.4.

Next, we prove the second part of Theorem 2.4.

CASE (1). K is a non-trivial 2-bridge knot in S^3.

By Theorem 8.2 of [15], every $(1, 1)$-splitting of a non-trivial 2-bridge knot is isotopic to that constructed as follows. For a non-trivial 2-bridge knot K, let $(B_1, a_1 \cup a_2) \cup_S (B_2, b_1 \cup b_2)$ be a 2-bridge decomposition. Put $V_1 = B_1 \cup N(b_2; B_2)$, $V_2 = \mathrm{cl}(B_2 - N(b_2; B_2))$, $t_1 = a_1 \cup a_2 \cup b_2$ and $t_2 = b_1$. Then $W_i := (V_i, t_i)$ is a pair of a solid torus V_i and a trivial arc t_i in V_i ($i = 1, 2$), and $(W_1, W_2; P)$ gives a $(1, 1)$-splitting of (S^3, K). In the following, we show that this $(1, 1)$-splitting has distance $= 2$.

Let D_i be a properly embedded disk in B_i such that D_i separates two trivial arcs in B_i ($i = 1, 2$). Then D_1 determines an ε_0-disk in W_1, and D_2 determines an ι-disk in W_2. Further, ∂D_1 and ∂D_2 are disjoint from an essential loop z in $\Sigma := P - K$, where z is one of the boundary components of the meridian disks $B_1 \cap N(b_2; B_2)$. Hence $d(W_1, W_2) \leq 2$. By Theorem 2.2 and Theorem 2.3, we have $d(W_1, W_2) = 2$.

CASE (2) and (3). K is a core knot in a lens space or a torus knot in M.

By Theorem C of [6] and Theorem 3 of [17], every $(1,1)$-splitting of (M, K) is isotopic to that constructed as follows. Let $(V_1, V_2; P)$ be a genus one Heegaard splitting of M such that $K \subset P$. Let p_1 and p_2 be distinct points in K. Then $p_1 \cup p_2$ cuts K into two arcs l_1 and l_2. Let t_i be the properly embedded arc by slightly pushing the interior of l_i into the interior of V_i, and put $W_i = (V_i, t_i)$ ($i = 1$ and 2). Then $(W_1, W_2; P)$ is a $(1,1)$-splitting of (M, K).

Let z be a core of the annulus $\mathrm{cl}(P - N(K; P))$. Then $\partial N(l_i; P)$ bounds an ι-disk in W_i ($i = 1, 2$), and $\partial N(l_1; P)$ and $\partial N(l_2; P)$ are disjoint from the essential loop z in P. So we have $d(W_1, W_2) \leq 2$. By Theorem 2.2 and Theorem 2.3, we obtain $d(W_1, W_2) = 2$.

CASE (4). $E(K)$ contains an essential torus.

Let $(W_1, W_2; P)$ be a $(1,1)$-splitting of (M, K). By Proposition 6.1, $(W_1, W_2; P)$ satisfies one of the conditions $(\#_a)$, $(\#_b)$ and $(\#_c)$.

Suppose that $(W_1, W_2; P)$ satisfies the condition $(\#_a)$. Let D_1 (D_2 resp.) be an ι-disk (an ε_1-disk resp.) in W_1 (W_2 resp.) such that $\partial D_1 \cap \partial D_2 = \emptyset$. By cutting $W_2 = (V_2, t_2)$ along D_2, we obtain a 2-string trivial tangle (B, τ). Let D_2^+ and D_2^- be the copy of D_2 in ∂B. Let D_2' be a disk properly embedded in B such that $D_2' \cap (D_2^+ \cup D_2^-) = \emptyset$ and D_2' separates a component of τ from the other. Then D_2' determines an ε_1-disk W_2, and D_2' is disjoint from D_2. Hence ∂D_1 and ∂D_2 give $d(W_1, W_2) \leq 2$.

We can easily see that the condition $(\#_b)$ directly gives $d(W_1, W_2) \leq 2$.

Finally, if the condition $(\#_c)$ is satisfied, then we can also obtain $d(W_1, W_2) \leq 2$ by using an argument similar to that in case of the condition $(\#_a)$. By Theorem 2.2 and Theorem 2.3, we obtain $d(W_1, W_2) = 2$.

We have completed the proof of Theorem 2.4. □

Proof of Corollary 2.6. By Thurston's hyperbolization theorem of Haken manifolds (see, for example, [13]), a knot K is hyperbolic if and only if $E(K)$ is irreducible, $E(K)$ contains no essential torus, and $E(K)$ is not a Seifert fibered space.

CASE 1. $E(K)$ is reducible.

By Proposition 2.9 of [2], $E(K)$ is reducible if and only if K is a trivial knot. Hence $d(W_1, W_2) = 0$ by Theorem 2.2.

CASE 2. $E(K)$ contains an essential torus.

Then by Theorem 2.6, $d(W_1, W_2) = 2$.

CASE 3. $E(K)$ is a Seifert fibered space whose regular fiber is not a meridian of K.

Then by Lemma 5.2 of [14], if $E(K)$ is a Seifert fibered space whose regular fiber is not a meridian of K and $\partial E(K)$ is incompressible in $E(K)$, then one of the following holds: (1) the base space is a disk with two singular points, where the regular fiber in $\partial E(K)$ intersects the meridian in one point, (2) the base space is a Möbius band with one singular point, where the regular fiber in $\partial E(K)$ intersects the meridian in one point, (3) $E(K)$ is a twisted S^1-bundle over a Möbius band. If $E(K)$ satisfies the condition (1) or (3), then K is a torus knot. If $E(K)$ satisfies the condition (2), then there is an essential torus in $E(K)$. Hence by Theorem 2.4, $d(W_1, W_2) = 2$.

Suppose that $\partial E(K)$ is compressible in $E(K)$. Then we obtain a 2-sphere S in $E(K)$ by compressing $\partial E(K)$. If S bounds a 3-ball in $E(K)$, then $E(K)$ is a solid torus and hence K is a trivial knot or a core knot. Otherwise, since S is essential in $E(K)$, K is a trivial knot by Proposition 2.9 of [2]. Hence by Theorems 2.2 and 2.3, we have $d(W_1, W_2) = 0$ or 1.

CASE 4. $E(K)$ is a Seifert fibered space whose regular fiber is a meridian of K.

Let B be the base orbifold of $E(K)$. Then $\pi_1(M) = \pi_1(E(K))/\langle f\rangle$, where f is the element of $\pi_1(E(K))$ represented by a regular fiber, is isomorphic to the orbifold fundamental group $\pi_1(B)$. Since M is a lens space, $\pi_1(B)$ is cyclic. It is known that such an orbifold is isomorphic to a disk with only one singular point (see, for example, Section 3 of [19]). Therefore $E(K)$ is a solid torus, and hence K is a core knot. Hence by Theorem 2.3, we have $d(W_1, W_2) = 1$.

Hence by Theorems 2.2–2.4 and the hypothesis of Proposition 2.6, $d(W_1, W_2) \leq 2$ if and only if $E(K)$ is a Seifert fibered space or contains an essential 2-sphere or torus. By Thurston's hyperbolization theorem, we obtain the desired result. $\square$

8. (1, 1)-splittings of distance ≥ 3

Theorem 2.7 can be proved by the arguments of J. Hempel in Section 2 of [11]. To this end, we first recall the covering distance introduced in [11].

Let S be a connected, compact, orientable surface. We say that a covering space $p\colon \tilde{S} \to S$ *separates* essential loops x and y in S if there are components $\tilde{x}$ of $p^{-1}(x)$ and $\tilde{y}$ of $p^{-1}(y)$ with $\tilde{x} \cap \tilde{y} = \emptyset$. A finite covering $p\colon \tilde{S} \to S$ is *sub-solvable* if p can be factored as a composition of cyclic coverings.

DEFINITION 8.1 ([11] Section 2). Let $[x]$ and $[y]$ be distinct vertices of $C(S)$, and let x (y resp.) be a representative of $[x]$ ($[y]$ resp.). Then we define the *covering distance* between $[x]$ and $[y]$ as follows.

$$cd([x],[y]) = 1 + \min\left\{ n \;\middle|\; \begin{array}{l} \text{there is a degree } 2^n \text{ sub-solvable covering of } S \\ \text{which separates } x \text{ and } y \end{array} \right\}.$$

As an analogy of Lemma 2.3 in [11], we obtain the following.

Lemma 8.2. *Let $[x]$ and $[y]$ be distinct vertices of $C(S)$. Then*

(1) $d([x],[y]) = 2$ *if and only if* $cd([x],[y]) = 2$ *and*

(2) $cd([x],[y]) \leq d([x],[y])$.

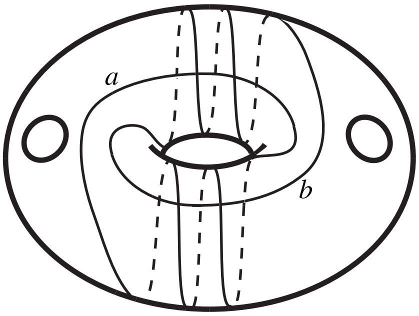

Fig. 9.

Proof. Let x (y resp.) be a representative of $[x]$ ($[y]$ resp.).

(1) Suppose that $d([x],[y]) = 2$, that is, $x \cap y \neq \emptyset$ and there is an essential loop z in S with $z \cap (x \cup y) = \emptyset$.

CASE 1. z is an ε-loop.

Since an ε-loop is a non-separating loop in S, $S' := \mathrm{cl}(S - N(z))$ is connected. We can construct a double cover $\tilde{S}$ of S by gluing two copies S'_1 and S'_2 of S' along z. Hence $\tilde{x}$ in S'_1 and $\tilde{y}$ on S'_2 can give $cd([x],[y]) = 2$.

CASE 2. z is an ι-loop.

Let γ be an essential arc which joins two punctures of S such that γ is disjoint from z. Then we can construct a double cover $\tilde{S}$ of S by gluing two copies of $\mathrm{cl}(S - N(\gamma))$. Therefore we can also get $cd([x],[y]) = 2$.

The converse follows from the proof of Lemma 2.3 in [11].

(2) The second assertion can also be proved by the same argument as that in the proof of Lemma 2.3 of [11].

This completes the proof of Lemma 8.2. □

By Lemma 8.2, we can get a lower estimation of the distance between distinct vertices on $C(S)$. For the covering distance, the following lemma is proved in [11].

Lemma 8.3 ([11] Theorem 2.5). *If $[x]$ and $[y]$ are vertices of $C(S)$ and $h\colon S \to S$ is a pseudo-Anosov homeomorphism, then* $\lim_{n\to\infty} cd([x],[h^n(y)]) = \infty$.

Proof of Theorem 2.7. We first construct a pseudo-Anosov map f of $\Sigma := P - K$ whose extension to P is isotopic to id. To this end, let a and b be essential loops on Σ illustrated in Fig. 9, and put $f = \tau_a^{-1} \circ \tau_b$, where τ_a (τ_b resp.) a right-hand Dehn twist along a (b resp.). Then f is pseudo-Anosov by Theorem 3.1 of [21], because $a \cup b$ fills Σ. Since a and b are isotopic in P, the extension $\hat{f}$ of f to P is isotopic to the identity.

Now let M be a 3-manifold with a genus one Heegaard splitting. Pick a

$(1,1)$-knot K in M and its $(1,1)$-splitting $(W_1, W_2; P)$. Let x (y resp.) be an ε-loop in Σ which bounds an ε_0-disk in W_1 (W_2 resp.). By Lemma 8.2 and Lemma 8.3 , for any positive integer n, there is an integer N such that $d([x],[f^N(y)]) > n+2$, where $[x]$ ($[f^N(y)]$ resp.) is represented by x ($f^N(y)$ resp.). Since $\hat{f} \simeq id$, the manifold obtained from M by cutting along P and regluing it after composing $\hat{f}^N$ is homeomorphic to M. Let $(W_1', W_2'; P)$ be a $(1,1)$-splitting obtained in the above way. Then by Proposition 3.8, we have $d(W_1', W_2') \geq d([x],[f^N(y)]) - 2 > n$.

We have completed the proof of Theorem 2.7. □

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