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Author(s)	Fukushima, Hiroshi
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## ON $p$ -RADICAL GROUPS $G$ AND THE NILPOTENCY INDICES OF $J(kG)$

HIROSHI FUKUSHIMA

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### 1. Introduction

Let  $kG$  be the group algebra of a finite group  $G$  over an algebraically closed field  $k$  of characteristic  $p > 0$ , and let  $P$  be a Sylow  $p$ -subgroup of  $G$ .

Following Motose and Ninomiya [9] we call  $G$   $p$ -radical if the induced module  $(k_p)^G$  of the trivial  $kP$ -module  $k_p$  is completely reducible as a right  $kG$ -module.

In [10], Okuyama has proved that  $p$ -radical groups are  $p$ -solvable. And Tsushima has characterized  $p$ -radical groups which are  $p$ -nilpotent by group theoretical properties (see Lemma 2.2). So it seems to be interesting to investigate the structure of  $p$ -radical groups of  $p$ -length 2 and in this paper we shall treat such a group with some additional properties.

Before describing our result we need to define some notations. Let  $F = GF(q^n)$  be a finite field of  $q^n$  elements for prime  $q$ . Let  $V$  be the additive group of  $F$ . Let  $T(q^n)$  be the set of semilinear transformations of the form  $v \rightarrow av^\sigma$  with  $v \in V$ ,  $0 \neq a \in F$ , and  $\sigma$  a field automorphism (see [11, p229]). Then we can consider the semidirect product  $VT(q^n)$  of  $V$  by  $T(q^n)$ . Let  $\lambda$  be a generator of the multiplicative group of  $F$  and  $v = \lambda^{q^{n/r}-1}$  for some integer  $r$  with  $r|n$ . Let  $T_0 = \{v \rightarrow av^\sigma \mid a \in \langle v \rangle, \sigma \in \text{Gal}(F/GF(q^{n/r}))\}$ . Then we define  $A_{q,n,r} = VT_0 \subseteq VT(q^n)$ .

**Theorem 1.** *Let  $G$  be a finite group with the following conditions.*

- (1)  $|G: O_{p',p,p'}(G)| = p$ ,  $O_p(G) = 1$  and  $O^{p'}(G) = G$ .
- (2)  $A$  Sylow  $p$ -subgroup  $P_0$  of  $O_{p',p}(G)$  is abelian.
- (3)  $V = [O_{p'}(G), P_0]$  is a minimal normal subgroup of  $G$ .

*Then  $G$  is  $p$ -radical if and only if the following conditions (A), (B) and (C) hold.*

- (A)  $\tilde{G} = G/VP_0$  is a Frobenius group with kernel  $O_{p'}(\tilde{G})$ .
- (B)  $V$  is an elementary abelian  $q$ -group for some prime  $q (\neq p)$ .
- (C) One of the following (1) and (2) holds.

(1) The following (i)-(vii) hold.

- (i)  $G = VN_G(P_0)$  and  $V \cap N_G(P_0) = 1$ .
- (ii)  $P_0 \triangleleft P_0 H \triangleleft P_0 H < s >$ , where  $|s| = p$  and  $H$  is a  $p'$ -group.
- (iii) By conjugation, we can regard  $V$  as an irreducible  $N_G(P_0)$ -module. Then  $V = V_1 \times \cdots \times V_p$ , where  $V_i, 1 \leq i \leq p$ , are the homogeneous components of  $V_{P_0}$ .
- (iv) Set  $P_i = C_{P_0}(V_1 \times \cdots \times V_{i-1} \times V_{i+1} \times \cdots \times V_p)$ ,  $1 \leq i \leq p$ .  
Then  $P_0 = P_1 \times \cdots \times P_p$ .
- (v)  $V_1^{s^i} = V_{i+1}$ ,  $P_1^{s^i} = P_{i+1}$ ,  $0 \leq i \leq p-1$ .
- (vi)  $V_i$  and  $P_i$  are  $H$ -invariant,  $1 \leq i \leq p$ , and  $VP_0 = (V_1 P_1) \times \cdots \times (V_p P_p)$ .
- (vii) Set  $r = |H/C_H(V_1)|$  and  $p^m = |P_1|$ ,  $q^n = |V_1|$ .  
Then  $r|n$  and  $\frac{q^n - 1}{q^{n/r} - 1} = p^m$  and  $V_i P_i H / C_H(V_i) \simeq A_{q, n, r}$ ,  $1 \leq i \leq p$ .

(2)  $C_G(v) \subseteq O_{p', p, p'}(G)$  for any element  $v$  of  $V^*$ .

$$\begin{array}{c}
 G \\
 \downarrow s \\
 p \\
 \downarrow p' \\
 H \\
 \downarrow p' \\
 P_0 \\
 \downarrow p' \\
 V \\
 \downarrow q' \\
 1
 \end{array}$$

Next, let  $t(G)$  be the nilpotency index of the radical  $J(kG)$  of  $kG$  and let  $p^a$  be the order of Sylow  $p$ -subgroups of  $G$ .

Wallace [14] proved that if  $G$  is  $p$ -solvable, then  $a(p-1)+1 \leq t(G) \leq p^a$ . If  $G$  has  $p$ -length 1, then by Motose and Ninomiya [8]  $t(G) = a(p-1)+1$  if and only if  $P$  is elementary abelian.

All known examples of  $p$ -solvable group  $G$  with  $t(G) = a(p-1)+1$  have  $p$ -length at most 2. Using Theorem 1, we can prove the following theorem.

**Theorem 2.** *If  $G$  is a  $p$ -radical group with  $t(G) = a(p-1)+1$ , then  $G = O_{p, p', p, p'}(G)$ . In particular, the  $p$ -length of  $G$  is at most 2.*

## 2. Preliminaries

In this section, we shall give some lemmas which will be used to prove the theorems.

**Lemma 2.1.** ([1, Theorem 6.5]). *Suppose that  $N \triangleleft G$ . Then the following (1)–(3) hold.*

- (1) *If  $G$  is  $p$ -radical, so are  $N$  and  $G/N$ .*
- (2) *If  $N$  is a  $p$ -group, then  $G$  is  $p$ -radical if and only if  $G/N$  is  $p$ -radical.*
- (3) *If  $G/N$  is a  $p'$ -group, then  $G$  is  $p$ -radical if and only if  $N$  is  $p$ -radical.*

**Lemma 2.2.** ([13, Theorem 2]). *Let  $G=PN$  be a  $p$ -nilpotent group with  $N=O_p(G)$ . Then  $G$  is  $p$ -radical if and only if  $[N, D] \cap C_N(D)=1$  for any  $p$ -subgroup  $D$  of  $G$ . In particular, if  $N$  is abelian, then  $G$  is  $p$ -radical.*

**Lemma 2.3.** *If  $G$  is  $p$ -radical, then  $O^{p'}(G)$  is solvable.*

Proof. Suppose it is false and let  $G$  be a minimal counterexample. Then we have  $G=O^{p'}(G)$ . By Theorem 1 of [10],  $G$  is  $p$ -solvable. If  $O_p(G) \neq 1$ , then  $G/O_p(G)$  is solvable since  $G/O_p(G)$  is  $p$ -radical. Hence  $G$  is solvable, a contradiction. Hence  $O_p(G)=1$ , and so  $O^{p'}(G) \neq 1$ . Let  $P$  be a Sylow  $p$ -subgroup of  $O_{p',p}(G)$ , and set  $W=[O_{p'}(G), x]$  for  $x \in \Omega_1(Z(P))$ . If  $W=1$ , then  $1 \neq \Omega_1(Z(P)) \subseteq C_G(O_{p'}(G)) \subseteq O_p(G)$ , a contradiction. Since  $G=O_p(G)N_G(P)$ ,  $1 \neq W \triangleleft G$ . Furthermore, for  $x \in \Omega_1(Z(P))$   $[O_{p'}(G), x]$  is a normal subgroup of  $O_{p'}(G)$  and is nilpotent by Thompson [12] as  $C_{O_{p'}(G)}(x) \cap [O_{p'}(G), x]=1$  (see Lemma 2.2). Hence  $W$  is solvable. Since  $G/W$  is  $p$ -radical,  $G/W$  is solvable. This implies that  $G$  is solvable, contrary to our choice of  $G$ .

Let  $B$  be a block of  $kG$ . We call  $B$  a  $p$ -radical block if  $k \otimes_p B$  is semisimple. Let  $N \triangleleft G$  and  $b_0$  a block of  $kN$  that is covered by  $B$ . Let  $T$  be the inertia group of  $b_0$ . Then there exists a unique block  $b$  of  $kT$  with  $b^G=B$ . We call  $b$  the Fong correspondent of  $B$  w.r.t.  $(K, T)$ . Then the next lemma holds.

**Lemma 2.4.** *The following (1) and (2) hold.*

- (1)  *$B$  is  $p$ -radical if and only if  $k \otimes_{P^y \cap T} b$  is a semisimple  $kT$ -module for any  $y \in G$ .*
- (2) ([13] Tsushima) *If  $|G:T|$  is a power of  $p$ , then  $B$  is  $p$ -radical if and only if  $b$  is  $p$ -radical.*

Proof. (1) Various facts are known about the relationship between  $B$  and  $b$ .  $B=(kG)b(kG)$  and  $J(B)=(kG)J(b)(kG)$ . Furthermore,  $(kG)b$  is a direct summand of  $B$  as a  $k(G \times T)$ -module. Hence  $k \otimes_p (kG)b$  is a direct summand of  $k \otimes_p B$  as a right  $kT$ -module. On the other hand,

$$k \otimes_p kG \simeq \bigoplus_{y \in P|G/T} k \otimes_{P^y \cap T} kT \text{ by Mackey decomposition, and so } k \otimes_p (kG)b \simeq \bigoplus_{y \in P|G/T} k \otimes_{P^y \cap T} b \text{ as right } kT\text{-modules.}$$

Assume that  $B$  is  $p$ -radical. Since  $(k \otimes_p B)J(b) \subseteq (k \otimes_p B)J(B)=0$ ,  $(k \otimes_{P^y \cap T} b)J(b)=0$ , and so  $k \otimes_{P^y \cap T} b$  is semisimple.

Conversely, assume that  $k \otimes_{P^y \cap T} b$  is a semisimple  $kT$ -module for any  $y \in G$ . Then  $k \otimes_P (kG)b$  is a semisimple  $kT$ -module. Therefore  $U = (k \otimes_P (kG)b) \otimes_T kG$  is semisimple by Fong's theory. Since  $k \otimes_P B$  is a natural homomorphic image of  $U$ , it is also semisimple.

(2) Since  $G = PT$  by assumption,  $P^y \cap T$  is a Sylow  $p$ -subgroup of  $T$  for any  $y \in G$ , and hence  $k \otimes_{P^y \cap T} b$  is semisimple if and only if  $b$  is  $p$ -radical. Therefore (2) follows from (1).

Let  $G \triangleright V$ . We let  $\text{Irr}(V)$  be the set of ordinary irreducible characters of  $V$  and let  $I_G(\varphi)$  be the inertia group of  $\varphi \in \text{Irr}(V)$ .

Furthermore, for a block  $B$  of  $kG$ , let  $\text{Irr}(B)$  be the set of irreducible characters of  $G$  belonging to  $B$ .

**Lemma 2.5.** *Let  $G = LV \triangleright V$ , where  $V$  is an abelian  $p'$ -group with  $L \cap V = 1$ . Let  $\varphi \in \text{Irr}(V)$  with  $I_G(\varphi) = G$ . Then  $L$  is  $p$ -radical if and only if all blocks which cover  $\varphi$  are  $p$ -radical.*

Proof. Let  $b$  be a block of  $kG$  which covers  $\varphi$ . Let  $\chi \in \text{Irr}(b)$ . Then each irreducible constituent of  $\chi_V$  is  $\varphi$ . Set  $U = \text{Ker } \varphi$ . Then we have  $\text{Ker } \chi \supseteq U$ , and so  $\text{Ker } b \supseteq U$ . Set  $\bar{G} = G/U$ , then  $\bar{G} = \bar{L} \times \bar{V}$ . Let  $e$  be a centrally primitive idempotent corresponding to  $\varphi$ . Then the sum of all blocks which cover  $\varphi$  is isomorphic to  $k\bar{L} \otimes_k ke \simeq kL$ . Then the lemma follows immediately.

Throughout this paper, we let  $\mathcal{F}$  be the family of all finite groups  $G$  such that  $t(G) = a(p-1) + 1$ , where  $p^a$  is the order of a Sylow  $p$ -subgroup of  $G$ .

**Lemma 2.6.** *Let  $G$  be a  $p$ -solvable group and  $N \triangleleft G$ . If  $G \in \mathcal{F}$ , then  $G/N \in \mathcal{F}$ .*

Proof. Let  $p^a, p^b$  be the orders of Sylow  $p$ -subgroups of  $G$  and  $N$ , respectively.

At first, assume  $N$  is a  $p'$ -group. By Theorem 2.2, 3.3 of [14],  $a(p-1) + 1 \leq t(G/N) \leq t(G) = a(p-1) + 1$ . Hence  $t(G/N) = a(p-1) + 1$ , and so  $G/N \in \mathcal{F}$ .

Next, assume  $N$  is a  $p$ -group. By Theorem 2.4, 3.3 of [14],  $b(p-1) + (a-b)(p-1) + 1 \leq t(N) + t(G/N) - 1 \leq t(G) = a(p-1) + 1$ . Hence  $t(N) = b(p-1) + 1$  and  $t(G/N) = (a-b)(p-1) + 1$ , and so  $G/N \in \mathcal{F}$ .

Now, we shall consider the general case. If  $N \neq 1$ ,  $0_p(N) \neq 1$  or  $0_p(N) \neq 1$  since  $N$  is  $p$ -solvable. By induction on  $|N|$ , the lemma follows.

**Lemma 2.7.** *Let  $G=LV \triangleright V$ , where  $V$  is a  $p$ -group and  $L$  is a  $p$ -radical  $p$ -nilpotent group with  $L \cap V=1$ . If  $G \in \mathcal{F}$ , then  $\langle VO_{p'}(L), s \rangle \in \mathcal{F}$  for any  $p$ -element  $s \in L$ .*

Proof. Set  $H=O_{p'}(L)$ . By Theorem 3.1 of [14],  $|s|=p$  and  $V$  is elementary abelian. Since  $L$  is  $p$ -radical,  $[H, s] \langle s \rangle$  is a Frobenius group by Lemma 2.2. By Theorem 2.7 of [5],  $gr \ kV$  is semisimple as a  $kG$ -module. Let  $N=V[H, s] \langle s \rangle \triangleleft G$ . Then  $gr \ kV$  is semisimple as a  $kN$ -module, and so  $N \in \mathcal{F}$  by Theorem 2.7 of [5]. Set  $M=\langle VO_{p'}(L), s \rangle$ . Since  $M \triangleright N$  and  $M/N$  is a  $p'$ -group,  $M \in \mathcal{F}$ .

### 3. Proof of Theorem 1

First we shall prove the if -part of Theorem 1. Let  $G$  be a finite group with conditions (1)–(3) in Theorem 1 and we assume that  $G$  is  $p$ -radical.

**Lemma 3.1.**  *$V=[O^{p'}(G), P_0]$  is an elementary abelian  $q$ -group for some prime  $q(\neq p)$  and  $N_G(P_0)$  is a complement of  $V$  in  $G$ .*

Proof. Since  $G$  is  $p$ -radical with  $O^{p'}(G)=G$ ,  $G$  is solvable by Lemma 2.3. In particular,  $V$  is solvable. Then  $V$  is an elementary abelian  $q$ -group for some prime  $q(\neq p)$  by the condition (3) in Theorem 1.

Next,  $G=VN_G(P_0)$  by the Frattini argument. Since  $[N_V(P_0), P_0] \subseteq V \cap P_0=1$ ,  $N_V(P_0) \subseteq C_V(P_0)=1$ . Thus  $N_G(P_0)$  is a complement of  $V$  in  $G$ .

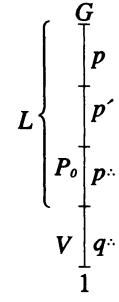
Now, set  $L=N_G(P_0)$  and let  $P$  be a Sylow  $p$ -subgroup of  $L$ .

**Lemma 3.2.**  *$\bar{L}=L/P_0$  is a Frobenius group with kernel  $O_{p'}(\bar{L})$  and complement  $\bar{P}$ .*

Proof. By the condition (1) in Theorem 1,  $O^{p'}(\bar{L})=\bar{L}$ . Then  $\bar{L}=[O_p(\bar{L}), \bar{P}]\bar{P}$  since  $\bar{L}=O_{p',p}(\bar{L})$ . Furthermore, since  $\bar{L} \simeq G/VP_0$ ,  $\bar{L}$  is  $p$ -radical by Lemma 2.1 (1), and so  $[O_{p'}(\bar{L}), \bar{P}] \cap C_{O_{p'}(\bar{L})}(\bar{P})=1$ . Thus  $\bar{L}$  is a Frobenius group.

By Lemma 3.2,  $G$  satisfies the condition (A) in Theorem 1.

**Lemma 3.3.** *Let  $\varphi \in \text{Irr}(V)$  with  $\varphi \neq 1_V$ . Then one of the following*



(1) and (2) holds.

- (1)  $I_G(\varphi) \subseteq O_{p',p,p'}(G)$ .
- (2)  $L = I_L(\varphi)P_0$  and  $I_L(\varphi) \cap P_0 = 1$ .

Proof. Set  $T = I_G(\varphi)$  and assume that  $T \not\subseteq O_{p',p,p'}(G)$ . Let  $b_0$  be a block of  $kV$  with  $\varphi \in b_0$ . Let  $B$  be a block of  $kG$  which covers  $b_0$  and let  $b$  be the Fong correspondent of  $B$  w.r.t.  $(V, T)$ . Let  $D$  be a defect group of  $b$  and let  $P^*$  be a Sylow  $p$ -subgroup of  $G$  which contains  $D$ . Since  $B$  is  $p$ -radical,  $k \otimes_{P^*y \cap T} b$  is semisimple for any  $y \in G$  by Lemma 2.4. On the other hand, there exists an irreducible  $kT$ -module in  $b$  with vertex  $D$  (see [1, Lemma 4.6]). Hence  $P^{*yt} \cap T \supseteq D$  for some  $t \in T$ , in particular  $P^{*yt} \cap T = D$ . Set  $\bar{G} = G/VP_0$ . Since  $D \not\subseteq O_{p',p,p'}(G)$ ,  $\bar{D}^{\bar{y}t} = \bar{P}^{*\bar{y}t} = \bar{D} \neq 1$ . Since  $\bar{G}$  is a Frobenius group,  $\bar{y}t \in \bar{P}^*$ . Hence  $y \in TP^* = TP_0$ . This implies that  $G = TP_0$ , and so  $L = (L \cap T)P_0$ .

Next, set  $Q = (L \cap T)P_0$ . Since  $P_0$  is abelian,  $Q \triangleleft L$ . Since  $[V, Q] \subseteq \ker \varphi \neq V$  and  $[V, Q]$  is  $L$ -invariant,  $[V, Q] = 1$  by the minimality of  $V$ , and so  $Q \subseteq O_p(G) = 1$ .

Now assume that  $C_G(v) \not\subseteq O_{p',p,p'}(G)$  for some  $v \in V^*$ . As  $V$  is a  $p'$ -group, this condition is equivalent to the condition that  $I_G(\varphi) \not\subseteq O_{p',p,p'}(G)$  for some  $\varphi \neq 1_V \in \text{Irr}(V)$  (see for example [3]§13). Then for such a  $\varphi$ ,  $I_L(\varphi)P_0 = L$  and  $I_L(\varphi)P_0 = 1$  by Lemma 3.3.

Let  $H$  be a Hall  $p'$ -subgroup of  $I_L(\varphi)$  (which is also a Hall  $p'$ -subgroup of  $L$ ). Then  $H \triangleleft I_L(\varphi)$  and  $I_L(\varphi) = H < s >$  for some element  $s$  of order  $p$  in  $P \setminus P_0$ .

We continue our discussion by assuming that there exists an element  $s$  of order  $p$  in  $N_P(H) \setminus P_0$  and shall prove that the condition (C)(1) in Theorem 1 holds.

**Lemma 3.4.** *If  $W$  is a subgroup of  $V$  with  $|V:W|=q$  and  $[s, V] \subseteq W$ , then there exists a Hall  $p'$ -subgroup  $H_1$  of  $L$  with  $[H_1, V] \subseteq W$ .*

Proof. Let  $\varphi$  be an irreducible character of  $V$  with kernel  $W$ . Then since  $[s, V] \subseteq W$ ,  $s \in I_G(\varphi)$ . Now the result follows by Lemma 3.3.

**Lemma 3.5.** *Assume that  $V = W \times W^s \times \cdots \times W^{s^{p-1}}$ , where  $W$  is  $P_0H$ -invariant. Then the following (1) and (2) hold.*

- (1) *If  $W_1$  is a subgroup of  $W$  with  $|W:W_1|=q$ , then there exists a Hall  $p'$ -subgroup  $H_1$  of  $L$  with  $[H_1, W] \subseteq W_1$ . In particular,  $W_1$  is  $H_1$ -invariant.*
- (2) *Let  $W_0$  be an irreducible  $P_0$ -module of  $W$ . Then  $W_0$  is  $H$ -invariant.*

Proof. (1) We can easily see that  $V = W \times [V, s]$  as  $V$  is an abelian  $p'$ -group. Put  $U = W_1 \times [V, s]$ . By Lemma 3.4, there exists a Hall  $p'$ -subgroup  $H_1$  of  $L$  with  $[H_1, V] \subseteq U$ . Since  $H_1 \subseteq P_0 H$ ,  $W$  is  $H_1$ -invariant, and so  $[H_1, W] \subseteq W \cap U = W_1$ . In particular,  $W_1$  is  $H_1$ -invariant.

(2) If  $W_0$  is not  $H$ -invariant, there exists an  $h \in H$  with  $W_0 \neq W_0^h$ . Let  $w \in W_0^*$ . Since  $W$  is elementary abelian, there exist subgroups  $W'$  and  $W''$  of  $W$  with  $W_0 = \langle w \rangle \times W'$  and  $W = W_0 \times W_0^h \times W''$ . Set  $W_1 = W' \times W_0^h \times W''$ , then  $|W:W_1|=q$  and  $W_0 \trianglelefteq W_1$ . By (1),  $W_1$  is  $H_1$ -invariant for some Hall  $p'$ -subgroup  $H_1$  of  $L$ . Since  $h \in H \subseteq P_0 H_1$ ,  $h = ah_1$  for some  $a \in P_0$  and  $h_1 \in H_1$ . Furthermore, since  $W_1 \supseteq W_0^h = W_0^{h_1}$ ,  $W_0 = (W_0^{h_1})^{h_1^{-1}} \subseteq W_1^{h_1^{-1}} = W_1$ , contrary to our choice of  $W_1$ .

Since  $L$  acts on  $V$  by conjugation, we can regard  $V$  as an  $L$ -module. Furthermore, since  $V$  is a minimal normal subgroup of  $G$ ,  $V$  is an irreducible  $L$ -module.

**Lemma 3.6.** *Let  $V = V_1 \times \cdots \times V_n$ , where  $V_i$ ,  $1 \leq i \leq n$ , are the homogeneous components of  $V$  with respect to  $P_0$ . Then  $n=p$  and we may take  $V_i^s = V_{i+1}$ ,  $0 \leq i \leq p-1$ . Furthermore,  $V_i$  is an irreducible  $P_0$ -module which is  $H$ -invariant,  $1 \leq i \leq p$ .*

Proof. We divide the proof of Lemma 3.6 into three steps.

STEP 1.  *$s$  induces a regular permutation representation on the set  $\{V_1, \dots, V_n\}$ .*

Proof. Suppose it is false, and let  $s$  fix  $V_1$ . Since  $P_0$  is abelian,  $\bar{P}_0 = P_0/C_{P_0}(V_1)$  is cyclic. Since  $\langle s \rangle \bar{P}_0 = \langle s \rangle P_0/C_{P_0}(V_1)$  has an abelian subgroup of  $(p, p)$  type, there exists an  $x \in \langle s \rangle P_0$  with  $C_{V_1}(x) \neq 1$  and  $\bar{x} \neq 1$ . If  $x \in P_0$ , then  $C_{V_1}(x) = 1$  since  $V_1$  is homogeneous as a  $P_0$ -module. Hence  $x \notin P_0$ . Let  $v \in C_{V_1}(x)^*$ , then  $x \in C_p(v)$ . Now by Lemma 3.3 and the remark before Lemma 3.4,  $C_{P_0}(v) = 1$ . So  $C_{P_0}(V_1) \subseteq C_{P_0}(v) = 1$  and  $P_0$  is cyclic. Since  $L \triangleright P_0$  and  $P_0$  is abelian,  $\bar{L} = L/P_0$  acts on  $P_0$  by conjugation. Since  $O^{p'}(\bar{L}) = \bar{L}$  and  $\text{Aut}(P_0)$  is abelian,  $O_{p'}(\bar{L})$  centralizes  $P_0$ . Hence  $L = O_{p', p}(L)$ , and so  $G = O_{p', p}(G)$ , contrary to the condition (1) of Theorem 1. Hence  $s$  acts regularly on the set  $\{V_1, \dots, V_n\}$ .

STEP 2. *Let  $\{V_1, \dots, V_t\}$  be an  $H$ -orbit of  $\{V_1, \dots, V_n\}$ , and set  $W = V_1 \times \cdots \times V_t$ . Then  $V = W \times W^s \times \cdots \times W^{s^{p-1}}$ .*

Proof. Suppose  $s$  fixes an  $H$ -orbit  $\{V_1, \dots, V_t\}$ . Since  $H$  is a  $p'$ -group,  $(p, t) = 1$ . Therefore  $s$  fixes  $V_i$  for some  $i$ , contrary to Step 1. Hence  $s$  doesn't fix an  $H$ -orbit  $\{V_1, \dots, V_t\}$  and this implies that  $W \times W^s \times \dots \times W^{s^{p-1}} \subseteq V$ . Furthermore, as  $W \times W^s \times \dots \times W^{s^{p-1}}$  is  $L$ -invariant, we have  $V = W \times W^s \times \dots \times W^{s^{p-1}}$ .

### STEP 3. Proof of Lemma 3.6.

Take an irreducible  $P_0$ -submodule  $W_0$  in  $V_1$ . Then by Lemma 3.5(2),  $W_0$  is  $P_0H$ -invariant. Therefore  $W_0 \times W_0^s \times \dots \times W_0^{s^{p-1}}$  is  $L$ -invariant and coincides with  $V$ . Thus  $W_0 = V_1$  (and  $t = 1$  in Step 2) and the lemma follows.

From Theorem 15.16 of [3], we obtain the following lemma.

**Lemma 3.7.** *Let  $P_0 \supseteq Q_1 \supseteq Q_2 \supseteq \Phi(P_0)$ , where  $Q_1$  and  $Q_2$  are  $H < s >$ -invariant subgroups of  $P_0$ . If  $H$  acts non-trivially on  $Q_1/Q_2$ , then  $|Q_1/Q_2| \geq p^p$ .*

**Lemma 3.8.** *Let  $\phi$  be a homomorphism of  $P_0$  into  $P_0/C_{P_0}(V_1) \times \dots \times P_0/C_{P_0}(V_p)$  which is defined by the rule  $\phi(x) = (C_{P_0}(V_1)x, \dots, C_{P_0}(V_p)x)$ . Then  $\phi$  is an isomorphism.*

Proof.  $\text{Ker } \phi = C_{P_0}(V) = 1$  and  $P_0$  is isomorphic to a subgroup of  $P_0/C_{P_0}(V_1) \times \dots \times P_0/C_{P_0}(V_p)$ . On the other hand,  $P_0/C_{P_0}(V_i)$  is cyclic,  $1 \leq i \leq p$ . Hence the rank of  $P_0$  is at most  $p$  and  $|P_0/\Phi(P_0)| \leq p^p$ , and so  $|P_0/\Phi(P_0)| = p^p$  by Lemma 3.7.

Suppose next that  $\phi$  is not any epimorphism. Let  $P_0$  have exponent  $p^m$ . Then  $p^m = |P_0/C_{P_0}(V_1)|$  as  $P_0/C_{P_0}(V_1)$  is cyclic. Set  $\Omega_{m-1}(P_0) = \{x \in P_0 \mid x^{p^{m-1}} = 1\}$ . Then  $\Omega_{m-1}(P_0)$  is  $< s > H$ -invariant and  $P_0 \not\supseteq \Omega_{m-1}(P_0) \not\supseteq \Phi(P_0)$ . By Lemma 3.7,  $H$  acts trivially on both  $P_0/\Omega_{m-1}(P_0)$  and  $\Omega_{m-1}(P_0)/\Phi(P_0)$  which is a contradiction.

Set  $P_i = C_P(V_1 \times \dots \times V_{i-1} \times V_{i+1} \times \dots \times V_p)$ ,  $1 \leq i \leq p$ . Then  $P_0 = P_1 \times \dots \times P_p$  by Lemma 3.8, and so  $VP = (V_1 P_1) \times \dots \times (V_p P_p)$ .  $P_i$  is cyclic as it acts on  $V_i$  irreducibly and faithfully.

**Lemma 3.9.**  *$C_{P_1H}(v)$  contains a Hall  $p'$ -subgroup of  $P_1H$  for any  $v \in V_1$ .*

Proof. Set  $K = P_1H$ . Then  $K$  acts on  $\text{Irr}(V_1)$  and on the set of elements of  $V_1$ . We claim that  $P_1$ -orbits coincide with  $K$ -orbits on

$V_1$ . Let  $A_1, \dots, A_m$  be  $P_1$ -orbits on  $\text{Irr}(V_1)$  and let  $B_1, \dots, B_n$  be  $P_1$ -orbits on the set of elements of  $V_1$ . By Corollary 6.3.3 of [3], we have  $m=n$ .

Let  $\chi$  be an irreducible character in  $A_1$ , and set  $W_1 = \text{Ker } \chi$ . Now  $V = V_1 \times V_1^s \times \dots \times V_1^{s^{p-1}}$ , and  $V_1$  is  $P_0H$ -invariant. Hence, by lemma 3.5 (1), there exists a Hall  $p'$ -subgroup  $H_1$  of  $L$  with  $[H_1, V_1] \subseteq W_1$ . As  $P_0 = P_1 C_{P_0}(V_1)$  by the remark before Lemma 3.9, we may take  $H_1$  in  $K$ . Since  $K = P_1 H = P_1 H_1$  and  $H_1$  fixes  $\chi$ ,  $A_1$  coincides with a  $K$ -orbit on  $\text{Irr}(V_1)$ . Similarly,  $A_i$  is a  $K$ -orbit on  $\text{Irr}(V_1)$ ,  $2 \leq i \leq m$ . By Corollary 6.3.3 of [3] again,  $B_i$  is a  $K$ -orbit on the set of elements of  $V_1$ ,  $1 \leq i \leq m$ , and our claim follows.

Let  $v$  be an element of  $V_1$ . For each element  $g \in K$ , we have  $v^g = v^x$  for some  $x \in P_1$  by the above claim. Hence  $gx^{-1} \in C_K(v)$ , and so  $g \in C_K(v)P_1$ . This implies that  $K = C_K(v)P_1$ . Hence  $C_K(v)$  contains a Hall  $p'$ -subgroup of  $K$ .

**Lemma 3.10.** *Set  $r = |H/C_H(V_1)|$  and  $p^m = |P_1|$ ,  $q^n = |V_1|$ . Then  $r|n$  and  $\frac{q^n - 1}{q^{n/r} - 1} = p^m$  and  $V_i P_i H / C_H(V_i) \simeq A_{q, n, r}$ ,  $1 \leq i \leq p$ .*

**Proof.** Set  $N = V_1 P_1 H$  and  $\bar{N} = V_1 P_1 H / C_H(V_1)$ . By Lemma 3.9,  $C_{\bar{N}}(\bar{v})$  contains a Hall  $p'$ -subgroup of  $\bar{N}$  for any  $v \in V_1$ . Hence every  $\bar{P}_1$ -orbit of  $V_1$  contains an element which is centralized by  $\bar{H}$ . Since  $P_1$  is cyclic, by Proposition 19.8 of [11] and Lemma 3.6, we can identify  $V_1$  with  $GF(q^n)$  in such a way that  $\bar{P}_1 \bar{H} \subseteq T(q^n) = \{v \rightarrow av^\sigma \mid a \in GF(q^n)^\ast, \sigma \in \text{Gal}(GF(q^n)/GF(q))\}$  and  $\bar{P}_1 \subseteq \{v \rightarrow av \mid a \in GF(q^n)^\ast\}$ .

As  $C_{\bar{H}}(\bar{P}_1)$  is contained in any Hall  $p'$ -subgroup of  $\bar{P}_1 \bar{H}$ ,  $C_{\bar{H}}(\bar{P}_1) = 1$ . Since  $\bar{P}_1$  is cyclic and  $\bar{H}$  is a  $p'$ -group,  $\bar{H}$  acts regularly on  $\bar{P}_1$ .

$$\begin{array}{c} \text{Gal}(GF(q^n)/GF(q)) \longrightarrow GF(q) \\ \downarrow \quad \quad \quad \downarrow \\ <\sigma> \longrightarrow GF(q^l) \\ \downarrow \quad \quad \quad \downarrow \\ 1 \longrightarrow GF(q^n) \end{array}$$

Let  $\bar{H} = <\eta>$ , where  $\eta(v) = bv^\sigma$ , for some  $b \in GF(q^n)^\ast$  and  $\sigma \in \text{Gal}(GF(q^n)/GF(q))$  for  $v \in V_1$ . By Proposition 19.8 of [11],  $\bar{H} \simeq \bar{P}_1 \bar{H}$ .  $C_{\bar{P}_1 \bar{H}}(\bar{P}_1) \subseteq \text{Gal}(GF(q^n)/GF(q))$ , and so  $r = |\bar{H}| = |\sigma|$  with  $r|n$ . Then  $<\sigma> = \text{Gal}(GF(q^n)/GF(q^l))$ , where  $n = lr$ . Let  $\bar{P}_1 = <x_a>$ , where  $x_a(v) = av$ ,  $a \in GF(q^n)^\ast$ . If  $a^i \in GF(q^l)$  for some  $i$ , then  $\eta x_{a^i}(v) = \eta(a^i v) = b(a^i v)^\sigma = (a^i)^\sigma b v^\sigma = a^i b v^\sigma$ . On the other hand,  $x_a \eta(v) = x_{a^i}(b v^\sigma) = a^i b v^\sigma$ . Hence  $\eta x_{a^i} = x_{a^i} \eta$ , and so  $\eta^{-1} x_{a^i} \eta = x_{a^i}$ . Since  $x_{a^i} = (x_a)^i$ ,  $\eta^{-1} (x_a)^i \eta = (x_a)^i$ . Thus  $a^i = 1$  as  $\bar{H}$  acts regularly on  $\bar{P}_1$ . Hence  $<\sigma> \cap GF(q^l)^\ast = 1$ .

Since  $\langle a \rangle \subseteq GF(q^n)$  is a  $P_1$ -orbits,  $\eta(a^i) = a^i$  for some  $i$ . Hence  $\eta(a^i) = b(a^i)^\sigma = a^i$ , and so  $b = (a^i)(a^{-i})^\sigma \in \langle a \rangle$ . Thus  $b = a^j$  for some  $j$ , and so  $\eta = x_{a^j}\sigma$ . Then  $\bar{P}_1\bar{H} = \langle x_a \rangle \langle x_{a^j}\sigma \rangle = \langle x_a \rangle \langle \sigma \rangle$ .

Let  $c$  be any element of  $GF(q^n)^\#$ . Since  $\{c, ca, \dots, ca^{p^m-1}\}$  is a  $\bar{P}_1$ -orbit,  $(ca^i)^\sigma = ca^i$  for some  $i$ . Hence  $ca^i \in GF(q^l)$ . This implies that  $GF(q^n)^\# = \langle a \rangle GF(q^l)^\#$ . Hence  $q^n - 1 = (q^l - 1)p^m = (q^{n/r} - 1)p^m$ . Furthermore, we have  $\overline{V_1 P_1 H} = V_1 \langle x_a \rangle \langle \sigma \rangle \simeq A_{q,n,r}$ .

Now we completed the proof of the if-part of Theorem 1 and we shall prove the “only-if-part” of the theorem.

**Lemma 3.11.** *Let  $G$  satisfy the conditions (1)–(3) in Theorem 1. If  $G$  has the conditions (A), (B) and (C) of Theorem 1, then  $G$  is  $p$ -radical.*

**Proof.** Let  $\varphi \in \text{Irr}(V)$  and let  $B$  be a block of  $kG$  which covers  $\varphi$ . Then we shall prove that  $B$  is  $p$ -radical.

CASE 1.  $\varphi = 1_V$ .

Since  $L/P_0$  is a Frobenius group,  $L$  is  $p$ -radical by Lemmas 2.1, 2.2. By lemma 2.5,  $B$  is  $p$ -radical.

CASE 2.  $\varphi \neq 1_V$ .

Set  $T = I_G(\varphi)$ . Suppose that  $G$  satisfies the condition (C)(2) in Theorem 1. By the remark before Lemma 3.4,  $T \subseteq O_{p',p,p}(G)$ . Set  $N = VP_0$ , then  $D \subseteq N$ , where  $D$  is a defect group of  $B$ . Then  $J(B) = J(kN)B$  by Theorem 2.3 of [1]. By Lemma 2.2,  $N$  is  $p$ -radical, and so  $(k \otimes_P B)J(B) = k \otimes_P J(kN)B \subseteq k \otimes_P J(kP_0)(kN)B = 0$ . Hence  $k \otimes_P B$  is semisimple, and so  $B$  is  $p$ -radical.

Suppose next that  $G$  satisfies the condition (C)(1) in Theorem 1. Let  $v$  be an element of  $V$ . Then  $v = v_1 \cdots v_p$ , where  $v_i \in V_i$ ,  $1 \leq i \leq p$ . Then, by (C)(2)(vii) in Theorem 1, there exist an  $x_i \in P_i$  with  $[v_i^{x_i}, H] = 1$ ,  $1 \leq i \leq p$ , where  $H$  is a Hall  $p'$ -subgroup of  $G$ . Set  $x = x_1 \cdots x_p$ . Then  $v^x = v_1^x \cdots v_p^x = v_1^{x_1} \cdots v_p^{x_p}$ , and so  $[v^x, H] = 1$ . This implies that  $C_G(v)$  contains a Hall  $p'$ -subgroup of  $G$  for each  $v \in V$ . By a similar argument in the proof of Lemma 3.9,  $T$  contains a Hall  $p'$ -subgroup.

Since  $V \subseteq T$  and  $G = VL$ , we have  $T = V(L \cap T)$ .  $T \cap L$  is  $p$ -closed or  $T \cap L / O_P(T \cap L)$  is a Frobenius group. In each case,  $T \cap L$  is  $p$ -radical by Lemmas 2.1, 2.2. Let  $b$  be the Fong correspondent of  $B$  w.r.t.  $(V, T)$ . Then  $b$  is  $p$ -radical by Lemma 2.5. Hence  $B$  is  $p$ -radical by Lemma 2.4(2).

#### 4. Proof of Theorem 2

In this section, we shall prove the following theorem from which Theorem 2 follows by using Lemma 2.6.

**Theorem 3.** *If  $G$  is a  $p$ -radical group with  $G/O_{p'}(G) \in \mathcal{F}$ , then  $G = O_{p,p',p,p'}(G)$ .*

Proof. Suppose it is false and let  $G$  be a minimal counterexample of Theorem 3. Now we divide the proof into several steps. At first, we shall prove that  $G$  satisfies the conditions (1)–(3) in Theorem 1.

STEP 1.  $O_p(G) = 1$  and  $O^{p'}(G) = G$ .

Proof. Suppose that  $O_p(G) \neq 1$ . Set  $\bar{G} = G/O_p(G)$ . Then  $\bar{G}$  is  $p$ -radical. Furthermore, since  $\bar{G}/O_{p'}(\bar{G})$  is a homomorphic image of  $G/O_p(G)$ ,  $\bar{G}/O_{p'}(\bar{G}) \in \mathcal{F}$  by Lemma 2.6. By the minimality of  $G$ ,  $\bar{G} = O_{p,p',p,p'}(\bar{G}) = O_{p',p,p'}(\bar{G})$ , and so  $G = O_{p,p',p,p'}(G)$ , contrary to our choice of  $G$ . Next we assume that  $O^{p'}(G) \subsetneq G$ . Set  $U = O^{p'}(G)$ , then since  $U$  is  $p$ -radical and  $U/O_{p'}(U) \in \mathcal{F}$ ,  $U = O_{p,p',p,p'}(U) = O_{p,p',p}(U)$ . Hence  $G = O_{p,p',p,p'}(G)$ , a contradiction.

Let  $V$  be a minimal normal subgroup of  $G$ . By Step 1 and Lemma 2.3,  $V$  is an abelian  $p'$ -group. Furthermore, let  $P_0$  be a  $p$ -subgroup of  $G$  with  $O_p(G/V) = P_0V/V$ . Then we show the following Step 2.

STEP 2.

- (1)  $G/V = O_{p,p',p}(G/V)$ , in particular,  $G = O_{p',p,p',p}(G)$ .
- (2)  $P_0 \neq 1$  is elementary abelian.
- (3) Let  $M$  be a Hall  $p'$ -subgroup of  $G$ . Then there exists an  $s \in N_G(M)$  with  $|s| = p$  and  $G = O_{p',p,p'}(G) \langle s \rangle$ .
- (4)  $V = [O_{p'}(G), P_0]$ .

Proof. Set  $\bar{G} = G/V$ . For (1), we note that  $\bar{G}$  is  $p$ -radical and  $\bar{G} \in \mathcal{F}$ . Since  $V \neq 1$ ,  $\bar{G} = O_{p,p',p,p'}(\bar{G}) = O_{p,p'}(\bar{G})$ , so that  $G = O_{p',p,p',p}(G)$ .

For (2) and (3), let  $P$  be a Sylow  $p$ -subgroup of  $G$  such that  $N_P(M)$  is a Sylow  $p$ -subgroup of  $N_G(M)$ . If  $P_0 = 1$ , then  $\bar{G} = O_{p',p}(\bar{G})$ , and so  $G = O_{p',p}(G)$ , contrary to our choice of  $G$ . Hence  $P_0 \neq 1$ . By Theorem 3.1 of [14],  $P_0$  is elementary abelian. Hence  $\bar{P}_0 = [\bar{P}_0, \bar{M}] \times C_{\bar{P}_0}(\bar{M})$ . Since  $\bar{G} = \bar{P}_0 N_{\bar{G}}(\bar{M}) = \bar{P}_0 N_{\bar{P}}(\bar{M}) \bar{M}$ ,  $\bar{G} = [\bar{P}_0, \bar{M}] \bar{M} N_{\bar{P}}(\bar{M})$ . Furthermore,  $[\bar{P}_0, \bar{M}] \bar{M} \triangleleft \bar{G}$  and  $[\bar{P}_0, \bar{M}] \bar{M} \cap N_{\bar{P}}(\bar{M}) = [\bar{P}_0, \bar{M}] \cap N_{\bar{P}}(\bar{M}) = [\bar{P}_0, \bar{M}] \cap C_{\bar{P}_0}(\bar{M}) = 1$ . Therefore  $N_{\bar{P}}(\bar{M})$  is a complement of  $[\bar{P}_0, \bar{M}] \bar{M}$  in  $\bar{G}$ . By Theorem

3.1 of [14],  $N_{\bar{P}}(\bar{M})$  is elementary abelian. If  $N_{\bar{P}}(\bar{M}) \subseteq \bar{P}_0$ , then  $\bar{G} = \bar{P}_0 \bar{M}$ . Hence  $G = O_{p',p,p'}(G)$ , contrary to our choice of  $G$ . Thus  $N_{\bar{P}}(\bar{M}) \not\subseteq \bar{P}_0$ . Therefore there exists an element  $s \in N_{\bar{P}}(\bar{M})$  with  $|s|=p$  and  $s \notin P_0$ .

Now set  $N = O_{p',p,p'}(G) < s >$ . Since  $N \triangleleft G$ ,  $N$  is  $p$ -radical by Lemma 2.1 (1). Furthermore,  $\bar{N} = N/O_{p'}(G) = N/O_{p'}(N) \in \mathcal{F}$  by Lemma 2.7. If  $N \subsetneq G$ , then  $N = O_{p',p,p'}(N)$ . Since  $N \triangleleft G$ ,  $O_p(N) \subseteq O_p(G) = 1$  and  $O_{p',p}(N) \subseteq O_{p',p}(G)$ . Hence  $N = O_{p',p,p'}(N)$ , and so  $s \in O_{p',p}(N) \subseteq O_{p',p}(G)$ , contrary to our choice of  $s$ . Thus  $G = O_{p',p,p'}(G) < s >$ .

For (4), we note  $[O_{p'}(G), P_0] \subseteq O_{p'}(G) \cap P_0 V = V$ .

### STEP 3. Proof of Theorem 3.

By Step 2, there exists an  $s$  with  $|s|=p$  and  $G = O_{p',p,p'}(G) < s >$ . By the remark before Lemma 3.4, the condition (C)(1) in Theorem 1 holds. We set  $\bar{G} = G/V$ , then  $\bar{G} \in \mathcal{F}$  by Lemma 2.6. Since  $\bar{P}_1$  is  $\bar{H}$ -invariant,  $\bar{P}_1$  is  $\bar{s}$ -invariant (see the proof of Lemma 11(7) of [7]), and we have a contradiction.

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Department of Mathematics  
Faculty of Education  
Gunma University  
Maebashi 371, Japan

