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ON p -RADICAL GROUPS G AND THE NILPOTENCY INDICES OF $J(kG)$

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1. Introduction

Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic $p > 0$, and let P be a Sylow p -subgroup of G .

Following Motose and Ninomiya [9] we call G p -radical if the induced module $(k_P)^G$ of the trivial kP -module k_P is completely reducible as a right kG -module.

In [10], Okuyama has proved that p -radical groups are p -solvable. And Tsushima has characterized p -radical groups which are p -nilpotent by group theoretical properties (see Lemma 2.2). So it seems to be interesting to investigate the structure of p -radical groups of p -length 2 and in this paper we shall treat such a group with some additional properties.

Before describing our result we need to define some notations. Let $F = GF(q^n)$ be a finite field of q^n elements for prime q . Let V be the additive group of F . Let $T(q^n)$ be the set of semilinear transformations of the form $v \rightarrow av^\sigma$ with $v \in V$, $0 \neq a \in F$, and σ a field automorphism (see [11, p229]). Then we can consider the semidirect product $VT(q^n)$ of V by $T(q^n)$. Let λ be a generator of the multiplicative group of F and $v = \lambda^{q^{n/r}-1}$ for some integer r with $r|n$. Let $T_0 = \{v \rightarrow av^\sigma | a \in \langle v \rangle, \sigma \in \text{Gal}(F/GF(q^{n/r}))\}$. Then we define $A_{q,n,r} = VT_0 \subseteq VT(q^n)$.

Theorem 1. *Let G be a finite group with the following conditions.*

- (1) $|G: O_{p',p,p'}(G)| = p$, $O_p(G) = 1$ and $O^{p'}(G) = G$.
- (2) A Sylow p -subgroup P_0 of $O_{p',p}(G)$ is abelian.
- (3) $V = [O_p(G), P_0]$ is a minimal normal subgroup of G .

Then G is p -radical if and only if the following conditions (A), (B) and (C) hold.

- (A) $\bar{G} = G/VP_0$ is a Frobenius group with kernel $O_{p'}(\bar{G})$.
- (B) V is an elementary abelian q -group for some prime $q (\neq p)$.
- (C) One of the following (1) and (2) holds.

(1) The following (i)-(vii) hold.

(i) $G = VN_G(P_0)$ and $V \cap N_G(P_0) = 1$.

(ii) $P_0 \triangleleft P_0 H \triangleleft P_0 H \langle s \rangle$, where $|s| = p$ and H is a p' -group.

(iii) By conjugation, we can regard V as an irreducible $N_G(P_0)$ -module. Then $V = V_1 \times \cdots \times V_p$, where $V_i, 1 \leq i \leq p$, are the homogeneous components of V_{P_0} .

(iv) Set $P_i = C_{P_0}(V_1 \times \cdots \times V_{i-1} \times V_{i+1} \times \cdots \times V_p)$, $1 \leq i \leq p$.

Then $P_0 = P_1 \times \cdots \times P_p$.

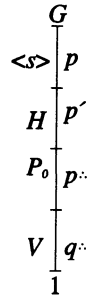
(v) $V_1^{s^i} = V_{i+1}$, $P_1^{s^i} = P_{i+1}$, $0 \leq i \leq p-1$.

(vi) V_i and P_i are H -invariant, $1 \leq i \leq p$, and $VP_0 = (V_1 P_1) \times \cdots \times (V_p P_p)$.

(vii) Set $r = |H/C_H(V_1)|$ and $p^m = |P_1|$, $q^n = |V_1|$.

Then $r|n$ and $\frac{q^n - 1}{q^{n/r} - 1} = p^m$ and $V_i P_i H/C_H(V_i) \simeq A_{q,n,r}$, $1 \leq i \leq p$.

(2) $C_G(v) \subseteq O_{p',p,p'}(G)$ for any element v of V^* .



Next, let $t(G)$ be the nilpotency index of the radical $J(kG)$ of kG and let p^a be the order of Sylow p -subgroups of G .

Wallace [14] proved that if G is p -solvable, then $a(p-1)+1 \leq t(G) \leq p^a$. If G has p -length 1, then by Motose and Ninomiya [8] $t(G) = a(p-1)+1$ if and only if P is elementary abelian.

All known examples of p -solvable group G with $t(G) = a(p-1)+1$ have p -length at most 2. Using Theorem 1, we can prove the following theorem.

Theorem 2. *If G is a p -radical group with $t(G) = a(p-1)+1$, then $G = O_{p,p',p,p'}(G)$. In particular, the p -length of G is at most 2.*

2. Preliminaries

In this section, we shall give some lemmas which will be used to prove the theorems.

Lemma 2.1. ([1, Theorem 6.5]). *Suppose that $N \triangleleft G$. Then the following (1)–(3) hold.*

(1) *If G is p -radical, so are N and G/N .*

(2) *If N is a p -group, then G is p -radical if and only if G/N is p -radical.*

(3) *If G/N is a p' -group, then G is p -radical if and only if N is p -radical.*

Lemma 2.2. ([13, Theorem 2]). *Let $G=PN$ be a p -nilpotent group with $N=O_p(G)$. Then G is p -radical if and only if $[N,D] \cap C_N(D)=1$ for any p -subgroup D of G . In particular, if N is abelian, then G is p -radical.*

Lemma 2.3. *If G is p -radical, then $O^{p'}(G)$ is solvable.*

Proof. Suppose it is false and let G be a minimal counterexample. Then we have $G=O^{p'}(G)$. By Theorem 1 of [10], G is p -solvable. If $O_p(G) \neq 1$, then $G/O_p(G)$ is solvable since $G/O_p(G)$ is p -radical. Hence G is solvable, a contradiction. Hence $O_p(G)=1$, and so $O_{p'}(G) \neq 1$. Let P be a Sylow p -subgroup of $O_{p'}(G)$, and set $W = \langle [O_{p'}(G), x] \mid x \in \Omega_1(Z(P)) \rangle$. If $W=1$, then $1 \neq \Omega_1(Z(P)) \subseteq C_G(O_{p'}(G)) \subseteq O_{p'}(G)$, a contradiction. Since $G=O_{p'}(G)N_G(P)$, $1 \neq W \triangleleft G$. Furthermore, for $x \in \Omega_1(Z(P))$ $[O_{p'}(G), x]$ is a normal subgroup of $O_{p'}(G)$ and is nilpotent by Thompson [12] as $C_{O_{p'}(G)}(x) \cap [O_{p'}(G), x] = 1$ (see Lemma 2.2). Hence W is solvable. Since G/W is p -radical, G/W is solvable. This implies that G is solvable, contrary to our choice of G .

Let B be a block of kG . We call B a p -radical block if $k \otimes_p B$ is semisimple. Let $N \triangleleft G$ and b_0 a block of kN that is covered by B . Let T be the inertia group of b_0 . Then there exists a unique block b of kT with $b^G = B$. We call b the *Fong correspondent* of B w.r.t. (K, T) . Then the next lemma holds.

Lemma 2.4. *The following (1) and (2) hold.*

- (1) *B is p -radical if and only if $k \otimes_{p y \cap T} b$ is a semisimple kT -module for any $y \in G$.*
- (2) ([13] Tsushima) *If $|G:T|$ is a power of p , then B is p -radical if and only if b is p -radical.*

Proof. (1) Various facts are known about the relationship between B and b . $B = (kG)b(kG)$ and $J(B) = (kG)J(b)(kG)$. Furthermore, $(kG)b$ is a direct summand of B as a $k(G \times T)$ -module. Hence $k \otimes_p (kG)b$ is a direct summand of $k \otimes_p B$ as a right kT -module. On the other hand,

$$k \otimes_p kG \simeq \bigoplus_{y \in P|G/T} k \otimes_{p y \cap T} kT \text{ by Mackey decomposition, and so } k \otimes_p (kG)b \simeq$$

$$\bigoplus_{y \in P|G/T} \otimes_{p y \cap T} b \text{ as right } kT\text{-modules.}$$

Assume that B is p -radical. Since $(k \otimes_p B)J(b) \subseteq (k \otimes_p B)J(B) = 0$, $(k \otimes_{p y \cap T} b)J(b) = 0$, and so $k \otimes_{p y \cap T} b$ is semisimple.

Conversely, assume that $k \otimes_{P^y \cap T} b$ is a semisimple kT -module for any $y \in G$. Then $k \otimes_P (kG)b$ is a semisimple kT -module. Therefore $U = (k \otimes_P (kG)b) \otimes_T kG$ is semisimple by Fong's theory. Since $k \otimes_P B$ is a natural homomorphic image of U , it is also semisimple.

(2) Since $G = PT$ by assumption, $P^y \cap T$ is a Sylow p -subgroup of T for any $y \in G$, and hence $k \otimes_{P^y \cap T} b$ is semisimple if and only if b is p -radical. Therefore (2) follows from (1).

Let $G \triangleright V$. We let $\text{Irr}(V)$ be the set of ordinary irreducible characters of V and let $I_G(\varphi)$ be the inertia group of $\varphi \in \text{Irr}(V)$.

Furthermore, for a block B of kG , let $\text{Irr}(B)$ be the set of irreducible characters of G belonging to B .

Lemma 2.5. *Let $G = LV \triangleright V$, where V is an abelian p' -group with $L \cap V = 1$. Let $\varphi \in \text{Irr}(V)$ with $I_G(\varphi) = G$. Then L is p -radical if and only if all blocks which cover φ are p -radical.*

Proof. Let b be a block of kG which covers φ . Let $\chi \in \text{Irr}(b)$. Then each irreducible constituent of χ_V is φ . Set $U = \text{Ker } \varphi$. Then we have $\text{Ker } \chi \supseteq U$, and so $\text{Ker } b \supseteq U$. Set $\bar{G} = G/U$, then $\bar{G} = \bar{L} \times \bar{V}$. Let e be a centrally primitive idempotent corresponding to φ . Then the sum of all blocks which cover φ is isomorphic to $k\bar{L} \otimes_k ke \simeq kL$. Then the lemma follows immediately.

Throughout this paper, we let \mathcal{F} be the family of all finite group G such that $t(G) = a(p-1) + 1$, where p^a is the order of a Sylow p -subgroup of G .

Lemma 2.6. *Let G be a p -solvable group and $N \triangleleft G$. If $G \in \mathcal{F}$, then $G/N \in \mathcal{F}$.*

Proof. Let p^a, p^b be the orders of Sylow p -subgroups of G and N , respectively.

At first, assume N is a p' -group. By Theorem 2.2, 3.3 of [14], $a(p-1) + 1 \leq t(G/N) \leq t(G) = a(p-1) + 1$. Hence $t(G/N) = a(p-1) + 1$, and so $G/N \in \mathcal{F}$.

Next, assume N is a p -group. By Theorem 2.4, 3.3 of [14], $b(p-1) + (a-b)(p-1) + 1 \leq t(N) + t(G/N) - 1 \leq t(G) = a(p-1) + 1$. Hence $t(N) = b(p-1) + 1$ and $t(G/N) = (a-b)(p-1) + 1$, and so $G/N \in \mathcal{F}$.

Now, we shall consider the general case. If $N \neq 1$, $0_p(N) \neq 1$ or $0_p(N) \neq 1$ since N is p -solvable. By induction on $|N|$, the lemma follows.

Lemma 2.7. *Let $G = LV \triangleright V$, where V is a p -group and L is a p -radical p -nilpotent group with $L \cap V = 1$. If $G \in \mathcal{F}$, then $\langle VO_{p'}(L), s \rangle \in \mathcal{F}$ for any p -element $s \in L$.*

Proof. Set $H = O_{p'}(L)$. By Theorem 3.1 of [14], $|s| = p$ and V is elementary abelian. Since L is p -radical, $[H, s] \langle s \rangle$ is a Frobenius group by Lemma 2.2. By Theorem 2.7 of [5], $\text{gr } kV$ is semisimple as a kG -module. Let $N = V[H, s] \langle s \rangle \triangleleft G$. Then $\text{gr } kV$ is semisimple as a kN -module, and so $N \in \mathcal{F}$ by Theorem 2.7 of [5]. Set $M = \langle VO_{p'}(L), s \rangle$. Since $M \triangleright N$ and M/N is a p' -group, $M \in \mathcal{F}$.

3. Proof of Theorem 1

First we shall prove the if -part of Theorem 1. Let G be a finite group with conditions (1)–(3) in Theorem 1 and we assume that G is p -radical.

Lemma 3.1. *$V = [O^{p'}(G), P_0]$ is an elementary abelian q -group for some prime $q (\neq p)$ and $N_G(P_0)$ is a complement of V in G .*

Proof. Since G is p -radical with $O^{p'}(G) = G$, G is solvable by Lemma 2.3. In particular, V is solvable. Then V is an elementary abelian q -group for some prime $q (\neq p)$ by the condition (3) in Theorem 1.

Next, $G = VN_G(P_0)$ by the Frattini argument. Since $[N_V(P_0), P_0] \subseteq V \cap P_0 = 1$, $N_V(P_0) \subseteq C_V(P_0) = 1$. Thus $N_G(P_0)$ is a complement of V in G .

Now, set $L = N_G(P_0)$ and let P be a Sylow p -subgroup of L .

Lemma 3.2. *$\bar{L} = L/P_0$ is a Frobenius group with kernel $O_{p'}(\bar{L})$ and complement \bar{P} .*

Proof. By the condition (1) in Theorem 1, $O^{p'}(\bar{L}) = \bar{L}$. Then $\bar{L} = [O_{p'}(\bar{L}), \bar{P}] \bar{P}$ since $\bar{L} = O_{p', p}(\bar{L})$. Furthermore, since $\bar{L} \simeq G/VP_0$, \bar{L} is p -radical by Lemma 2.1 (1), and so $[O_{p'}(\bar{L}), \bar{P}] \cap C_{O_{p'}(\bar{L})}(\bar{P}) = 1$. Thus \bar{L} is a Frobenius group.

By Lemma 3.2, G satisfies the condition (A) in Theorem 1.

Lemma 3.3. *Let $\varphi \in \text{Irr}(V)$ with $\varphi \neq 1_V$. Then one of the following*

$$L \left\{ \begin{array}{l} G \\ p \\ p' \\ P_0 \\ p'' \\ V \\ q \\ 1 \end{array} \right.$$

(1) and (2) holds.

$$(1) \quad I_G(\varphi) \subseteq O_{p',p,p'}(G).$$

$$(2) \quad L = I_L(\varphi)P_0 \text{ and } I_L(\varphi) \cap P_0 = 1.$$

Proof. Set $T = I_G(\varphi)$ and assume that $T \not\subseteq O_{p',p,p'}(G)$. Let b_0 be a block of kV with $\varphi \in b_0$. Let B be a block of kG which covers b_0 and let b be the Fong correspondent of B w.r.t. (V, T) . Let D be a defect group of b and let P^* be a Sylow p -subgroup of G which contains D . Since B is p -radical, $k \otimes_{p^{*y} \cap T} b$ is semisimple for any $y \in G$ by Lemma 2.4. On the other hand, there exists an irreducible kT -module in b with vertex D (see [1, Lemma 4.6]). Hence $P^{*y} \cap T \supseteq D$ for some $t \in T$, in particular $P^{*y} \cap T = D$. Set $\bar{G} = G/VP_0$. Since $D \not\subseteq O_{p',p,p'}(G)$, $\bar{D}^{y\bar{t}} = \bar{P}^{*y\bar{t}} = \bar{D} \neq 1$. Since \bar{G} is a Frobenius group, $y\bar{t} \in \bar{P}^*$. Hence $y \in TP^* = TP_0$. This implies that $G = TP_0$, and so $L = (L \cap T)P_0$.

Next, set $Q = (L \cap T)P_0$. Since P_0 is abelian, $Q \triangleleft L$. Since $[V, Q] \subseteq \ker \varphi \neq V$ and $[V, Q]$ is L -invariant, $[V, Q] = 1$ by the minimality of V , and so $Q \subseteq O_p(G) = 1$.

Now assume that $C_G(v) \not\subseteq O_{p',p,p'}(G)$ for some $v \in V^*$. As V is a p' -group, this condition is equivalent to the condition that $I_G(\varphi) \not\subseteq O_{p',p,p'}(G)$ for some $\varphi \neq 1_V \in \text{Irr}(V)$ (see for example [3]§13). Then for such a φ , $I_L(\varphi)P_0 = L$ and $I_L(\varphi)P_0 = 1$ by Lemma 3.3.

Let H be a Hall p' -subgroup of $I_L(\varphi)$ (which is also a Hall p' -subgroup of L). Then $H \triangleleft I_L(\varphi)$ and $I_L(\varphi) = H \langle s \rangle$ for some element s of order p in $P \setminus P_0$.

We continue our discussion by assuming that there exists an element s of order p in $N_P(H) \setminus P_0$ and shall prove that the condition (C)(1) in Theorem 1 holds.

Lemma 3.4. *If W is a subgroup of V with $|V:W| = q$ and $[s, V] \subseteq W$, then there exists a Hall p' -subgroup H_1 of L with $[H_1, V] \subseteq W$.*

Proof. Let φ be an irreducible character of V with kernel W . Then since $[s, V] \subseteq W$, $s \in I_G(\varphi)$. Now the result follows by Lemma 3.3.

Lemma 3.5. *Assume that $V = W \times W^s \times \cdots \times W^{s^{p-1}}$, where W is P_0H -invariant. Then the following (1) and (2) hold.*

(1) *If W_1 is a subgroup of W with $|W:W_1| = q$, then there exists a Hall p' -subgroup H_1 of L with $[H_1, W] \subseteq W_1$. In particular, W_1 is H_1 -invariant.*

(2) *Let W_0 be an irreducible P_0 -module of W . Then W_0 is H -invariant.*

Proof. (1) We can easily see that $V = W \times [V, s]$ as V is an abelian p' -group. Put $U = W_1 \times [V, s]$. By Lemma 3.4, there exists a Hall p' -subgroup H_1 of L with $[H_1, V] \subseteq U$. Since $H_1 \subseteq P_0 H$, W is H_1 -invariant, and so $[H_1, W] \subseteq W \cap U = W_1$. In particular, W_1 is H_1 -invariant.

(2) If W_0 is not H -invariant, there exists an $h \in H$ with $W_0 \neq W_0^h$. Let $w \in W_0^*$. Since W is elementary abelian, there exist subgroups W' and W'' of W with $W_0 = \langle w \rangle \times W'$ and $W = W_0 \times W_0^h \times W''$. Set $W_1 = W' \times W_0^h \times W''$, then $|W:W_1| = q$ and $W_0 \not\subseteq W_1$. By (1), W_1 is H_1 -invariant for some Hall p' -subgroup H_1 of L . Since $h \in H \subseteq P_0 H_1$, $h = ah_1$ for some $a \in P_0$ and $h_1 \in H_1$. Furthermore, since $W_1 \supseteq W_0^h = W_0^{h_1}$, $W_0 = (W_0^{h_1})^{h_1^{-1}} \subseteq W_1^{h_1^{-1}} = W_1$, contrary to our choice of W_1 .

Since L acts on V by conjugation, we can regard V as an L -module. Furthermore, since V is a minimal normal subgroup of G , V is an irreducible L -module.

Lemma 3.6. *Let $V = V_1 \times \cdots \times V_n$, where V_i , $1 \leq i \leq n$, are the homogeneous components of V with respect to P_0 . Then $n = p$ and we may take $V_i^{s^i} = V_{i+1}$, $0 \leq i \leq p-1$. Furthermore, V_i is an irreducible P_0 -module which is H -invariant, $1 \leq i \leq p$.*

Proof. We divide the proof of Lemma 3.6 into three steps.

STEP 1. s induces a regular permutation representation on the set $\{V_1, \dots, V_n\}$.

Proof. Suppose it is false, and let s fix V_1 . Since P_0 is abelian, $\bar{P}_0 = P_0/C_{P_0}(V_1)$ is cyclic. Since $\langle s \rangle \bar{P}_0 = \langle s \rangle P_0/C_{P_0}(V_1)$ has an abelian subgroup of (p, p) type, there exists an $x \in \langle s \rangle P_0$ with $C_{V_1}(x) \neq 1$ and $\bar{x} \neq 1$. If $x \in P_0$, then $C_{V_1}(x) = 1$ since V_1 is homogeneous as a P_0 -module. Hence $x \notin P_0$. Let $v \in C_{V_1}(x)^*$, then $x \in C_P(v)$. Now by Lemma 3.3 and the remark before Lemma 3.4, $C_{P_0}(v) = 1$. So $C_{P_0}(V_1) \subseteq C_{P_0}(v) = 1$ and P_0 is cyclic. Since $L \triangleright P_0$ and P_0 is abelian, $\bar{L} = L/P_0$ acts on P_0 by conjugation. Since $O_{p'}(\bar{L}) = \bar{L}$ and $\text{Aut}(P_0)$ is abelian, $O_{p'}(\bar{L})$ centralizes P_0 . Hence $L = O_{p', p}(L)$, and so $G = O_{p', p}(G)$, contrary to the condition (1) of Theorem 1. Hence s acts regularly on the set $\{V_1, \dots, V_n\}$.

STEP 2. Let $\{V_1, \dots, V_t\}$ be an H -orbit of $\{V_1, \dots, V_n\}$, and set $W = V_1 \times \cdots \times V_t$. Then $V = W \times W^s \times \cdots \times W^{s^{p-1}}$.

Proof. Suppose s fixes an H -orbit $\{V_1, \dots, V_t\}$. Since H is a p' -group, $(p, t) = 1$. Therefore s fixes V_i for some i , contrary to Step 1. Hence s doesn't fix an H -orbit $\{V_1, \dots, V_t\}$ and this implies that $W \times W^s \times \dots \times W^{s^{p-1}} \subseteq V$. Furthermore, as $W \times W^s \times \dots \times W^{s^{p-1}}$ is L -invariant, we have $V = W \times W^s \times \dots \times W^{s^{p-1}}$.

STEP 3. Proof of Lemma 3.6.

Take an irreducible P_0 -submodule W_0 in V_1 . Then by Lemma 3.5(2), W_0 is $P_0 H$ -invariant. Therefore $W_0 \times W_0^s \times \dots \times W_0^{s^{p-1}}$ is L -invariant and coincides with V . Thus $W_0 = V_1$ (and $t = 1$ in Step 2) and the lemma follows.

From Theorem 15.16 of [3], we obtain the following lemma.

Lemma 3.7. *Let $P_0 \supseteq Q_1 \supseteq Q_2 \supseteq \Phi(P_0)$, where Q_1 and Q_2 are $H \langle s \rangle$ -invariant subgroups of P_0 . If H acts non-trivially on Q_1/Q_2 , then $|Q_1/Q_2| \geq p^p$.*

Lemma 3.8. *Let ϕ be a homomorphism of P_0 into $P_0/C_{P_0}(V_1) \times \dots \times P_0/C_{P_0}(V_p)$ which is defined by the rule $\phi(x) = (C_{P_0}(V_1)x, \dots, C_{P_0}(V_p)x)$. Then ϕ is an isomorphism.*

Proof. $\text{Ker } \phi = C_{P_0}(V) = 1$ and P_0 is isomorphic to a subgroup of $P_0/C_{P_0}(V_1) \times \dots \times P_0/C_{P_0}(V_p)$. On the other hand, $P_0/C_{P_0}(V_i)$ is cyclic, $1 \leq i \leq p$. Hence the rank of P_0 is at most p and $|P_0/\Phi(P_0)| \leq p^p$, and so $|P_0/\Phi(P_0)| = p^p$ by Lemma 3.7.

Suppose next that ϕ is not an epimorphism. Let P_0 have exponent p^m . Then $p^m = |P_0/C_{P_0}(V_1)|$ as $P_0/C_{P_0}(V_1)$ is cyclic. Set $\Omega_{m-1}(P_0) = \{x \in P_0 \mid x^{p^{m-1}} = 1\}$. Then $\Omega_{m-1}(P_0)$ is $\langle s \rangle H$ -invariant and $P_0 \not\supseteq \Omega_{m-1}(P_0) \not\supseteq \Phi(P_0)$. By Lemma 3.7, H acts trivially on both $P_0/\Omega_{m-1}(P_0)$ and $\Omega_{m-1}(P_0)/\Phi(P_0)$ which is a contradiction.

Set $P_i = C_P(V_1 \times \dots \times V_{i-1} \times V_{i+1} \times \dots \times V_p)$, $1 \leq i \leq p$. Then $P_0 = P_1 \times \dots \times P_p$ by Lemma 3.8, and so $VP = (V_1 P_1) \times \dots \times (V_p P_p)$. P_i is cyclic as it acts on V_i irreducibly and faithfully.

Lemma 3.9. *$C_{P_1 H}(v)$ contains a Hall p' -subgroup of $P_1 H$ for any $v \in V_1$.*

Proof. Set $K = P_1 H$. Then K acts on $\text{Irr}(V_1)$ and on the set of elements of V_1 . We claim that P_1 -orbits coincide with K -orbits on

V_1 . Let A_1, \dots, A_m be P_1 -orbits on $\text{Irr}(V_1)$ and let B_1, \dots, B_n be P_1 -orbits on the set of elements of V_1 . By Corollary 6.3.3 of [3], we have $m=n$.

Let χ be an irreducible character in A_1 , and set $W_1 = \text{Ker} \chi$. Now $V = V_1 \times V_1^s \times \dots \times V_1^{s^{p-1}}$, and V_1 is $P_0 H$ -invariant. Hence, by lemma 3.5 (1), there exists a Hall p' -subgroup H_1 of L with $[H_1, V_1] \subseteq W_1$. As $P_0 = P_1 C_{P_0}(V_1)$ by the remark before Lemma 3.9, we may take H_1 in K . Since $K = P_1 H = P_1 H_1$ and H_1 fixes χ , A_1 coincides with a K -orbit on $\text{Irr}(V_1)$. Similarly, A_i is a K -orbit on $\text{Irr}(V_1)$, $2 \leq i \leq m$. By Corollary 6.3.3 of [3] again, B_i is a K -orbit on the set of elements of V_1 , $1 \leq i \leq m$, and our claim follows.

Let v be an element of V_1 . For each element $g \in K$, we have $v^g = v^x$ for some $x \in P_1$ by the above claim. Hence $gx^{-1} \in C_K(v)$, and so $g \in C_K(v)P_1$. This implies that $K = C_K(v)P_1$. Hence $C_K(v)$ contains a Hall p' -subgroup of K .

Lemma 3.10. Set $r = |H/C_H(V_1)|$ and $p^m = |P_1|$, $q^n = |V_1|$. Then $r|n$ and $\frac{q^n - 1}{q^{n/r} - 1} = p^m$ and $V_i P_i H / C_H(V_i) \simeq A_{q,n,r}$, $1 \leq i \leq p$.

Proof. Set $N = V_1 P_1 H$ and $\bar{N} = V_1 P_1 H / C_H(V_1)$. By Lemma 3.9, $C_{\bar{N}}(\bar{v})$ contains a Hall p' -subgroup of \bar{N} for any $v \in V_1$. Hence every \bar{P}_1 -orbit of V_1 contains an element which is centralized by \bar{H} . Since P_1 is cyclic, by Proposition 19.8 of [11] and Lemma 3.6, we can identify V_1 with $GF(q^n)$ in such a way that $\bar{P}_1 \bar{H} \subseteq T(q^n) = \{v \rightarrow av^a | a \in GF(q^n)^*, \sigma \in \text{Gal}(GF(q^n)/GF(q))\}$ and $\bar{P}_1 \subseteq \{v \rightarrow av | a \in GF(q^n)^*\}$.

As $C_{\bar{H}}(\bar{P}_1)$ is contained in any Hall p' -subgroup of $\bar{P}_1 \bar{H}$, $C_{\bar{H}}(\bar{P}_1) = 1$. Since \bar{P}_1 is cyclic and \bar{H} is a p' -group, \bar{H} acts regularly on \bar{P}_1 .

$$\begin{array}{ccc} \text{Gal}(GF(q^n)/GF(q)) & & GF(q) \\ \left\langle \sigma \right\rangle \downarrow & \text{---} & \downarrow GF(q^l) \\ 1 & & GF(q^n) \end{array}$$

Let $\bar{H} = \langle \eta \rangle$, where $\eta(v) = bv^\sigma$, for some $b \in GF(q^n)^*$ and $\sigma \in \text{Gal}(GF(q^n)/GF(q))$ for $v \in V_1$. By Proposition 19.8 of [11], $\bar{H} \simeq \bar{P}_1 \bar{H} / C_{\bar{P}_1 \bar{H}}(\bar{P}_1) \subseteq \text{Gal}(GF(q^n)/GF(q))$, and so $r = |\bar{H}| = |\sigma|$ with $r|n$. Then $\langle \sigma \rangle = \text{Gal}(GF(q^n)/GF(q^l))$, where $n = lr$. Let $\bar{P}_1 = \langle x_a \rangle$, where $x_a(v) = av$, $a \in GF(q^n)^*$. If $a^i \in GF(q^l)$ for some i , then $\eta x_{a^i}(v) = \eta(a^i v) = b(a^i v)^\sigma = (a^i)^\sigma b v^\sigma = a^i b v^\sigma$. On the other hand, $x_{a^i} \eta(v) = x_{a^i}(bv^\sigma) = a^i b v^\sigma$. Hence $\eta x_{a^i} = x_{a^i} \eta$, and so $\eta^{-1} x_{a^i} \eta = x_{a^i}$. Since $x_{a^i} = (x_a)^i$, $\eta^{-1} (x_a)^i \eta = (x_a)^i$. Thus $a^i = 1$ as \bar{H} acts regularly on \bar{P}_1 . Hence $\langle a \rangle \cap GF(q^l)^* = 1$.

Since $\langle a \rangle \subseteq GF(q^n)$ is a P_1 -orbit, $\eta(a^i) = a^i$ for some i . Hence $\eta(a^i) = b(a^i)^\sigma = a^i$, and so $b = (a^i)(a^{-i})^\sigma \in \langle a \rangle$. Thus $b = a^j$ for some j , and so $\eta = x_a \sigma$. Then $\bar{P}_1 \bar{H} = \langle x_a \rangle \langle x_a \sigma \rangle = \langle x_a \rangle \langle \sigma \rangle$.

Let c be any element of $GF(q^n)^*$. Since $\{c, ca, \dots, ca^{p^m-1}\}$ is a \bar{P}_1 -orbit, $(ca^i)^\sigma = ca^i$ for some i . Hence $ca^i \in GF(q^l)$. This implies that $GF(q^n)^* = \langle a \rangle GF(q^l)^*$. Hence $q^n - 1 = (q^l - 1)p^m = (q^{n/r} - 1)p^m$. Furthermore, we have $\overline{V_1 P_1 H} = V_1 \langle x_a \rangle \langle \sigma \rangle \simeq A_{q,n,r}$.

Now we completed the proof of the if-part of Theorem 1 and we shall prove the "only-if-part" of the theorem.

Lemma 3.11. *Let G satisfy the conditions (1)–(3) in Theorem 1. If G has the conditions (A), (B) and (C) of Theorem 1, then G is p -radical.*

Proof. Let $\varphi \in \text{Irr}(V)$ and let B be a block of kG which covers φ . Then we shall prove that B is p -radical.

CASE 1. $\varphi = 1_V$.

Since L/P_0 is a Frobenius group, L is p -radical by Lemmas 2.1, 2.2. By lemma 2.5, B is p -radical.

CASE 2. $\varphi \neq 1_V$.

Set $T = I_G(\varphi)$. Suppose that G satisfies the condition (C)(2) in Theorem 1. By the remark before Lemma 3.4, $T \subseteq O_{p', p, p'}(G)$. Set $N = VP_0$, then $D \subseteq N$, where D is a defect group of B . Then $J(B) = J(kN)B$ by Theorem 2.3 of [1]. By Lemma 2.2, N is p -radical, and so $(k \otimes_p B)J(B) = k \otimes_p J(kN)B \subseteq k \otimes_p J(kP_0)(kN)B = 0$. Hence $k \otimes_p B$ is semisimple, and so B is p -radical.

Suppose next that G satisfies the condition (C)(1) in Theorem 1. Let v be an element of V . Then $v = v_1 \cdots v_p$, where $v_i \in V_i$, $1 \leq i \leq p$. Then, by (C)(2)(vii) in Theorem 1, there exist $x_i \in P_i$ with $[v_i^{x_i}, H] = 1$, $1 \leq i \leq p$, where H is a Hall p' -subgroup of G . Set $x = x_1 \cdots x_p$. Then $v^x = v_1^{x_1} \cdots v_p^{x_p} = v_1^{x_1} \cdots v_p^{x_p}$, and so $[v^x, H] = 1$. This implies that $C_G(v)$ contains a Hall p' -subgroup of G for each $v \in V$. By a similar argument in the proof of Lemma 3.9, T contains a Hall p' -subgroup.

Since $V \subseteq T$ and $G = VL$, we have $T = V(L \cap T)$. $T \cap L$ is p -closed or $T \cap L/O_p(T \cap L)$ is a Frobenius group. In each case, $T \cap L$ is p -radical by Lemmas 2.1, 2.2. Let b be the Fong correspondent of B w.r.t. (V, T) . Then b is p -radical by Lemma 2.5. Hence B is p -radical by Lemma 2.4(2).

4. Proof of Theorem 2

In this section, we shall prove the following theorem from which Theorem 2 follows by using Lemma 2.6.

Theorem 3. *If G is a p -radical group with $G/O_p(G) \in \mathcal{F}$, then $G = O_{p,p',p,p'}(G)$.*

Proof. Suppose it is false and let G be a minimal counterexample of Theorem 3. Now we divide the proof into several steps. At first, we shall prove that G satisfies the conditions (1)–(3) in Theorem 1.

STEP 1. $O_p(G) = 1$ and $O^{p'}(G) = G$.

Proof. Suppose that $O_p(G) \neq 1$. Set $\bar{G} = G/O_p(G)$. Then \bar{G} is p -radical. Furthermore, since $\bar{G}/O_{p'}(\bar{G})$ is a homomorphic image of $G/O_p(G)$, $\bar{G}/O_{p'}(\bar{G}) \in \mathcal{F}$ by Lemma 2.6. By the minimality of G , $\bar{G} = O_{p,p',p,p'}(\bar{G}) = O_{p',p,p'}(\bar{G})$, and so $G = O_{p,p',p,p'}(G)$, contrary to our choice of G . Next we assume that $O^{p'}(G) \subsetneq G$. Set $U = O^{p'}(G)$, then since U is p -radical and $U/O_p(U) \in \mathcal{F}$, $U = O_{p,p',p,p'}(U) = O_{p,p',p}(U)$. Hence $G = O_{p,p',p,p'}(G)$, a contradiction.

Let V be a minimal normal subgroup of G . By Step 1 and Lemma 2.3, V is an abelian p' -group. Furthermore, let P_0 be a p -subgroup of G with $O_p(G/V) = P_0V/V$. Then we show the following Step 2.

STEP 2.

- (1) $G/V = O_{p,p',p}(G/V)$, in particular, $G = O_{p',p,p,p'}(G)$.
- (2) $P_0 \neq 1$ is elementary abelian.
- (3) Let M be a Hall p' -subgroup of G . Then there exists an $s \in N_G(M)$ with $|s| = p$ and $G = O_{p',p,p'}(G) \langle s \rangle$.
- (4) $V = [O_{p'}(G), P_0]$.

Proof. Set $\bar{G} = G/V$. For (1), we note that \bar{G} is p -radical and $\bar{G} \in \mathcal{F}$. Since $V \neq 1$, $\bar{G} = O_{p,p',p,p'}(\bar{G}) = O_{p,p',p}(\bar{G})$, so that $G = O_{p',p,p,p'}(G)$.

For (2) and (3), let P be a Sylow p -subgroup of G such that $N_P(M)$ is a Sylow p -subgroup of $N_G(M)$. If $P_0 = 1$, then $\bar{G} = O_{p',p}(\bar{G})$, and so $G = O_{p',p}(G)$, contrary to our choice of G . Hence $P_0 \neq 1$. By Theorem 3.1 of [14], P_0 is elementary abelian. Hence $\bar{P}_0 = [\bar{P}_0, \bar{M}] \times C_{\bar{P}_0}(\bar{M})$. Since $\bar{G} = \bar{P}_0 N_{\bar{G}}(\bar{M}) = \bar{P}_0 N_{\bar{P}}(\bar{M}) \bar{M}$, $\bar{G} = [\bar{P}_0, \bar{M}] \bar{M} N_{\bar{P}}(\bar{M})$. Furthermore, $[\bar{P}_0, \bar{M}] \bar{M} \triangleleft \bar{G}$ and $[\bar{P}_0, \bar{M}] \bar{M} \cap N_{\bar{P}}(\bar{M}) = [\bar{P}_0, \bar{M}] \cap N_{\bar{P}}(\bar{M}) = [\bar{P}_0, \bar{M}] \cap C_{\bar{P}_0}(\bar{M}) = 1$. Therefore $N_{\bar{P}}(\bar{M})$ is a complement of $[\bar{P}_0, \bar{M}] \bar{M}$ in \bar{G} . By Theorem

3.1 of [14], $N_{\bar{P}}(\bar{M})$ is elementary abelian. If $N_{\bar{P}}(\bar{M}) \subseteq \bar{P}_0$, then $\bar{G} = \bar{P}_0 \bar{M}$. Hence $G = O_{p', p, p'}(G)$, contrary to our choice of G . Thus $N_{\bar{P}}(\bar{M}) \not\subseteq \bar{P}_0$. Therefore there exists an element $s \in N_p(M)$ with $|s| = p$ and $s \notin P_0$.

Now set $N = O_{p', p, p'}(G) \langle s \rangle$. Since $N \triangleleft G$, N is p -radical by Lemma 2.1 (1). Furthermore, $\bar{N} = N/O_{p'}(G) = N/O_{p'}(N) \in \mathcal{F}$ by Lemma 2.7. If $N \subsetneq G$, then $N = O_{p, p', p, p'}(N)$. Since $N \triangleleft G$, $O_p(N) \subseteq O_p(G) = 1$ and $O_{p', p}(N) \subseteq O_{p', p}(G)$. Hence $N = O_{p', p, p'}(N)$, and so $s \in O_{p', p}(N) \subseteq O_{p', p}(G)$, contrary to our choice of s . Thus $G = O_{p', p, p'}(G) \langle s \rangle$.

For (4), we note $[O_{p'}(G), P_0] \subseteq O_{p'}(G) \cap P_0 V = V$.

STEP 3. Proof of Theorem 3.

By Step 2, there exists an s with $|s| = p$ and $G = O_{p', p, p'}(G) \langle s \rangle$. By the remark before Lemma 3.4, the condition (C)(1) in Theorem 1 holds. We set $\bar{G} = G/V$, then $\bar{G} \in \mathcal{F}$ by Lemma 2.6. Since \bar{P}_1 is \bar{H} -invariant, \bar{P}_1 is \bar{s} -invariant (see the proof of Lemma 11(7) of [7]), and we have a contradiction.

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