1. Introduction. Asymptotic sufficiency of maximum likelihood (m.l.) estimator in regular cases has been studied by many authors (see Wald [17], LeCam [2], Pfanzagl [12], Michel [8], Suzuki [14], [15], and so on).

In [6], Matsuda showed that for \( k \in \mathbb{N} = \{1, 2, \ldots \} \) a statistic \( T_{nk} = (T_n, G_2^1(z_n, T_n), \ldots, G_k^k(z_n, T_n)) \) is asymptotically sufficient up to order \( O(n^{-k/2}) \). Here \( \{T_n\} \) is a sequence of asymptotic m.l. estimators and \( G_k^m(z_n, \theta) \) denotes the \( m \)-th derivative relative to \( \theta \) of the log-likelihood function. In the case \( k = 1 \), \( T_{n,1} \) means \( T_n \).

The purpose of this paper is to investigate asymptotic sufficiency of a statistic constructed by m.l. estimators in the following cases. Let \( x_1, \ldots, x_n \) be independent and identically distributed random variables with common density \( p(x - \theta), -\infty < x, \theta < \infty \), where \( \theta \) is an unknown translation parameter and \( p(x) \) is uniformly continuous and positive only on the interval \((0, \infty)\). We shall consider here two cases.

Case (i): \( p(x) \sim ax \) as \( x \to +0 \), where \( a > 0 \).

Case (ii): \( p(x) \sim ax^{1+\beta} \) as \( x \to +0 \), where \( a, \beta > 0 \).

It is assumed that in Case (i) Fisher's information number is infinite. Let \( \hat{\theta}_n \) denote m.l. estimator of \( \theta \) for the sample size \( n \). In this case, Takeuchi [16] and Woodroofe [20] proved the asymptotic normality of \( \sqrt{n} (\hat{\theta}_n - \theta) \) and the speed of convergence to the standard normal distribution was given by Matsuda [4]. Moreover, it was shown by Takeuchi [16] and Weiss and Wolfowitz [19] that \( \hat{\theta}_n \) is an asymptotically efficient estimator of \( \theta \).

In Case (ii), it is well known that if Fisher's information number \( J \) is finite, then the distribution of \( \sqrt{n} (\hat{\theta}_n - \theta) \) converges weakly to the standard normal distribution. The order of convergence to normality is \( o(n^{-\nu/2}) \) for every \( \nu < \beta \) if \( \beta \leq 1 \) and \( O(n^{-1/2}) \) if \( \beta > 1 \) (see Matsuda [3] and cf. also Pfanzagl [11]).

In both cases, Mita [9] showed that m.l. estimator is asymptotically sufficient up to order \( o(1) \). For \( n, k \in \mathbb{N} \) define \( \hat{\theta}_{nk} = (\hat{\theta}_n, G_2^1(z_n, \hat{\theta}_n), \ldots, G_k^k(z_n, \hat{\theta}_n)) \), where \( \hat{\theta}_{n,1} \) means \( \hat{\theta}_n \). We shall show that in Case (i) the statistic \( \hat{\theta}_{nk} \) is asymptotically sufficient up to order \( o((\log n)^{-\nu}) \) for every \( \nu < (k+1)/(k+3) \) and that in Case (ii) \( \hat{\theta}_{nk} \) is asymptotically sufficient up to order \( o(n^{-\nu}) \) for every
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\[ \nu < \frac{\beta(k+1)}{2(k+\beta+3)} \text{ if } \beta \leq k(k+3) \text{ and is asymptotically sufficient up to order } O(n^{-k/2}) \]
\[ \text{if } \beta > k(k+3). \]

In Section 2 we introduce the results of von Bahr and Esseen [1] and Nagaev [10] which are useful to estimate probabilities of deviations for sums of independent and identically distributed random variables with a restricted moment. Section 3 is devoted to the problem of asymptotic sufficiency in Case (i) and Section 4 to the one in Case (ii).

2. Probabilities of deviations. Let \( Y_1, \ldots, Y_n \) be a sequence of random variables (r.v.'s) and put \( S_m = \sum_{i=1}^{m} Y_i \), \( 1 \leq m \leq n \). Using the elementary inequality
\[ E|S_m|' \leq \sum_{i=1}^{m} E|Y_i|', \quad r \leq 1, \]
it follows from Markov's inequality that for \( x > 0 \)
\[ P\{|S_m| \geq x\} \leq x^{-r} \sum_{i=1}^{m} E|Y_i|', \quad r \leq 1. \]

If the r.v.'s satisfy the relations
\[ E(Y_{m+1}|S_m) = 0 \text{ a.s.} \quad 1 \leq m \leq n-1, \]
than von Bahr and Esseen [1] showed that
\[ E|S_m|' \leq 2 \sum_{i=1}^{m} E|Y_i|', \quad 1 \leq r \leq 2. \]
The condition (2.2) is fulfilled if the r.v.'s are independent and have zero means. In this case, (2.3) together with Markov's inequality implies the following inequality
\[ P\{|S_m| \geq x\} \leq 2x^{-r} \sum_{i=1}^{m} E|Y_i|', \quad 1 \leq r \leq 2, \]
for \( x > 0 \).

Let \( Y_1, \ldots, Y_n \) be a sequence of identically distributed independent r.v.'s and \( E(Y_1) = 0, E(Y_i^2) = 1 \). In [10], Nagaev proved the following theorem (cf. Lemma in Michel [8]).

Theorem 2.1. If \( \xi_r = E|Y_i|' < \infty, r > 2, \) then
\[ P\{|S_m| > x\} < c \xi_r n x^{-r} \]
for \( x > 4 \sqrt{n \max \left[ \log n^{r-2} \frac{1}{K_r \xi_r}, 0 \right]} \),
where \( K_r = 1 + (r+1)^{r+2} \exp(-r) \) and \( c \) is an absolute constant depending only on \( r \).

Remark 1. It is obvious that Theorem 2.1 remains valid even if \( E(Y_i^2) = 0 \).
As a consequence of Theorem 2.1 we obtain a result on probabilities of moderate deviations: If $\xi_r < \infty$, $r > 2$, then there is a positive constant $c$ such that

$$P\{|S_n| > c\sqrt{n \log n} \} = o(n^{-(r-2)/2}).$$

It is remarked that Theorem 4 in Michel [7] implies the above result under the same condition and Lemma 1 in Pfanzagl [13] gives a uniform version of this result when $r \geq 3$.

3. Asymptotic sufficiency: Case (i). For $\theta \in \mathcal{R}$, let $P_{\theta}$ be a probability measure on the Borel real line $(\mathcal{R}, \mathcal{B})$. It is assumed that every $P_{\theta}$ is absolutely continuous with respect to the Lebesgue measure $\mu$ on $\mathcal{R}$ and $dP_{\theta}/d\mu = p(x - \theta)$. For each $n \in \mathcal{N} = \{1, 2, \ldots\}$, let $(\mathcal{R}^n, \mathcal{B}^n)$ be the Cartesian product of $n$ copies of $(\mathcal{R}, \mathcal{B})$ and $P_{n, \theta}$ be the product measure of $n$ copies of $P_{\theta}$. Furthermore, let $\mu_n$ denote the product measure of $n$ copies of $\mu$ and set $p_n(x_n, \theta) = dP_{n, \theta}/d\mu_n$ for $\theta \in \mathcal{R}$ and $x_n = (x_1, \ldots, x_n) \in \mathcal{R}^n$.

Let $k$ be a positive integer equal to or greater than 2. We shall impose the following Condition $A_k$ on $p(x)$.

Condition $A_k$

(i) $p(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0)$ and is positive on $(0, \infty)$.

(ii) $p(x)$ is $(k + 1)$-times continuously differentiable on $(0, \infty)$.

Let $g(x) = \log p(x)$ for $x > 0$ and $g^{(m)}(x)$ be the $m$-th derivative of $g(x)$.

(iii) For some $\alpha \in (0, \infty)$ and $\gamma \in (0, \infty)$

$$p(x) = ax^\alpha + O(x^{\gamma-1}), \quad g^{(1)}(x) = -x^{-1} + O(x^{\gamma-1}), \quad g^{(2)}(x) = -x^{-2} + O(x^{\gamma-2}),$$

$$g^{(3)}(x) = O(x^{-3}) \quad \text{and} \quad g^{(k+1)}(x) = O(x^{-k-1}) \quad \text{as} \quad x \to +0.$$

(iv) For every $t \geq 0$, there exists $\eta > 0$ such that

$$\int_0^\infty |g(x+t)|^{1+\eta} p(x) \, d\mu < \infty.$$

(v) For some $M > 0$

$$\int_M^\infty |g^{(3)}(x)|^2 p(x) \, d\mu < \infty.$$

(vi) For every $a > 0$, there exist $\delta > 0$ and $\eta > 0$ such that

(a) $$\int_0^\infty \sup_{x \in [|t| + \delta]} |g^{(2)}(x+u)|^{1+\eta} p(x) \, d\mu < \infty,$$

(b) $$\int_0^\infty \sup_{x \in [|t| + \delta]} |g^{(3)}(x+u)| \, p(x) \, d\mu < \infty,$$

(c) $$\int_0^\infty \sup_{x \in [|t| + \delta]} |g^{(k+1)}(x+u)| \, p(x) \, d\mu < \infty.$$
Let $M_n = \min(x_1, \ldots, x_n)$ and $G_n(x_n, t) = \sum_{i=1}^{\Lambda} g(x_i - t)$ for $t < M_n$. Condition (i) insures that m.l. estimators of $\theta$ for the sample size $n$ exist in the interval $(-\infty, M_n)$. Let $\{\hat{\theta}_n; n \in \mathbb{N}\}$ be a sequence of m.l. estimators. Woodroofe [20] remarked that condition (i) and

$$
\int_{-\infty}^{\Lambda} -g(x) p(x) \, d\mu < \infty
$$

imply all assumptions of Wald [18] and that, moreover, if $g(x)$ is continuously differentiable, then $\{\hat{\theta}_n\}$ will form a consistent sequence of roots of the likelihood equation

$$
\hat{\theta}_n < M_n \quad \text{and} \quad G_n^{(1)}(x_n, \hat{\theta}_n) = 0.
$$

We shall use $a_n = \sqrt{\frac{1}{2} \alpha n (\log n + \log \log n)}$ rather than $\sqrt{\frac{1}{2} \alpha n \log n}$ as the convergence order of m.l. estimator to the true parameter $\theta$ (see [4] and [5]).

Since $\theta$ is a translation parameter, we restrict our attention to the case that $\theta = 0$. The following lemma is the same as Lemma 5 in [5] except that conditions (v) (b) and (v) (c) in [5] can be replaced by weaker conditions (vi) (a) and (vi) (b) because of (2.4).

**Lemma 3.1.** Let conditions (i)–(iii), (vi) (a) and (vi) (b) be satisfied for $k=2$. Then for every $s \in (0, 1)$ there exists $c > 0$ such that

$$
P_n \left\{ \sup_{t \leq \Lambda} |a_n^{-2}G_n^{(2)}(x_n, t) + 1| \geq c(\log n)^{-1} \right\} = O((\log n)^{-1}),
$$

where $b_n(s) = a_n^{-1}(\log n)^{s/2}$.

**Remark 2.** It seems to be impossible to improve Lemma 3.1 (see Remark in [5]).

Using (2.4) instead of Chebyshev's inequality in the proofs of Lemma 1 and Lemma 2 of [4], we obtain the following Lemma 3.2 and Lemma 3.3, respectively.

**Lemma 3.2.** Let conditions (i)–(iii) and (vi) (a) be satisfied for $k=2$. Then for sufficiently small $\varepsilon > 0$, there are events $D_n$, $n \in \mathbb{N}$ for which $P_{n,0}\{(D_n)\} = o((\log n)^{-1})$ and $z_n \in D_n$ implies

$$
\sup_{-\Lambda < t < \Lambda} n^{-1}G_n^{(2)}(x_n, t) < -1.
$$

**Lemma 3.3.** Let conditions (i)–(iii) and (iv) be satisfied for $k=2$. Then for every $\varepsilon > 0$

$$
P_{n,0}\{|\hat{\theta}_n| \geq \varepsilon\} = o((\log n)^{-1}).
$$

**Lemma 3.4.** Let Condition $A_2$ be satisfied. Then for every $s \in (0, 1)$
\[P_{n,0}\{a_n | \hat{\theta}_n| \geq \log \log n\} = O((\log n)^{-1}).\]

Proof. We shall use ideas related to Woodroofe [20]. It follows from Lemma 3.2 and Lemma 3.3 that

\begin{align*}
(3.1) \quad & P_{n,0}\{a_n \hat{\theta}_n \leq -\log \log n\} = P_{n,0}\left\{\sum_{i=1}^{n} g^{(1)}(x_i + a_n^{-1}\log \log n) \geq 0\right\} + o((\log n)^{-1}), \\
(3.2) \quad & P_{n,0}\{a_n \hat{\theta}_n \geq \log \log n\} = P_{n,0}\left\{\sum_{i=1}^{n} g^{(1)}(x_i - a_n^{-1}\log \log n) \leq 0\right\},
\end{align*}

Using Lemma 3.1 and the equality

\[\sum_{i=1}^{n} g^{(1)}(x_i + a_n^{-1}\log \log n) = \sum_{i=1}^{n} g^{(1)}(x_i) + a_n^{-1}\log \log n \sum_{i=1}^{n} g^{(2)}(x_i + u_i)\]

with \(u_i \in (0, a_n^{-1}\log \log n)\), (3.1) implies that

\[P_{n,0}\{a_n \hat{\theta}_n \leq -\log \log n\} \leq P_{n,0}\left\{a_n^{-1}\sum_{i=1}^{n} g^{(1)}(x_i) \geq \frac{1}{2} \log \log n\right\} + O((\log n)^{-1}).\]

A similar argument shows that (3.2) implies

\[P_{n,0}\{a_n \hat{\theta}_n \geq \log \log n\} \leq P_{n,0}\left\{a_n^{-1}\sum_{i=1}^{n} g^{(1)}(x_i) \leq -\frac{1}{2} \log \log n\right\} + O((\log n)^{-1}).\]

Lemma 3 in [4], together with the fact

\[\Phi\left(-\frac{1}{2} \log \log n\right) = o((\log n)^{-1}),\]

leads to the desired result. Here \(\Phi(x)\) denotes the cumulative distribution function of the standard normal distribution.

Lemma 3.1 and Lemma 3.4 yield the following lemma.

**Lemma 3.5** (cf. Lemma 6 in [5]). Let Condition \(A_2\) be satisfied. Then for every \(s \in (0, 1)\) there exists \(c > 0\) such that

\[P_{n,0}\left\{\sup_{|t| \leq 2b_n(s)} |a_n^{-2}G_n^{(2)}(x_n, \hat{\theta}_n + t) + 1| \geq c(\log n)^{-1}\right\} = O((\log n)^{-1}).\]

We shall investigate an asymptotic behavior of \(a_n^{k-1}G_n^{(k+1)}(x_n, t), |t - \hat{\theta}_n| \leq b_n(s)\), with \(k \geq 2\).

**Lemma 3.6.** Let conditions (i)–(iii) and (vi) (c) be satisfied for some \(k \geq 2\). Then for every \(s \in (0, 1)\) there exists \(c > 0\) such that

\[P_{n,0}\left\{\sup_{|t| \leq 2b_n(s)} |a_n^{k-1}G_n^{(k+1)}(x_n, t)| \geq c(\log n)^{-(k+1)}\right\} = O((\log n)^{-1}).\]

Proof. Since \(P_{n,0}\{M_n \leq 2b_n(s)\} = O((\log n)^{-1})\), we may assume that \(M_n > 2b_n(s)\). Then \(G_n^{(k+1)}(x_n, t) = (-1)^{k+1} \sum_{i=1}^{n} g^{(k+1)}(x_i - t)\) for \(|t| \leq 2b_n(s)\). Let \(a > 0\) be so small that \(p(x) \leq 2ax\) and \(|g^{(k+1)}(x)| \leq Lx^{-k-1}\) for \(0 < x < 2a\) where \(L\) is a positive constant, and choose \(\delta > 0\) to satisfy condition (vi) (c). Then we have
\( (3.3) \quad |a_n^{-k-1}G_n^{(k+1)}(x,n,t)| \leq L a_n^{-k-1} \sum_{i=1}^n (x_i - 2b_n(s))^{-k-1} + a_n^{-k-1} \sum_{i=1}^n \sup_{|u| \leq \delta} |g^{(k+1)}(x_i + u)| \)

for \(|t| \leq 2b_n(s)\) and all sufficiently large \(n\). Here \(\sum_n\) denotes the summation over \(i \leq n\) satisfying \(u \leq x_i < v\).

To evaluate the first term above define \(\{Y_{ni}; \ i = 1, \ldots, n\}\) by

\[
Y_{ni} = (x_i - 2b_n(s))^{-k-1}, \quad \text{if} \quad 3b_n(s) \leq x_i < a,
\]

\[
= 0, \quad \text{if} \quad x_i < 3b_n(s) \text{ or } a \leq x_i.
\]

Since \(E(Y_{ni}^2) = O(b_n(s)^{-2k})\), it follows from Chebyshev's inequality that

\[
P_{n,0}\{|a_n^{-k-1} \sum_{i=1}^n (Y_{ni} - E(Y_{ni}))| \geq \frac{1}{2} (\log n)^{-(k+1)/2} \} = O((\log n)^{-1}).
\]

Moreover, using \(a_n^{-k-1} \sum_{i=1}^n E(Y_{ni}) = o((\log n)^{(k+1)/2})\) we obtain

\[
P_{n,0}\{|a_n^{-k-1} \sum_{i=1}^n Y_{ni} | \geq (\log n)^{-(k+1)/2} \} = O((\log n)^{-1}).
\]

This, together with the fact \(P_{n,0}\{\sum_{i=1}^n Y_{ni} \neq 0\} = o((\log n)^{(k+1)/2})\) implies

\[
P_{n,0}\{|a_n^{-k-1} \sum_{i=1}^n (x_i - 2b_n(s))^{-k-1} \| \geq (\log n)^{-(k+1)/2} \} = O((\log n)^{-1}).
\]

It remains to estimate the second term on the righthand side of (3.3). It follows from Markov's inequality and condition (vi) (c) that

\[
P_{n,0}\{a_n^{-k-1} \sum_{i=1}^n |g^{(k+1)}(x_i + u)| \geq (\log n)^{-(k+1)/2} \} = o((\log n)^{-1}).
\]

This completes the proof.

**Remark 3.** It is easily seen that in the case \(k \geq 2\) the distribution of \(n^{-(k+1)/2} \sum_{i=1}^n g^{(k+1)}(x_i)\) converges weakly to a stable law with characteristic exponent \(2/(k+1)\). By the same reason as of Remark 2, we cannot expect to improve Lemma 3.6.

The following lemma immediately follows from Lemma 3.4 and Lemma 3.6.

**Lemma 3.7.** Let Condition \(A_k\) be satisfied for some \(k \geq 2\). Then for every \(s \in (0,1)\) there exists \(c > 0\) such that

\[
P_{n,0}\{\sup_{|i| \leq n, x(i)} |a_n^{-k-1}G_n^{(k+1)}(x_i, \hat{\theta}_n + t)| \geq c(\log n)^{-(k+1)/2} \} = O((\log n)^{-1}).
\]

At first we study asymptotic sufficiency of m.l. estimator \(\hat{\theta}_n\).
Theorem 3.1. If Condition $A_2$ holds, then there exists a sequence of families of probability measures $\{Q_{n,\theta}; \theta \in \mathbb{R}\}$, $n \in \mathbb{N}$, such that

(a) for each $n \in \mathbb{N}$, $\hat{\theta}_n$ is sufficient for $\{Q_{n,\theta}; \theta \in \mathbb{R}\}$
(b) for every $\nu < 1/2$

$$\sup_{\theta \in \mathbb{R}} ||P_{n,\theta} - Q_{n,\theta}|| = o((\log n)^{-\nu}).$$

Proof. Let $I_A(\cdot)$ be the indicator function of a set $A$. We define

$$q_n(z_n, \theta) = I_{V_n,\theta \cap W_n}(z_n) \exp \left\{ G_n(z_n, \hat{\theta}_n) - \frac{1}{2} a_n^2(\hat{\theta}_n - \theta)^2 \right\},$$

$$q_n(z_n, \theta) = c_n(\theta) q_n^*(z_n, \theta),$$

where

$$V_n,\theta = \{ z_n \in \mathbb{R}^s; a_n|\hat{\theta}_n - \theta| < \log \log n \},$$

$$W_n = \{ z_n \in \mathbb{R}^s; \sup_{|t| \leq \log^{1/2} n} |a_n^2 G_n(z_n, \hat{\theta}_n + t) + 1| < c(\log n)^{-1/2} \}$$

and $c_n(\theta) = \left[ \int_{\mathbb{R}^s} q_n^*(z_n, \theta) d\mu_n \right]^{-1}$. Here the constant $c$ in $W_n$ is determined by Lemma 3.5 with $s=1/2$.

For every $A \in \mathcal{B}$, let $Q_{n,\theta}(A) = \int_A q_n(z_n, \theta) d\mu_n$ and $Q_{n,\theta}^*(A) = \int_A q_n^*(z_n, \theta) d\mu_n$. According to the factorization theorem, for each $n \in \mathbb{N}$ $\hat{\theta}_n$ is sufficient for $\{Q_{n,\theta}; \theta \in \mathbb{R}\}$.

Next we have

$$||P_{n,\theta} - Q_{n,\theta}^*|| = \int_{\mathbb{R}^s} |p_n(z_n, \theta) - q_n^*(z_n, \theta)| d\mu_n \leq \int_{V_n,\theta \cap W_n} |1 - q_n^*(z_n, \theta)| p_n(z_n, \theta) d\mu_n + P_{n,\theta}(\{V_n,\theta\}) + P_{n,\theta}(\{W_n\}) \cdot (3.4)$$

It follows from Lemma 3.4 that $\sup_{\theta \in \mathbb{R}} P_{n,\theta}(\{V_n,\theta\}) = o((\log n)^{-\nu})$. And Lemma 3.5 implies that $\sup_{\theta \in \mathbb{R}} P_{n,\theta}(\{W_n\}) = O((\log n)^{-1/2})$. It remains to estimate the first term on the righthand side of (3.4). Since

$$G_n(z_n, \theta) = G_n(z_n, \hat{\theta}_n) + \frac{1}{2} (\theta - \hat{\theta}_n)^2 G_n^{(2)}(z_n, \theta^*_n)$$

with $|\theta^*_n - \hat{\theta}_n| \leq |\hat{\theta}_n - \theta|$, we have for $z_n \in V_n,\theta \cap W_n$

$$\left| \frac{1 - q_n^*(z_n, \theta)}{p_n(z_n, \theta)} \right| = |1 - \exp \left\{ -G_n(z_n, \theta) - G_n(z_n, \hat{\theta}_n) + \frac{1}{2} a_n^2(\hat{\theta}_n - \theta)^2 \right\}| \leq c(\log \log n)^2(\log n)^{-1/2} = o((\log n)^{-\nu}).$$
Here we used the inequality $|1 - \exp(x)| \leq 2|x|$ for sufficiently small $x$. Thus we obtain
\[
\sup_{\theta \in R} \|P_{\theta} - Q_{\theta}\| = o((\log n)^{-\gamma}).
\]
Since
\[
\sup_{\theta \in R} |1 - c_{n}(\theta)^{-1}| = \sup_{\theta \in R} |P_{\theta} - Q_{\theta}| - Q_{\theta} + Q_{\theta} - Q_{\theta} - Q_{\theta} - Q_{\theta} = o((\log n)^{-\gamma}),
\]
we have
\[
\sup_{\theta \in R} ||P_{\theta} - Q_{\theta}|| \leq \sup_{\theta \in R} ||P_{\theta} - Q_{\theta}\| + \sup_{\theta \in R} ||Q_{\theta} - Q_{\theta}\|
\leq o((\log n)^{-\gamma}) + \sup_{\theta \in R} |1 - c_{n}(\theta)^{-1}|
= o((\log n)^{-\gamma}).
\]
This completes the proof.

We can also show higher order asymptotic sufficiency of the statistic $\hat{\theta}_{n,k} = (\hat{\theta}_{n}, G^{(2)}_{n}(\hat{\theta}_{n}), \cdots, G^{(k)}_{n}(\hat{\theta}_{n})).

**Theorem 3.2.** If Condition A holds for some $k \geq 2$, then there exists a sequence of families of probability measures $\{R_{n,\theta}; \theta \in R\}, n \in N$, such that
(a) for each $n \in N$, the statistic $\hat{\theta}_{n,k}$ is sufficient for $\{R_{n,\theta}; \theta \in R\}$
(b) for every $\nu < (k+1)/(k+3)$
\[
\sup_{\theta \in R} ||P_{n,\theta} - R_{n,\theta}|| = o((\log n)^{-\gamma}).
\]

**Proof.** Let
\[
r_{n}(z_{n}, \theta) = I_{V_{n,\theta} \cap W_{n,\theta}}(z_{n}) \exp \left\{ G_{n}(z_{n}, \hat{\theta}_{n}) + \sum_{m=0}^{k} \frac{(\theta - \hat{\theta}_{n})^{m}}{m!} G_{n}^{(m)}(z_{n}, \hat{\theta}_{n}) \right\},
\]
\[
r_{n}(z_{n}, \theta) = c_{n}(\theta) r_{n}(z_{n}, \theta),
\]
where
\[
W_{n,\theta} = \left\{ z_{n} \in R^{n}; \sup_{|t| \leq b_{n}(2/(k+3))} |G^{(k+1)}_{n}(z_{n}, \hat{\theta}_{n} + t)| < c(\log n)^{-\gamma/(k+3)} \right\},
\]
\[
\tilde{c}_{n}(\theta) = \left[ \int_{R^{n}} r_{n}(z_{n}, \theta) d\mu_{n} \right]^{-1}
\]
and $V_{n,\theta}$ is the same as in the proof of Theorem 3.1. Here the constant $c$ in $W_{n,\theta}$ is determined by Lemma 3.7 with $s = 2/(k+3)$. Moreover, define $R_{n,\theta}(A) = \int_{A} r_{n}(z_{n}, \theta) d\mu_{n}$ and $R^{*}_{n,\theta}(A) = \int_{A} r_{n}(z_{n}, \theta) d\mu_{n}$ for every $A \in \mathcal{B}^{n}$. Then it follows from the factorization theorem that $\hat{\theta}_{n,k}$ is sufficient for $\{R_{n,\theta}; \theta \in R\}$.

Using the Taylor expansion
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with \( \theta_n^* - \theta_n \leq \frac{1}{n} \), we have for \( z_n \in V_{n, \theta} \cap W_{n, k} \):

\[
1 - \frac{r_n(z_n, \theta)}{\rho_n(z_n, \theta)} = 1 - \exp \left\{ -\left[ G_n(z_n, \theta) - G_n(z_n, \theta_n) - \sum_{m=0}^{k+1} \frac{(\theta-n)^{(m+1)}}{m!} G_n^{(m)}(z_n, \theta_n) \right] \right\} \\
\leq \frac{2c}{(k+1)!} \left( \log \log n \right)^{(k+1)/2} \left( \log n \right)^{(k+1)/(k+3)} \\
= o(\log n)^{-2}.
\]

Hence an argument analogous to the proof of Theorem 3.1 shows that Lemma 3.4 and Lemma 3.7 imply

\[
\sup_{\theta \in \mathcal{R}} \| P_{n, \theta} - R_{n, \theta}^* \| = o(\log n)^{-2},
\]

which leads to

\[
\sup_{\theta \in \mathcal{R}} \| P_{n, \theta} - R_{n, \theta} \| = o(\log n)^{-2}.
\]

This completes the proof.

**EXAMPLES** (Woodroofe [20]). Let

\[
f(x) = r_{\Gamma(2/r)}^{-1} x \exp(-x'), \quad x > 0 \quad \text{for some } r > 0,
\]

or

\[
f(x) = (r(1+r))^{-1} x (1+x)^{-2} - r, \quad x > 0 \quad \text{for some } r > 0,
\]

then Condition \( A_k \) is satisfied for every \( k \geq 2 \).

4. **Asymptotic sufficiency: Case (ii).** We continue to use the same notations as in Section 3. We shall need the following Condition \( B(\beta, k) \) on \( \rho(x) \) where \( \beta \) is a positive number and \( k \) is a positive integer.

Condition \( B(\beta, k) \)

(i) \( \rho(x) \) is a uniformly continuous density which vanishes on \( (-\infty, 0) \) and is positive on \( (0, \infty) \).

(ii) \( \rho(x) \) is \((k+2)\)-times continuously differentiable on \((0, \infty)\).

(iii) \( \rho(x) = O(x^{\beta+1}), g^{(0)}(x) = O(x^{-1}), g^{(1)}(x) = O(x^{-2}), g^{(2)}(x) = O(x^{-3}), g^{(k+1)}(x) = O(x^{-k-1}) \) and \( g^{(k+2)}(x) = O(x^{-k-2}) \) as \( x \to +0 \).

Moreover, \( g^{(k)}(x) \leq 0 \) for sufficiently small \( x > 0 \).

Let \( \rho_1 = \min \left[ \frac{k+2}{2}, \frac{(\beta+2)(k+3)}{2(k+\beta+3)} \right], \rho_2 = \min \left[ \frac{k+2}{2}, \frac{(\beta+2)(k+3)}{k+\beta+6} \right] \) and \( \rho_3 = \min \left[ \frac{k+2}{2}, \frac{(\beta+2)(k+3)}{k+\beta+6} \right] \). It is clear that \( \rho_2 \leq 2 \rho_3 \leq 2 \rho_1 \).

(iv) For every \( t \geq 0 \)

\[
\int_0^\infty |g(x+t)|^\rho \rho(x) \, d\mu < \infty.
\]
(v) There exists $M > 0$ such that

(a) $\int_M^{\infty} |g^{(i)}(x)|^{2p_i} p(x) \, d\mu < \infty$,

(b) $\int_M^{\infty} |g^{(i)}(x)|^{s_i} p(x) \, d\mu < \infty$,

(c) $\int_M^{\infty} |g^{(i+1)}(x)|^{s_i} p(x) \, d\mu < \infty$.

(vi) For every $a > 0$, there exists $\delta > 0$ such that

(a) $\int_M^{a} \sup_{|\xi| \leq \delta} |g^{(2)}(x+\xi)|^{2p} p(x) \, d\mu < \infty$,

(b) $\int_M^{a} \sup_{|\xi| \leq \delta} |g^{(i+2)}(x+\xi)|^{s_i} p(x) \, d\mu < \infty$.

Let $J = -\int_0^{\infty} g^{(2)}(x) p(x) \, d\mu$. Conditions (i)–(iii) and (v) (b) guarantee that $J$ is finite. Moreover, we need

(vii) $J > 0$.

According to condition (iii), it may be expected that $g^{(k+1)}(x)$ has the absolute moment of order $r$ for every $r < (\beta+2)/(k+1)$, but we will not always require this. That is, conditions (i)–(iii) and (v) (c) insure that

\begin{equation}
E|g^{(k+1)}(\cdot)|^r < \infty \quad \text{for every } r < (\beta+2)/(k+1), \quad \text{if } \beta \leq k(k+3),
\end{equation}

\begin{equation}
E|g^{(k+1)}(\cdot)|^{k+2} < \infty, \quad \text{if } \beta > k(k+3).
\end{equation}

We define $\lambda^*$ and $\nu^*$ as follows

\begin{equation}
\lambda^* = \frac{(k+1)(k+3)}{2(k+\beta+3)}, \quad \text{for } \beta < k(k+3),
\end{equation}

\begin{equation}
\nu^* = \min \left[ \frac{k}{2}, \frac{\beta(k+1)}{2(k+\beta+3)} \right], \quad \text{for } \beta > 0, \, k \in \mathbb{N}.
\end{equation}

It is noticed that $\lambda^* > 1/2$, and that $\nu^* < k/2$ for $\beta < k(k+3)$ and $\nu^* = k/2$ for $\beta \geq k(k+3)$.

(4.1) together with (2.1), (2.4) and (2.5) implies

\begin{equation}
P_{n,0} \{ \sum_{i=1}^{n} g^{(k+1)}(x_i) \geq n^{\lambda^*} \} = o(n^{-\lambda^*}), \quad 0 < \beta \leq k-1,
\end{equation}

\begin{equation}
P_{n,0} \{ \sum_{i=1}^{n} [g^{(k+1)}(x_i) - E(g^{(k+1)}(\cdot))] \geq n^{\nu^*} \} = o(n^{-\nu^*}), \quad k-1 < \beta < k(k+3),
\end{equation}

\begin{equation}
P_{n,0} \{ \sum_{i=1}^{n} [g^{(k+1)}(x_i) - E(g^{(k+1)}(\cdot))] \geq c\sqrt{n \log n} \} = o(n^{-\nu^*}), \quad \beta = k(k+3),
\end{equation}

\begin{equation}
P_{n,0} \{ \sum_{i=1}^{n} [g^{(k+1)}(x_i) - E(g^{(k+1)}(\cdot))] \geq c\sqrt{n \log n} \} = o(n^{-k/2}), \quad \beta > k(k+3),
\end{equation}

\begin{equation}
P_{n,0} \{ \sum_{i=1}^{n} [g^{(k+1)}(x_i) - E(g^{(k+1)}(\cdot))] \geq c\sqrt{n \log n} \} = o(n^{-k/2}), \quad \beta > k(k+3),
\end{equation}
where $\nu < \nu^*$ and $c$ is a positive constant independent of $\nu$.

Hereafter, we shall use $c > 0$ as a generic constant independent of $n \in \mathbb{N}$.

**Lemma 4.1.** Let conditions (i)–(iii) and (vi) (b) be satisfied for some $\beta > 0$ and $k \in \mathbb{N}$. Then for every $L > 0$ there exists $c > 0$ such that for $\nu < \nu^*$

$$P_{\nu, 0} \{ \sup_{|t| \leq Ld_n} \left| \sum_{i=1}^{n} [g^{(k+1)}(x_i + t) - g^{(k+1)}(x_i)] \right| \geq cn^{\nu} \} = o(n^{-\gamma}), \quad \beta < k(k+3),$$

and

$$P_{\nu, 0} \{ \sup_{|t| \leq Ld_n} \left| \sum_{i=1}^{n} [g^{(k+1)}(x_i + t) - g^{(k+1)}(x_i)] \right| \geq cn^{\nu} \} = O(n^{-k/2}), \quad \beta \geq k(k+3),$$

where $d_n = n^{-1/2} \sqrt{\log n}$ and the supremum is understood to be infinite if $M_n \leq Ld_n$.

**Proof.** Let $a > 0$ be so small that $p(x) < cx^{\beta+1}$ and $|g^{(k+2)}(x)| < cx^{-k-2}$ for $0 < x < 2a$. Since $P_{\nu, 0} \{ M_n \leq 2Ld_n \} = o(n^{-\gamma})$, we may assume that $M_n > 2Ld_n$.

Using the equality

$$\sum_{i=1}^{n} g^{(k+1)}(x_i + t) = \sum_{i=1}^{n} g^{(k+1)}(x_i) + \sum_{i=1}^{n} \int_{0}^{t} g^{(k+2)}(x_i + u) \, du$$

we have for sufficiently large $n$ and $|t| \leq Ld_n$

$$|\sum_{i=1}^{n} [g^{(k+1)}(x_i + t) - g^{(k+1)}(x_i)]| \leq |t| [c \sum_{i=1}^{n} \delta(x_i - Ld_n)^{-k-2} + \sum_{i=1}^{n} \sup_{|u| \leq \delta} |g^{(k+2)}(x_i + u)|],$$

where $\delta$ is determined by condition (vi) (b). For $i = 1, \ldots, n$ let us define

$$U_{ni} = (x_i - Ld_n)^{-k-2}, \quad 2Ld_n \leq x_i < a,$$

$$= 0, \quad \text{otherwise.}$$

Since $E|U_{ni}|^r \leq c < \infty$ for every $n \in \mathbb{N}$ and $r < (\beta+2)/(k+2)$, it follows from (2.1), (2.4) or (2.5) that

$$P_{\nu, 0} \left\{ \sum_{i=1}^{n} U_{ni} \geq d_n^{-1} n^{\nu} \right\} = o(n^{-\gamma}), \quad \beta \leq k,$$

$$P_{\nu, 0} \left\{ \left| \sum_{i=1}^{n} [U_{ni} - E(U_{ni})] \right| \geq d_n^{-1} n^{\nu} \right\} = o(n^{-\gamma}), \quad k < \beta < k(k+3),$$

$$P_{\nu, 0} \left\{ \left| \sum_{i=1}^{n} [U_{ni} - E(U_{ni})] \right| \geq n \right\} = o(n^{-k/2}), \quad \beta \geq k(k+3).$$

This implies that

$$P_{\nu, 0} \{ \sum_{i=1}^{n} \delta(x_i - Ld_n)^{-k-2} \geq 2d_n^{-1} n^{\nu} \} = o(n^{-\gamma}), \quad \beta < k(k+3),$$

$$P_{\nu, 0} \{ \sum_{i=1}^{n} \delta(x_i - Ld_n)^{-k-2} \geq cn \} = o(n^{-k/2}), \quad \beta \geq k(k+3).$$

Taking account of condition (vi) (b), a similar argument shows that

$$P_{\nu, 0} \{ \sup_{|t| \leq Ld_n} |g^{(k+2)}(x_i + u)| \geq 2d_n^{-1} n^{\nu} \} = o(n^{-\gamma}), \quad \beta < k(k+3),$$
Thus, (4.5) implies the desired assertion.

**Remark 4.** If \( \beta \geq k(k+3) \), then for every \( L > 0 \) there are events \( H_n, n \in N \) for which \( P_{n,0} \{(H_n)^c\} = O(n^{-k/2}) \) and \( z_n \in H_n \) implies

\[
\sup_{|t| \leq \delta} \left| \sum_{i=1}^{n} \left[ g^{(k+1)}(x_i + t) - g^{(k+1)}(x_i) \right] \right| \leq cn|t|.
\]

**Lemma 4.2.** Let conditions (i)–(iii), (v) (b), (vi) (a) and (vii) be satisfied for some \( \beta > 0 \) and \( k \in N \). Then for sufficiently small \( \varepsilon > 0 \) there are events \( D_n^* \), \( n \in N \) for which \( P_{n,0} \{(D_n^*)^c\} = O(n^{-\varepsilon}) \) and \( z_n \in D_n^* \) implies

\[
\sup_{-t \leq x \leq \Lambda_n} G^{(2)}(z_n, t) < -Jn/5.
\]

Proof. Let \( \alpha > 0 \) be so small that \( g^{(3)}(x) \leq 0 \) for \( 0 < x < 2\alpha \) and \( \int_0^\alpha g^{(2)}(x) p(x) d\mu > -J/5 \). Then the event \( M_n \leq \varepsilon \) implies that

\[
\sup_{-t \leq x \leq \Lambda_n} G^{(2)}(z_n, t) \leq \sup_{-t \leq x \leq \Lambda_n} \sum_{i=1}^{n} g^{(3)}(x_i - t) \leq \sum_{i=1}^{n} g^{(3)}(x_i) + \varepsilon \sum_{i=1}^{n} \sup_{|u| \leq \delta} |g^{(3)}(x_i + u)|.
\]

Let \( \varepsilon = \min \left[ \delta, \frac{J}{5} \left( \int_{|x| \leq \delta} |g^{(3)}(x+u)| p(x) d\mu \right)^{-1} \right] \) with \( \delta < \alpha \), satisfying condition (vi) (a). Then

\[
M_n \leq \varepsilon, \quad \sum_{i=1}^{n} g^{(3)}(x_i) - n \int_{\delta}^{\alpha} g^{(2)}(x) p(x) d\mu < Jn/5
\]

and

\[
\sum_{i=1}^{n} \sup_{|u| \leq \delta} |g^{(3)}(x_i + u)| - n \int_{\delta}^{\alpha} \sup_{|u| \leq \delta} |g^{(3)}(x+u)| p(x) d\mu < \frac{Jn}{5\varepsilon}
\]

imply

\[
\sup_{-t \leq x \leq \Lambda_n} G^{(2)}(z_n, t) < -Jn/5.
\]

Since \( \rho_i > 1 \), it follows from (2.4) or (2.5) that

\[
P_{n,0} \left\{ \sum_{i=1}^{n} g^{(3)}(x_i) - n \int_{\delta}^{\alpha} g^{(2)}(x) p(x) d\mu \geq Jn/5 \right\} = O(n^{-\varepsilon}),
\]

\[
P_{n,0} \left\{ \sum_{i=1}^{n} \sup_{|u| \leq \delta} |g^{(3)}(x_i + u)| - n \int_{\delta}^{\alpha} \sup_{|u| \leq \delta} |g^{(3)}(x+u)| p(x) d\mu \geq \frac{Jn}{5\varepsilon} \right\} = O(n^{-\varepsilon}).
\]

Lemma 4.2 follows easily.

The following lemma is proved in the same manner as Lemma 2 in [4] except that (2.4) or (2.5) is used instead of Chebyshev's inequality.

**Lemma 4.3.** Let conditions (i)–(iii) and (iv) be satisfied for some \( \beta > 0 \) and \( k \in N \). Then for every \( \varepsilon > 0 \)
Lemma 4.4. Let conditions (i)–(iii), (v) (b), (vi) (a) and (vii) be satisfied for some $\beta>0$ and $k \in \mathbb{N}$. Then for every $L > 0$

$$P_{n,0}\{\hat{\theta}_n \geq \varepsilon\} = O(n^{-\nu}).$$

Proof. Since $E|g^{(0)}(\cdot)|^{\nu+1} < \infty$, (2.4) or (2.5) implies that

$$P_{n,0}\left\{\sum_{i=1}^{n} [g^{(0)}(x_i) + J] \geq Jn/6\right\} = O(n^{-\nu}).$$

It remains to estimate the right side of (4.5) with $k=1$. Let $U_{ni}$ be the same as in (4.6) with $k=1$. It is easy to see that

$$E|U_{ni}|^{r} \leq c < \infty \quad \text{for} \quad r < (\beta + 2)/3 \leq 1, \quad \text{if} \quad \beta \leq 1,$$

$$E|U_{ni}|^{r} \leq c < \infty \quad \text{for} \quad 1 < r < (\beta + 2)/3, \quad \text{if} \quad 1 < \beta < k(k+3),$$

$$E|U_{ni}|^{(k+3)/3} \leq c < \infty, \quad \text{if} \quad \beta \geq k(k+3).$$

Accordingly, from (2.1), (2.4) or (2.5) we obtain

$$P_{n,0}\{cLd_n \sum_{i=1}^{n} U_{ni} \geq Jn/6\} = o(n^{-\nu}),$$

so that

$$P_{n,0}\{cLd_n \sum_{i=1}^{n} (x_i - Ld_a)^{-2} \geq Jn/6\} = o(n^{-\nu}).$$

Moreover, condition (vi) (a) together with (2.4) or (2.5) gives us

$$P_{n,0}\{Ld_n \sum_{i=1}^{n} \sup_{|u| \leq d} |g^{(0)}(x_i + u)| \geq Jn/6\} = o(n^{-\nu}).$$

Thus the lemma follows.

Lemma 4.5. Let conditions (i)–(iv), (v) (a), (v) (b), (vi) (a) and (vii) be satisfied for some $\beta>0$ and $k \in \mathbb{N}$. Then there exists $L > 0$ such that

$$P_{n,0}\{\hat{\theta}_n \geq Ld_a\} = O(n^{-\nu}).$$

Proof. It follows from Lemma 4.2 and Lemma 4.3 that for every $L > 0$

$$P_{n,0}\{\hat{\theta}_n \leq -Ld_a\} = P_{n,0}\{\sum_{i=1}^{n} g^{(0)}(x_i + Ld_a) \geq 0\} + O(n^{-\nu}),$$

$$P_{n,0}\{\hat{\theta}_n \geq Ld_a\} = P_{n,0}\{\sum_{i=1}^{n} g^{(0)}(x_i - Ld_a) \leq 0, M_a > Ld_a\} + O(n^{-\nu}).$$

Using the equality

$$\sum_{i=1}^{n} g^{(0)}(x_i + Ld_a) = \sum_{i=1}^{n} g^{(0)}(x_i) + Ld_a \sum_{i=1}^{n} g^{(0)}(x_i + u_n)$$

with $u_n \in (0, Ld_a)$, Lemma 4.4 implies that

$$P_{n,0}\{\sum_{i=1}^{n} g^{(0)}(x_i + Ld_a) \geq 0\} \geq P_{n,0}\{\sum_{i=1}^{n} g^{(0)}(x_i) \geq JL/2 \sqrt{n \log n}\} + O(n^{-\nu}).$$
Since $E|g^{(l)}(\cdot)|^{2^l} < \infty$, it follows from (2.5) that
\[(4.10) \quad P_{n,0}\left\{ \sum_{i=1}^{n} g^{(l)}(x_i) \geq \frac{1}{2} L \sqrt{n \log n} \right\} = o(n^{-\nu}) \]
for some large $L > 0$. Thus relations (4.7), (4.9) and (4.10) imply
\[P_{n,0}\{ \hat{\theta}_n \leq -Ld_n \} = O(n^{-\nu}).\]
By a similar argument, (4.8) implies
\[P_{n,0}\{ \hat{\theta}_n \geq Ld_n \} = O(n^{-\nu}).\]

The following lemma is an immediate consequence of (4.4), Lemma 4.1 and Lemma 4.5.

**Lemma 4.6.** Suppose that Condition B($\beta$, $k$) holds for some $\beta > 0$ and $k \in \mathbb{N}$.

If $\beta \leq k(k+3)$, then there exist $L > 0$ and $c > 0$ such that for $0 < \nu < \nu^*$
\[P_{n,0}\left\{ \sup_{\theta \in \mathcal{D}_n} \left| G^{(k+1)}(x_n, \hat{\theta}_n + t) - E(G^{(k+1)}(\cdot, 0)) \right| \leq c n^{-\gamma} \right\} = o(n^{-\gamma}), \quad 0 < \beta \leq k-1,\]
\[P_{n,0}\left\{ \sup_{\theta \in \mathcal{D}_n} \left| G^{(k+1)}(x_n, \hat{\theta}_n + t) - E(G^{(k+1)}(\cdot, 0)) \right| \leq c \sqrt{n \log n} \right\} = o(n^{-\gamma}), \quad k-1 < \beta < k(k+3),\]
\[P_{n,0}\left\{ \sup_{\theta \in \mathcal{D}_n} \left| G^{(k+1)}(x_n, \hat{\theta}_n + t) - E(G^{(k+1)}(\cdot, 0)) \right| \leq c \sqrt{n \log n} \right\} = o(n^{-\gamma}), \quad \beta = k(k+3),\]
where $\lambda^*$ and $\nu^*$ are defined by (4.2) and (4.3), respectively.

If $\beta > k(k+3)$, then there exist $L > 0$ and $c > 0$ such that
\[P_{n,0}\left\{ \sup_{\theta \in \mathcal{D}_n} \left| G^{(k+1)}(x_n, \hat{\theta}_n + t) - E(G^{(k+1)}(\cdot, 0)) \right| \leq c \sqrt{n \log n} \right\} = O(n^{-\nu/2}).\]

Now we shall discuss asymptotic sufficiency of the statistic $\hat{\theta}_{n,k} = (\hat{\theta}_n, G_{n}^{(k)}(x_n, \hat{\theta}_n), \ldots, G_{n}^{(k)}(x_n, \hat{\theta}_n)), k \in \mathbb{N}$, where $\hat{\theta}_{n,1}$ means $\hat{\theta}_n$.

**Theorem 4.1.** If Condition B($\beta$, $k$) holds for some $\beta > 0$ and $k \in \mathbb{N}$, then there exists a sequence of families of probability measures $\{Q_{n,\theta}^{k}; \theta \in \mathcal{R}\}, n \in \mathbb{N},$ such that

(a) for each $n \in \mathbb{N}$, the statistic $\hat{\theta}_{n,k}$ is sufficient for $\{Q_{n,\theta}^{k}; \theta \in \mathcal{R}\}$

(b) $\sup_{\theta \in \mathcal{R}} ||P_{n,\theta} - Q_{n,\theta}|| = o(n^{-\gamma})$ for $0 < \nu < \nu^*$,

where $\nu^* = \min\left\{ \frac{k}{2}, \frac{\beta(k+1)}{2(k+\beta+3)} \right\}$.

Proof. The proof of the theorem is analogous to those of Theorem 3.1 and Theorem 3.2. Let
\[q_{n,\theta}^{k}(x_n, \theta) = I_{\nu_{n,\theta}^{k}, \nu_{n,\theta}^{k}}(x_n) \exp \left\{ G_{n}(x_n, \hat{\theta}_n) - \frac{J}{2} n(\theta - \hat{\theta}_n)^2 \right\}, \quad \text{if } k = 1,\]
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\[ I_{x; n, \theta} \exp \left\{ G_n(z_n, \theta_n) + \sum_{m=2}^{k} \left( \frac{\theta - \theta_n}{m} \right)^m G_m(z_n, \theta_n) \right\}, \]

if \( k \geq 2 \) and \( 0 < \beta \leq k - 1 \),

\[ = I_{x; n, \theta} \exp \left\{ G_n(z_n, \theta_n) + \sum_{m=2}^{k} \left( \frac{\theta - \theta_n}{m} \right)^m G_m(z_n, \theta_n) \right\} + \frac{(\theta - \theta_n)^{k+1}}{(k+1)!} E(G_n^{(k+1)}(\cdot, 0)) \]

where

\[ V_{x; \theta} = \{ |\theta_n - \theta| < Ld_n \} , \]

\[ W_{x; \theta} = \{ \sup \{ G_n^{(k+1)}(z_n, \theta_n), t \} < cn^\alpha \} , \]

\[ = \{ \sup \{ G_n^{(k+1)}(z_n, \theta_n), t \} - E(G_n^{(k+1)}(\cdot, 0)) < cn^\alpha \} , \]

\[ = \{ \sup \{ G_n^{(k+1)}(z_n, \theta_n), t \} - E(G_n^{(k+1)}(\cdot, 0)) < c \sqrt{n \log n} \} , \]

if \( \beta \geq k+3 \).

Here \( L > 0 \) and \( c > 0 \) are determined by Lemma 4.5 and Lemma 4.6. Moreover, let \( q_n, k(z_n, \theta) \) be the normalizing of \( q_n, k(z_n, \theta) \) and let \( Q_n, n(\cdot, \theta) \) be a probability measure with the density \( q_n, k(z_n, \theta) \). Since the remaining part of the proof runs parallel to the lines of the corresponding part of the proofs of Theorem 3.1 and Theorem 3.2, we shall omit it.

In the case \( \beta > k(k+3) \), we can improve Theorem 4.1, if condition \((v) (b)\) in \( B(\beta, k) \) is replaced by a stronger condition \((v) (b)'\).

\((v) (b)'\) There exists \( M > 0 \) such that

\[ \int_{-\infty}^{\infty} |g^{(2)}(x)|^{k+2} p(x) d\mu < \infty . \]

In this case, condition \((v) (b)'\) with conditions (i)–(iii) implies \( E |g^{(2)}(\cdot)|^{k+2} < \infty \). This leads to a stronger result than Lemma 4.4: For every \( L > 0 \) there exists \( c > 0 \) such that

\[ P_p \{ \sup \left\{ \sum_{i=1}^{n} [g^{(2)}(x_i) + f] \right\} \geq c \sqrt{n \log n} \} = o(n^{-1/2}) . \]

Lemma 4.7. If Condition \( B(\beta, k) \) with \((v) (b)'\) replacing \((v) (b)\) holds for \( \beta > k(k+3) \), then

\[ P_p \{ |\theta_n + J^{-1} n^{-1} \sum_{i=1}^{n} g^{(1)}(x_i) - \theta_n| \geq n^{-1}(\log n)^{1/2} \} = O(n^{-1/2}) . \]

Proof. By a Taylor expansion we obtain

\[ \theta_n + J^{-1} n^{-1} \sum_{i=1}^{n} g^{(1)}(x_i) = \theta_n + J^{-1} n^{-1} \left\{ \sum_{i=1}^{n} [g^{(1)}(x_i) - g^{(1)}(x_i - \theta_n)] \right\} \]

\[ = J^{-1} n^{-1} \theta_n + \sum_{i=1}^{n} [g^{(2)}(x_i - \theta_n) + f] \]
with \(|\theta_n^* - \theta| \leq |\hat{\theta}_n - \theta|\). This, together with Lemma 4.5 and (4.11), implies the desired assertion.

**Theorem 4.2.** If Condition \(B(\beta, k)\) with \((v)\) \((b)\)' replacing \((v)\) \((b)\) holds for \(\beta > k(k+3)\), then Theorem 4.1 holds with the following \((b)\)' instead of \((b)\):

\[
(b)' \quad \sup_{\theta \in \mathcal{R}} ||P_{n, \theta} - Q_{n, \theta}^k|| = O(n^{-k/2}).
\]

Because of Remark 4 and Lemma 4.7, Theorem 4.2 can be shown in quite the same way as in the proof of Theorem 2 in [6].

**Remark 5.** Suppose \(k=1\). From Theorem 4.1 and 4.2 it follows that if \(\beta \leq 4\), then m.l. estimator \(\hat{\theta}_n\) is asymptotically sufficient up to order \(o(n^{-\nu})\) for every \(\nu < \beta/(4+\beta)\) and that if \(\beta > 4\), then \(\hat{\theta}_n\) is asymptotically sufficient up to order \(O(n^{-1/2})\).

**Remark 6.** Theorems 4.1 and 4.2 still hold even if a sequence of m.l. estimators \(\{\hat{\theta}_n\}\) is replaced by \(\{T_n\}\) with the following properties: There exist positive constants \(\pi_1\) and \(\pi_2\) (depending on \(\nu^*\)) such that

1. \[
\sup_{\theta \in \mathcal{R}} P_{n, \theta} \{z_n \in \mathbb{R}^n; n^{1/2} |T_n(z_n) - \theta| \geq (\log n)^{\pi_1}\} = O(n^{-\nu'})
\]
2. \[
\sup_{\theta \in \mathcal{R}} P_{n, \theta} \{z_n \in \mathbb{R}^n; n^{\nu'} \left| \sum_{i=1}^s g^{(1)}(x_i - T_n(z_n)) \right| \geq (\log n)^{\pi_2}\} = O(n^{-\nu'}).
\]

\(\{T_n\}\) with properties (1) and (2) is called a sequence of asymptotic m.l. estimators of order \(O(n^{-\nu'})\) (see Michel [8] and Matsuda [6]).

**References**


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