Hirasawa, M. and Murasugi, K.
Osaka J. Math.
44 (2007), 11–70

FIBRED DOUBLE TORUS KNOTS WHICH ARE BAND-SUMS OF TORUS KNOTS

MIKAMI HIRASAWA and KUNIO MURASUGI

(Received May 2, 2005, revised February 23, 2006)

Abstract

A double torus knot K is a knot embedded in a Heegaard surface H of genus 2, and K is non-separating if $H \setminus K$ is connected. In this paper, we determine the genus of a non-separating double torus knot that is a band-connected sum of two torus knots. We build a bridge between an algebraic condition and a geometric requirement (Theorem 5.5), and prove that such a knot is fibred if (and only if) its Alexander polynomial is monic, i.e. the leading coefficient is ± 1. We actually construct fibre surfaces, using T. Kobayashi's geometric characterization of a fibred knot in our family. Separating double torus knots are also discussed in the last section.

1. Introduction

A knot (or link) K in S^3 is called a *double torus knot* (*or link*) if K can be embedded in the Heegaard surface H of genus 2 (i.e., a standardly embedded closed surface of genus 2). In [6] and [7], such knots are extensively studied. Double torus knots form a large family of knots that contains torus knots, 2-bridge knots, knots with $(1, 1)$-decomposition (i.e., genus one bridge one knots) and tunnel number one knots. However, the class of double torus knots is not excessively large, with some 3-bridge knots outside the category. (Also, double torus knots have tunnel number at most 2.)

Other interesting examples of double torus knots are Berge's doubly primitive knots [1], and Dean's twisted torus knots [3] [11]. The class of Berge's knots is conjectured (cf. [5]) to cover all knots which yield lens spaces via Dehn surgery. They are known to be fibred knots [14], [7]. Some twisted torus knots yield small Seifert fibred spaces via Dehn surgery, and so far, all examples which yield those with finite fundamental groups are known to be fibred.

In general, it is not easy to decide whether or not a given knot is a fibred knot, and to study fibred knots, both algebraic and geometric methods have been used.

In this paper, we study double torus knots of type $(1, 1)$, (or simply, $(1, 1)$-double torus knots). These should not be confused with $g1$-$b1$ knots. For a detailed description of such knots, see §2 or [7]. We characterize completely fibred $(1, 1)$-double torus

2000 Mathematics Subject Classification. 57M25.

The first author was partially supported by MEXT, Grant-in-Aid for Young Scientists (B) 14740048. The second author was partially supported by NSERC Grant A 4034.

knots, and establish a method to determine whether or not a given $(1,1)$-double torus knot is fibred. We also determine the genera of $(1,1)$-double torus knots.

If K is embedded in a double torus H so that $H \setminus K$ is connected (resp. disconnected), then K is called *non-separating* (resp. *separating*). Note that a non-separating K may be embedded in H in a different way so that K is also called separating, and vice versa.

Double torus knots of type $(1,1)$ are, in general, satellite knots, and their pattern knots are also of type $(1,1)$ which are, if non-separating, ribbon knots and hence slice knots [7].

Our main theorem proved in this paper is as follows;

Theorem A. *A non-separating double torus knot K_0 of type $(1,1)$ is fibred if and only if its Alexander polynomial $\Delta_{K_0}(t)$ is monic, i.e., the leading coefficient of $\Delta_{K_0}(t)$ is ± 1.*

In Section 14, we deal with separating (1,1)-double torus knots. They are of genus at most one, and we determine which of them are fibred (i.e., determine when they are the unknot, the trefoil knot or the figure-eight knot).

The Alexander polynomial rarely determines the fibredness of knots, except for limited situations like alternating knots. Although our theorem does not hold in general for all double torus knots, there are other classes of double torus knots for which a similar theorem holds. As one of such classes, satellite knots of tunnel number one are discussed in [8].

Recall that if a knot K is fibred, then $\Delta_K(t)$ is monic [2, Proposition 8.16]. Theorem A is proved as follows: First we note that $(1,1)$-double torus knot K_0 is, in general, a satellite knot of a satellite knot, where the companions are torus knots and the final pattern knot, denoted by $K = K(n, p \mid \alpha, \beta)$, is also a non-separating $(1,1)$-double torus knot (Proposition 2.2).

Then we prove:

Theorem A$'$. *The knot $K(n, p \mid \alpha, \beta)$ is fibred if and only if its Alexander polynomial is monic.*

This is the first and crucial step toward the proof of Theorem A.[1] In fact, the most of this paper is devoted to the study of $K(n, p \mid \alpha, \beta)$. To prove Theorem A$'$, we define the graph H and the notion of the graph being *admissible*.

We establish a quick algorithm that calculates $\Delta_K(t)$ using this graph $H(K)$ for $K = K(n, p \mid \alpha, \beta)$. This algorithm also works for all 2-bridge knots (Corollary 4.9),

[1] For the fibredness, Theorem A does not immediately follow from Theorem A$'$. See [2, Corollary 4.15 and the following Remark].

and we see that if $H(K)$ is admissible, then $\Delta_K(t)$ is monic. Then our proof of Theorem A′ splits into two parts: one is algebraic and the other is geometric in nature. In the algebraic part, we show:

Theorem B. *If $\Delta_K(t)$ is monic, then $H(K)$ is admissible.*

In the geometric part, we define (in Section 12) the notion of (*admissible*) *word* $W(K)$ of K, and show that $W(K)$ is admissible if and only if $H(K)$ is admissible (Proposition 12.9). When $W(K)$ is admissible, we actually construct a fibre surface for them. Using T. Kobayashi's theory of *pre-fibre surfaces* (see [10], or Section 12 for definitions) we show the following theorem which completes the proof of Theorem A′:

Theorem C. *If $W(K)$ is admissible, then K is fibred.*

Using the fibre surface for $K(n, p \mid \alpha, \beta)$, we construct a fibre surface for the satellite knot K_0, and complete the proof of Theorem A.

Then, we construct a minimal genus Seifert surface for $(1, 1)$-double torus knots, and prove the following:

Theorem D. *Let K_0 be a non-separating double torus knot of type* $(1, 1)$. *Then the genus of K_0 is exactly half of the degree of the Alexander polynomial of K_0.*

If K is separating, neither Theorem A nor Theorem D holds. In fact, the genus of a separating double torus knot is at most one. However, in the last section, we determine the genus of such knots and characterize fibred knots.

This paper is organized as follows: Section 2 begins with a brief description of double torus knots, particularly, those of type (1,1). In Section 3, we prove some basic properties of non-separating double torus knots of type $(1, 1)$. In Section 4, we define the graph $H(K)$ of $K = K(n, p \mid \alpha, \beta)$, and the notion of $H(K)$ being admissible. Then we provide an easy algorithm to calculate, using $H(K)$, the Alexander polynomial of K, and also that of any 2-bridge knot. The following five sections, §5–9, are devoted to a proof of Theorem 5.5, called the Non-Cancellation Theorem, which is one of the key theorems in our paper. Theorem B, proved in Section 10, is an easy consequence of Theorem 5.5. In Section 11, we classify $K(n, p \mid \alpha, \beta)$ into six classes according to the monicity of their Alexander polynomials $\Delta_K(t)$ (i.e., $\Delta_K(0) = \pm 1$). In Section 12, we review basic tools to prove fibredness (Stallings twists, and K-banding of pre-fibre surfaces), and define the word $W(K)$. In Section 13, we first construct fibre surfaces for the pattern knots $K(n, p \mid \alpha, \beta)$ whose word $W(K)$ is admissible (Subsection 13.1), thus proving Theorem C. In 13.2, we construct fibre surfaces for all non-separating $(1, 1)$-double torus knots and prove Theorem A. We then construct minimal genus Seifert surfaces for $(1, 1)$-double torus knots whose word is not necessarily admissible (Subsection 13.3), thus proving Theorem D. In Section 14, we study separating (1,1)-double

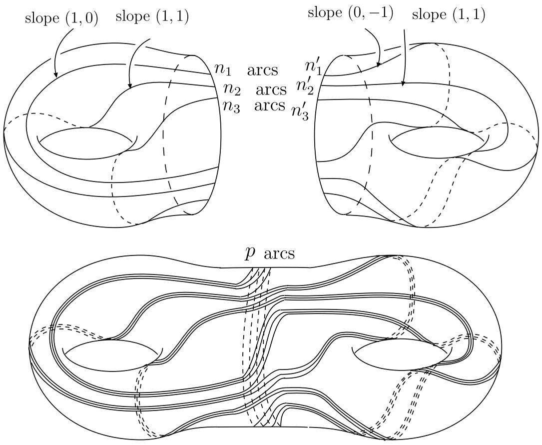


Fig. 2.1. $K = \{(3, 3, 3; 3, 3, 3 \mid 4)(1, 0, 1, 1)(0, -1, 1, 1)\}$.

torus knots. Using theorems proved in [6], we determine which separating (1,1)-double torus knots are fibred knots (i.e., the unknot or 3_1 or 4_1).

2. Preliminaries

In this section we state some properties of (1, 1)-double torus knots. For the self-containedness, we begin with a few necessary definitions. However, for details, we refer to [6] and [7].

Throughout this paper, we consider almost exclusively knots, not links, unless specified otherwise. Also, we do not consider orientations of knots.

Let K be a knot embedded in a standard double torus H. As in Fig. 2.1, we regard H as being obtained by glueing two once-punctured tori T_L (on the left side) and T_R (on the right side) along the circle $\mathcal{O}$. Then, K is cut by $\mathcal{O}$ into parallel classes of arcs properly embedded in T_L and T_R. If K misses one of the tori, K is a torus knot. Therefore, we assume $\mathcal{O}$ cuts K non-trivially.

On each torus, $K \setminus \mathcal{O}$ consists of at most three parallel classes. Then as in Fig. 2.1, we denote by $(n_1, n_2, n_3, n_1', n_2', n_3')$ the numbers of constituent arcs. Of course we have the equality $n_1 + n_2 + n_3 = n_1' + n_2' + n_3' := n$. Denote by $(r, s), (u, v)$ the slopes of the first and second parallel classes of arcs in T_L, and the slope of the third is automatically $(-r + u, -s + v)$. Also denote by $(r', s'), (u', v')$ the slopes of the two of the parallel classes in T_R. The convention of ordering the arcs and that of the slope should be inferred from Fig. 2.1. Finally, in gluing the arcs along $\mathcal{O}$, we have a choice, which is denoted by $-n < p \leq n$. Then by arranging the above numbers as in $K = \{(n_1, n_2, n_3; n_1', n_2', n_3' \mid p)(r, s, u, v)(r', s', u', v')\}$ we can express a double torus knot K. When K has only one parallel class of arcs on both T_L and T_R, K can be denoted by

$$K = \{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(r', s', -, -)\}, \tag{2.1}$$

and we say that K is *of type* (1,1), or simply a (1,1)-double torus knot. As other types, we have (1, 2)-, (1, 3)-, (2, 2)-, (2, 3)- and (3, 3)-types. We say that K is *separating* if $H \setminus K$ is disconnected, and otherwise, K is *non-separating*.

If $n = 1$, K is a connected sum of two torus knots and hence K is fibred. Therefore, we assume hereafter that $n > 1$.

The following is the starting points of the study of (1, 1)-double torus knots.

Proposition 2.1 ([7, Proposition 4.5]). *Let K be a* (1,1)*-double torus knot. Then*
(1) $\gcd(n, p) = 1$, *and*
(2) *K is non-separating if and only if n is odd.*

For $K = \{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(r', s', -, -)\}$, if any of r, s, r', s' equals 0, then K is a torus knot (or a trivial knot). Since all torus knots are fibred, we assume $rsr's' \neq 0$.

Since our knots are, in general, satellite knots, we first review when our knots are satellite knots and what the pattern knots are.

Proposition 2.2 ([7, Theorem 4.4]). *Let $K=\{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$, $n \geq 2$ be a* (1, 1)*-double torus knot. If $|r|, |s|, |r'|, |s'| \geq 2$, then K is a satellite knot. To be more precise*;
(1) *If $|r| \geq 2$ and $|s| \geq 2$, then K is a satellite knot with companion a torus knot $T(r, s)$, and its pattern knot K' is a* (1, 1)*-double torus knot of the form*:

$$K' = \{(n, 0, 0; n, 0, 0 \mid p)(1, rs, -, -)(r', s', -, -)\}.$$

(2) *If further $|r'| \geq 2$ and $|s'| \geq 2$, then K' is a satellite knot with companion a torus knot $T(r', -s')$, and its pattern knot K'' is a* (1, 1)*-double torus knot of the form*:

$$K'' = \{(n, 0, 0; n, 0, 0 \mid p)(1, rs, -, -)(1, r's', -, -)\}.$$

REMARK 2.3. By Proposition 2.4 below, it is justified to assume that the pattern knot of a (1,1)-double torus knot is of the following form, where α and β are non-zero integers.

$$K = \{(n, 0, 0; n, 0, 0 \mid p)(1, \alpha, -, -)(1, \beta, -, -)\},$$

For simplicity, the knot of the above form will be denoted by $K(n, p \mid \alpha, \beta)$. (If $\alpha = 0$ or $\beta = 0$, then K is a trivial knot.)

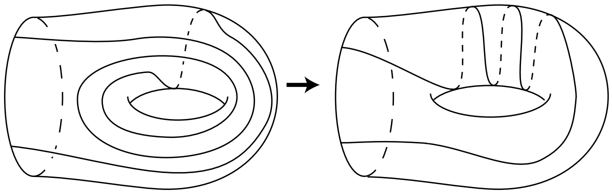


Fig. 2.2.

Proposition 2.4. *If* $|r| = 1$ *or* $|s| = 1$, *then* K *is ambient isotopic to* $K' = \{(n, 0, 0; n, 0, 0 \mid p)(1, rs, -, -)(r', s', -, -)\}$. *If* $|r'| = 1$ *or* $|s'| = 1$, *then* K *is ambient isotopic to* $K'' = \{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(1, r's', -, -)\}$.

Proof. General cases are understood by a typical deformation in Fig. 2.2. □

3. Non-separating (1, 1)-double torus knots

We assume, from this section through Section 13, that our (1, 1)-double knots are non-separating, and hence we assume:

(3.1) $$n \quad \text{is odd and} \quad \gcd(n, p) = 1.$$

(Separating knots are discussed in the last section, Section 14.)

In this section, we prove some basic properties of non-separating (1,1)-double torus knots. As is found in [7], $K = \{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(r', s', -, -)\}$ is obtained by a band-connected sum of a split union of two torus knots of type (r, s) and $(r', -s')$. See Fig. 3.1 for a special case. If any of r, s, r', s' equals 0, then K is a torus knot. Since all torus knots are fibred, we assume $rsr's' \neq 0$.

Now the pattern knot $K(n, p \mid \alpha, \beta)$ is obtained from the split union of two unknots by banding. See in Figs. 3.1 and 3.2 for example, $K(5, 2 \mid 2, 2)$ and $K(7, 4 \mid 2, 2)$ in a schematic form, where the band is depicted by an arc. Moreover, we can prove that the full-twists of the arcs can be removed without affecting the fibredness, while preserving the Alexander polynomials. (See Proposition 3.5 and §13.)

By rotating, twisting T_R, taking mirror images, and isotopies, we have the following:

Proposition 3.1. *We have the following equivalences, where* $-K$ *means the mirror image of* K. $K(n,p|\alpha,\beta) \cong K(n,-p|\alpha,\beta) \cong K(n,n-p|\alpha,\beta) \cong K(n,p|-\beta,-\alpha) \cong -K(n, p \mid \beta, \alpha)$.

Proof. We can deform $K(n, p \mid \alpha, \beta)$ into $K(n, n + p \mid \alpha, \beta)$ by twisting T_R (right half of the double torus) by π. In particular, we have the second equivalence. Other

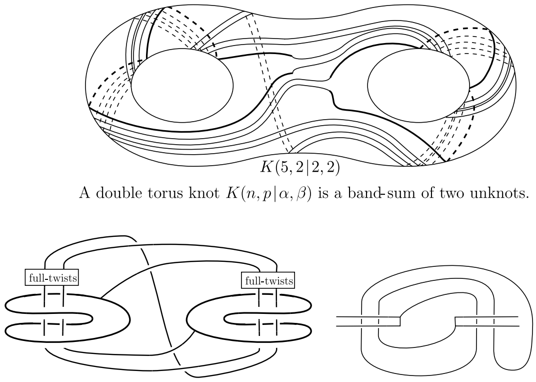


Fig. 3.1. $K(5, 2 \mid 2, 2)$.

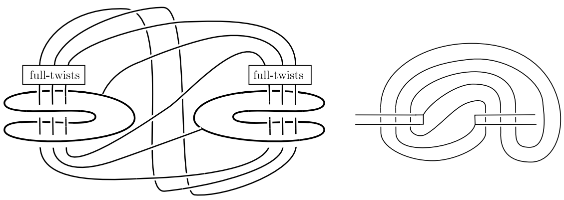


Fig. 3.2. $K(7, 4 \mid 2, 2)$.

equivalences are demonstrated by Fig. 3.3, where 'rotation' means a π-rotation along the axis vertical to the paper, and 'mirror' means the simultaneous crossing changes. Note that (α, β) becomes $(-\beta, -\alpha)$ by a rotation, because of the difference of the convention of positive twists in T_L and T_R. Refer to Figs. 2.1 and 3.1. □

As a consequence of Remark 2.3 and Proposition 3.1, we have:

Corollary 3.2. *Let K be a non-trivial, non-separating* $(1, 1)$-*double torus knot* $K(n, p \mid \alpha, \beta)$. *Then $n \geq 3$, n is odd,* $\gcd(n, p) = 1$ *and $\alpha\beta \neq 0$. Furthermore, without loss of generality, we may assume that $n > p > 0$ and $\alpha \geq |\beta| > 0$.*

REMARK 3.3. In drawing figures and just calculating the Alexander polynomials, it is sometimes convenient to assume that p is even.

The Alexander polynomial of $(1, 1)$-double torus knots has been obtained in the following proposition. (In Section 4, we see that the polynomial $f(t)$ below is shown to coincide with $h(t)$ defined in Definition 4.4.) We denote by $B(n, p)$ the 2-bridge knot of type (n, p) using Schubert's notation.

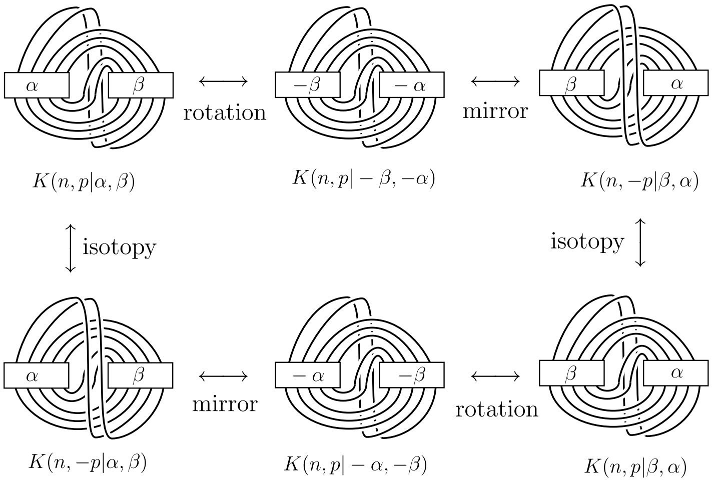


Fig. 3.3. Deformation of $K(5, 2 \mid \alpha, \beta)$.

Proposition 3.4 ([7, Theorem 4.7]). *Let* $K = \{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$. *Then K is a band sum of a split union of two torus knots $T(r, s)$ and $T(r', -s')$, and the Alexander polynomial of K is of the form*:

$$\Delta_K(t) = \Delta_{T(r,s)}(t)\Delta_{T(r',-s')}(t) f(t) f(t^{-1})$$

for some $f(t)$. Moreover, if $rs = r's' = \alpha$, then $f(t) = \Delta_{B(n,p)}(t^{\alpha})$.

This is the first place where we can see an algebraic relationship between the 2-bridge knot $B(n, p)$ and the knot $\{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(r', s', -, -)\}$. Inspired by this, T. Nakamura has given a geometric interpretation of these relationships for $K(n, p \mid 1, 1)$ [12].

Proposition 3.5 ([12, Proposition 3.3]). *The double torus knot $K = K(n, p \mid 1, 1)$ can be deformed by 'twistings of bands' into the connected sum of $B(n, p)$ and its mirror image $-B(n, p)$. Moreover the twistings preserve the Alexander polynomial.*

Fig. 3.4 illustrates the former half of Proposition 3.5. Note that by constructing Seifert surfaces in Section 13, we can see that the twisting of bands are realized by Stallings twists on minimal genus Seifert surfaces. However, in general, the Seifert surface obtained by smoothing the ribbon singularities of the ribbon disk is not of minimal genus.

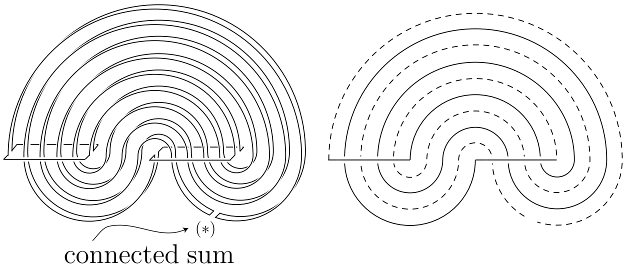


Fig. 3.4. The connected sum of $B(5,4)$ and $-B(5,4)$. Put $B(5,4)$ on this side of the sheet and $-B(5,4)$ on the other side. Making the connected sum is equivalent to cutting 'the band' at $(*)$ and we obtain a ribbon knot consisting of two unknots connected by a band along 'the half' of Schubert's diagram. Compare to Fig. 3.1 regarding $\alpha = \beta = 1$.

4. The Alexander polynomial of $K(n, p \mid \alpha, \beta)$ and the sequence of signs

For a (1,1)-double torus knot $K(n,p|\alpha,\beta)$, we define the key notions of this paper, namely, the sequence of signs for (n, p), the graph $H(K)$, and the polynomial $h(t) = h_{(n,p|\alpha,\beta)}(t)$. We assume n is odd.

4.1. Sequence of signs. Given a pair of co-prime integers (n, p) with $n > p > 0$, consider a sequence $\tilde{S} = \{p, 2p, \ldots, (n-1)p\}$. Choose the representative $\overline{kp}$, $(1 \leq k \leq n-1) \bmod 2n$ so that $-n < \overline{kp} < n$, and define a new sequence of integers $\overline{S} = \left\{\overline{p}, \overline{2p}, \ldots, \overline{(n-1)p}\right\}$. Let ε_k be the sign of $\overline{kp}$, i.e., $\varepsilon_k = \overline{kp}\big/\left|\overline{kp}\right|$. *The sequence of signs for the pair* (n, p) is defined to be $S = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}\}$. In the following, when we refer to the pair of integers (n, p), we always assume $\gcd(n, p) = 1$ and $n > p > 0$, unless otherwise specified. The following is an important fact which relates $K(n, p \mid \alpha, \beta)$ and the 2-bridge knot $B(n, p)$.

Fact 4.1. *The sequence $\overline{S}$, (or more generally, the sequence S) for the pair (n, p) recovers the Schubert normal form of the diagram for the 2-bridge knot $B(n, p)$.*

Now we prove two simple propositions on S.

Proposition 4.2. (1) *If p is even, then S is skew-symmetric, i.e., $\varepsilon_k = -\varepsilon_{n-k}$.* (2) *If p is odd, then S is symmetric, i.e., $\varepsilon_k = \varepsilon_{n-k}$.*

Proof. (1) Suppose p is even, i.e., $p = 2r$. (i) If $\varepsilon_k = 1$, then $\overline{kp} > 0$, and hence, for some m, $kp = 2mn+q$, where $0 < q < n$. Then, $(n-k)p = 2rn-(2mn+q) = 2n(r-m)-q$. Since $0 < q < n$, we have $\overline{(n-k)p} < 0$ and hence $\varepsilon_{n-k} = -1$. (ii) If $\varepsilon_k = -1$, then $kp = 2mn - q$, where $0 < q < n$. Thus $(n-k)p = 2n(r-m) + q$, and hence $\varepsilon_{n-k} = 1$. (2) Suppose p is odd, i.e., $p = 2r+1$. (i) If $\varepsilon_k = 1$, then $kp = 2mn + q$, for some m, where $0 < q < n$. Thus, $(n-k)p = n(2r+1) - (2mn+q) = 2n(r-m) + (n-q)$. Since $0 < n-q < n$, we have $\varepsilon_{n-k} = 1$. (ii) If $\varepsilon_k = -1$, then $kp = 2mn - q$, where $0 < q < n$. Thus $(n-k)p = 2n(r-m) + (n+q) = 2n(r-m+1) - (n-q)$, and hence $\varepsilon_{n-k} = -1$. □

We can relate the sequence of signs for (n, p) and $(n, n-p)$ as follows:

Proposition 4.3. *Let $S = \{\varepsilon_1, \dots, \varepsilon_{n-1}\}$ be the sequence of signs for the pair (n, p), and $S = \{\varepsilon'_1, \dots, \varepsilon'_{n-1}\}$ for $(n, n-p)$. Then $\varepsilon_k = (-1)^{k+1}\varepsilon'_k$.*

Proof. We may assume, without loss of generality, that p is odd. (1) Suppose k is even, i.e., $k = 2a$. (i) If $\varepsilon_k = 1$, then $kp = 2mn + q$, for some m, where $0 < q < n$. Therefore $k(n-p) = 2an - (2mn+q) = 2n(a-m) - q$, and hence $\varepsilon'_k = -1$. (ii) If $\varepsilon_k = -1$, then $kp = 2mn - q$, where $0 < q < n$. Therefore $k(n-p) = 2n(a-m) + q$, and hence $\varepsilon'_k = 1$. (2) Suppose k is odd, i.e., $k = 2a+1$. (i) If $\varepsilon_k = 1$, then $kp = 2mn + q$, where $0 < q < n$. Therefore $k(n-p) = (2a+1)n - (2mn+q) = 2n(a-m) + (n-q)$. Since $0 < n-q < n$, we have $\varepsilon'_k = 1$. (ii) If $\varepsilon_k = -1$, then $kp = 2mn - q$, where $0 < q < n$. Thus similarly, $k(n-p) = 2n(a-m) + n + q = 2n(a-m+1) - (n-q)$, and hence, $\varepsilon'_k = -1$. □

4.2. Graph $H(K)$ and polynomial $h(t)$ of K. In this subsection, we introduce two basic tools.

DEFINITION 4.4. Let $S = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}\}$ be the sequence of signs for (n, p), where n is odd, p is even and $n > p > 0$.

Let $Q = \{q_i\} = \{0, -\beta\varepsilon_1, \alpha\varepsilon_2, -\beta\varepsilon_3, \dots, -\beta\varepsilon_{n-2}, \alpha\varepsilon_{n-1}\}$, and $R = \{r_i\} = \left\{\sum_{k=1}^{i} q_k\right\}$. Define the polynomial by:

$$h(t) = h_{(n,p\,|\,\alpha,\beta)}(t) = \sum_{i=1}^{n} (-1)^i t^{r_i}.$$

DEFINITION 4.5. The graph $H(K)$ of $K(n, p\,|\,\alpha, \beta)$, where p is even, is defined as follows: $H(K)$ consists of n vertices with coordinates $(0, r_1), (1, r_2), (2, r_3), \dots, (n-1, r_n)$ in the xy-plane and edges connecting adjacent vertices, where $\{r_i\}$ is defined in Definition 4.4. The vertices of $H(K)$ are bi-colored, black and white alternately, so that the first and the last (the n-th) are black. We say that the graph $H(K)$ is *admissible* if $H(K)$ has exactly one highest vertex and one lowest vertex.

Note that we can read off $H(K)$ from the half of Schubert's diagram of $B(n, p)$, as in Fig. 4.1, by following one underpath from the left end-point and recording from

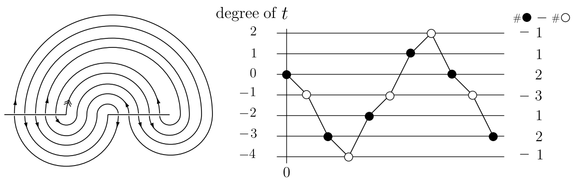


Fig. 4.1.

which direction (above or below) one goes under the overpath. At each step, the y-coordinate of the vertices changes by β or α alternately.

REMARK 4.6. We can read off $h(t)$ from $H(K)$:

$$\begin{aligned}(\text{the coefficient for } t^j) &= \#(\text{black vertices on the line } y = j) \\ &\quad - \#(\text{white vertices on the same line}),\end{aligned}$$

where # indicates the number of elements.

See Fig. 4.1 for example where $K(n, p \mid \alpha, \beta) = K(11, 8 \mid 2, 1)$.

$$\overline{S} = \{8, -6, 2, 10, -4, 4, -10, -2, 6, -8\},$$

$$S = \{1, -1, 1, 1, -1, 1, -1, -1, 1, -1\},$$

$$Q = \{0, -1, -2, -1, 2, 1, 2, 1, -2, -1, -2\},$$

$$R = \{0, -1, -3, -4, -2, -1, 1, 2, 0, -1, -3\}.$$

$$h(t) = -t^{-4} + 2t^{-3} + t^{-2} - 3t^{-1} + 2 + t - t^2.$$

EXAMPLE 4.7. See Figs. 4.2, 4.3 for the cases of $(n, p) = (7, 2)$ and $(7, 4)$ for various (α, β).

Now we state some applications of $h(t)$. Using $h(t)$, we can calculate the Alexander polynomial of $K(n, p \mid \alpha, \beta)$.

Theorem 4.8. *Suppose that $n \geq 3$, $n > p > 0$, and that p is even. Then, for $K = K(n, p \mid \alpha, \beta)$, we have $\Delta_K(t) \doteq h(t)h(t^{-1})$.*

A proof will be given in Section 10.

By Theorem 4.8 and Proposition 3.5, we can calculate the Alexander polynomial of $B(n, p)$ as follows:

Corollary 4.9. *For a 2-bridge knot $K = B(n, p)$ with p even, we have $\Delta_K(t) \doteq h_{K'}(t)$, where $K' = K(n, p \mid 1, 1)$.*

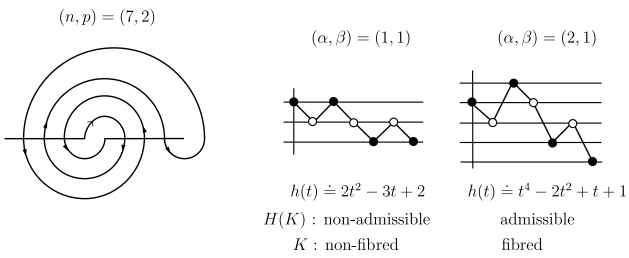


Fig. 4.2.

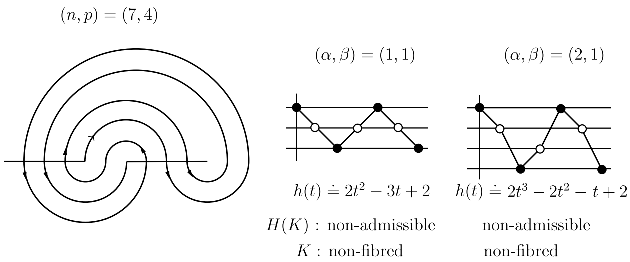


Fig. 4.3.

REMARK 4.10. If $(\alpha, \beta) = (1, 1)$, cancellations of the terms in $h(t)$ never occur, because the vertices of the graph $H(K)$ with the same y-coordinate have the same color. However in general, as seen in $K(7, 2 \mid 2, 1)$, some terms of $h(t)$ may cancel each other. Moreover cancellations among terms of local maximum may happen: try for example, $K(11, 2 \mid 2, 1)$. Cancellations among terms of the highest degree may yield a 'non-fibred knot with a monic Alexander polynomial.' However, Theorem B asserts it never happens.

REMARK 4.11. As 2-bridge knots, $B(7, 2) = B(7, 4)$, and hence they have the same Alexander polynomial. However, as seen in the above example, the ways terms appear are different. This difference causes the following interesting fact. The 2-bridge knot $B(7, 2) = B(7, 4)$ is non-fibred, and hence $K(7, 2 \mid 1, 1)$ and $K(7, 4 \mid 1, 1)$ are non-fibred. However, $K(7, 2 \mid 2, 1)$ is fibred, while $K(7, 4 \mid 2, 1)$ is non-fibred. See Section 11 for a further discussion.

Finally, we can use the diagrammatic calculations of $\Delta_K(t)$ to have a straight forward explanation to the facts found in [7]. For example, the latter half of Theorem 3.4 is understood as follows: For $K = K(n, p \mid \alpha, \beta)$ with p even, the graph $H(K)$ is obtained by expanding each edge of the graph for $K' = K(n, p \mid 1, 1)$ α times. Therefore $h_K(t) \doteq h_{K'}(t^\alpha)$. Meanwhile, by Proposition 3.5, we have $h_{K'}(t) \doteq \Delta_{B(n,p)}(t)$. Therefore, $\Delta_{K(n,p|1,1)}(t)$ is monic if and only if $\Delta_{B(n,p)}(t)$ is monic.

We give one more application. The following was pointed out in [7, p.636]. We understand this by seeing that the Alexander polynomial is not monic.

Proposition 4.12. *For any non-zero α, $K(n, p \mid \alpha, -\alpha)$ is not a fibred knot.*

Proof. Assume p is even. Then the y-coordinates of the vertices $v_0, \dots, v_{n-1}$ are as follows:

$$\{0, \alpha\varepsilon_1, \alpha\varepsilon_1 + \alpha\varepsilon_2, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \alpha\varepsilon_3, \dots, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \cdots + \alpha\varepsilon_{n-1}\}.$$

By the skew-symmetry of $\{\varepsilon_1, \dots, \varepsilon_{n-1}\}$, (Proposition 4.2), we see that the above is equal to

$$\{0, \alpha\varepsilon_1, \alpha\varepsilon_1 + \alpha\varepsilon_2, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \alpha\varepsilon_3, \dots, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \alpha\varepsilon_3, \alpha\varepsilon_1 + \alpha\varepsilon_2, \alpha\varepsilon_1, 0\}.$$

Since the number of vertices of $H(K)$ is odd $(= n)$, this means that the bi-colored graph $H(K)$ is symmetric with respect to a vertical line which goes through the center vertex $v_{(n-1)/2}$. In other words, each vertex other than $v_{(n-1)/2}$ has its counterpart of the same color at the same y-coordinate. Therefore, $h_K(t)$ is not monic, and hence by Theorem 4.8, neither is $\Delta_K(t)$. □

REMARK 4.13. After our first manuscript was completed, Y. Marumoto told us that T. Yasuda had introduced a graph similar to $H(K)$ in [15]. In fact, Yasuda studied in [15] ribbon n-knots with m-fusions in S^{n+2}, $n \geq 2$, and he calculated the Alexander polynomial $\Delta_{K^n}(t)$ using his graph. Our graph $H(K)$ coincides with his graph when $m = 1$, and therefore, it is shown that $h_{(n,p|1,1)}(t) = \Delta_{K^2}(t)$ for some ribbon 2-knot K^2 with 1-fusion.

5. Non-Cancellation theorem

In this section, we state Theorem 5.5, which is the key theorem to prove Theorem B. A proof of Theorem 5.5 will be given in Sections 5 through 9. Given a pair of co-prime integers (n, p) with $n > p > 0$, consider a sequence of signs defined in Section 4: $S = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}\}$. We associate with S a graph $G(n, p)$ on the xy-plane, called the *graph of a pair* (n, p). The graph $G(n, p)$ consists of n vertices $P_0, P_1, \dots, P_{n-1}$, and $(n-1)$ edges connecting P_k and P_{k+1}, $0 \leq k \leq n-2$, where P_k has the following coordinates:

$$P_0 = (0, 0), \quad P_k = (k, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k), \quad 1 \leq k \leq n-1.$$

REMARK 5.1. The graph $G(n, p)$ is related to the graph $H(K(n, p \mid \alpha, \beta))$ defined in Section 4 as follows:

$$G(n, p) = \begin{cases} H(K(n, n-p \mid -1, -1)), & \text{if } p \text{ is odd}, \\ H(K(n, p \mid 1, -1)), & \text{if } p \text{ is even}. \end{cases}$$

DEFINITION 5.2. Let α and β be positive integers with $\alpha > \beta > 0$. Then for a vertex P_k of the graph $G(n, p)$, the *level* of P_k, $\text{lev}(P_k)$, is defined as its y-coordinate, i.e., $\text{lev}(P_k) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k$, and the *index* of P_k, $\text{ind}(P_k)$, is defined as $\text{ind}(P_k) = \varepsilon_1 + \varepsilon_3 + \cdots + \varepsilon_{\bar{k}}$, where $\bar{k}$ is the maximal odd integer not exceeding k.

To each vertex P_k, we associate the term t^{d_k}, where $d_k = \text{ind}(P_k)\alpha + [\text{lev}(P_k) - \text{ind}(P_k)]\beta$. Therefore, $d_0 = 0$ and, if k is even, $d_k = (\varepsilon_1 + \varepsilon_3 + \cdots + \varepsilon_{k-1})\alpha + (\varepsilon_2 + \varepsilon_4 + \cdots + \varepsilon_k)\beta$, and if k is odd, $d_k = (\varepsilon_1 + \varepsilon_3 + \cdots + \varepsilon_k)\alpha + (\varepsilon_2 + \varepsilon_4 + \cdots + \varepsilon_{k-1})\beta$.

Using these terms, we define the polynomial $\phi_{(n,p|\alpha,\beta)}(t)$ by

$$\phi_{(n,p\,|\,\alpha,\beta)}(t) = \sum_{k=0}^{n-1}(-1)^k t^{d_k}. \tag{5.1}$$

In other words, each vertex of $G(n, p)$ corresponds to a term in $\phi_{(n,p|\alpha,\beta)}(t)$. Since the degree d_k corresponds to vertex P_k, it is called the *degree* of P_k. The degree is determined by $\text{ind}(P_k)$ and $\text{lev}(P_k)$. An edge of $G(n, p)$ connecting P_k and P_{k+1} is called an *odd* edge (resp. an *even* edge) if $k + 1$ is odd (resp. even). Therefore, the first edge is an odd edge.

We will show later that the polynomial $\phi_{(n,p|\alpha,\beta)}(t)$ determines the Alexander polynomial of $K(n, p \mid \alpha, \beta)$ (Proposition 10.4). Also, we will show a relationship between $\phi_{(n,p|\alpha,\beta)}(t)$ and $h_{(n,p|\alpha,\beta)}(t)$ (Proposition 10.2).

EXAMPLE 5.3 (see Fig. 5.1). For $(n, p) = (7, 3)$, we have:

$$\begin{array}{lllllllll}
\tilde{S} = \{3, 6, \;\; 9, \; 12, 15, 18\} & & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\
\overline{S} = \{3, 6, -5, -2, \;\; 1, \;\; 4\}, & \text{level} & 0 & 1 & 2 & 1 & 0 & 1 & 2, \\
S = \{1, 1, -1, -1, \;\; 1, \;\; 1\} & \text{index} & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}$$

and

$$\phi_{(7,3|\alpha,\beta)}(t) = 2 - 2t^{\alpha} - t^{\beta} + 2t^{\alpha+\beta}.$$

EXAMPLE 5.4 (see Fig. 5.1). For $(n, p) = (7, 5)$, we have:

$$\begin{array}{lllllllll}
\tilde{S} = \{5, \; 10, 15, 20, \; 25, 30\} & & P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\
\overline{S} = \{5, -4, \;\; 1, \;\; 6, -3, \;\; 2\}, & \text{level} & 0 & 1 & 0 & 1 & 2 & 1 & 2, \\
S = \{1, -1, \;\; 1, \;\; 1, -1, \;\; 1\} & \text{index} & 0 & 1 & 1 & 2 & 2 & 1 & 1
\end{array}$$

and

$$\begin{aligned}
\phi_{(7,5|\alpha,\beta)}(t) &= 1 - 2t^{\alpha} + t^{\alpha-\beta} - t^{2\alpha-\beta} + t^{2\alpha} + t^{\alpha+\beta} \\
&\doteq -t^{2\alpha-2\beta} + t^{2\alpha-\beta} + t^{\alpha-2\beta} - 2t^{\alpha-\beta} + t^{\alpha} + t^{-\beta}.
\end{aligned}$$

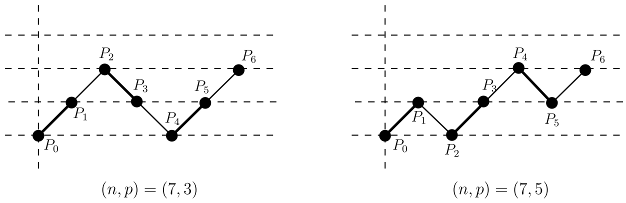


Fig. 5.1.

Throughout the rest of this paper, the graph $G(n, p)$ may be denoted by G, if no confusion occurs. One of the key theorems to prove Theorem B is as follows:

Theorem 5.5 (Non-cancellation Theorem). *Let $G(n, p)$ be the graph of (n, p). Assume $\alpha > \beta > 0$. Let $P_{j_1}, P_{j_2}, \ldots, P_{j_l}$ be the vertices of G with the highest level. Then for any vertex P_k $(0 \leq k \leq n-1)$, $\max\{\mathrm{ind}(P_{j_1}), \mathrm{ind}(P_{j_2}), \ldots, \mathrm{ind}(P_{j_l})\} \geq \mathrm{ind}(P_k)$. Similarly, let $P_{m_1}, P_{m_2}, \ldots, P_{m_q}$ be the vertices of G with the lowest level. Then for any vertex P_k $(0 \leq k \leq n-1)$, $\min\{\mathrm{ind}(P_{m_1}), \mathrm{ind}(P_{m_2}), \ldots, \mathrm{ind}(P_{m_q})\} \leq \mathrm{ind}(P_k)$.*

A proof of Theorem 5.5 will be given in Sections 5 through 9. One of the immediate consequence of Theorem 5.5 is the following:

Corollary 5.6. *The terms of the highest degree in $\phi_{(n,p|\alpha,\beta)}(t)$ correspond to vertices of the maximal index in the highest level. Similarly, the terms of the lowest degree in $\phi_{(n,p|\alpha,\beta)}(t)$ correspond to vertices of the minimal index in the lowest level.*

Proof of Corollary 5.6. Let $\mathrm{ind}(P_{j_i}) = r_i$ and $\mathrm{lev}(P_{j_i}) = m$. Let $r_c = \max\{r_1, r_2, \ldots, r_l\}$. Then $d_{j_c} = r_c\alpha + (m - r_c)\beta$, and $d_{j_i} = r_i\alpha + (m - r_i)\beta$. Since $\alpha > \beta > 0$, and $r_c \geq r_i$, we see $d_{j_c} - d_{j_i} = (r_c - r_i)\alpha + (r_i - r_c)\beta = (r_c - r_i)(\alpha - \beta) \geq 0$ and that the equality holds if and only if $r_c = r_i$. Now let P_k be a vertex of a non-highest level, and $\mathrm{ind}(P_k) = s$ and $\mathrm{lev}(P_k) = q$. Then since $q < m$ and $s \leq r_c$, we see $d_{j_c} - d_k = r_c\alpha + (m - r_c)\beta - (s\alpha + (q - s)\beta) = (r_c - s)\alpha + (m - q - r_c + s)\beta = (r_c - s)(\alpha - \beta) + (m - q)\beta > 0$. Hence the degree of a vertex of a non-highest level is strictly less than d_{j_c}. The statement for the terms with the lowest degree is proved analogously. □

REMARK 5.7. Theorem 5.5 and Corollary 5.6 show that the terms with the highest (resp. lowest) degree in $\phi_{(n,p|\alpha,\beta)}(t)$ correspond to some vertices of the highest (resp. lowest) level. Since two vertices of the same level have the same parity on their indices, the terms corresponding to the pair never cancel each other. This is a reason Theorem 5.5 is called the Non-cancellation theorem.

From Remark 5.7, we see immediately the following corollary.

Corollary 5.8. *For any* $\alpha > \beta > 0$ *and* $n > p > 0$, $\phi_{(n,p|\alpha,\beta)}(t)$ *is monic if and only if there exist exactly one vertex* P_0 *and one vertex* Q_0 *in* $G(n, p)$ *such that both* $\mathrm{lev}(P_0)$ *and* $\mathrm{ind}(P_0)$ (*resp.* $\mathrm{lev}(Q_0)$ *and* $\mathrm{ind}(Q_0)$) *are maximum* (*resp. minimum*).

6. Proof of Theorem 5.5 (I): The sequence of signs

The following several sections will be devoted to the proof of Theorem 5.5.

However, we concentrate on the proof of the first statement of Theorem 5.5. The proof of the second statement will be done simultaneously, and we omit it to avoid unnecessary complications. But to help readers, we provide enough information for the proof of the second statement.

As the first step, we study the sequence S of signs of a pair (n, p), and prove several basic properties for S. First we introduce new notations.

Let S be an arbitrary sequence of signs, i.e., of $+1$ or -1. In S, the consecutive sequence of $+1$ (or -1) k times is denoted by $\langle k\rangle$ (or $\langle -k\rangle$). For example, $\{1, 1, 1, -1, -1\} = \{\langle 3\rangle\langle -2\rangle\}$. Using this notation, S can be written as

$$S = \{\langle a_1\rangle\langle -b_1\rangle\langle a_2\rangle\langle -b_2\rangle \cdots \langle a_{l-1}\rangle\langle -b_{l-1}\rangle\langle a_l\rangle\langle -b_l\rangle\}$$

where $a_1, a_2, \ldots, a_l > 0$ and $b_1, b_2, \ldots, b_l > 0$, but a_1 or $-b_l$ may be missing. By abuse of notations, we also call $\langle a_i\rangle$ or $\langle -b_i\rangle$ terms of S

Proposition 6.1. *Let* n *be an odd integer,* $n > p > 0$ *and* $\gcd(n, p) = 1$. *Write* $n = mp + r$, *where* $m \geq 1$ *and* $0 < r < p$. *Let* $S = \{\langle a_1\rangle\langle -b_1\rangle\langle a_2\rangle\langle -b_2\rangle \cdots \langle a_l\rangle\langle -b_l\rangle\}$ *be the sequence of signs of the pair* (n, p), *where only* b_l *may be missing. Then we have the following*:

(1) a_i *and* b_j *are either* m *or* $m + 1$,

(2) $a_1 = m$ *and the last term of* S *is* $\pm m$,

(3) *The number of times* $\langle \pm m\rangle$ *appears in* S *is* $p - r + 1$, *and the number of times* $\langle \pm(m + 1)\rangle$ *appears in* S *is* $r - 1$. *The total number of* $\langle \pm m\rangle$ *and* $\langle \pm(m + 1)\rangle$ *in* S *is* p.

Thus, if p *is odd, then* $a_l = m$ *and* b_l *is missing, while if* p *is even,* $b_l = m$.

Proof. Let $A_i = (ip)$, $1 \leq i \leq n - 1$ be $n - 1$ points in the open interval $(0, np)$, and B_k, $1 \leq k \leq p$, the first point in $\{A_i\}$ appeared in the interval $((k - 1)n, kn)$. (Note that A_i determines ε_i.) The x-coordinate of B_k is written as $x_k + (k - 1)n$, $0 < x_k < p$. Since $n = mp + r$, $0 < r < p$, each interval $((k - 1)n, kn)$ contains m or $m + 1$ points in $\{A_i\}$. This proves (1). Further, the interval $((k - 1)n, kn)$ contains $m + 1$ points if and only if $0 < x_k < r$. Therefore, the number of such intervals is exactly $r - 1$, and hence $\langle \pm(m + 1)\rangle$ appears $r - 1$ times in S. Consequently, $\langle \pm m\rangle$ appears $p - r + 1$ times. This proves (3). Finally, a_1 (and the last term a_l or b_l) cannot be $m + 1$, since $n = mp + r$. Hence $a_1 = m$. This proves (2). The last conclusion follows from Proposition 4.2. □

Proposition 6.2. *Suppose* $n = mp + r$, *where* $m \geq 1$ *and* $0 < r < p$. *Let* S *be the sequence of signs for* (n, p). *Then we have the following*:
(1) *Suppose that in* S, $\langle m \rangle$ *is followed by* $\langle -m \rangle$, *or* $\langle -m \rangle$ *is followed by* $\langle m \rangle$, *like* $\cdots \langle m \rangle \langle -m \rangle \cdots$ *or* $\cdots \langle -m \rangle \langle m \rangle \cdots$. (*We say* $\langle \pm m \rangle$*'s occur consecutively.*) *Then* $\langle m + 1 \rangle$ *cannot be followed by* $\langle -(m+1) \rangle$, *or* $\langle -(m+1) \rangle$ *cannot be followed by* $\langle m+1 \rangle$. *We say then* $\langle m + 1 \rangle$*'s are isolated.*
(2) *Analogously, if* $\langle m + 1 \rangle$ *is followed by* $\langle -(m + 1) \rangle$ (*or* $\langle -(m + 1) \rangle$ *is followed by* $\langle m + 1 \rangle$), *then* $\langle \pm m \rangle$*'s are isolated.*

Proof. (1) Suppose $\langle m \rangle$ is followed by $\langle -m \rangle$. Then $(m - 1)p + (m - 1)p + p < 2n < (2m + 1)p$, namely, (i) $(2m - 1)p < 2n < (2m + 1)p$. On the other hand, if $\langle m + 1 \rangle$ is followed by $\langle -(m + 1) \rangle$, (or $\langle -(m + 1) \rangle$ is followed by $\langle m + 1 \rangle$), then the same argument shows that (ii) $(2m + 1)p < 2n < (2m + 3)p$. However, (i) and (ii) cannot hold simultaneously. The proof of (2) is analogous, and is omitted. □

Proposition 6.3. *Let* $n = mp + r$, *where* $m \geq 1$ *and* $0 < r < p$. *Somewhere in* S, *suppose* $\langle \pm m \rangle$*'s occur consecutively* k *times* (*maximally*). *Then at any other places,* $\langle \pm m \rangle$*'s occur at least* $(k - 1)$ *times consecutively. The same is true when* $\langle \pm m \rangle$ *and* $\langle \pm(m + 1) \rangle$ *are interchanged.*

Proof. Suppose m points in $\{A_i\}$ appear in each open interval $(ln, (l+1)n), \ldots, ((l+k-1)n, (l+k)n)$. Then we have

$$(mk - 1)p < kn < (mk + 1)p. \tag{6.1}$$

Suppose there are exactly $k - 2$ $\langle \pm m \rangle$'s between a pair of $\langle \pm(m + 1) \rangle$'s.

Consider consecutive k intervals consisting of $k-2$ intervals containing these $\langle \pm m \rangle$ and two intervals, before and after these $(k - 2)$ intervals. Then we apply on these k intervals the same argument as above, and obtain $(m(k-2)-1)p+(m+1)p+(m+1)p < kn$, and hence $mkp + p < kn$, which contradicts (6.1). □

Proposition 6.4. *Let* $n = mp + r$, *where* $m \geq 1$ *and* $0 < r < p$.
(1) *Suppose* S *begins with the following form*

$$S = \{\langle m \rangle \underbrace{\langle -(m + 1) \rangle \langle m + 1 \rangle \cdots \langle \pm(m + 1) \rangle}_{k \text{ times } k \geq 1} \langle \mp m \rangle \cdots \}.$$

Then in S, $\langle \pm(m + 1) \rangle$*'s always occur at least* k *times consecutively.*
(2) *Suppose* S *begins with the following form*

$$S = \{\underbrace{\langle m \rangle \langle -m \rangle \cdots \langle \pm m \rangle}_{k \text{ times } k \geq 1} \langle \mp(m + 1) \rangle \cdots \}.$$

Then in S, $\langle \pm m \rangle$'s occur at most k times consecutively.

Proof. (1) By Proposition 6.3, $\langle \pm(m+1) \rangle$'s occur either $(k+1)$ times or $(k-1)$ times consecutively. Suppose $\langle \pm(m+1) \rangle$'s occur $(k-1)$ times consecutively somewhere in S. Then there are $k+1$ consecutive intervals such that the first and the last interval contain m points from $\{A_i\}$. Therefore, we have:

$$[(k-1)(m+1)+2m-1]p < (k+1)n < [(k-1)(m+1)+2m+1]p. \tag{6.2}$$

On the other hand, the original assumption on S yields the following inequality:

$$[(m+1)k+m]p < (k+1)n, \tag{6.3}$$

since each of k consecutive intervals $(n, 2n), (2n, 3n), \ldots, (kn, (k+1)n)$ contains $(m+1)$ points. The second inequality of (6.2) and inequality (6.3) are inconsistent. (2) A proof is analogous and hence is omitted. □

7. Proof of Theorem 5.5 (II): Reductions

In this section, we introduce two reduction operations τ_1 and τ_2 on the sequence of signs S, and study their effects on the graph.

7.1. Reduction operations. Let $S = \{\langle c_1 \rangle \langle -c_2 \rangle \langle c_3 \rangle \cdots \langle \pm c_q \rangle\}$ be the sequence of signs of (n, p). Let $n = mp + r$, where $m \geq 1$ and $0 < r < p$.

CASE (A). Suppose $m \geq 2$, or $m = 2$ and $r > 1$. By Proposition 6.1, $c_i \geq 2$ for $1 \leq i \leq q$. Then we define a new sequence of signs S_1^* by $S_1^* = \{\langle c_1^* \rangle \langle -c_2^* \rangle \langle c_2^* \rangle \cdots \langle \pm c_q^* \rangle\}$, where $c_i^* = c_i - 2$. If the first s terms in S_1^* are 0 and the $(s+1)^{\text{st}}$ term is non-zero, then by Proposition 4.2, the last s terms of S_1^* are also 0 and the $(q-s-1)^{\text{st}}$ term is non-zero. In this case, we delete these $2s$ zeros from S_1^*. Further, if $c_j^* = 0$, $s+1 < j < q-s-1$, then $\langle \pm c_{j-1}^* \rangle \langle 0 \rangle \langle \pm c_{j+1}^* \rangle$ is written as $\langle \pm(c_{j-1}^* + c_{j+1}^*) \rangle$. We repeat these removals of zeros until no zeros are left. The final form thus obtained is our new sequence S^*. This reduction $S \to S^*$ is the first operation and is denoted by τ_1. (See Examples 7.3 to 7.5 below.)

CASE (B). Suppose $m = 1$, i.e., $n = p + r$, and $0 < r < p - 1$. The second reduction is a bit complex. Note that S contains only $\langle \pm 1 \rangle$ or $\langle \pm 2 \rangle$. Since $n = p + r$, $c_1 = c_q = 1$. First, we remove these $\langle c_1 \rangle$ and $\langle c_q \rangle$. Next, we replace every $\langle \pm 2 \rangle$ by $\langle 0 \rangle$ so that we obtain a new sequence S_1^* consisting of only $\langle \pm 1 \rangle$ and $\langle 0 \rangle$. On this new sequence, we apply the process of removing $\langle 0 \rangle$'s defined in the case (A) until no zeros are left. Finally, we change every sign in the resulting sequence. The sequence thus obtained is denoted by S^*, and the reduction $S \to S^*$ is our second operation τ_2.

It should be noted that we do not define τ_1 if $n = 2p + 1$, and τ_2 if $n = 2p - 1$. If $n = 2p + 1$, S consists of only $\langle \pm 2 \rangle$, and if $n = 2p - 1$, then all c_j's are 2, except c_1 and c_q (both of which are 1). Also, we do not define either reduction when $p = 1$.

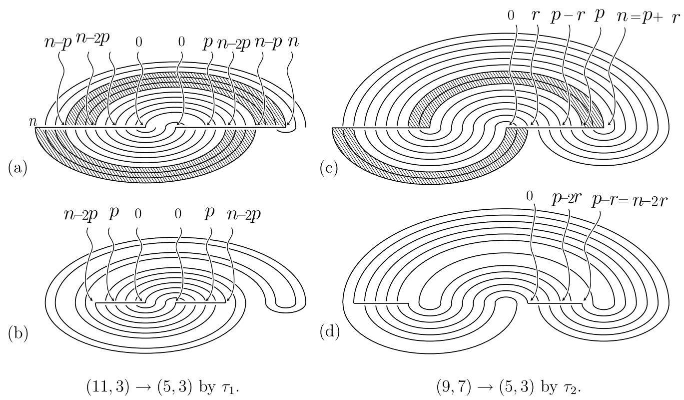


Fig. 7.1. Reductions by τ_1 and τ_2.

Now we will show that S^* is the sequence of signs for some pair of co-prime integers n^*, p^*, where $n^* > |p^*| > 0$.

Proposition 7.1. *Write $n = mp + r$, where $m \geq 1$ and $p > r > 0$. Let S be the sequence of signs of (n, p). Then we have the following*:

CASE (A) $m \geq 2$. *$S^* = \tau_1(S)$ is the sequence of signs of $(n^*, p^*) = (n - 2p, p)$. Here, if $n - 2p < p$, then $(n - 2p, p)$ is interpreted as $(n - 2p, p - 2(n - 2p))$.*

CASE (B) $m = 1$. *$S^* = \tau_2(S)$ is the sequence of signs of $(n^*, p^*) = (n - 2r, p - 2r)$.*

REMARK 7.2. Even if $p^* < 0$, the original definition in Section 4 is applied to obtain the sequence of signs. In this case, $\varepsilon_1 = -1$.

Proof of Proposition 7.1. To prove Proposition 7.1, we use a well-known fact that the sequence S of signs of (n, p) is obtained from Schubert's normal form of a 2-bridge knot $B(n, p)$. We note that the i-th sign ε_i of S is +1 if and only if the curve joining two points $\overline{(i-1)p}$ and $\overline{ip}$ underpasses the right (resp. left) bridge from the upper part (resp. lower part). Consider Schubert's normal form of a 2-bridge knot $B(n, p)$.

Case (A). We see the removal of consecutive two same signs $\{+1, +1\}$ or $\{-1, -1\}$ from each block $\langle \pm m\rangle$ and $\langle \pm(m + 1)\rangle$ in S corresponds to the delation of arcs on the boundaries of the shaded regions in Fig. 7.1 (a). The result is shown in Fig. 7.1 (b), where the partial overpath, i.e., the bridges, connecting the points $n - 2p$ to n have been removed. We can naturally connect the remaining arcs. Then (n^*, p^*) is easily determined as follows. By the above operation τ_1, $2p$ points are removed from each overpath of the old normal form, and hence $n^* = n - 2p$. Further, since 0 is connected to p in the new form (and the old form, too), it follows that $p^* = p$. This proves Proposition 7.1 for Case (A).

Case (B). Removal of $\langle 2\rangle, \langle -2\rangle$ and $\langle c_1\rangle, \langle \pm c_q\rangle$ from S corresponds to elimination of the arcs on the boundary of the shaded regions in Fig. 7.1 (c). The result is

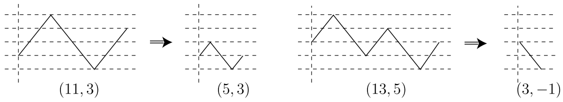


Fig. 7.2.

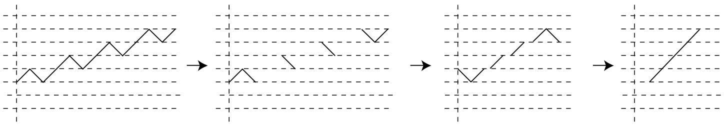


Fig. 7.3.

Fig. 7.1 (d), where partial overpaths connecting points 0 to r, and p to n (= $p+r$) have also been removed. We can naturally connect the remaining arcs.

Before we determine (n^*, p^*) for Case (B), we first note the following fact. Let $S = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}\}$ be the sequence of signs for (n, p), where ε_i is defined at the beginning of the proof, when we follow the underpath starting at 0. However, if we follow the underpath starting at the end point n $(= p+r)$ of the right bridge, then the sequence $\hat{S} = \{\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_{n-1}\}$ of signs obtained in the same way as before is equal to $-S$. Because, for any $i = 1, 2, \ldots, n-1$, we see $\varepsilon'_i = \overline{n+ip}\big/\big|\overline{n+ip}\big|$ and $\varepsilon_i = \overline{ip}\big/\big|\overline{ip}\big|$.

Now we return to the proof for Case (B). Since $2r$ points are removed from each bridge, we have $n^* = n - 2r = p - r$. Further, the old point $p - r$ in the old normal form now corresponds to $p - 2r$ in the new normal form, and hence $p^* = p - 2r$. Finally, from the above remark, it is evident that the new sequence of signs for (n^*, p^*) is exactly $\tau_2(S)$. □

EXAMPLE 7.3. $(n, p) = (11, 3)$, $n = 11 = 3 \times 3 + 2$, $S = \{\langle 3\rangle\langle -4\rangle\langle 3\rangle\}$. Therefore, $\tau_1 \colon S \to S^* = \{\langle 1\rangle\langle -2\rangle\langle 1\rangle\}$, $(n^*, p^*) = (5, 3)$. On the deformation of the graph, see Proposition 7.8 in Subsection 7.2 and Fig. 7.2.

EXAMPLE 7.4. $(n, p) = (13, 5)$, $n = 13 = 2 \times 5 + 3$, $S = \{\langle 2\rangle\langle -3\rangle\langle 2\rangle\langle -3\rangle\langle 2\rangle\}$. Therefore, $\tau_1 \colon S \to S^* = \{\langle 0\rangle\langle -1\rangle\langle 0\rangle\langle -1\rangle\langle 0\rangle\} = \{\langle -2\rangle\}$, $(n^*, p^*) = (3, 5) \to (3, -1)$. See Fig. 7.2.

EXAMPLE 7.5. $(n, p) = (13, 9)$, $n = 13 = 1 \times 9 + 4$, $S = \{\langle 1\rangle\langle -1\rangle\langle 2\rangle\langle -1\rangle\langle 2\rangle\langle -1\rangle\langle 2\rangle \langle -1\rangle\langle 1\rangle\}$. Therefore, $\tau_2 \colon S \to \{\langle 1\rangle\langle -1\rangle\langle 0\rangle\langle -1\rangle\langle 0\rangle\langle -1\rangle\langle 0\rangle\langle -1\rangle\langle 1\rangle\} \to \{\langle 1\rangle\langle -4\rangle\langle 1\rangle\} \to S^* = \{\langle 4\rangle\}$ $(n^*, p^*) = (5, 1)$. See Fig. 7.3.

7.2. Reduction on $G(n,p)$. In this subsection, we study the change of the graph $G(n, p)$ by applications of τ_1 and τ_2. This is an important step to our inductive proof of Theorem 5.5.

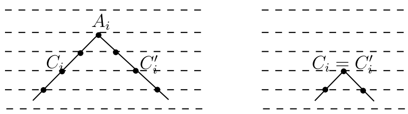


Fig. 7.4.

First, we introduce a few terminologies.

DEFINITION 7.6. A vertex P of $G(n, p)$ is called a *peak* if P is a local maximal vertex. Two peaks P, P' are called *consecutive* if the path in G connecting P and P' has no peaks other than P and P'. We call P and P' *neighboring peaks*. Every peak P has two neighboring peaks unless P is the end of G. A set of consecutive peaks, $P_1, P_2, \ldots, P_k$ of the same level is called a *block* (*of peaks*) if neither the preceding peak to P_1 nor the following peak from P_k is on the same level as P_i. Or equivalently, $\{P_1, \ldots, P_k\}$ is not a proper subset of another set of consecutive peaks containing $\{P_1, \ldots, P_k\}$. In particular, if the peak preceding to P_1 and the following peak from P_k have strictly lower level, then the block is said to be of *maximal type*. A vertex V is called *even* (resp. *odd*) if $\mathrm{lev}(V)$ is even (resp. odd).

Analogously, we call a local minimal vertex a *bottom*. A *block of bottoms*, and a *block of minimal type* are also easily understood.

Proposition 7.7. *Let $n = mp + r$, where $m \geq 2$ and $0 < r < p$. Let S be a sequence of signs of (n, p) and $G(n, p)$ the graph of (n, p). Then the graph of $\tau_1(S) = S^*$ is obtained as follows: Let $A_1, A_2, \ldots, A_a$ be all the peaks of $G(n, p)$. First remove two consecutive edges before and after A_i $(i = 1, 2, \ldots, a)$, and then identify two vertices C_i and C_i', the ends of the edges.* (*See* Fig. 7.4.) *The graph thus obtained is the graph of $G(n^*, p^*)$.*

Proof. Evident from the construction. $\square$

Proposition 7.8. *Under the same notation in* Proposition 7.7, *assume that $m = 1$ i.e., $n = p + r$. Then $G(n^*, p^*)$, the graph of $\tau_2(S) = S^*$, is obtained as follows:*
(i) *Remove all two consecutive (upgoing or downgoing) edges so that $G(n, p)$ is decomposed into several connected components $G_1, G_2, \ldots, G_d$.*
(ii) *Each connected component G_i is in between, say, level l and level $l + 1$ (see* Fig. 7.5). *Then each G_i is reflected along the central horizontal line between level l and $l + 1$, to get $\tilde{G}_i$. Therefore, the initial vertex of $\tilde{G}_i$ and the last vertex of $\tilde{G}_{i-1}$ are on the same level, and hence they are identified on this level. Thus we obtain a connected graph $\tilde{G}$.*
(iii) *Remove the first and the last edges from $\tilde{G}$. The graph thus obtained is $G(n^*, p^*)$.*

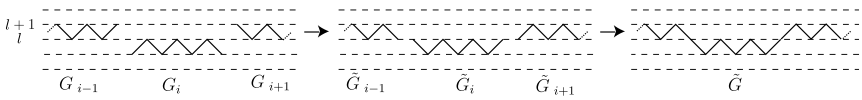


Fig. 7.5.

Proof. This also follows from the construction. (Recall Examples 7.3–7.5.) □

8. Proof of Theorem (IV): Index

In this section, we prove a few lemmas on the index of peaks and bottoms.

To each proposition about peaks, we can also prove, in a similar fashion, the corresponding proposition about bottoms. Therefore, we only state the propositions about bottoms without proof.

We begin with the following easy lemma without proof.

Lemma 8.1. *Let V be a non-peak vertex of $G(n,p)$, and let P be the peak before or after V. Then we have*:

$$\mathrm{lev}(V) < \mathrm{lev}(P) \quad \textit{and} \quad \mathrm{ind}(V) \leq \mathrm{ind}(P).$$

Lemma 8.2. *If B is a vertex, not a bottom, of $G(n, p)$ and if Q is a bottom before or after B, then* $\mathrm{lev}(B) > \mathrm{lev}(Q)$ *and* $\mathrm{ind}(B) \geq \mathrm{ind}(Q)$.

Let S be the sequence of signs of (n, p): $S = \{\langle a_1\rangle\langle -b_1\rangle \cdots \langle a_l\rangle\langle -b_l\rangle\}$. We note that a peak P of $G(n,p)$ is a turning point from an up-going path corresponding to, say $\langle a_i\rangle$ to down-going path corresponding to $\langle -b_i\rangle$. To illustrate this, we write $G(n, p)$ as

$$G = \langle a_1\rangle P_1\langle -b_1\rangle\langle a_2\rangle P_2\langle -b_2\rangle \cdots \langle a_l\rangle P_l\langle -b_l\rangle,$$

where $P_1, P_2, \ldots, P_l$ are peaks.

Lemma 8.3. *Let $\mathcal{P} = \{P_i, P_{i+1}, \ldots, P_j\}$ be a block of peaks of $G(n, p)$. Write the part of $G(n, p)$ involving $\mathcal{P}$ as $\cdots P_{i-1}\langle -b_{i-1}\rangle\langle a_i\rangle P_i\langle -b_i\rangle \cdots \langle a_j\rangle P_j\langle -b_j\rangle\langle a_{j+1}\rangle P_{j+1}\cdots$.*

Then we have the following:

(1) *If b_i is even, then* $\mathrm{ind}(P_i) = \mathrm{ind}(P_{i+1}) = \cdots = \mathrm{ind}(P_j)$.

(2) *Suppose b_i is odd. Then*

(2-1) *If P_i (and hence all $P_k, 1 \leq k \leq j$) is an even peak, then* $\mathrm{ind}(P_k) = \mathrm{ind}(P_i) - (k-i)$, *for any k $(= i, i+1, \ldots, j)$.*

(2-2) *If P_i is an odd peak, then* $\mathrm{ind}(P_k) = \mathrm{ind}(P_i) + (k-i)$, *for any k $(= i, i+1, \ldots, j)$.*

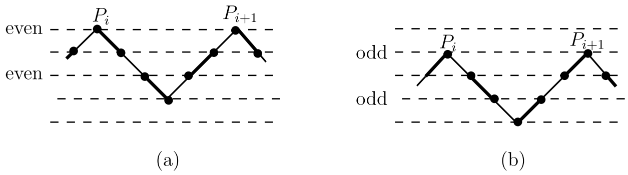


Fig. 8.1. Thick edges indicate odd edges.

Lemma 8.4. *Let $\mathcal{Q} = \{Q_l, Q_{l+1}, \ldots, Q_q\}$ be a block of bottoms. Write the part of $G(n, p)$ involving $\mathcal{Q}$ as $Q_{l-1}\langle a_l\rangle\langle -b_l\rangle Q_l\langle a_{l+1}\rangle \cdots \langle -b_q\rangle Q_q\langle a_{q+1}\rangle$. Then we have the following*:

(1) *If b_l is even, then* $\mathrm{ind}(Q_l) = \mathrm{ind}(Q_{l+1}) = \cdots = \mathrm{ind}(Q_q)$.

(2) *Suppose b_l is odd, then we have*:

(2-1) *If Q_l (and hence all $Q_k, l \le k \le q$) is an even bottom then*

$$\mathrm{ind}(Q_k) = \mathrm{ind}(Q_l) + (k - l), \quad \textit{for any} \quad k \ (= l, l+1, \ldots, q).$$

(2-2) *If Q_l is an odd bottom, then*

$$\mathrm{ind}(Q_k) = \mathrm{ind}(Q_l) - (k - l), \quad \textit{for any} \quad k \ (= l, l+1, \ldots, q).$$

Proof of Lemma 8.3. First we note that $b_i = a_{i+1} = \cdots = b_{j-1} = a_j$. Now (1) is obvious.

(2-1) In the path joining P_i to P_{i+1}, there are exactly $(b_i + 1)/2$ downward odd edges and $(a_{i+1} - 1)/2$ upward odd edges, and hence $\mathrm{ind}(P_{i+1}) = \mathrm{ind}(P_i) - 1$. (See Fig. 8.1 (a).) Inductively we obtain (2-1).

(2-2) If P_i is an odd peak, then the numbers of downward (resp. upward) odd edges is $(b_i - 1)/2$ (resp. $(a_{i+1} + 1)/2$). and hence $\mathrm{ind}(P_{i+1}) = \mathrm{ind}(P_i) + 1$. (See Fig. 8.1 (b).) Inductively, we obtain (2-2). □

Now suppose that Theorem 5.5 does not hold. Namely, there exists a vertex Y or a vertex Z of $G(n, p)$ such that

$$(8.1) \qquad \begin{cases} \text{(i)} & \mathrm{lev}(Y) \text{ is not maximum in } G(n, p), \\ \text{(ii)} & \mathrm{ind}(Y) \ge \mathrm{ind}(Y'), \text{ for any vertex } Y' \text{ of } G(n, p), \text{ and in particular,} \\ \text{(iii)} & \mathrm{ind}(Y) > \mathrm{ind}(Y') \text{ if } Y' \text{ is on the highest level.} \end{cases}$$

$$(8.2) \qquad \begin{cases} \text{(i)} & \mathrm{lev}(Z) \text{ is not minimum in } G(n, p) \\ \text{(ii)} & \mathrm{ind}(Z) \le \mathrm{ind}(Z'), \text{ for any vertex } Z' \text{ of } G(n, p), \text{ and in particular,} \\ \text{(iii)} & \mathrm{ind}(Z) < \mathrm{ind}(Z') \text{ if } Z' \text{ is on the lowest level.} \end{cases}$$

We see from Lemma 8.1 or 8.2 that Y must be a peak and that Z must be a bottom. Let Y_0 be a peak satisfying (8.1), and suppose that Y_0 is on the highest level among peaks satisfying (8.1) (Y_0 is one of the counter-examples to Theorem 5.5.)

Similarly, let Z_0 be a bottom satisfying (8.2) and suppose that Z_0 is on the lowest level among the bottoms satisfying (8.2).

We will prove such a peak Y_0 (or a bottom Z_0) does not exist. Our proof will be done by induction on the number of peaks of $G(n, p)$. However, first we prove Theorem 5.5 for special cases.

Lemma 8.5. Theorem 5.5 *holds for the following three special cases*: $n = mp+1$, $n = 2p - 1$ *and* $p = 1$.

Proof. For the first two cases, it follows from Proposition 6.1 (3), all peaks (and bottoms) except the ends have the same level. Therefore, a peak (resp. a bottom) satisfying (8.1) (resp. (8.2)) does not exist. For the last case, Theorem 5.5 holds trivially. □

Next, we study some properties Y_0 (or Z_0) should have.

Lemma 8.6. *Let* Y_0 *be a peak and a counter-example to* Theorem 5.5. *Let* $\mathcal{P} = \{P_i, P_{i+1}, \dots, P_j\}$ *be the block of peaks containing* Y_0. *Then* $\mathcal{P}$ *is of maximal type.*

Lemma 8.7. *Let* $\mathcal{Q} = \{Q_l, Q_{l+1}, \dots, Q_q\}$ *be the block of bottoms containing* Z_0. *Then* $\mathcal{Q}$ *is of minimal type.*

Proof of Lemma 8.6. First G contains a part,

$$P_{i-1}\langle -b_{i-1}\rangle\langle a_i\rangle P_i\langle -b_i\rangle \cdots \langle a_j\rangle P_j\langle -b_j\rangle\langle a_{j+1}\rangle P_{j+1},$$

and Y_0 is one of P_k above. Note that $b_i = a_{i+1} = \cdots = a_j$. Suppose that $\mathcal{P}$ is not of maximal type. Then the following two cases can occur.

(I) $\mathrm{lev}(P_{i-1}) > \mathrm{lev}(P_i)$.

(II) $\mathrm{lev}(P_j) < \mathrm{lev}(P_{j+1})$.

Since a proof is analogous, we only show that case (II) leads to a contradiction.

CASE (A). Suppose a_j is even. By Lemma 8.3, we see that $\mathrm{ind}(P_i) = \mathrm{ind}(P_{i+1}) = \cdots = \mathrm{ind}(P_j)$ and hence, we may assume $Y_0 = P_j$. Now since $\mathrm{lev}(P_j) < \mathrm{lev}(P_{j+1})$, the following two cases can occur.

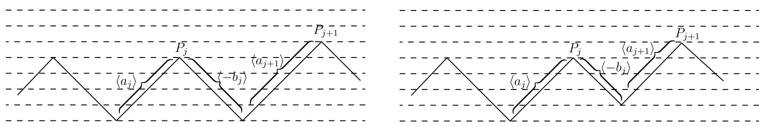
(A-1) $a_j = b_j$ and $a_{j+1} = a_j + 1$.

(A-2) $b_j = a_j - 1$ and $a_{j+1} = a_j$.

In either case, $\mathrm{ind}(P_{j+1}) \geq \mathrm{ind}(P_j) = \mathrm{ind}(Y_0)$. This contradicts the choice of Y_0. (Y_0 should be a counter-example to Theorem 5.5 with the maximal level.)

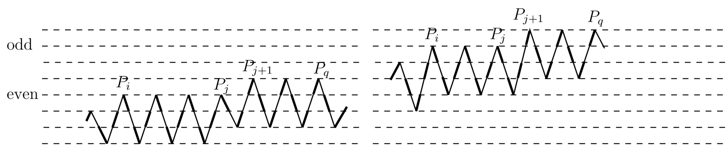
CASE (B). Suppose a_j is odd.

(1) If P_j is an odd peak, then $Y_0 = P_j$, since $\mathrm{ind}(P_i) < \mathrm{ind}(P_{i+1}) < \cdots < \mathrm{ind}(P_j)$. Then there are two cases to be considered.



(1-a) (1-b)

Fig. 8.2.



(2-a) (2-b)

Fig. 8.3.

(1-a) $b_j = a_j$ and $a_{j+1} = a_j + 1$.
(1-b) $b_j = a_j - 1$ and $a_{j+1} = a_j$.

For (1-a), $\mathrm{ind}(P_{j+1}) > \mathrm{ind}(P_j)$, and for (1-b), $\mathrm{ind}(P_{j+1}) = \mathrm{ind}(P_j)$. (See Fig. 8.2.)

Therefore, in either case, it contradicts the choice of Y_0.

Finally, we consider the case (2): P_j is an even peak. In this case, $Y_0 = P_i$, since $\mathrm{ind}(P_i) > \mathrm{ind}(P_{i+1}) > \cdots > \mathrm{ind}(P_j)$. This case requires a more careful observation.

Now we consider the block $\mathcal{P}'$ immediately after $\mathcal{P}$. Write $\mathcal{P}' = \{P_{j+1}, P_{j+2}, \ldots, P_q\}$. Then G has the following part:

$$\cdots P_{i-1}\langle -b_{i-1}\rangle\langle a_i\rangle P_i\langle -b_i\rangle \cdots \langle a_j\rangle P_j\langle -b_j\rangle\langle a_{j+1}\rangle P_{j+1}\langle -b_{j+1}\rangle \cdots \langle a_q\rangle P_q\langle -b_q\rangle \cdots .$$

There are two cases (as we considered before). (See Fig. 8.3.)

(2-a) $a_j > b_j$ and $a_j = a_{j+1} = b_{j+1}$:
(2-b) $a_j = b_j < a_{j+1}$ and $b_j = b_{j+1}$.

First we consider the case (2-a): If $a_i < b_i$, then $b_{i-1} = b_i$, and hence $\mathrm{ind}(P_{i-1}) \geq \mathrm{ind}(P_i) = \mathrm{ind}(Y_0)$, a contradiction. Therefore, $a_i = b_i$, and then $\langle a_i\rangle\langle -b_i\rangle \cdots \langle a_j\rangle$ in S represents $2(j-i)+1$ consecutive $\langle \pm(a_i)\rangle$'s. By Proposition 6.3, it must be followed by at least $2(j-i)$ consecutive $\langle \pm(a_i)\rangle$'s, that is $\langle a_{j+1}\rangle\langle -b_{j+1}\rangle \cdots \langle -b_q\rangle \cdots$, and hence, $q - j \geq j - i$. Since $\mathrm{ind}(P_{j+1}) = \mathrm{ind}(P_j) + 1$ and P_{j+1} is an odd peak in $\mathcal{P}'$, it follows that $\mathrm{ind}(P_q) = \mathrm{ind}(P_{j+1}) + (q - j - 1) = \mathrm{ind}(P_i) - (j - i) + (q - j) \geq \mathrm{ind}(P_i) = \mathrm{ind}(Y_0)$. This contradicts the choice of Y_0.

Next we consider the case (2-b): $\langle -b_i\rangle\langle a_{i+1}\rangle \cdots \langle -b_j\rangle$ in S represents also $2(j-i)+1$ consecutive $\langle \pm(a_{i+1})\rangle$'s, and it is followed by at least $2(j-i)$ consecutive $\langle \pm(a_{i+1})\rangle$'s. They are $\langle -b_{j+1}\rangle\langle a_{j+2}\rangle \cdots \langle a_q\rangle \cdots$ and hence $q - j - 1 \geq j - i$. Since P_{j+1} is an odd peak in $\mathcal{P}'$, we have $\mathrm{ind}(P_q) = \mathrm{ind}(P_{j+1}) + (q - j - 1)$, and also $\mathrm{ind}(P_{j+1}) = \mathrm{ind}(P_j)$. Therefore $\mathrm{ind}(P_q) = \mathrm{ind}(P_j) + (q - j + 1) = \mathrm{ind}(P_i) + (i - j) + (q - j - 1) \geq$

$\mathrm{ind}(P_i) = \mathrm{ind}(Y_0)$, a contradiction. □

Using Lemmas 8.6 and 8.7, we can prove Theorem 5.5 for another special case.

Proposition 8.8. *Suppose that both* $\langle m\rangle$ *and* $\langle m+1\rangle$ *are isolated. Then* Theorem 5.5 *holds*.

Proof. Since $\langle m\rangle$ and $\langle m+1\rangle$ are isolated, there is only one block of peaks of maximal type, (and one block of bottoms of minimal type) and hence Theorem 5.5 holds trivially. □

9. Proof of Theorem 5.5 (V): Conclusion

In this section, Theorem 5.5 is finally proved by induction on the number χ of peaks in $G(n,p)$.

In the initial case, where $\chi = 1$, it is obvious by Lemma 8.1 (or Lemma 8.2).

Suppose, for induction, Theorem 5.5 holds for any graph $G(n,p)$ with $\chi(G) < d$. Suppose $G(n,p)$ has exactly d peaks, $P_1, P_2, \dots, P_d$. Our proof is divided into two cases:

CASE A: $n = mp + r$, $m \geq 2$, $0 < r < p$

CASE B: $n = p + r$, $0 < r < p$.

Case A is further divided into two subcases:

CASE (A-1): $m \geq 3$,

CASE (A-2): $m = 2$.

First, consider subcase (A-1). By applying τ_1 on $G(n,p)$, we obtain a new graph $G^* = G(n^*, p^*)$, where $n^* = (m-2)p + r$, $m - 2 \geq 1$. Let $S = \{\langle a_1\rangle\langle -b_1\rangle \cdots \langle a_l\rangle\langle -b_l\rangle\}$ be the sequence of (n,p). Then the sequence S^* of the new pair (n^*, p^*) is $S^* = \{\langle a_1 - 2\rangle\langle -(b_1-2)\rangle\langle a_2 - 2\rangle \cdots \langle a_l - 2\rangle\langle -(b_l - 2)\rangle\}$. Since $a_i, b_j \geq 3$ for $1 \leq i,\ j \leq l$, $G(n^*, p^*)$ has as many peaks as $G(n,p)$ has.

Write:

$$G = \langle a_1\rangle P_1 \langle -b_1\rangle\langle a_2\rangle P_2 \langle -b_2\rangle \cdots \langle a_l\rangle P_l \langle -b_l\rangle,$$

and

$$G^* = \langle a_1^*\rangle P_1^* \langle -b_1^*\rangle\langle a_2^*\rangle P_2^* \langle -b_2^*\rangle \cdots \langle a_l^*\rangle P_l^* \langle -b_l^*\rangle,$$

where $a_i^* = a_i - 2$ and $b_i^* = b_i - 2$, $1 \leq i \leq l$.

Then $\mathrm{lev}(P_i) = \sum_{k=1}^{i} a_k + \sum_{k=1}^{i-1}(-b_i)$, while $\mathrm{lev}(P_i^*) = \sum_{k=1}^{i}(a_k - 2) + \sum_{k=1}^{i-1}(-(b_i - 2)) = \sum_{k=1}^{i} a_k - 2i + \sum_{k=1}^{i-1}(-b_k) + 2(i-1) = \mathrm{lev}(P_i) - 2$.

On the other hand, $\mathrm{ind}(P_i)$ is the sum of the 'odd terms,' i.e., the sum of all the i-th terms with i odd. in S, and $\mathrm{ind}(P_i^*)$ is the sum of the odd terms in S^*. Since one positive term in $\langle a_k\rangle$ is cancelled with one negative term in $\langle -b_k\rangle$, it follows that

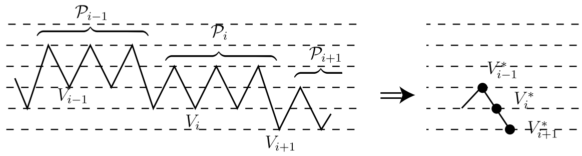


Fig. 9.1.

$\mathrm{ind}(P_i)-\mathrm{ind}(P_i^*)$ is equal to the difference between the number of positive signs in $\langle a_i\rangle$ and that of $\langle a_i^*\rangle$, that is 1, and hence $\mathrm{ind}(P_i^*) = \mathrm{ind}(P_i) - 1$. Since Y_0 ($= P_s$ for some s) is a counter-example of Theorem 5.5 for $G(n, p)$, $Y_0^* = P_s^*$ is also a counter-example for G^*.

(It is evident that if Q_k is a bottom of $G(n, p)$, then $\mathrm{lev}(Q_k^*) = \mathrm{lev}(Q_k)$ and $\mathrm{ind}(Q_k^*) = \mathrm{ind}(Q_k)$, and hence Z_0^* is also a counter-example to Theorem 5.5 for G^*.)

Repeated applications of τ_1 reduce m to 1 or 2.

SUBCASE (A-2): $m = 2$, i.e., $n = 2p + r$.

For the subcase (A-2), and the case B, the number of peaks $\chi(G)$ may be decreased by applications of τ_1 or τ_2. Therefore, we need a careful examination of the levels and indices of the peaks for G and for $G^* = \tau_1(G)$ or $\tau_2(G)$.

Now we are concerned about the highest peaks and the particular peak Y_0. These peaks belong to some blocks of maximal type. Therefore, we study how theses peaks behave under τ_1 and τ_2. First we divide the set of peaks of G into blocks. Let us denote them by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$.

DEFINITION 9.1. We define the level and index of a block $\mathcal{P}_i$ as

$$\begin{cases} \mathrm{lev}(\mathcal{P}_i) = \mathrm{lev}(P), & \text{for any} \quad P \in \mathcal{P}_i \\ \mathrm{ind}(\mathcal{P}_i) = \max_{P\in\mathcal{P}_i}\{\mathrm{ind}(P)\}. & \end{cases} \tag{9.1}$$

Now we consider subcase (A-2):

SUBCASE (A-2): There are two cases to be considered.

CASE (A-2-1): $\langle\pm3\rangle$ is isolated, and hence $\langle\pm2\rangle$ occurs consecutively. (The case where both $\langle2\rangle$ and $\langle3\rangle$ are isolated is excluded by Proposition 8.8) Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$ be the blocks of peaks of $G(n, p)$. Then, an application of τ_1 collapses all peaks in each $\mathcal{P}_i$ to one vertex V_i^* (not necessarily a peak) of G^*. See Fig. 9.1.

(Note that if $\mathcal{P}_i$ is of maximal type, then V_i^* is a peak.) Further in this case, we have $\mathrm{ind}(\mathcal{P}_i) = \mathrm{ind}(P)$ and $\mathrm{lev}(\mathcal{P}_i) = \mathrm{lev}(P)$ for any $P \in \mathcal{P}_i$, and $\mathrm{lev}(V_i^*) = \mathrm{lev}(\mathcal{P}_i) - 2$. Also, we see that $\mathrm{ind}(V_i^*) = \mathrm{ind}(\mathcal{P}_i) - 1$.

Since the highest peak P_h of $G(n, p)$ collapses to a highest peak P_h^* of G^*, and since Y_0 also collapses to a peak Y_0^* of G^*, it follows again that Y_0^* is a counter-example to Theorem 5.5 for G^*. However $\chi(G^*) < \chi(G)$, (since each block contains at least two peaks), and hence by induction hypothesis, such a peak Y_0^* does not exist, a contradiction.

Fig. 9.2.

(For the bottoms, similarly, we see that some bottoms $Q_l, \dots, Q_q$ collapse to one bottom Q_l^*, but $\mathrm{lev}(Q_l^*) = \mathrm{lev}(\mathcal{Q})$ and $\mathrm{ind}(\mathcal{Q}_l^*) = \mathrm{ind}(\mathcal{Q})$. Therefore, Z_0^* is again a counter-example to Theorem 5.5, a contradiction.)

CASE (A-2-2): $\langle \pm 2 \rangle$ is isolated.

Let $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_t$ be blocks of peaks of $G(n, p)$. Suppose that $\mathcal{P}_i$ is of maximal type. It is, for example, of the form:

$$\cdots \langle 3\rangle\langle -2\rangle\langle 3\rangle\langle -3\rangle \cdots \langle 3\rangle\langle -3\rangle\langle 2\rangle\langle -3\rangle \cdots .$$

By an application of τ_1, it is reduced to

$$\begin{aligned} &\cdots \langle 1\rangle\langle 0\rangle\langle 1\rangle\langle -1\rangle\langle 1\rangle\langle -1\rangle\langle 1\rangle\langle -1\rangle\langle 0\rangle\langle -1\rangle \cdots \\ &= \cdots \langle 2\rangle\langle -1\rangle\langle 1\rangle\langle -1\rangle\langle 1\rangle\langle -2\rangle \cdots . \end{aligned}$$

Other cases are similar, and we see $\mathrm{lev}(\mathcal{P}_i^*) = \mathrm{lev}(\mathcal{P}_i) - 2$. (See Fig. 9.2.)

The number of peaks in $\mathcal{P}_i$ is equal to that of $\mathcal{P}_i^*$, but two peaks, the one that precedes the first peak and the other that follows the last peak of $\mathcal{P}_i$, are not peaks in G^*. Thus the total number of peaks in G is decreased at least by 2. Since elimination of edges of G always occurs in pairs, we see $\mathrm{ind}(\mathcal{P}_i^*) = \mathrm{ind}(\mathcal{P}_i) - 1$. Therefore, $\tau_1(Y_0) = Y_0^*$ is also a counter-example for G^*. However, since $\chi(G^*) < \chi(G)$, such a peak Y_0^* does not exist, by induction hypothesis.

(For the bottoms, we see again $\mathrm{lev}(\mathcal{Q}_i^*) = \mathrm{lev}(\mathcal{Q}_i)$ and $\mathrm{ind}(\mathcal{Q}_i^*) = \mathrm{ind}(\mathcal{Q}_i)$, and hence induction works.)

Finally, we consider the case B: $n = p + r$.

In S, a_i and b_i are either 1 or 2. There are two subcases.

SUBCASE (B-1). $\langle \pm 1 \rangle$ is isolated.

SUBCASE (B-2). $\langle \pm 2 \rangle$ is isolated.

Consider subcase (B-1). Since $\langle \pm 1 \rangle$ is isolated, $\langle \pm 2 \rangle$ appears consecutively. Let $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_t$ be blocks of peaks for $G(n, p)$. Then we can write S as: $S = \langle \delta_0\rangle \mathcal{P}_0 \langle \delta_1\rangle \mathcal{P}_1 \langle \delta_2\rangle \mathcal{P}_2 \langle \delta_3\rangle \cdots \mathcal{P}_t \langle \delta_{t+1}\rangle$, where $\delta_0 = 1$ and $\delta_j = \pm 1$, $j = 1, \dots, t$. Applying τ_2 on S, we obtain a new sequence: $S^* = \langle -\delta_1\rangle\langle -\delta_2\rangle\langle -\delta_3\rangle \cdots \langle -\delta_t\rangle$, because each $\mathcal{P}_i$ involves only $\langle \pm 2 \rangle$. Now we study the level and the index of peaks in G and G^*. Suppose $\mathcal{P}_i$ is of maximal type. (See Fig. 9.3.) Then $\delta_i = -1$ and $\delta_{i+1} = +1$. Also, for any peak P in $\mathcal{P}_i$, we have

$$\begin{cases} \mathrm{lev}(\mathcal{P}_i) = \mathrm{lev}(P) \quad \text{and} \\ \mathrm{ind}(\mathcal{P}_i) = \mathrm{ind}(P). \end{cases}$$

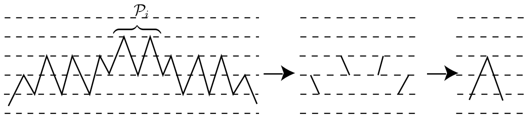
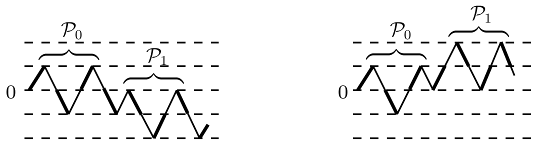


Fig. 9.3.



(a) $\delta_1 = 1$ (b) $\delta_1 = -1$

Fig. 9.4.

Now by τ_2, all peaks in $\mathcal{P}_i$ collapse to one vertex V_i^* in G^*. Since $-\delta_i = 1$ and $-\delta_{i+1} = -1$, V_i^* is in fact a peak of G^*. Further, obviously, $\mathrm{lev}(V_i^*) = \mathrm{lev}(\mathcal{P}_i) - 1$.

We evaluate the index of $\mathcal{P}_i$.

Lemma 9.2. (1) $\mathrm{ind}(\mathcal{P}_0) = 1$.
(2) *For* $1 \leq k$, $\mathrm{ind}(\mathcal{P}_{2k-1}) = \mathrm{ind}(\mathcal{P}_{2k}) = 1 - (\delta_1 + \delta_3 + \cdots + \delta_{2k-1})$.

Proof. (1) is evident. To prove (2), suppose $\delta_1 = 1$. Since $\mathrm{ind}(\mathcal{P}_0) = 1$ and $\mathrm{lev}(\mathcal{P}_1) = \mathrm{lev}(\mathcal{P}_0) - 1$, we have $\mathrm{ind}(\mathcal{P}_1) = 1 - 1 = 0$. See Fig. 9.4 (a). If $\delta_1 < 0$, then $\mathrm{lev}(\mathcal{P}_1) = \mathrm{lev}(\mathcal{P}_0) + 1$, and hence $\mathrm{ind}(\mathcal{P}_1) = 2$. See Fig. 9.4 (b).

Next, we compute $\mathrm{ind}(\mathcal{P}_2)$. Since δ_2 is an odd term, δ_2 is always counted, but it is cancelled with the odd term in $\langle 2\rangle$ that follows $\langle 2\rangle$ or precedes $\langle \delta_2\rangle$. Therefore $\mathrm{ind}(\mathcal{P}_2) = \mathrm{ind}(\mathcal{P}_1) = 1 - \delta_1$. Now exactly the same argument proves the formula for general case. $\square$

Now by Lemma 9.2, $\mathrm{ind}(\mathcal{P}_i) = 1 - \delta_1 - \delta_3 - \cdots - \delta_{i'}$, where $i' = i$ or $i - 1$ so that i' is odd. Since V_i^* is a vertex of G^* on which $\mathcal{P}_i$ collapses, we have $\mathrm{ind}(V_i^*) = -\delta_1 - \delta_3 - \cdots - \delta_{i'}$. Therefore, $\mathrm{ind}(V_i^*) = \mathrm{ind}(\mathcal{P}_i) - 1$, and hence, $\tau_2(Y_0) = Y_0^*$ is also a counter-example for G^*. Since $\chi(G^*) < \chi(G)$, it is impossible.

(For the bottoms, we see $\mathrm{lev}(W_i^*) = \mathrm{lev}(\mathcal{Q}_i) + 1$, while $\mathrm{ind}(W_i^*) = \mathrm{ind}(\mathcal{Q}_i)$, and we can apply induction hypothesis.)

This eliminates the subcase (B1).

Finally, we consider the case (B-2): $\langle \pm 2\rangle$ is isolated. We compare S with S^*. Consider for example:

$$S = \underbrace{\langle 1\rangle\langle -1\rangle\langle 1\rangle\langle -1\rangle \cdots \langle 1\rangle\langle -1\rangle}_{\mathcal{P}_1}\langle 2\rangle\,\underbrace{\langle -1\rangle\langle 1\rangle\langle -1\rangle\langle 1\rangle \cdots \langle -1\rangle}_{\mathcal{P}_2}\langle 2\rangle\langle -1\rangle \cdots .$$

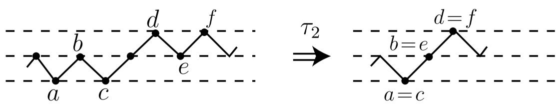


Fig. 9.5.

Then

$$S^* = \underbrace{\langle 1\rangle\langle -1\rangle \cdots \langle 1\rangle\langle -1\rangle\langle 2\rangle}_{\mathcal{P}_1^*} \underbrace{\langle -1\rangle\langle 1\rangle \cdots \langle -1\rangle\langle 2\rangle}_{\mathcal{P}_2^*}\langle -1\rangle \cdots,$$

where the number of peaks in $\mathcal{P}_i^*$ is exactly one less than that of $\mathcal{P}_i$. (If $\langle 2\rangle$ is replaced by $\langle -2\rangle$, the same argument works.)

Therefore, S^* is obtained from S by deleting $\langle +1\rangle\langle -1\rangle$ or $\langle -1\rangle\langle +1\rangle$ from each $\mathcal{P}_i$. This interpretation of τ_2 simplifies our proof considerably. Let G and G^* be the graphs of (n, p) and (n^*, p^*) respectively. The following lemma is evident:

Lemma 9.3. *If i is odd (resp. even), then $\mathcal{P}_i$ is of odd (resp. even) type.*

Therefore, we have: (1) $\mathrm{lev}(\mathcal{P}_i) = \mathrm{lev}(P)$ for any $P \in \mathcal{P}_i$, and (2) if i is odd (resp. even), then $\mathrm{ind}(\mathcal{P}_i)$ is equal to the index of the last peak (resp. first peak) of $\mathcal{P}_i$.

Now to obtain S^* from S, we drop the last pair $\langle +1\rangle\langle -1\rangle$ or $\langle -1\rangle\langle 1\rangle$ from $\mathcal{P}_{2i-1}$ and the first pair $\langle 1\rangle\langle -1\rangle$ or $\langle -1\rangle\langle 1\rangle$ from $\mathcal{P}_{2i}$. Graphically, it means that the last peak of $\mathcal{P}_{2i-1}$ and the first peak of $\mathcal{P}_{2i}$ are eliminated. Let P_i^* be a peak obtained from P_i by this 'new operation' τ_2. If $P_i = P_i^*$ under τ_2, i.e., P_i is not affected by τ_2, then $\mathrm{lev}(P_i) = \mathrm{lev}(P_i^*)$ and $\mathrm{ind}(P_i) = \mathrm{ind}(P_i^*)$. (See Fig. 9.5.) In fact, if ab is an odd edge of G, then de is also an odd edge.

If bc is an odd edge then so is ef. Thus these two edges are not counted in the evaluation of the index. By the same reasoning, we have $\mathrm{lev}(\mathcal{P}_i^*) = \mathrm{lev}(\mathcal{P}_i)$ and $\mathrm{ind}(\mathcal{P}_i^*) = \mathrm{ind}(\mathcal{P}_i) - 1$.

Now Y_0 belongs to some $\mathcal{P}_i$, and $\mathrm{ind}(Y_0) = \mathrm{ind}(\mathcal{P}_i)$. Therefore, Y_0 is eliminated by τ_2. However, a new block $\mathcal{P}_i^*$ also contains a Y_0^* with $\mathrm{ind}(Y_0^*) = \mathrm{ind}(\mathcal{P}_i^*)$, and hence $\mathrm{ind}(Y_0^*) = \mathrm{ind}(Y_0) - 1$. While, a peak P_0 in $\mathcal{P}_k$ on the highest level that has maximal index is also eliminated by τ_2. However, $\mathcal{P}_k^*$ contains another peak P_0^* on the highest level having the maximal index, and $\mathrm{ind}(P_0^*) = \mathrm{ind}(P_0) - 1$. Therefore Y_0^* is again a counter-example to Theorem 5.5 for G^*. Since $\chi(G^*) < \chi(G)$, such a Y_0^* does not exist, a contradiction.

(For the bottoms, we see $\mathrm{lev}(\mathcal{Q}^*) = \mathrm{lev}(\mathcal{Q})$ and that $\mathrm{ind}(\mathcal{Q}^*) = \mathrm{ind}(\mathcal{Q})$, and hence we can apply induction hypothesis.)

This proves Theorem 5.5. $\square$

10. Proof of Theorem B

In this section, we prove Theorem B and also Theorem 4.8. To prove them, we first study two polynomials $\phi_{(n,p\,|\,\alpha,\beta)}(t)$ and $h_{(n,p\,|\,\alpha,\beta)}(t)$ defined in §§4–5.

Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}\}$ be the sequence of signs for (n, p). We define two sequences of integers:

$$\begin{cases} \lambda_1 = 0 \\ \lambda_k = \varepsilon_1 + \varepsilon_3 + \cdots + \varepsilon_{2k-3}, \quad \text{for} \quad k = 2, 3, \ldots, l+1 = \dfrac{n+1}{2}. \end{cases}$$

$$\begin{cases} \mu_1 = 0 \\ \mu_k = \varepsilon_2 + \varepsilon_4 + \cdots + \varepsilon_{2k-2}, \quad \text{for} \quad k = 2, 3, \ldots, l+1 = \dfrac{n+1}{2}. \end{cases}$$

Now we recall that, for an arbitrary integer $p > 0$,

$$\phi_{(n,p\,|\,\alpha,\beta)}(t) = \sum_{k=1}^{l} (1 - t^{\alpha\varepsilon_{2k-1}}) t^{\alpha\lambda_k + \beta\mu_k} + t^{\alpha\lambda_{l+1} + \beta\mu_{k+1}}, \tag{10.1}$$

and for an even integer $p > 0$,

$$h_{(n,p\,|\,\alpha,\beta)}(t) = \sum_{k=1}^{l} (1 - t^{-\beta\varepsilon_{2k-1}}) t^{-\beta\lambda_k + \alpha\mu_k} + t^{-\beta\lambda_{l+1} + \alpha\mu_{l+1}}. \tag{10.2}$$

Most of the following propositions are immediate consequences of these definitions.

Proposition 10.1. *For any n and p with $n > p > 0$, and for any α and β, we have*:

$$\phi_{(n,p\,|\,\alpha,-\beta)}(t) = \phi_{(n,n-p\,|\,\alpha,\beta)}(t). \tag{10.3}$$

Proof. If $S = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}\}$ is the sequence of signs for (n, p), then the sequence $\widetilde{S}$ of signs for $(n, n-p)$, is $\widetilde{S} = \{\varepsilon_1, -\varepsilon_2, \varepsilon_3, -\varepsilon_4, \ldots, -\varepsilon_{n-1}\}$ (see Proposition 4.3). Therefore, (10.3) follows immediately. □

A similar argument proves the next proposition.

Proposition 10.2. *For any n and even q with $n > q > 0$, and for any α and β, we have*:

$$h_{(n,q\,|\,\alpha,\beta)}(t) = \phi_{(n,n-q\,|\,-\beta,-\alpha)}(t), \tag{10.4}$$

or equivalently

$$h_{(n,q\,|\,\alpha,\beta)}(t^{-1}) = \phi_{(n,n-q\,|\,\beta,\alpha)}(t). \tag{10.5}$$

Proof. If $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}\}$ is the sequence of signs for (n, q), then $\{\varepsilon_1, -\varepsilon_2, \varepsilon_3, \dots, -\varepsilon_{n-1}\}$ is the sequence of signs for $(n, n-q)$. Therefore, we have

$$\phi_{(n,n-q\,|\,-\beta,-\alpha)}(t) = \sum_{k=1}^{l}(1 - t^{-\beta\varepsilon_{2k-1}})t^{-\beta\lambda_k+\alpha\mu_k} + t^{-\beta\lambda_{l+1}+\alpha\mu_{l+1}} = h_{(n,q\,|\,\alpha,\beta)}(t). \qquad \square$$

REMARK 10.3. Proposition 10.2 claims not only that $h_{(n,q\,|\,\alpha,\beta)}(t)$ is equal to $\phi_{(n,n-q\,|\,-\beta,-\alpha)}(t)$, but also from Remarks 4.6 and 5.1 that the terms of the two polynomials are in one to one correspondence. Therefore, the Non-cancellation Theorem 5.5 for $G(n, p)$ implies that no cancellations occur among the highest and lowest vertices in $H(n, q \mid \alpha, \beta)$ with q even.

Proposition 10.4. *Suppose p is odd. Then for any α and β, we have*:

$$\Delta_{K(n,p\,|\,\alpha,\beta)}(t) \doteq \phi_{(n,p\,|\,\alpha,\beta)}(t)\phi_{(n,p\,|\,\alpha,\beta)}(t^{-1}) \tag{10.6}$$

$$\Delta_{K(n,p\,|\,\alpha,-\beta)}(t) \doteq \phi_{(n,n-p\,|\,\alpha,\beta)}(t)\phi_{(n,n-p\,|\,\alpha,\beta)}(t^{-1}) \tag{10.7}$$

Proof. For an odd integer p, it is shown in [7, Proposition 4.7] that $\Delta_{K(n,p\,|\,\alpha,\beta)}(t) = f_{(n,p)}(t)\, f_{(n,p)}(t^{-1})$, where $f_{(n,p)}$ is exactly $-\phi_{(n,p\,|\,\alpha,\beta)}(t)$. This proves (10.6). To prove (10.7), we note that, for p odd, $\Delta_{K(n,p\,|\,\alpha,-\beta)}(t) \doteq \phi_{(n,p\,|\,\alpha,-\beta)}(t)\phi_{(n,p\,|\,\alpha,-\beta)}(t^{-1})$. Now (10.7) follows from (10.3). $\square$

As an immediate consequence of Proposition 10.4, we obtain:

Proposition 10.5. *Suppose q is even. Then for any α and β, we have*:

$$\Delta_{K(n,q\,|\,\alpha,\beta)}(t) = \phi_{(n,n-q\,|\,\alpha,\beta)}(t)\phi_{(n,n-q\,|\,\alpha,\beta)}(t^{-1}) \tag{10.8}$$

$$\Delta_{K(n,q\,|\,\alpha,-\beta)}(t) = \phi_{(n,q\,|\,\alpha,\beta)}(t)\phi_{(n,q\,|\,\alpha,\beta)}(t^{-1}) \tag{10.9}$$

Proof. Since $n-q$ is odd, it follows from Propositions 3.1 and 10.4 that

$$\Delta_{K(n,q\,|\,\alpha,\beta)}(t) = \Delta_{K(n,n-q\,|\,\alpha,\beta)}(t) = \phi_{(n,n-q\,|\,\alpha,\beta)}(t)\phi_{(n,n-q\,|\,\alpha,\beta)}(t^{-1}).$$

This proves (10.8). Similarly, we have:

$$\begin{aligned}\Delta_{K(n,q\,|\,\alpha,-\beta)}(t) &= \Delta_{(n,n-q\,|\,\alpha,-\beta)}(t)\\ &= \phi_{(n,n-q\,|\,\alpha,-\beta)}(t)\phi_{(n,n-q\,|\,\alpha,-\beta)}(t^{-1}) = \phi_{(n,q\,|\,\alpha,\beta)}(t)\phi_{(n,q\,|,\alpha,\beta)}(t^{-1}). \qquad \square\end{aligned}$$

Proof of Theorem 4.8. By Proposition 3.1 we know that $K(n, p \,|\, \alpha, \beta) \cong K(n, n-q \mid \beta, \alpha) \cong -K(n, q \mid \beta, \alpha)$, and hence, $\Delta_{K(n,q\,|\,\alpha,\beta)}(t) = \Delta_{K(n,q\,|\,\beta,\alpha)}(t) = \Delta_{K(n,n-q\,|\,\beta,\alpha)}(t)$. Since $n-q$ is odd, we see further by Propositions 10.4 and (10.4) that

$$\Delta_{K(n,n-q\,|\,\beta,\alpha)}(t) = \phi_{(n,n-q\,|\,\beta,\alpha)}(t)\phi_{(n,n-q\,|\,\beta,\alpha)}(t^{-1}) = h_{(n,q\,|\,\alpha,\beta)}(t^{-1})h_{(n,q\,|\,\alpha,\beta)}(t). \qquad \square$$

Now we return to the proof of Theorem B. Suppose $\Delta_{K(n,p|\alpha,\beta)}(t)$, with p even, is monic. First assume $\alpha > \beta > 0$. By Proposition 3.1, $\Delta_{K(n,p|-\beta,-\alpha)}(t)$ is monic. Then Proposition 10.2, $h_{(n,p|-\beta,-\alpha)}(t) = \phi_{(n,n-p|\alpha,\beta)}(t)$ and by Theorem 4.8, they are monic, i.e., the terms with the highest and the lowest degree have coefficient ± 1. By Remark 5.7, in $\phi_{(n,n-p|\alpha,\beta)}(t)$, no cancellations occur among the terms with the highest or the lowest degree. Therefore, by Remark 10.3 and Remark 4.6, $H(K)$ is admissible. Next assume $\alpha > 0 > \beta$. Then by (10.4) and (10.3), $h_{(n,p|\alpha,\beta)}(t) = \phi_{(n,n-p|-\beta,-\alpha)}(t) = \phi_{(n,p|-\beta,\alpha)}(t)$, and by Theorem 4.8, they are monic. In the same manner as above, we see that $H(K)$ is admissible. This proves Theorem B. $\square$

11. Fibredness of $K(n, p \mid \alpha, \beta)$ via 2-bridge knot $B(n, p)$

11.1. Classification. In Section 3, we saw that for any $\alpha \neq 0$, $\Delta_{K(n,p|\alpha,-\alpha)}(t)$ is not monic, while $\Delta_{K(n,p|\alpha,\alpha)}(t)$ is monic if and only if $\Delta_{B(n,p)}(t)$ is monic, i.e., the 2-bridge knot $B(n, p)$ is fibred. However, even if $\Delta_{B(n,p)}(t)$ is not monic, it can happen that $\Delta_{K(n,p|\alpha,\beta)}(t)$ is monic for some α, β (see Remark 4.11).

In this section, we study how the fibredness of $K(n, p \mid \alpha, \beta)$ behaves if we fix (n, p) and take various (α, β)'s. From this view point, we classify $K(n, p \mid \alpha, \beta)$ into six types.

For more detailed discussions, the following propositions are useful.

Proposition 11.1. *Suppose $\alpha > 0$ and $\beta > 0$. Then we have*:
(1) $\Delta_{K(n,p|\alpha,\beta)}(t)$ *is monic if and only if* $\Delta_{K(n,p|2,1)}(t)$ *is monic*.
(2) $\Delta_{K(n,p|\alpha,-\beta)}(t)$ *is monic if and only if* $\Delta_{K(n,p|2,-1)}(t)$ *is monic*.

Proposition 11.2. *If $B(n, p)$ is fibred, then for any $\alpha > 0$ and $\beta > 0$, $\Delta_{K(n,p|\alpha,\beta)}(t)$ is monic*.

Proof of Proposition 11.1. Theorem 5.5 has been proved for arbitrary α, β with $\alpha > \beta > 0$, and $\alpha > 0 > \beta$. Therefore, if $\Delta_{K(n,p|2,1)}$ is monic, $\Delta_{K(n,p|\alpha,\beta)}$ must be monic for any α, β, with $\alpha > \beta > 0$. The same holds for $\alpha > 0 > \beta$. $\square$

Proof of Proposition 11.2. Since $B(n, p)$ is fibred, $\Delta_{K(n,p|1,1)}$ is monic. It means that the graph $G(n, p)$ has only one peak with the highest level, and one bottom with the lowest level. This follows from the fact that when $\alpha = \beta = 1$, the number of vertices of G with level h is the absolute value of the coefficient of t^h in $\Delta_{B(n,p)}(t)$. Therefore, by Theorem 5.5, for any $\alpha, \beta > 0$, $\Delta_{K(n,p|\alpha,\beta)}$ is monic. $\square$

By Proposition 3.1, we assume without loss of generality that p is always even in $K(n, p \mid \alpha, \beta)$. We see that the monicity of the Alexander polynomial (and hence fibredness) of our knots $K(n, p \mid \alpha, \beta)$ behaves in one of the six patterns with respect to various values of α and β. By Proposition 11.1 and 11.2, the pattern is determined by the pair (n, p), listed below, where in each pattern, we give an example of a pair (n, p).

Class (A): $B(n, p)$ is fibred.

(1) $K(n, p \mid \alpha, \beta)$ is fibred $\Longleftrightarrow \alpha \neq -\beta$. e.g., $(n, p) = (25, 18)$.

(2) $K(n, p \mid \alpha, \beta)$ is fibred $\Longleftrightarrow \alpha\beta > 0$. e.g., $(n, p) = (5, 2)$.

Class (B): $B(n, p)$ is not fibred.

(1) $K(n, p \mid \alpha, \beta)$ is fibred $\Longleftrightarrow \alpha \neq \pm\beta$. e.g., $(n, p) = (7, 2)$.

(2) $K(n, p \mid \alpha, \beta)$ is fibred $\Longleftrightarrow \alpha\beta > 0$, $\alpha \neq \beta$. e.g., $(n, p) = (9, 2)$.

(3) $K(n, p \mid \alpha, \beta)$ is fibred $\Longleftrightarrow \alpha\beta < 0$, $\alpha \neq -\beta$. e.g., $(n, p) = (17, 10)$.

(4) $K(n, p \mid \alpha, \beta)$ is not fibred for any α, β e.g., $(n, p) = (9, 4)$.

REMARK 11.3. In [7, Proposition 5.3], it was proved that $K(n, n-1 \mid \alpha, \beta)$ belongs to Class A1, by studying the fundamental groups of explicitly constructed Seifert surfaces and their complement.

Note that even if (n, p) and (n', p') are different, as 2-bridge knots $B(n, p)$ may be equivalent to $B(n', p')$, but $K(n, p \mid \alpha, \beta)$ and $K(n', p' \mid \alpha, \beta)$ may belong to different classes (see Remark 4.11).

At the end of this section, we present an algorithm to determine to which class a knot $K(n, p \mid \alpha, \beta)$ belongs. Using this algorithm, we can characterize the pairs (n, p) for each class. (See Theorem 11.15.)

11.2. Classes [I] and [II]. We begin with a definition of two new classes [I] and [II].

DEFINITION 11.4. We say that the pair (n, p) belongs to [I] (resp. [II]) if $\Delta_{K(n,p \mid \alpha,\beta)}(t)$ is monic for any $\alpha, \beta > 0$, $\alpha \neq \beta$ (resp. $\Delta_{K(n,p \mid \alpha,-\beta)}(t)$ is monic for any $\alpha, \beta > 0$, $\alpha \neq \beta$).

A pair (n, p) may belong to both [I] and [II], or neither [I] nor [II]. For example, if $B(n, p)$ is fibred, then (n, p) at least belongs to [I], and probably to [II] as well. If a fibred knot $B(n, p)$ belongs to both [I] and [II], then $K(n, p \mid \alpha, \beta)$ belongs to Class A1.

REMARK 11.5. Definition 11.4 can be rephrased in terms of $\phi_{(n,q \mid \alpha,\beta)}(t)$ as follows: Suppose q is an even integer. Then (n, q) belongs to [I] (simply $(n, q) \in$ [I]), if $\phi_{(n,n-q \mid \alpha,\beta)}(t)$ is monic for any $\alpha, \beta > 0$ with $\alpha \neq \beta$. On the other hand, $(n, q) \in$ [II], if $\phi_{(n,q \mid \alpha,\beta)}(t)$ is monic for any $\alpha, \beta > 0$ with $\alpha \neq \beta$.

Now one of the important consequences of the Non-cancellation theorem is that if the graph $G(n, p)$ satisfies certain conditions, the monic property of the polynomial $\phi_{(n,p \mid \alpha,\beta)}(t)$ is preserved under reduction operations τ_1 and τ_2. More precisely, we obtain:

Proposition 11.6. *Let p be an arbitrary positive integer.*
(I) *Let $n = mp + r$, where $m \geq 2$ and $0 < r < p$. Let $\tau_1 \colon (n, p) \to (n^*, p^*)$ be the first reduction operation where $(n^*, p^*) = (n - 2p, p)$.*

(1) *Assume $m \geq 3$. Then for any $\alpha > \beta > 0$, $\phi_{(n,p|\alpha,\beta)}(t)$ is monic if and only if $\phi_{(n^*,p^*|\alpha,\beta)}(t)$ is monic.*

(2) *Assume $m = 2$.*

(i) *Suppose $\langle \pm 2 \rangle$ is isolated in $S(n, p)$, the sequence of signs of the pair (n, p). Then for any $\alpha > \beta > 0$, $\phi_{(n,p|\alpha,\beta)}(t)$ is monic if and only if $\phi_{(n^*,p^*|\alpha,\beta)}(t)$ is monic.*

(ii) *Suppose $\langle \pm 2 \rangle$ is not isolated in $S(n,p)$ (and hence $\langle \pm 3 \rangle$ is isolated). Then for any $\alpha > \beta > 0$, $\phi_{(n,p|\alpha,\beta)}(t)$ is not monic.*

(II) *Let $n = p + r$, where $0 < r < p$. Let $\tau_2 \colon (n, p) \to (n^*, p^*)$ be the second reduction operation, where $n^* = n - 2r$ and $p* = p - 2r$.*

(i) *Suppose that a $\langle \pm 2 \rangle$ is isolated in $S(n, p)$, or the first three or four terms of $S(n, p)$ are of the form $S = \{\langle 1 \rangle \langle -2 \rangle \langle 1 \rangle \cdots\}$, or $S = \{\langle 1 \rangle \langle -2 \rangle \langle 2 \rangle \langle -1 \rangle \cdots\}$. Then for any $\alpha > \beta > 0$, $\phi_{(n,p|\alpha,\beta)}(t)$ is monic if and only if $\phi_{(n^*,p^*|\alpha,\beta)}(t)$ is monic.*

(ii) *Suppose $\langle \pm 2 \rangle$ always appears in $S(n, p)$ three (or more) times consecutively, or equivalently that S is of the form*

$$\{\langle 1 \rangle \underbrace{\langle -2 \rangle \langle 2 \rangle \cdots \langle \pm 2 \rangle}_{k \ \text{times} \ k \geq 3} \langle \mp 1 \rangle \cdots\}.$$

Then $\phi_{(n,p|\alpha,\beta)}(t)$ is not monic.

A proof is obtained easily from a careful study of the graph $G(n, p)$ as we did in the proof of Theorem 5.5, and hence the details are omitted.

From Proposition 6.4, we see that if $m \leq 2$, then $S(n, p)$ has one of the forms stated in Proposition 11.6.

We need a few more notations. Let q be an even integer with $n > q > 0$. The (even) continued fraction of n/q:

$$\frac{n}{q} = a_1 - \cfrac{1}{b_1 - \cfrac{1}{a_2 - \cfrac{1}{b_2 - \ddots - \cfrac{1}{a_s - \cfrac{1}{b_s}}}}}$$

will be denoted by $[a_1, b_1, a_2, b_2, \ldots, a_s, b_s]$, where a_i, b_i $(1 \leq i \leq s)$ are even $(\neq 0)$.

EXAMPLE 11.7.

$$\frac{101}{28} = [4, 2, -2, -6] = 4 - \cfrac{1}{2 - \cfrac{1}{(-2) - \cfrac{1}{(-6)}}},$$

$$\frac{141}{32} = [4, -2, 2, -6] = 4 - \cfrac{1}{(-2) - \cfrac{1}{2 - \cfrac{1}{(-6)}}}.$$

Let n/q and n_0/q_0, q, q_0 being even, be two rational numbers represented by an (even) continued fraction $A = [a_1, b_1, a_2, b_2, \dots, a_s, b_s]$ and its shorter fraction $B = [a_2, b_2, \dots, a_s, b_s]$. For convenience, we write $[a_1, b_1, a_2, b_2, \dots, a_s, b_s] = n/q$ and $[a_2, b_2, \dots, a_s, b_s] = n_0/q_0$.

Further, since we consider only even continued fractions, whenever we write

$[a_1, b_1, a_2, b_2, \dots, a_s, b_s] = n/q$, we always assume that q is even. We note, from the definition, that $nq > 0$ if and only if $a_1 > 0$. Using Proposition 11.6, we prove a series of propositions.

Proposition 11.8. *Let* $n/q = [a_1, b_1, a_2, b_2, \dots, a_s, b_s]$. *Suppose* $a_1 = 2k \geq 4$ *and let* $\hat{A} = [a_1 - 2, b_1, a_2, b_2, \dots, a_s, b_s] = \hat{n}/\hat{q}$. *Then we have*: $(n, q) \in$ [I] (*resp.* [II]) *if and only if* $(\hat{n}, \hat{q}) \in$ [I] (*resp.* [II]).

Proof. Let $n'/q' = [b_1, a_2, b_2, \dots, a_s, b_s]$. Suppose $b_1 > 0$. Then $n/q = 2k - q'/n' = (2kn' - q')/n'$ and $\hat{n}/\hat{q} = ((2k-2)n' - q')/n'$. Now $n = 2kn' - q' = (2k-1)n' + (n' - q') = (2k-1)q + (n' - q')$ and $0 < n' - q' < q$ $(= n')$. Since $2k - 1 \geq 3$ and $\tau_1(n, q) = (\hat{n}, \hat{q})$, we have: $(n, q) \in$ [II] if and only if $(\hat{n}, \hat{q}) \in$ [II].

For the case $b_1 < 0$, the same argument works. Next, we show that $(n, q) \in$ [I] if and only if $(\hat{n}, \hat{q}) \in$ [I], or equivalently, that $\phi_{(n,n-q \,|\, \alpha,\beta)}(t)$ is monic if and only if $\phi_{(\hat{n},\hat{n}-\hat{q} \,|\, \alpha,\beta)}(t)$ is monic. Now since $n/q = (2kn' - q')/n'$, we see $n/(n - q) = (2kn' - q')/((2k-1)n' - q')$. Suppose $b_1 > 0$. Then $n'q' > 0$. Since $k \geq 2$, it follows that $(2k-2)n' > q'$ and hence $(2k-1)n' - q' > n'$. Therefore, we can write $n = (n - q) + q$, $0 < q$ $(= n') < n - q$ $(= (2k-1)n' - q')$. We claim that $\langle \pm 2 \rangle$ is isolated in $S(n, n - q)$, or equivalently, that $\langle 1 \rangle$ is not isolated. However, it is now evident that $S = \{\langle 1 \rangle \langle -1 \rangle \cdots\}$, since $2n < 3(n - q)$. Therefore, by Proposition 11.6, $(n, q) \in$ [I] if and only if $(\hat{n}, \hat{q}) \in$ [I]. Since a similar argument works for the case $b_1 < 0$, the details will be omitted. □

By Proposition 11.8, we may assume, without loss of generality, that $a_1 = 2$ in order to decide whether or not $(n, p) \in$ [I] or [II]. Therefore, hereafter, we will use the following notations. For an arbitrary non-zero integer k, $A = [2, 2k, a_2, b_2, \dots, a_s, b_s] = n_k/q_k$, $B = [a_2, b_2, \dots, a_s, b_s] = n_0/q_0$.

Proposition 11.9. *Assume $k > 0$ and $a_2 > 0$. Then we have the following*:

(1) (a) $(n_k, q_k) \notin [\mathrm{I}]$, *if* $k \geq 2$.
(b) $(n_1, q_1) \in [\mathrm{I}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{I}]$.

(2) (a) $(n_k, q_k) \notin [\mathrm{II}]$, *if* $k \geq 3$.
(b) $(n_2, q_2) \in [\mathrm{II}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{II}]$.
(c) $(n_1, q_1) \in [\mathrm{II}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{II}]$.

Proof. First we see that, for $k > 0$, $n_k/q_k = 2 - 1/(2k - q_0/n_0) = ((4k-1)n_0 - 2q_0)/(2kn_0 - q_0)$, and hence $n_k = q_k + (2k-1)n_0 - q_0$, where $0 < (2k-1)n_0 - q_0 < q_k$ $(= 2kn_0 - q_0)$.

CASE (2). If $k \geq 3$, we can show $S = \{\langle 1\rangle\langle -2\rangle\langle 2\rangle\langle -2\rangle \cdots\}$, or equivalently, that $7q_k < 4n_k$. In fact, $4n_k - 7q_k = (16k-4)n_0 - 8q_0 - (14kn_0 - 7q_0) = (2k-4)n_0 - q_0 > 0$, since $k \geq 3$. Therefore $(n_k, q_k) \notin [\mathrm{II}]$ by Proposition 11.6 (II) (ii).

If $k = 2$, then $4n_2 < 7q_2$, but $3n_2 < 6q_2 < 4n_2$, and hence $S = \{\langle 1\rangle\langle -2\rangle\langle 2\rangle\langle -1\rangle \cdots\}$. Therefore, by Proposition 11.6 (I) (i), $(n_2, q_2) \in [\mathrm{II}]$ if and only if $(n_0, q_0) \in [\mathrm{II}]$.

If $k = 1$, then $n_1 = 3n_0 - 2q_0$ and $q_1 = 2n_0 - q_0$. Then $S = \{\langle 1\rangle\langle -1\rangle \cdots\}$, since $2n_1 < 3q_1$. Thus $\langle 2\rangle$ is isolated, and Proposition 11.6 (II) (i) shows that $(n_1, q_1) \in [\mathrm{II}]$ if and only if $(n_0, q_0) \in [\mathrm{II}]$.

Now to consider the case (1), we use $n_k/(n_k - q_k) = ((4k-1)n_0 - 2q_0)/((2k-1)n_0 - q_0)$. We write $n_k = 2(n_k - q_k) + n_0$, $0 < n_0 < n_k - q_k$. Then we see that if $k \geq 2$, $S = \{\langle 2\rangle\langle -2\rangle \cdots\}$, since $4(n_k - q_k) < 2n_k < 5(n_k - q_k)$, and hence, $(n_k, q_k) \notin [\mathrm{I}]$ if $k \geq 2$, by Proposition 11.6 (I) (ii).

If $k = 1$, then $n_1 = 3n_0 - 2q_0$ and $n_1 - q_1 = n_0 - q_0$. Write $q_0 = \lambda(n_0 - q_0) + r$, $0 < r < n_0 - q_0$, for some $\lambda \geq 0$ and r. Then $n_1 = (3 + \lambda)(n_0 - q_0) + r$ and hence $(n_1, q_1) \in [\mathrm{I}]$ if and only if $(n_0, q_0) \in [\mathrm{I}]$. $\square$

Proposition 11.10. *Suppose $k > 0$ and $a_2 < 0$. Then we have the following*:

(1) (a) $(n_k, q_k) \notin [\mathrm{I}]$, *if* $k \geq 2$.
(b) $(n_1, q_1) \in [\mathrm{I}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{I}]$.

(2) (a) $(n_k, q_k) \notin [\mathrm{II}]$, *if* $k \geq 2$.
(b) $(n_1, q_1) \in [\mathrm{II}] \Longleftrightarrow (n_0, -q_0) \in [\mathrm{II}]$.

Proof. We write $A_k = [2, 2k, -a_2, b_2, \ldots, a_s, b_s] = n_k/q_k$, $a_2 > 0$, and $B = [-a_2, b_2, \ldots, a_s, b_s] = -n_0/q_0$. Then $n_k/q_k = ((4k-1)n_0 + 2q_0)/(2kn_0 + q_0)$, and $n_k = q_k + (2k-1)n_0 + q_0$, $0 < (2k-1)n_0 + q_0 < q_k$. If $k \geq 2$, then $S = \{\langle 1\rangle\langle -2\rangle\langle 2\rangle\langle -2\rangle \cdots\}$, since $7q_k < 4n_k$. Thus, $(n_k, q_k) \notin [\mathrm{II}]$, if $k \geq 2$.

If $k = 1$, $n_1 = 3n_0 + 2q_0$ and $q_1 = 2n_0 + q_0$, and hence, $n_1 = q_1 + (n_0 + q_0)$, $0 < n_0 + q_0 < q_1$. Inequalities $3q_1 < 2n_1 < 4q_1 < 3n_1 < 5q_1$ imply that $S = \{\langle 1\rangle\langle -2\rangle\langle 1\rangle \cdots\}$ and hence, since $\tau_2(n_1, q_1) = (n_0, -q_0)$, $(n_1, q_1) \in [\mathrm{II}]$ if and only if $(n_0, -q_0) \in [\mathrm{II}]$,

Next, consider $n_k/(n_k - q_k)$, where $n_k = (4k-1)n_0 + 2q_0$ and $n_k - q_k = (2k-1)n_0 + q_0$. Thus $n_k = 2(n_k - q_k) + n_0$, $0 < n_0 < n_k - q_k$. If $k \geq 2$, then $S = \{\langle 2\rangle\langle -2\rangle \cdots\}$, since $4(n_k - q_k) < 2n_k < 5(n_k - q_k)$. Therefore $\langle \pm 2 \rangle$ is not isolated in $S(n_k, n_k - q_k)$ and $(n_k, q_k) \notin [\mathrm{I}]$, if $k \geq 2$.

If $k = 1$, then $n_1 = 3n_0 + 2q_0$ and $n_1 - q_1 = n_0 + q_0$, and hence $n_1 = 2(n_1 - q_1) + n_0$, $0 < n_0 < n_1 - q_1$. Since $5(n_1 - q_1) < 5n_1$, we see that $S = \{\langle 2 \rangle \langle -3 \rangle \cdots\}$, and hence by Proposition 6.4, $\langle \pm 2 \rangle$ is isolated in $S(n_1, n_1 - q_1)$. Therefore, from Proposition 11.6 (II) (i), it follows that $(n_1, q_1) \in [\mathrm{I}]$ if and only if $(n_0, q_0) \in [\mathrm{I}]$. □

Proposition 11.11. *Assume $a_2 > 0$ and $k > 0$. Then we have the following*:

(1) (a) $(n_{-k}, q_{-k}) \notin [\mathrm{I}]$, *if* $k \geq 2$.

(b) $(n_{-1}, q_{-1}) \in [\mathrm{I}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{I}]$.

(2) $(n_{-k}, q_{-k}) \notin [\mathrm{II}]$, *if* $k \geq 1$.

Proof. Note that $n_{-k}/q_{-k} = ((4k+1)n_0 + 2q_0)/(2kn_0 + q_0)$ and $n_{-k} = 2q_{-k} + n_0$, $0 < n_0 < q_{-k}$.

If $k \geq 1$, $S = \{\langle 2 \rangle \langle -2 \rangle \cdots\}$, since $2n_{-k} < 5q_{-k}$, and hence $(n_{-k}, q_{-k}) \notin [\mathrm{II}]$, if $k \geq 1$, by Proposition 11.6 (I) (ii).

Now consider $n_{-k}/(n_{-k} - q_{-k}) = ((4k+1)n_0 + 2q_0)/((2k+1)n_0 + q_0)$ and $n_{-k} = (n_{-k} - q_{-k}) + (2kn_0 + q_0)$, $0 < 2kn_0 + q_0 < n_{-k} - q_{-k}$.

If $k \geq 2$, then $S(n_{-k}, n_{-k} - q_{-k}) = \{\langle 1 \rangle \langle -2 \rangle \langle 2 \rangle \langle -2 \rangle \cdots\}$, since $7(n_{-k} - q_{-k}) < 4n_{-k}$. Therefore, $(n_{-k}, q_{-k}) \notin [\mathrm{I}]$ if $k \geq 2$.

If $k = 1$, then $n_{-1} = 5n_0 + 2q_0$ and $n_{-1} - q_{-1} = 3n_0 + q_0$, and $n_{-1} = (n_{-1} - q_{-1}) + (2n_0 + q_0)$, $0 < 2n_0 + q_0 < n_{-1} - q_{-1}$. Then $S = \{\langle 1 \rangle \langle -2 \rangle \langle 2 \rangle \langle -1 \rangle \cdots\}$, since $5(n_{-1} - q_{-1}) < 3n_{-1} < 6(n_{-1} - q_{-1}) < 4n_{-1} < 7(n_{-1} - q_{-1})$. Since $\tau_2(n_{-1}, n_{-1} - q_{-1}) = (n_0, -n_0 - q_0) = (n_0, n_0 - q_0)$, it follows that $(n_{-1}, q_{-1}) \in [\mathrm{I}]$ if and only if $(n_0, q_0) \in [\mathrm{I}]$. □

Proposition 11.12. *Assume $a_2 < 0$ and $k > 0$. Then we have the following*:

(1) (a) $(n_{-k}, q_{-k}) \notin [\mathrm{I}]$, *if* $k \geq 2$.

(b) $(n_{-1}, q_{-1}) \in [\mathrm{I}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{I}]$.

(2) (a) $(n_{-k}, q_{-k}) \notin [\mathrm{II}]$, *if* $k \geq 2$.

(b) $(n_{-1}, q_{-1}) \in [\mathrm{II}] \Longleftrightarrow (n_0, q_0) \in [\mathrm{II}]$.

Proof. As in the proof of Proposition 11.10, we write $A_k = [2, -2k, -a_2, b_2, \ldots, a_s, b_s]$, where $a_2 > 0$ and $k > 0$. Then $n_{-k}/q_{-k} = ((4k+1)n_0 - 2q_0)/(2kn_0 - q_0)$ and $n_{-k} = 2q_{-k} + n_0$. First, we see that if $k \geq 2$, then $S = \{\langle 2 \rangle \langle -2 \rangle \cdots\}$, since $2n_{-k} < 5q_{-k}$, and hence $\langle \pm 2 \rangle$ is not isolated. Therefore $(n_{-k}, q_{-k}) \notin [\mathrm{II}]$, if $k \geq 2$.

If $k = 1$, then $n_{-1} = 5n_0 - 2q_0$ and $q_{-1} = 2n_0 - q_0$, and $n_{-1} = 2q_{-1} + n_0$, $0 < n_0 < q_{-1}$. Then we see $S = \{\langle 2 \rangle \langle -3 \rangle \cdots\}$, since $5q_{-1} < 2n_{-1}$, and hence $\langle \pm 2 \rangle$ is isolated in $S(n_{-1}, q_{-1})$. Therefore, $(n_{-1}, q_{-1}) \in [\mathrm{II}]$ if and only if $(n_0, q_0) \in [\mathrm{II}]$, since $\tau_1(n_{-1}, q_{-1}) = (n_0, 2n_0 - q_0) = (n_0, -q_0)$.

Now we consider $n_{-k}/(n_{-k}-q_{-k}) = ((4k+1)n_0-2q_0)/((2k+1)n_0-q_0)$ and $n_{-k} = (n_{-k}-q_{-k})+(2kn_0-q_0), 0 < 2kn_0-q_0 < n_{-k}-q_{-k}$. First we see that if $k \geq 2$, then $S = \{\langle 1\rangle\langle -2\rangle\langle 2\rangle\langle -2\rangle\cdots\}$, since $7(n_{-k}-q_{-k}) < 4n_{-k}$. Thus $(n_{-k}, q_{-k}) \notin$ [II], if $k \geq 2$.

Suppose $k = 1$. Then $n_{-1} = 5n_0 - 2q_0$ and $n_{-1}-q_{-1} = 3n_0-q_0$, and $n_{-1} = (n_{-1}-q_{-1})+(2n_0-q_0)$, $0 < 2n_0-q_0 < n_{-1}-q_{-1}$. Now we see that $\langle\pm 2\rangle$ is isolated in $S(n_{-1}, n_{-1}-q_{-1})$ or $S(n_{-1}, n_{-1}-q_{-1}) = \{\langle 1\rangle\langle -2\rangle\langle 1\rangle\cdots\}$, since $4(n_{-1}-q_{-1}) < 3n_{-1} < 5(n_{-1}-q_{-1})$. Thus $(n_{-1}, q_{-1}) \in$ [I] if and only if $(n_0, q_0) \in$ [I]. □

Finally, we consider the case $A = [2, \pm 2k]$.

Proposition 11.13. *We have the following*:
(1) (a) $(4k-1, 2k) \notin$ [I], *if* $k \geq 2$.
(b) $(3, 2) \in$ [I].
(2) (a) $(4k-1, 2k) \notin$ [II], *if* $k \geq 2$.
(b) $(3, 2) \in$ [II].

Proof. Note that $[2, 2k] = (4k-1)/2k$. It is easily seen that

$$S(4k-1, 2k) = \{\langle 1\rangle \underbrace{\langle -2\rangle\langle 2\rangle \ldots \langle 2\rangle}_{(2k-2)\ \text{times}}\langle -1\rangle\}.$$

Hence, if $k \geq 2$, then $(4k-1, 2k) \notin$ [II]. On the other hand, $S(4k-1, 2k-1)$ consists of $\langle\pm 2\rangle$, and hence $(4k-1, 2k) \notin$ [I], if $k \geq 2$. Further, (1) (b) and (2) (b) are obvious. □

Proposition 11.14. *We have the following*:
(1) (a) $(4k+1, 2k) \notin$ [I], *if* $k \geq 2$.
(b) $(5, 2) \in$ [I].
(2) $(4k+1, 2k) \notin$ [II], *if* $k \geq 1$.

Proof. Note that $[2, -2k] = (4k+1)/2k$. Since $S(4k+1, 2k)$ consists of only $\langle\pm 2\rangle$, $(4k+1, 2k) \notin$ [II], if $k \geq 1$. On the other hand,

$$S(4k+1, 2k+1) = \{\langle 1\rangle \underbrace{\langle -2\rangle \cdots \langle -2\rangle}_{(2k-1)\ \text{times}}\langle 1\rangle\}.$$

Hence, if $k \geq 2$, then $(4k+1, 2k) \notin$ [I]. However, if $k = 1$, then $(5, 2) \in$ [I]. □

11.3. Characterization. A series of Propositions 11.8–11.14 provide an algorithm that decides to which class (n, p) belongs. Using this algorithm, we can now characterize knots $K(n, p \mid \alpha, \beta)$ in each class by the pair (n, p).

Theorem 11.15. *Let (n, p) be a pair, where p is even and $0 < p < n$. Denote by $[a_1, b_1, a_2, b_2, \dots, a_s, b_s]$ the continued fraction expansion of n/p, where all a_i and b_i are even $(1 \le i \le s)$. Then the fibredness of $K = K(n, p \mid \alpha, \beta)$ is determined by the pair (n, p) as follows*:

CASE 1: *The 2-bridge knot $B(n, p)$ is fibred, and hence all a_i's and b_j's are ± 2.*

- $K \in$ Class A1 (*i.e.*, $(n, p) \in [\mathrm{I}] \cap [\mathrm{II}]$) *if and only if* (11.10) *below is satisfied.*

$$(11.10) \quad \begin{cases} \text{(a)} \quad a_i b_i > 0 \text{ for all } i \ (1 \le i \le s) \text{ or} \\ \text{(b)} \quad a_s b_s > 0, \text{ and whenever } a_i b_i < 0 \ (1 \le i < s), \text{ we have } a_i a_{i+1} < 0. \end{cases}$$

CASE 2: *$B(n, p)$ is not fibred.*

SUBCASE 2 (i): *$b_j = \pm 2$ for all j $(1 \le j \le s)$.*

- $K \in$ Class B1 *or* Class B2.
- $K \in$ Class B1 *if and only if n/p satisfies* (11.10).

SUBCASE 2 (ii): *Each b_j is either ± 2 or ± 4, with some b_j being ± 4.*

- $K \in$ Class B3 *or* Class B4.
- $K \in$ Class B3 *if and only if* (11.11) *below is satisfied.*

(11.11)
$$\begin{cases} \text{(a)} \quad |b_s| = 2 \text{ and } a_s b_s > 0, \\ \text{(b)} \quad b_i = \pm 2 \ (1 \le i < s) \Longrightarrow \text{(i) } a_i b_i > 0 \text{ or (ii) } a_i b_i < 0 \text{ and } a_i a_{i+1} < 0, \text{ and} \\ \text{(c)} \quad b_i = \pm 4 \ (1 \le i < s) \Longrightarrow \text{(i) } a_i b_i > 0 \text{ and (ii) } a_i a_{i+1} > 0. \end{cases}$$

SUBCASE 2 (iii): *There exists b_j with $|b_j| \ge 6$.*

- $K \in$ Class B4.

EXAMPLES. Here we denote (n, p) by n/p, and say $n/p \in X$ for some class X if $K(n, p \mid \alpha, \beta)$ belongs to the class X.

(1) $[2, 2, -2, 2, 2, -2, -2, -2] = 177/112 \in$ A1.
(2) $[2, 2, -2, 2, 2, 2, -2, 2] = 265/168 \in$ A2.
(3) $[4, 2, -2, 2, 6, -2, -4, -2] = 3181/888 \in$ B1.
(4) $[4, 2, -2, 2, 6, 2, -4, 2] = 4412/1232 \in$ B2.
(5) $[4, 4, 2, 4, 2, -2, -2, -2] = 875/236 \in$ B3.
(6) $[4, 2, -2, 6, 6, -2, -4, -2] = 8869/2468 \in$ B4.
(7) $[4, 4, -2, 4, 2, -2, -2, -2] = 1525/404 \in$ B4.

12. Preliminary for the construction of fibre surfaces

In this section, we first review two methods to prove that a Seifert surface is a fibre surface. Then we introduce a key notion to construct a Seifert (fibre) surface for $K(n, p \mid \alpha, \beta)$.

12.1. Tools to prove fibredness. In this subsection, we review two important methods to prove that a Seifert surface is a fibre surface. One is *Stallings twists* and the other *Kobayashi's banding on pre-fibre surfaces*.

A *Stallings twist* is an operation to produce a new fibre surface from a fibre surface under a certain condition: Let c be an unknotted oriented circle embedded in a surface F in S^3. Suppose the linking number $\mathrm{lk}(c, c') = 0$, where c' is a push off of c in a normal direction of F. Then apply ± 1-surgery along c. Briefly, the operation is to cut F by a disk spanned by c' and then glue it back after a full-twist. Obviously, the new ambient manifold is S^3, but we have a new Seifert surface for a (different) link. However, J. Stallings [13] showed the following:

Proposition 12.1 ([13, Theorem 4]). *Suppose a Seifert surface F' is obtained from F by a Stallings twist. Then F' is a fibre surface if and only if F is too.*

In [9], T. Kobayashi introduced the notion of *pre-fibre* (*Seifert*) *surface* for links and, using that notion, determined when a band connected sum of links is a fibred link [10]. In this subsection, we summarize his main results. For the notion of a *sutured manifold*, we refer to [9] or [4].

Let L be a link with a Seifert surface F. Denote by $F_E = F \cap E(L)$ the restriction of F in the link exterior $E(L) = \mathrm{cl}(S^3 - N(L))$. The sutured manifold $(N, \delta) = (F_E \times I, \partial F_E \times I)$ is a *product sutured manifold*, where $R_+(\delta)$ and $R_-(\delta)$ are respectively $F_E \times \{1\}$ and $F_E \times \{0\}$. The sutured manifold $(N^c, \delta^c) = (\mathrm{cl}(E(L) - N), \mathrm{cl}(\partial E(L) - \delta))$, where $R_\pm(\delta^c) = R_\mp(\delta)$, is called the *complementary sutured manifold* for F.

DEFINITION 12.2. A Seifert surface S is a *pre-fibre surface* if there exist pairwise disjoint compressing disks D^+ and D^- in N^c for $R_+(\delta^c)$ and $R_-(\delta^c)$ respectively such that $(\overline{N}, \delta^c)$ is homeomorphic to a (not necessarily connected) product sutured manifold, where $\overline{N}$ denotes the manifold obtained from N^c by cutting along $D^+ \cup D^-$. Then there is a pair of compressing disks $\overline{D}^+$ and $\overline{D}^-$ for S such that $\overline{D}^\pm \cap N^c = D^\pm$, which we call *a pair of canonical compressing disks* for S.

To determine when a band connected sum of two links are fibred, the following notion is essential. Kobayashi called the following banding a *band of type F*, but now after Kobayashi, we call it a *K-band*.

Let S be a pre-fibre surface with a pair of canonical compressing disks $D^+ \cup D^-$. Let p_+ and p_- be properly embedded arcs in S sharing exactly one end point $e \subset \partial S$. Their interiors may intersect each other in S. Push p_+ (resp. p_-) in the positive (resp. negative) normal direction of S, and then push $e = p_+ \cap p_-$ off S so that we obtain an arc α in S^3 such that $\alpha \cap S = \partial\alpha \subset \partial S$. Suppose α intersects each of D^+ and D^- in exactly one point.

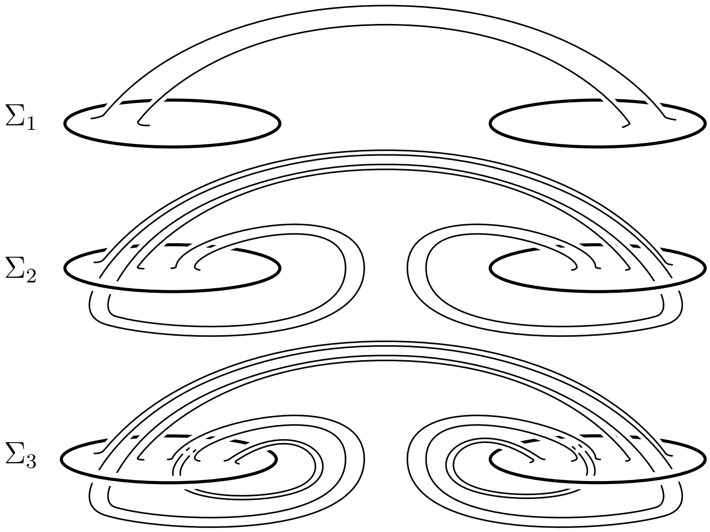


Fig. 12.1. Pre-fiber surfaces for the 2-component trivial link.

DEFINITION 12.3. Let S be a pre-fibre surface and β a band whose ends are attached to ∂S and whose interior misses S. We call β a *K-band* if its core γ (fixing its end points) is isotopic to an arc α obtained by the above construction.

Kobayashi obtained the following:

Proposition 12.4 ([10, Proposition A]). *Let F be a Seifert surface obtained from a pre-fibre surface S by adding a band β. Then F is a fibre surface if and only if β is a K-band.*

REMARK 12.5. Note that the twisting of β is irrelevant because that can be generated by Stallings twists using D^+.

EXAMPLE 12.6. The following sequence of Seifert surfaces $\Sigma_1, \Sigma_2, \dots$ in Fig. 12.1 are examples of pre-fibre surfaces. First, Σ_1 is an annulus, which is obtained by tubing two disks. Second, Σ_2 is obtained from Σ_1 by another tubing, where the new tube goes through the first tube. Next, Σ_3 is obtained from Σ_2 again by adding a tube which goes though the innermost tube of Σ_2. Inductively, we can construct Σ_i's. By [10, Theorem 3], any pre-fibre surface for the 2-component trivial link is isotopic to Σ_i for some i, where the pair of canonical compressing disks comes from the innermost disks among $Q - (Q \cap \Sigma_i)$, where Q is the separating 2-sphere for the trivial link, positioned naturally so that each tube meets Q in one essential circle.

Actually, Kobayashi characterized pre-fibre surfaces for split links as follows:

Proposition 12.7 ([10, Theorem 3]). *Let $L = L_1 \cup L_2$ be a split link with a 2-sphere separating L_1 and L_2 in S^3. Then L bounds a pre-fibre surface S if and only if both L_1 and L_2 are fibred. Moreover the pre-fibre surface S is constructed as follows.*
(1) *Take disjoint fibre surfaces F_1 and F_2 for L_1 and L_2.*
(2) *Take a 2-sphere Q bounding a ball B which meets each of F_1 and F_2 in a disk.*
(3) *Apply tubing to the two disks in B as in* Example 12.6.

12.2. The word for $K(n, p \mid \alpha, \beta)$. In this subsection, we introduce a notion of the *word* for the pair (n, p), which is the basic tool to construct a minimal genus Seifert surface for $K(n, p \mid \alpha, \beta)$.

Given a pair of co-prime integers $n > p > 0$, consider the sequence of signs defined in Section 4: $S = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}\}$. For $K = K(n, p \mid \alpha, \beta)$, *the primitive word of K*, denoted by $\widetilde{W}(K)$, composed of x, y's and their inverses is defined by:

$$\widetilde{W}(K) = y^{\beta\varepsilon_1} x^{\alpha\varepsilon_2} y^{\beta\varepsilon_3} x^{\alpha\varepsilon_4} \cdots y^{\beta\varepsilon_{n-2}} x^{\alpha\varepsilon_{n-1}}.$$

We call the words of the form $x^i y^i$ or $y^j x^j$ *standard syllables*. The *degree* of standard syllables is defined to be $\deg(x^i y^i) := i, \deg(y^j x^j) := -j$. A word T composed of standard syllables is called a *normalized word of K* if it satisfies the following:
(1) T is reduced to a word of the form $x^k \widetilde{W}(K) y^l$ for some integers k and l (possibly 0).
(2) The number of standard syllables in T is exactly n.
(3) Syllables of the form $x^h y^h$ and $y^i x^i$ appear alternately. (As a result, $x^0 y^0$ or $y^0 x^0$ may appear as syllables, which may be abbreviated as 1.)
(4) T starts with $x^k y^k$ and ends with $x^l y^l$. (If $k = 0$ (resp. $l = 0$), then T starts with (resp. end with) $x^0 y^0$, which we never omit.)

Note that a normalized word of K is uniquely determined by each choice of the degree k of the initial syllable, and we can easily normalize a word from left to right. For example, let $K = K(11, 8 \mid 2, 1)$. Then $\widetilde{W}(K) = y^1 x^{-2} y^1 x^2 y^{-1} x^2 y^{-1} x^{-2} y^1 x^{-2}$. If we set $k = 0$, then we have

$$x^0 y^0 \cdot yx \cdot x^{-3} y^{-3} \cdot y^4 x^4 \cdot x^{-2} y^{-2} \cdot yx \cdot xy \cdot y^{-2} x^{-2} \cdot x^0 y^0 \cdot yx \cdot x^{-3} y^{-3}.$$

In a normalized word, adjacent syllables have different degrees. The list of degrees of syllables in the above word is $\{0, -1, -3, -4, -2, -1, 1, 2, 0, -1, -3\}$, which by construction, coincides with the sequence R for $(n, p \mid \alpha, \beta) = (11, 8 \mid 2, 1)$ defined in Section 4. Therefore, we can also easily read the primitive word for $K(n, p \mid \alpha, \beta)$ from the Schubert diagram for $B(n, p)$, by travelling along the underpath recording from which direction (above or below) one goes under the overpath. See Fig. 4.1.

Note that if we change the initial degree, say from k to $k+h$, then the new normalized word is obtained by changing the degrees of all syllables by h. This corresponds to the slide of $H(K)$ along the y-axis. Now consider sliding the graph so that the vertices of the minimal degree have the y-coordinate 0. A normalized word is called *strictly normalized* if the initial degree is chosen so that the minimal degree of syllables is 0. We denote by $W(K)$ the strictly normalized word of K. For example, by sliding up by 4, we see $W(K)$ for $K = K(11, 8 \mid 2, 1)$ is

$$x^4y^4 \cdot y^{-3}x^{-3} \cdot x^1y^1 \cdot y^0x^0 \cdot x^2y^2 \cdot y^{-3}x^{-3} \cdot x^5y^5 \cdot y^{-6}x^{-6} \cdot x^4y^4 \cdot y^{-3}x^{-3} \cdot x^1y^1,$$

where the list of degrees is $\{4, 3, 1, 0, 2, 3, 5, 6, 4, 3, 1\}$.

DEFINITION 12.8. Let $K = K(n, p \mid \alpha, \beta)$. We say that the strictly normalized word $W(K)$ is *admissible* if $W(K)$ has exactly one syllable of the maximal degree and one syllable of the minimal degree. Accordingly, we say that the primitive word $\widetilde{W}(K)$ or its normalized word is admissiable, if its strictly normalized word is admissible.

Then we have the following:

Proposition 12.9. *For* $K = K(n, p \mid \alpha, \beta)$, *the graph* $H(K)$ *is admissible if and only if* $W(K)$ *is admissible. Moreover,* $(1/2) \deg \Delta_K(t) = \max\{$*degrees of syllables of* $W(K)\}$.

Proof. The first statement is obvious from the construction. The second statement follows from Theorem 4.8 and Remark 10.3. □

13. Minimal genus (fibre) Seifert surface for K

In this section, we first construct a fibre surface for $K(n, p \mid \alpha, \beta)$ with an admissible word, using K-bandings of a pre-fibre surface and prove Theorem C, and hence Theorem A′ is proved. Then we construct fibre surfaces for satellite $(1, 1)$-double torus knots whose pattern knot $K(n, p \mid \alpha, \beta)$ has an admissible word, and prove Theorem A. Finally, we construct a minimal genus Seifert surface for non-separating $(1, 1)$-double torus knots, and prove Theorem D.

By Corollary 3.2 we may assume:

$(*)$ $n \geq 3$ is odd, p is even, $n > p > 0$, $\gcd(n, p) = 1$ and $\alpha \geq |\beta| > 0$.

13.1. $K(n, p \mid \alpha, \beta)$ with an admissible word. Suppose that the strictly normalized word $W(K)$ for $K = K(n, p \mid \alpha, \beta)$ is admissible. Denote by $\deg(W)$ the maximal degree of the syllables of $W(K)$.

Basically, the construction of a Seifert surface for K is as follows: K is obtained by attaching a band $\mathcal{B}$ to connect two split unknots spanning disks D_L and D_R. The band transversely intersects D_L and D_R, and we eliminate the intersections by removing small disks from D_L and D_R, and then connecting the resulting small holes in D_L and D_R with a tube (annulus). However, this cannot be done immediately. (Otherwise,

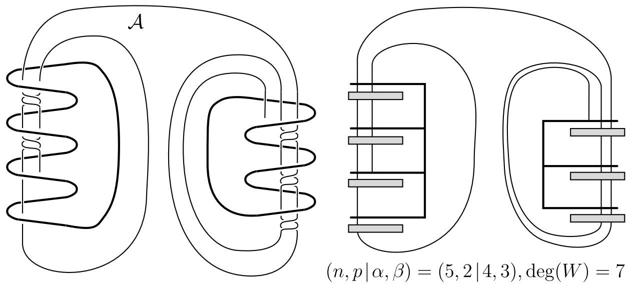


Fig. 13.1. Construction of $\tilde{\Sigma}$ for the 2-component trivial link.

we obtain a non-orientable surface, or a surface of non-minimal genus.) We must first isotope the band to increase the intersection points.

As we explain below, the process is divided into the following two steps:

STEP 1: Disregarding the band $\mathcal{B}$, construct a Seifert surface Σ for the 2-component trivial link L by tubing two disks D_L and D_R. We will see that Σ is a pre-fibre surface having $\deg(W)$ tubes.

STEP 2: Using $W(K)$, bend the band $\mathcal{B}$ 'nicely' so that $\mathcal{B}$ meets Σ only at its ends. Then we will prove that $\Sigma \cup \mathcal{B}$ is a K-banding on Σ, and hence by Theorem 12.4, $\Sigma \cup \mathcal{B}$ is a fibre surface for K.

First we deal with the case $\beta > 0$.

STEP 1: Let L be the 2-component trivial link spanning two disjoint disks D_L and D_R shaped as in Fig. 13.1.

Let $\mathcal{A}$ be an arc as in Fig. 13.1 having its endpoints in $\mathrm{int}(D_L \cup D_R)$ such that $\#(\mathcal{A} \cap D_L) = \#(\mathcal{A} \cap D_R) = \deg(W)$. As a convention, the thin box indicates a full-twist of whatever goes through, in positive way near D_L and negative way near D_R (consistent with the convention of twisting direction in Section 2). Now denote by $\{l_{\deg(W)}, \ldots, l_2, l_1, r_1, r_2, \ldots, r_{\deg(W)}\}$ the intersection points of $\mathcal{A}$ and $D_L \cup D_R$ named from the left end of $\mathcal{A}$. Apply a tubing along the subarc (l_1, r_1), then apply another tubing along the subarc (l_2, r_2) through the first tube. By repeating this operation of tubing $\deg(W)$ times, we obtain a Seifert surface for L, denoted by Σ. Now we have the following:

Proposition 13.1. *The tubed surface Σ is a pre-fibre surface.*

Proof. Use the sequence of isotopies depicted in Fig. 13.2. The first step is to twist the subdisks, and the second and third steps are to slide the end of $\mathcal{A}$ along the boundary of the disk. This induces an isotopy of the tubes of Σ toward the standard form of a pre-fibre surface in Fig. 12.1, and we see that Σ is a pre-fibre surface for L. $\square$

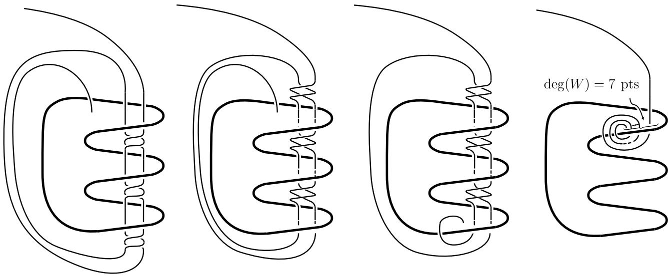


Fig. 13.2. Isotoping $\tilde{\Sigma}$ into the standard form of a pre-fibre surface.

Denote by $T_1, T_2, \ldots T_{\deg(W)}$ the tubes so that T_1 is the first widest one containing the other tubes and that $T_{\deg(W)}$ is the last narrowest tube. Define the *depth* of T_i by $\mathrm{dep}(T_i) = i$.

CONVENTION. By convention, the outside (resp. inside) of the tube T_1 touches the face (resp. back) of Σ. At this time, though D_L and D_R do not exist any more, we still assume virtually they are there. So we could say: we penetrate D_L and D_R several times as we travel along the band. We always assume that the arc $\mathcal{B}$ (representing the band connecting the two unknots) is oriented from D_L to D_R. As seen in Fig. 13.1, D_L (resp. D_R) has α (resp. β) subdisks $D_{L_1}, D_{L_2}, \ldots, D_{L_\alpha}$ (resp. $D_{R_1}, D_{R_2}, \ldots, D_{R_\beta}$) named so that T_1 connects D_{L_1} and D_{R_1}. We say that two tubes T_i, T_j are *locally adjacent to each other on* D_L (resp. D_R) if the difference of their depths is α (resp. β). Note that the feet of locally adjacent tubes lie next to each other in a subdisk of D_L or D_R.

Before proceeding to the next step of construction of a fibre surface, let us briefly explain our way using $K = K(5, 2 \mid 2, 1)$ as an example. (Fig. 13.3).

The primitive word is $yx^2y^{-1}x^{-2}$, and hence the strictly normalized word is $W(K) = xy \cdot 1 \cdot x^2y^2 \cdot y^{-3}x^{-3} \cdot xy$. Fig. 13.3 (a) depicts K, where the band is depicted by an arc $\mathcal{B}$. Note that each box contains a full twist. In Fig. 13.3 (b) we slid the ends $\mathcal{B}$ along the boundaries of D_L and D_R. Then we forget the twist boxes, and continue the construction for the new knot. We remark that this corresponds to Stallings twists, and once we have constructed a fibre surface, we can easily modify it to a fibre surface for K. Actually, these untwistings and twistings are introduced just to simplify the figures. Now isotope $\mathcal{B}$ as in Fig. 13.3 (c) so that we can read $W(K)$ from the itinerary, and that we can superimpose the pre-fibre surface as in Fig. 13.3 (d). Then we show that the band is a K-band.

STEP 2. We start with a figure of $K = K(n, p \mid \alpha, \beta)$ as in Fig. 13.4, where K is expressed as the union of two unknots and an arc $\mathcal{B}$ representing the band. Regard the unknots and $\mathcal{B}$ as lying slightly above the double torus H. The intersection $\mathcal{B} \cap$

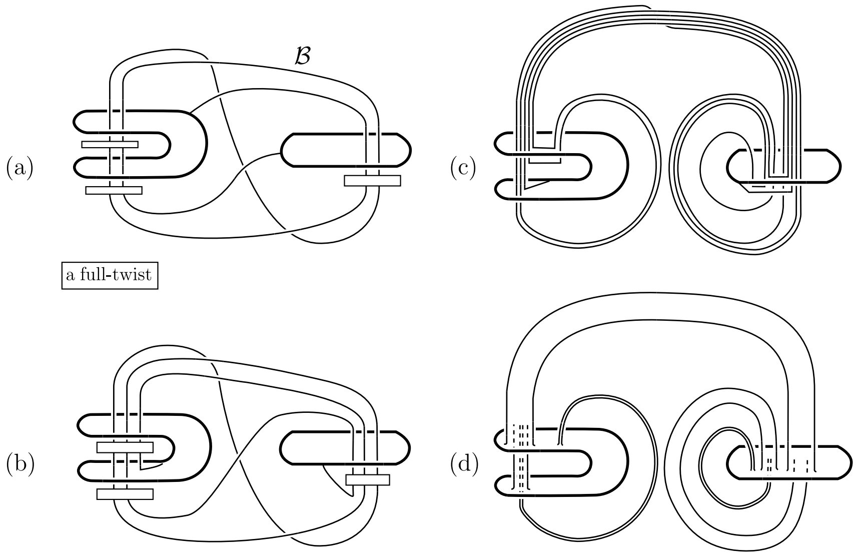


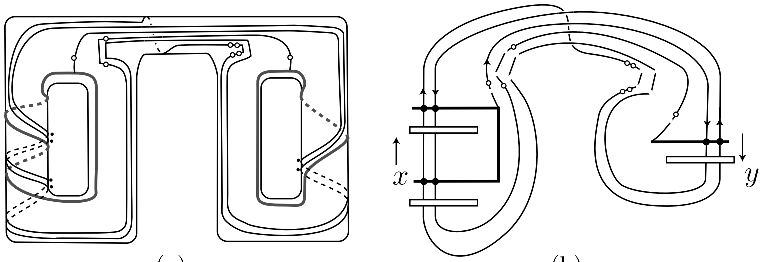
Fig. 13.3. Construction of a fiber surface for $K(5, 2 \mid 2, 1)$.

$(D_L \cup D_R)$, which happens in the longitudinal disks of H, is indicated by dots. We can read the primitive word $\widetilde{W}(K)$ of K by following $\mathcal{B}$ from the left end and record from which way one penetrates D_L and D_R. Note that the number of letters in $\widetilde{W}(K)$ is equal to the number of dots. The hollow dots, explained later, will indicate new intersection points to be created so that the total number of dots and hollow dots equals that of the letters in the strictly normalized word $W(K)$. Remark that there are exactly n subarcs connecting D_L and D_R. This corresponds to the facts that the length of the sequence of signs for (n, p) (defined in Section 4) is exactly $n-1$, and that the strictly normalized word $W(K)$ has exactly n syllables.

Now we subdivide $\mathcal{B}$ into $2n + 1 = 2 + n + (n - 1)$ subarcs classified into three classes as in Fig. 13.4 (b): The both ends are *end-arcs*, which are short. Between the end-arcs, there are n *long-arcs* and $n-1$ *flat-arcs* appearing alternately. The long-arcs run between D_R and D_L, and flat-arcs are short straight arcs between long-arcs.

When we show that we have a K-banding, we will push end-arcs and flat-arcs (resp. long-arcs) to the disk part (resp. tube part) of Σ. Assign the j^{th} syllable of $W(K)$ to the j^{th} long-arc counted from the left end of $\mathcal{B}$. Then define the *degree* of each long-arc as that of the corresponding syllable.

Our strategy is to use $W(K)$ as a guide to slide the ends of $\mathcal{B}$ and drag the flat-arcs stretching the long-arcs (in H) so that we can read the strictly normalized word $W(K)$ from the new form of $\mathcal{B}$. We see this is possible, because $W(K)$ is obtained from $\widetilde{W}(K)$ by prepending x^k, appending y^l (recall $k > 0$, $l \geq 0$) and by inserting $x^i x^{-i}$ and $y^j y^{-j}$ for various i, j's. This corresponds to adding the hollow dots to $\mathcal{B}$ as in Fig. 13.4 (a), which indicate newly created intersections of $\mathcal{B}$ and $D_L \cup D_R$.



(a) (b)

Fig. 13.4. $K = K(5, 2 \mid 2, 1)$, $\widetilde{W}(K) = yx^2y^{-1}x^{-2}$, $W(K) = xy \cdot 1 \cdot_\partial x^2y^2 \cdot y^{-3}x^{-3} \cdot xy$.

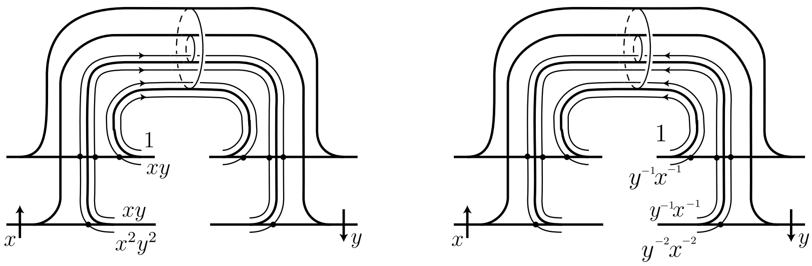


Fig. 13.5.

Note that the dots and hollow dots are put only on the long-arcs. Now we establish a correspondence between a syllable in $W(K)$ and dots along a long-arc. Suppose we have already stretched and pushed a long-arc, say γ, onto the back (resp. face) side of a tube T of depth i, then we read x^iy^i or $y^{-i}x^{-i}$ (resp. $x^{i-1}y^{i-1}$ or $y^{-(i-1)}x^{-(i-1)}$). (See Fig. 13.5.) Note that γ reads 1 if and only if γ is pushed onto the face side of the tube of depth 1, and that γ reads $x^{\deg(W)}y^{\deg(W)}$ or $y^{-\deg(W)}x^{-\deg(W)}$ if and only if γ is pushed onto the back side of the narrowest tube. By the assumption that $W(K)$ is admissible, each case occurs exactly once. Therefore, each component of the pair of canonical compressing disks of Σ is penetrated by $\mathcal{B}$ exactly once. (This is necessary to have a K-banding.)

Also note that γ reads x^iy^i or $y^{-i}x^{-i}$, $(1 < i < \deg(W))$ in two cases: on the back side of the tube of depth i and on the face side of the tube of depth $i+1$. Both cases may occur several times. By definition, the strictly normalized word has no syllables of negative degree, i.e., x^gy^g and $y^{-g}x^{-g}$ with $g < 0$. Hence we do not need a tube attached to the back of D_L and D_R.

Now we have to move $\mathcal{B}$ properly so that other conditions of a K-banding are satisfied. First, we specify to which tube and to which side we push the subarcs, and specify one particular flat-arc (marked ∂) which passes the boundary of D_L or D_R exactly once. We put the mark ∂ on the unique flat-arc just before (resp. after) the long-

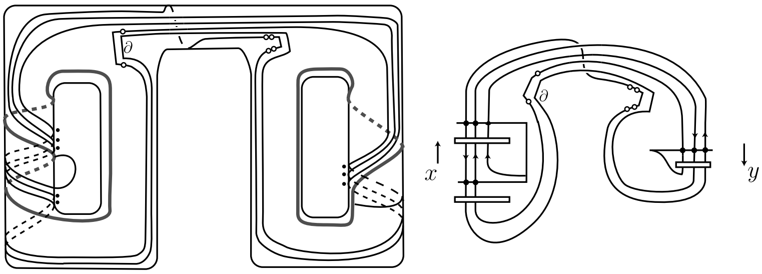


Fig. 13.6. Sliding the end of $\mathcal{B}$ in $K = K(5, 2 \mid 2, 1)$. we now read $x \cdot yx^2y^{-1}x^{-2} \cdot y$.

arc which corresponds to the unique syllable 1, if the syllable 1 appears after (resp. before) the unique syllable of the maximal degree.

Let $\mathcal{B} \setminus \partial = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) contains the long-arc of degree 0 (resp. degree deg(W)). Then $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is pushed into the face side (resp. back side) of Σ. Accordingly, we call the arcs in $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) *top arcs* (resp. *bottom arcs*). For example, for $K(5,2 \mid 2,1)$, $\mathcal{B}_1$ is the first half of $\mathcal{B} \setminus \partial$, i.e., the part that contains the left end of $\mathcal{B}$. For $K(5,4 \mid 2,1)$, $\mathcal{B}_1$ is the second half of $\mathcal{B} \setminus \partial$ and contains the right end of $\mathcal{B}$. (Recall $\mathcal{B}$ is oriented from the left end.) See Fig. 13.7. By the observation above, we see that a top (bottom) long-arc of degree i is pushed onto the face (reps. back) side of the tube of depth $i + 1$ (resp. depth i).

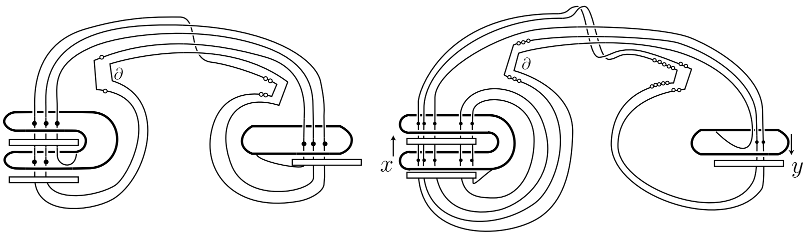
We simplify the figures by 'untwists' which will turn to be Stallings twists, but before that we must slide the ends of $\mathcal{B}$ as shown in Fig. 13.6. Recall that the strictly normalized word is of the form $x^k y^k \cdots x^l y^l$, with $k > 0$ and $l \geq 0$. For $K = K(5, 2 \mid 2, 1)$, $k = l = 1$. Slide the endpoint of $\mathcal{B}$ along D_L (resp. D_R) so that $|\mathcal{B} \cap D_L|$ (resp. $|\mathcal{B} \cap D_R|$) increases by k (resp. l). Note that since $k, l \geq 0$, we slide $\mathcal{B}$ counterclockwise (resp. clockwise) along ∂D_L (resp. ∂D_R).

Fig. 13.7 depicts actual examples of $K(5,2|2,1)$ and $K(5,4|2,1)$, with twist boxes. Note that in the former, the left end-arc is a top arc, and in the latter, the left end-arc is a bottom arc.

See Fig. 13.8 for general situations for D_L. For D_R, just reflect the figure by a vertical line in the paper.

Fig. 13.8 (a) (resp. (b)) depicts the situation where the left end-arc is a top arc (resp. bottom arc), and hence will be pushed to the top side of the subdisk D_{L_3} (resp. the back side of the subdisk D_{L_2}). Note also that in Fig. 13.8 (a) (resp. (b)), the long-arc γ connected to the left end-arc corresponds to the first syllable $x^k y^k$ in $W(K)$ and will be pushed to the top side of the tube of degree $k + 1$ (resp. back side of the tube of degree k).

Here, to simplify the figures, we forget the twist boxes in Fig. 13.8, denoting the results by $\mathcal{B}'$ and Σ'. If we draw figures on the double torus, we do not need such an operation, but the figures become too complicated. Obviously, Σ' inherits the pair



(a) $K(5,2|2,1)$
$xy \cdot 1 \cdot_\partial x^2y^2 \cdot y^{-3}x^{-3} \cdot xy.$

(b) $K(5,4|2,1)$
$x^6y^6 \cdot y^{-5}x^{-5} \cdot x^3y^3 \cdot y^{-2}x^{-2} \cdot_\partial 1$

Fig. 13.7. Sliding of the end-arcs.

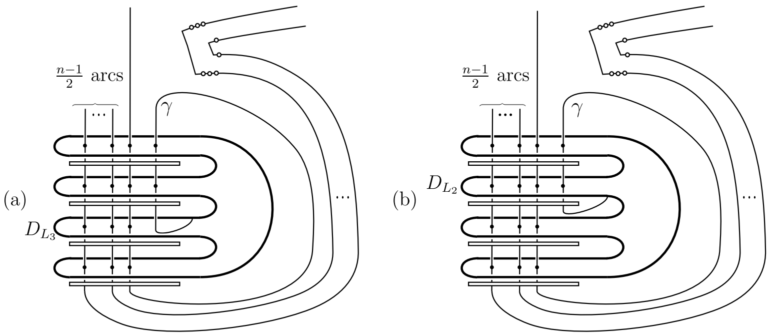


Fig. 13.8. Sliding of the end-arc, where $k = 6$.

of canonical compressing disks and we will arrange $\mathcal{B}'$ so that it is a K-band on Σ'. Then we apply Stallings twists on $\mathcal{B}' \cup \Sigma'$ so that it becomes a K-banding of Σ. Our Stallings twists will use simple closed curves which are closely parallel to each boundary of the subdisks $D_{L_1}, \dots, D_{L_\alpha}, D_{R_1}, \dots D_{R_\beta}$.

Now we drag the flat-arcs through D_L and D_R and stretch long-arcs so that we can read $W(K)$ from $\mathcal{B}'$. As in Fig. 13.7, we add hollow dots to $\mathcal{B}'$ to indicate newly created intersections of $\mathcal{B}'$ and $D_L \cup D_R$, i.e., to see how far we should drag each flat-arcs. Note that, except for those hollow dots corresponding to prepending x^k and appending y^l to $\mathcal{B}$, the hollow dots which indicate new intersection points to be created are arranged in pairs near the flat-arcs (see Fig. 13.7). We drag flat-arcs along the guide line indicated by the broken line in Fig. 13.9 (a). To be more precise, when we drag a flat-arc, say γ, that is a bottom arc, then we will stop it below a subdisk immediately after the hollow dots are replaced by newly created intersection points. On the other hand, if γ is a top arc, then γ is dragged further so that it lies on the top side of the next subdisk. See Figs. 13.9 (b) and (c) respectively.

When we finish dragging a flat-arc γ, we put it near the boundary of a sub-disk of D_L or D_R so that if 'inner flat-arc' should be dragged further, then the connected

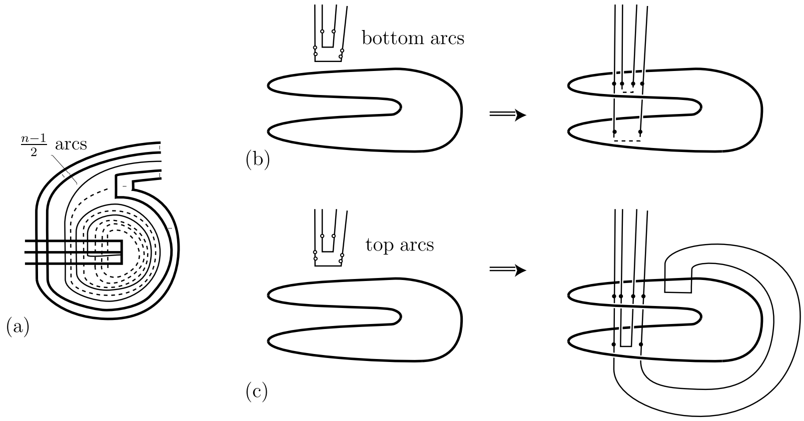


Fig. 13.9. Drag flat-arcs so that we can read $W(K)$.

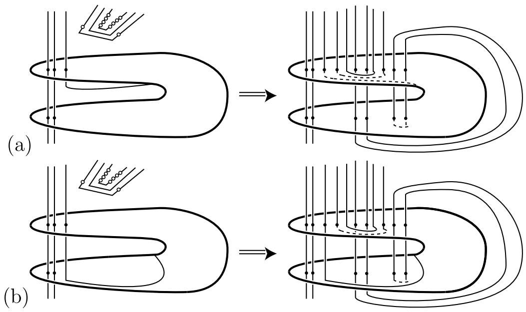


Fig. 13.10. Arrangement of dragged flat-arcs.

long-arcs go under γ. Moreover, the end-arcs are also placed so that other long-arcs go under them. See Fig. 13.10 (a) (resp. Fig. 13.10 (b)) for D_L with the left end-arc being a bottom (resp. top) arc, where all flat-arcs depicted are bottom arcs.

Claim 1. *At this stage, we can superimpose Σ' and $\mathcal{B}'$ so that $\Sigma' \cap \mathcal{B}' = \partial \mathcal{B}'$.*

Proof. First place the feet of tubes as in Fig. 13.11 (left). Then we can push each flat-arc (except for the one with the mark ∂) and end-arcs to the face or back side of the disk part of Σ'. Because the long-arcs adjacent to a flat-arc near D_L (resp. D_R) have the gap of the degree exactly α (resp. β). We can also push the ends of long-arcs properly as in Fig. 13.11 (right).

Secondly we arrange the long-arcs along the tubes. All the long-arcs run parallel to each other, except for the neighborhood of the middle of the tubes. Now place the long-arcs near the middle of the tubes as in Fig. 13.2. Then radially move each long-arc toward the core of tubes until it sits in the proper side of the proper tube.

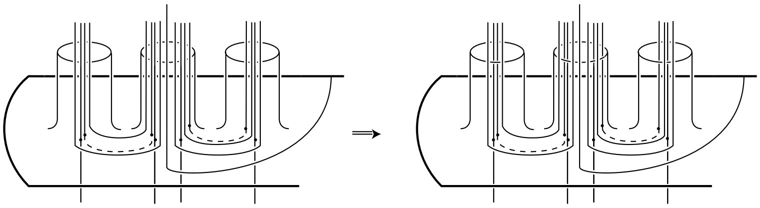


Fig. 13.11.

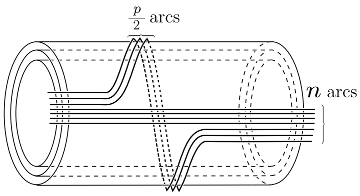


Fig. 13.12.

By extending this projection toward the end of the tubes, we can place the long-arcs properly. Therefore, we have Claim 1. □

Next, we show the following:

Claim 2. *We can push $\mathcal{B}'$ to Σ' so that $\mathcal{B}'$ is a K-banding.*

Proof. Since our purpose is to construct a K-banding on Σ by $\mathcal{B}$, we can technically omit the proof that Σ' is a pre-fibre surface, though actually it is. Recall that $W(K)$ is admissible and that each component of the pair of canonical compressing disks of Σ' is penetrated exactly once by $\mathcal{B}'$. In the proof of Claim 1, we have already seen that each long-arc can be pushed onto the face or back side of the tubes without intersections among them. Now we push flat-arcs to the face or back side of a subdisk of D_L or D_R. By construction, there are no intersections among them as shown in Fig. 13.13, where the rectangle depicts a subdisk of D_L or D_R. (The intersection in the middle of Fig. 13.13 does not violate the condition of K-banding.)

Therefore, we are left with only the flat-arc f with the mark ∂. See Fig. 13.14. By symmetry, we may suppose that f is on the subdisk D_{L_1}. Note that f lies on the face side of D_{L_1} because one long-arc, labeled δ', connected to it corresponds to the syllable 1 in $W(K)$. We can move the part of the long-arc, labeled δ, as in Fig. 13.14 (b). To prove that it is always possible, it suffices to show that neither flat-arc nor end-arcs interfere. Fig. 13.14 (c) depicts the case flat-arcs γ_1 and γ_2 which interfere.

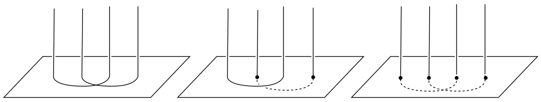


Fig. 13.13.

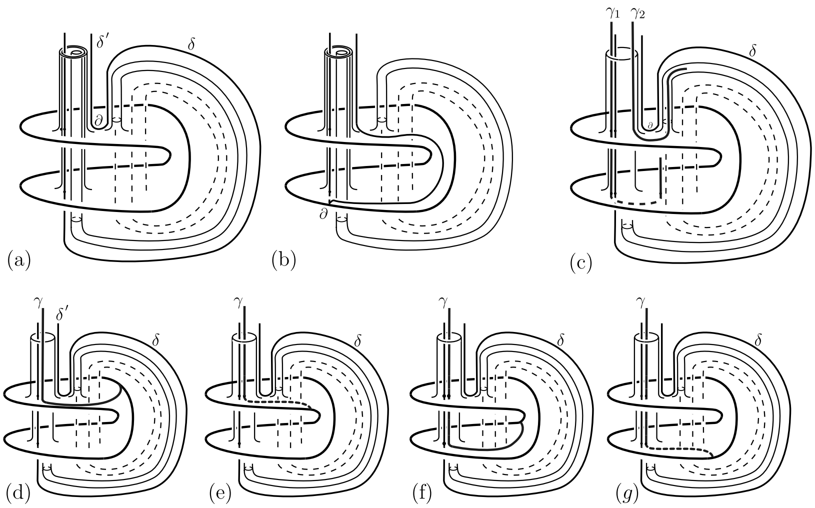
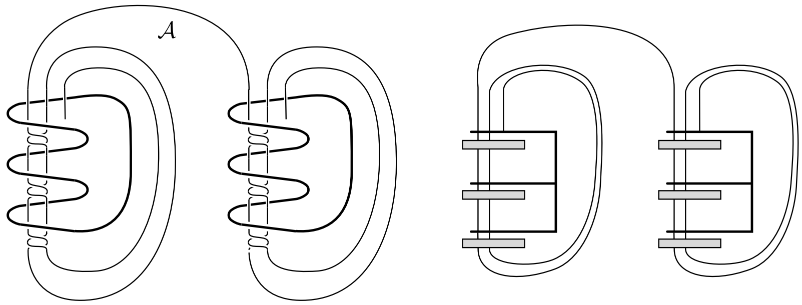


Fig. 13.14.

First, from Fig. 13.8, we see that the dots on the long-arc labeled δ are not the newly created ones. However, by construction, there are no newly created dots to the left of the original ones, and hence the arc γ_1 never exists. Moreover, by reading $W(K)$, we see that the arc γ_2 corresponds to the unique syllable 1. Since $W(K)$ is admissible, we do not have γ_2. Now Fig. 13.14 (d)–(g) covers all the cases of end-arcs. By sliding the endpoint of $\mathcal{B}'$, we see that they do not interfere with each other. (Note that Fig. 13.14 (d) never occurs, since both γ and δ' corresponds to the syllable 1, which is impossible since $W(K)$ is admissible. We have proved Claim 2. $\square$

Now we have seen that $\mathcal{B}' \cup \Sigma'$ is a K-banding. To recover the twist boxes as in Fig. 13.8, apply Stallings twists along circles closely parallel to each boundary of the subdisks $D_{L_1}, \ldots, D_{L_\alpha}, D_{R_1}, \ldots, D_{R_\beta}$. By construction, these circles are not 'linked' with the moved flat-arcs, and hence we can recover a Seifert surface $\mathcal{B} \cup \Sigma$ for K. Moreover, since $\mathcal{B}'$ is pushed onto Σ nicely to satisfy all requirements for a K-band, we see that $\mathcal{B}$ is also a K-band on Σ. Therefore, by Proposition 12.4, K is a fibred knot. Therefore, Theorem C is now proved for the case $\beta > 0$ in $K(n, p \mid \alpha, \beta)$.

Finally we describe the case $\beta < 0$. Since we can handle this case almost in the same manner as in the case $\beta > 0$, we only show one example and explain the



$\alpha = 3, \beta = -3, \deg(W) = 7.$

Fig. 13.15. Pre-fiber surface used in the case $\beta < 0$.

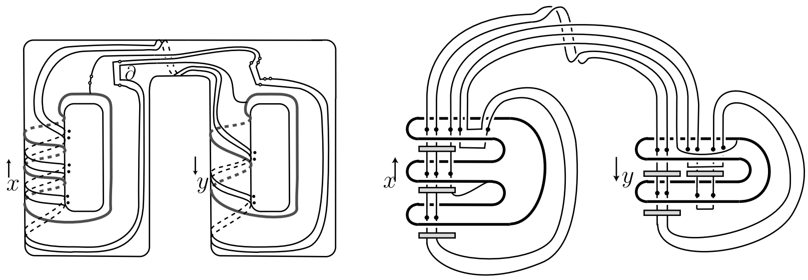


Fig. 13.16. $K = K(5,4\,|\,3,-2)$. $W(K) = x^2y^2 \cdot y^{-4}x^{-4} \cdot xy \cdot y^{-3}x^{-3} \cdot_\partial 1$.

slight difference in the figure. When $\beta < 0$, we use the pre-fibre surface constructed as before by the guide line $\mathcal{A}$, but this time it appears as in Fig. 13.15, and the right disk D_R is facing the same was as D_L. Fig. 13.16 depicts $K = K(5, 4\,|\,3, -2)$. The strictly normalized word $W(K) = x^2y^2 \cdot y^{-4}x^{-4} \cdot xy \cdot y^{-3}x^{-3} \cdot_\partial 1$. Using $W(K)$, we add hollow dots and stretch $\mathcal{B}$ as in Fig. 13.16. Then as we proved before, we have a K-banding.

This completes the proof of Theorem C. □

13.2. Fibre surfaces for satellite (1, 1)-double torus knots whose pattern knot has an admissible word. In the previous subsection we constructed a fibre surface F for $K = K(n, p\,|\,\alpha, \beta)$ with an admissible word. In this subsection, we first construct a Seifert surface $\widehat{F}$ for the satellite double torus knot $\widehat{K} = K\{(n, 0, 0; n, 0, 0\,|\,p)(1, \alpha, -, -)(r', s', -, -)\}$, whose pattern knot is K and companion knot is the torus knot $T = T(r', -s')$, where $\beta = r's'$. Then we show that $\widehat{F}$ is a fibre surface. Consider a loop L in $E(K)$ such that S^3 is split into two solid tori $N(L)$ and $E(L)$ as depicted in Fig. 13.17.

We regard K as a knot in the solid torus $E(L)$, and $N(L) \cap F$ is a meridian disk for $N(L)$. Then construct a Seifert surface $\widehat{F}$ for $\widehat{K}$ as follows. Let φ be a homeo-

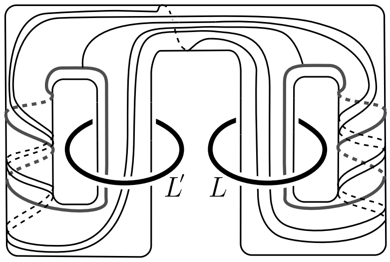


Fig. 13.17. A pattern $K(n, p \mid \alpha, r's')$ and a meridian of the companion torus.

morphism which takes $E(L)$ to $N(T)$ so that $\varphi(F \cap \partial E(L))$ is null-homologous in $E(T)$. Then let $\widehat{F}$ be a Seifert surface for $\widehat{K}$ obtained by capping $\varphi(F \cap E(L))$ with a fibre surface, say F_T for T along $\varphi(F \cap \partial E(L))$.

Now we prove the following:

Claim 13.2. *The Seifert surface $\widehat{F}$ is a fibre surface for $\widehat{K}$.*

Proof. It suffices to show that $\widehat{F}$ is a K-banding of a pre-fibre surface for the split link consisting of an unknot D_L and T, see Fig. 13.1. Recall the pre-fibre surface Σ for the 2-component trivial link constructed in the previous section. Note that F is of the form $(\Sigma \cap E(L)) \cup (\Sigma \cap N(L)) \cup \mathcal{B}$, where $(\Sigma \cap N(L))$ is a disk and $\mathcal{B}$ is a K-band. In the construction of Σ in the proof of Proposition 13.1, we may assume that the attached tubes are contained in $E(L)$. Then the isotopy of the tubes depicted in Fig. 13.2 is not interfered by L. Therefore, we see that $\varphi(\Sigma \cap E(L)) \cup F_T = \widehat{F} \setminus \varphi(\mathcal{B})$ bounds the split link $T \cup \partial D_L$, and it is a pre-fibre surface by Proposition 12.7. In the proof that $\mathcal{B}$ is a K-band on Σ, we may assume that $\mathcal{B}$ is completely contained in $E(L)$ and pushed onto Σ in $E(L)$. Therefore, $\varphi(\mathcal{B})$ is a K-band on the pre-fibre surface $\widehat{F} \setminus \varphi(\mathcal{B})$. Claim 13.2 is now proved. $\square$

Once we construct a fibre surface $\widehat{F}$, we can construct, using L' in Fig. 13.17 in the same manner, a fibre surface for the original satellite $(1,1)$-double torus knot $K\{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(r', s', -, -)\}$, where $rs = \alpha$.

Therefore, we have the following:

Proposition 13.3. $K\{(n, 0, 0; n, 0, 0|p)(r, s, -, -)(r', s', -, -)\}$ *is fibred if its final pattern knot* $K(n, p \mid rs, r's')$ *is fibred.*

Proof of Theorem A. We only prove the 'if' part, since the 'only if' part is a well-known fact. By Proposition 8.23 in [2] on the Alexander polynomial of satellite knots in terms of that of the pattern and companion knots, we have the following:

Proposition 13.4. *Let K be a non-separating* $(1,1)$*-double torus knot, and K', K'' its pattern knots* (*as in* Section 2). *If one of K, K' and K'' has a monic Alexander polynomial, then the other two also have a monic Alexander polynomial.*

Suppose that $K = K\{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$ has monic Alexander polynomial. Then the pattern knot $K(n, p \mid rs, r's')$ has a monic Alexander polynomial. By Theorem B, Proposition 12.9, and Theorem C, $K(n, p \mid rs, r's')$ is fibred. Finally by Proposition 13.3, K is fibred. □

13.3. Minimal genus Seifert surfaces for those without admissible words. In this subsection, we first construct a minimal genus Seifert surface for $K(n, p \mid \alpha, \beta)$ which does not necessarily have an admissible word. Then we construct a minimal genus Seifert surface for a general $(1,1)$-double torus knot, and prove Theorem D.

We construct a Seifert surface $\mathcal{B} \cup \Sigma$ similarly as we did in Subsection 13.1. The only difference is that the strictly normalized word $W(K)$ may have many syllables of the maximal degree and/or the minimal degree. Recall that the maximal syllable $(x^{\deg(W)}y^{\deg(W)})^{\pm 1}$ (resp. minimal syllable 1) can be read only when the corresponding long-arc runs along the back (resp. face) side of the tube. Therefore, if $(x^{\deg(W)}y^{\deg(W)})^{\pm 1}$ and 1 alternates several times, $\mathcal{B}$ must be switched from the face side to the back side several times. Since we only need to construct a (minimal genus) Seifert surface and do not need to push $\mathcal{B}$ onto Σ, we only specify all long-arcs, except for those corresponding to the syllable 1, to lie on the back side of the tubes. So we put the mark ∂ exactly before and after each syllable 1. Then as in the proof of Claim 1 in Subsection 13.1, we can bring $\mathcal{B}$ so that we can superimpose Σ. Now we have the following.

Proposition 13.5. *The Seifert surfaces $F = \Sigma \cup \mathcal{B}$ for $K(n, p \mid \alpha, \beta)$ thus constructed is of minimal genus. Furthermore, the genus of $K(n, p \mid \alpha, \beta)$ is exactly half of the degree of its Alexander polynomial.*

Proof. Since F is constructed from two disks by adding $\deg(W)$ tubes, we have $g(F) = \deg(W)$, By construction, $\deg(W) = \deg(h(t))$ defined in Section 4. By Proposition 10.2 and Remark 5.7, we have $2\deg(W) = \deg(\Delta_{K(n,p \mid \alpha,\beta)}(t))$. Since the degree of the Alexander polynomial of a knot does not exceed the twice of the genus, we have the proposition. □

Next we construct a minimal genus Seifert surface for a satellite $(1,1)$-double torus knot $K_0 = \{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$, with n odd, defined in Section 2.

In Proposition 2.2 we saw that K_0 is obtained from the pattern knot $K(n,p \mid \alpha,\beta) = K(n, p \mid rs, r's')$ along the companion torus knots $T(r,s)$ and $T(r',-s')$. Let V be a handlebody of genus 2 such that ∂V carries $K(n, p \mid rs, r's')$ and $(S^3 \setminus V) \cap F$ consists of two disks D_1 and D_2, where F is the minimal genus Seifert surface we have constructed for $K(n, p \mid rs, r's')$ and the two disks D_1 and D_2 come from D_L and D_R. By knotting each 1-handle of V along $T(r,s)$ and $T(r',-s')$, and replacing each of D_1 and D_2 by the fibre surface for $T(r,s)$ and $T(r',-s')$ respectively, we obtain a Seifert surface S for K_0, with $g(S) = g(T(r,s)) + g(T(r',-s')) + g(K(n, p \mid rs, r's'))$. Then we see that $g(S) = g(K_0)$ by using the following proposition due to Schubert, where $l = 1$.

Proposition 13.6 ([2, Proposition 2.10]). *Let K be a satellite knot with pattern knot K_p and companion knot K_c. Then we have $g(K) \geq |l| g(K_c) + g(K_p)$, where l is the linking number between K and the meridian of the tubular neighborhood of K_c.*

Finally we prove Theorem D.

By the argument in Section 10 and Proposition 3.4, we see that the degree d of the Alexander polynomial of $K_0 = \{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$ is the product of those of $T(r,s), T(r',-s')$ and $K(n, p \mid rs, r's')$. Therefore, we have $(1/2)d = g(K_0)$. □

14. Separating (1, 1)-double torus knots

In this section, we study separating (1, 1)-double torus knots, and prove Theorem 14.1 which is a direct consequence of Propositions 14.2, 14.3 and 14.4. Let $K = \{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$ be a separating (1,1)-double torus knot. Then by Proposition 2.1, n is even. Since K, embedded in the double torus H, is separating, each component of $H \setminus K$ is a genus 1 Seifert surface for K, and only fibred knots with genus at most one are the unknot, the trefoil, and the figure-eight knot.

Theorem 14.1. *Let $K = \{(n,0,0;n,0,0 \mid p)(r,s,-,-)(r',s',-,-)\}$ be a separating (1, 1)-double torus knot, where n is even. Then we have*:

(1) $g(K) = (1/2) \deg \Delta_K(t)$ *if and only if $rsr's' = 0$, or $lrsr's' \neq 0$, where l is the linking number of the 2-bridge link $B(n, p)$ with an arbitrary orientation.*

(2) *K is the unknot if and only if $rsr's' = 0$.*

(3) *K is a non-trivial fibred knot if and only if $n = 2$, and $|rsr's'| = 1$.*

Conclusion (2) in Theorem 14.1 is guaranteed by Theorem 3.27 in [6]. Furthermore, the Alexander polynomial of K is calculated as follows:

Proposition 14.2 ([6, Proposition 3.16, 3.18]). *Let K be a separating double torus knot as above. Denote by l the linking number of the 2-bridge link $B(n, p)$ with an arbitrary orientation. Then $\Delta_K(t) \doteq \theta t^2 + (1 - 2\theta)t + \theta$, where $\theta = -l^2\alpha\beta$, $\alpha = rs$, and $\beta = r's'$.*

Then, Conclusion (1) follows from (2) and Proposition 14.2.

Now to prove (3), we first prove the following proposition on the pattern knots.

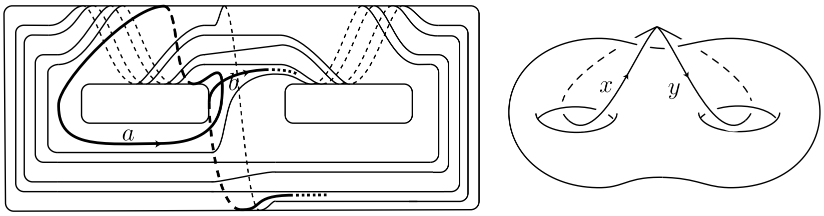


Fig. 14.1. $K(n, p \mid \alpha, \beta) = K(4, 1 \mid 1, 1)$.

Proposition 14.3. *Let $K = K(n, p \mid \alpha, \beta)$ be a non-trivial separating $(1, 1)$-double torus knot. Then K is fibred if and only if $n = 2$ and $|\alpha\beta| = 1$.*

Proof. Suppose $n = 2$ and hence $p = 1$ since $\gcd(n, p) = 1$ and $0 < p < n$. Then $K(2, 1 \mid 1, 1)$ is the figure-eight knot and $K(2, 1 \mid 1, -1)$ is the trefoil knot, which are both fibred.

Next we show that if $n \geq 4$, then K is not fibred. Suppose the contrary, where $n \geq 4$ and K is fibred. By Proposition 14.2 we may assume $\alpha = 1$ and $\beta = \pm 1$. Since K separates the double torus H into two Seifert surfaces F_0 and F_1 for K, $H = F_0 \cup F_1$ separates S^3 into two handlebodies V_0 and V_1 of genus 2. If K is fibred, F_0 is isotopic to F_1 and both V_0 and V_1 should be homeomorphic to $F_1 \times [0, 1]$. We claim that V_1, containing the 'outside' of H, is not homeomorphic to $F_1 \times [0,1]$. To do this, it suffices to show that an inclusion map $\phi\colon \pi_1(F_1) \hookrightarrow \pi_1(V_1)$ is not surjective. Now we choose the set of free generators, $\{a, b\}$ and $\{x, y\}$, for $\pi_1(F_1)$ and $\pi_1(V_1)$ respectively as in Fig. 14.1.

Then it is easy to see that $\phi(a) = x^{-1}$, and $\phi(b) = y^{\varepsilon_1} x^{\varepsilon_2} y^{\varepsilon_3} x^{\varepsilon_4} \cdots y^{\varepsilon_{n-1}} x^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$, $1 \leq i \leq n$. Since $n \geq 4$ and $\phi(b)$ is a reduced word in the free group $\pi_1(V_1)$, it follows that y cannot be expressed in terms of $\phi(a)$ and $\phi(b)$, and hence ϕ is not surjective. This proves Proposition 14.3. $\square$

Proposition 14.4. *Let $K = \{(n, 0, 0; n, 0, 0 \mid p)(r, s, -, -)(r', s', -, -)\}$ be a non-trivial separating $(1, 1)$-double torus knot, where n is even. If $|r|$ and $|s| \geq 2$ (or $|r'|$ and $|s'| \geq 2$), then K is not a fibred knot.*

Proof. Suppose $\theta \neq 0$. If $|r|$ and $|s| \geq 2$, then the Alexander polynomial of the pattern knot $K(n, p \mid rs, r's')$ is not monic by Proposition 14.2, and hence K is not fibred. If $\theta = 0$, then the Alexander polynomial is 1, but K is not trivial and hence is not fibred. $\square$

Then, Conclusion (3) follows from Propositions 14.3 and 14.4. The proof of Theorem 14.2 is now completed. $\square$

ACKNOWLEDGEMENTS. We would like to express our appreciation to P. Hill, M. Sakuma, Y. Marumoto, T. Yasuda and, in particular, to T. Kobayashi, for their invaluable comments and suggestions.

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Mikami Hirasawa
Department of Mathematics
Gakushuin University
Tokyo 171–8588
Japan

Current address:
Department of Mathematics
Nagoya Institute of Technology
Gokiso, Nagoya 466–8555
Japan
e-mail: hirasawa.mikami@nitech.ac.jp

Kunio Murasugi
Department of Mathematics
University of Toronto
Toronto, ON M5S2E4
Canada
e-mail: murasugi@math.toronto.edu