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A REFINEMENT OF JOHNSON'S BOUNDING FOR THE STABLE GENERA OF HEEGAARD SPLITTINGS

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Abstract

For each integer $k \geq 2$, Johnson gave a 3-manifold with Heegaard splittings of genera $2k$ and $2k-1$ such that any common stabilization of these two surfaces has genus at least $3k-1$. We modify his argument to produce a 3-manifold with two Heegaard splitings of genus $2k$ such that any common stabilization of them has genus at least $3k$.

1. Introduction

A *genus g Heegaard splitting* for a closed 3-manifold M is a triple (Σ, H^-, H^+) where H^-, H^+ are genus g handlebodies such that $H^- \cup H^+ = M$ and $H^- \cap H^+ = \partial H^- = \partial H^+ = \Sigma$. The genus g surface Σ is called the *Heegaard surface*. Any closed, orientable, connected 3-manifold has Heegaard splittings. Two Heegaard splittings for the same 3-manifold are called *isotopic* if there is an ambient isotopy taking one of the Heegaard surfaces to the other.

Suppose α is a properly embedded arc in H^+ parallel to Σ. Add a regular neighborhood of α to H^- and delete it from H^+. Then the result is a new Heegaard splitting whose genus is one greater than that of the original. A *stabilization* of a Heegaard splitting is another splitting obtained by a finite sequence of such processes. Any two Heegaard splittings of the same 3-manifold have a common stabilization [12], [17]. That is to say, there is a third Heegaard splitting which is isotopic to a stabilization of each of the initial splittings. The *stable genus* of two Heegaard splittings is the minimal genus of their common stabilizations.

It had been conjectured that the stable genus of any two Heegaard splittings is at most $p+1$, where p is the larger of the two initial genera, which is called *the stabilization conjecture*. This conjecture has been verified for many classes of 3-manifolds, including Seifert fibered spaces [15], most genus-two 3-manifolds [14] (see also [2]) and most graph manifolds [4] (see also [16]).

Johnson [9] gave a counterexample for this conjecture. For each $k \geq 2$, he constructed an irreducible toroidal 3-manifold with Heegaard splittings of genera $2k-1$ and $2k$ such that the stable genus of these two splittings is $3k-1$. In fact, we can see

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that the stable genus is at most $3k-1$ by a simple observation, and the point is the bounding from below. His construction can be easily modified to produce an atoroidal 3-manifold with Heegaard splittings of genera $2k-n$ and $2k$ whose stable genus is $3k-n$, where n is larger than 1. However, the larger n is, the closer the stable genus is to the genus of the original. If n is larger than $k-2$, it does not give a counterexample for the conjecture. We modify his construction to the opposite direction and refine the bounding for the stable genus from bellow as the following:

Theorem 1. *For every $k \geq 2$, there exists a 3-manifold with two Heegaard splittings of genus $2k$ whose stable genus is $3k$.*

This 3-manifold is reducible. Actually, we get it by taking connected sum of two closed 3-manifolds with Heegaard splittings of genus k with high Hempel distance (see Section 6). It may be a strong point of this paper that we can construct a counterexample for the stabilization conjecture from genus-two 3-manifolds by substituting 2 for k. There are fairly many studies on genus-two 3-manifolds. For instance, Kobayashi [10] gave a complete list of genus-two 3-manifolds admitting nontrivial torus decompositions.

Prior to Johnson [9], a counterexample for the "oriented version" of the stabilization conjecture was given by Hass, Thompson and Thurston [5]. In the "oriented version", two Heegaard splittings are called isotopic only if the isotopy preserves the order of the handlebodies. For a Heegaard splitting, the minimal genus of its stabilizations where the handlebodies can be interchanged by an isotopy is called the *flip genus*. They showed that there is a Heegaard splitting whose flip genus is twice the initial genus.

For the oriented version, Johnson [8] gave an estimate for general Heegaard splittings. He showed that the flip genus of any Heegaard splitting of genus k with Hempel distance d is at least $\min\{2k, (1/2)d\}$. His counterexample in [9] and ours for the non-oriented version can be viewed as applications of this estimation.

Bachman [1] also gave several counterexamples using different techniques. One is for the oriented version, and another is for the non-oriented version.

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2. Heegaard splittings

To begin with, we will define Heegaard splittings for compact 3-manifolds possibly with boundaries. A *compression body* is a connected 3-manifold H which can be obtained from $S \times [0, 1]$ by attaching finitely many 1-handles to $S \times \{1\}$ where S is a closed, orientable, possibly disconnected surface. We will use the notations like $\partial_- H = S \times \{0\}$ and $\partial_+ H = \partial H \setminus \partial_- H$. Handlebodies are regarded as the extreme cases of compression bodies, i.e. $\partial_- H = \emptyset$. A *Heegaard splitting* for a compact 3-manifold M is a triple (Σ, H^-, H^+) where H^-, H^+ are compression bodies such that $H^- \cup H^+ = M$ and $H^- \cap H^+ = \partial_+ H^- = \partial_+ H^+ = \Sigma$. The *genus* of (Σ, H^-, H^+) is the genus of Σ.

In addition to stabilizations, we will use some sorts of operations to construct new Heegaard splittings from given Heegaard splittings. Now, we will define such operations in the next three paragraphs:

Suppose (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+) are Heegaard splittings for compact 3-manifolds M_1 and M_2, respectively. Let B_i be a ball in M_i such that $\Sigma_i \cap B_i$ is an equatorial plane of B_i for each $i = 1, 2$. Suppose $\varphi\colon \partial B_1 \to \partial B_2$ is a homeomorphism such that $\varphi(H_1^- \cap \partial B_1) = H_2^- \cap \partial B_2$ and $\varphi(H_1^+ \cap \partial B_1) = H_2^+ \cap \partial B_2$. Let M be the 3-manifold obtained by gluing the closures of $M_1 \setminus B_1$ and $M_2 \setminus B_2$ by φ, namely, the connected sum of M_1 and M_2. Let H^- be the compression body obtained by gluing the closures of $H_1^- \setminus B_1$ and $H_2^- \setminus B_2$ by φ and let H^+ be the compression body obtained by gluing the closures of $H_1^+ \setminus B_1$ and $H_2^+ \setminus B_2$ by φ. Then (Σ, H^-, H^+) is a Heegaard splitting for M where $\Sigma = \partial_+ H^- = \partial_+ H^+$. It is called the *connected sum* of (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+).

Suppose $M_1, M_2, (\Sigma_1, H_1^-, H_1^+)$ and (Σ_2, H_2^-, H_2^+) are as above. Suppose $\partial_- H_1^+$ is non-empty and homeomorphic to $\partial_- H_2^+$. Let M be the union of M_1 and M_2 identifying $\partial_- H_1^+$ with $\partial_- H_2^+$ by some homeomorphism. Since H_i^+ is a compression body, it can be decomposed into a product manifold $\partial_- H_i^+ \times [0,1]$ and a collection of 1-handles for each $i = 1, 2$. The part $(\partial_- H_1^+ \times [0, 1]) \cup (\partial_- H_2^+ \times [0, 1])$ of M can be collapsed without changing the topology of M. Then we can regard the 1-handles which belonged to H_1^+ are attached to H_2^-, forming a new compression body H^+. Similarly, H_1^- and the 1-handles which belonged to H_2^+ form another compression body H^-. Then (Σ, H^-, H^+) is a Heegaard splitting for M where $\Sigma = \partial_+ H^- = \partial_+ H^+$. We will say that (Σ, H^-, H^+) is the *amalgamation* of (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+). Note that $H_1^- \subset H^-, H_2^- \subset H^+$ and (Σ, H^+, H^-) is the amalgamation of (Σ_2, H_2^-, H_2^+) and (Σ_1, H_1^-, H_1^+).

Suppose M is a compact 3-manifold with a single boundary component, and (Σ, H^-, H^+) is a Heegaard splitting for M such that $\partial_- H^+ = \partial M$. Decompose H^+ into a product manifold $\partial_- H^+ \times [0, 1]$ and a collection of 1-handles. Let α be a vertical arc in $\partial_- H^+ \times [0, 1]$. Add a neighborhood of the union of α and $\partial_- H^+$ to H^-, to obtain a compression body H'^+. Then the closure of the complement of H'^+ in M is homeomorphic to the union of $(\partial_- H^+ \setminus (\text{an open disk})) \times [0, 1]$ and 1-handles. This is a handlebody, denoted by H'^-. We will call (Σ', H'^-, H'^+) the *boundary stabilization* of (Σ, H^-, H^+) where $\Sigma' = \partial H'^- = \partial_+ H'^+$. We are afraid the labels of H'^- and H'^+ are confusing, but we would like to keep the condition that ∂M is contained in the latter compression body.

Johnson's counterexample was constructed by amalgamations along the torus boundaries. All his arguments in [9] can be applied also if the boundaries have genus more than one. We will make the same construction changing the place of torus boundaries by sphere boundaries. Though it is common in theories on Heegaard splittings to assume that the 3-manifolds do not have sphere boundaries, we do not have to do so at least in

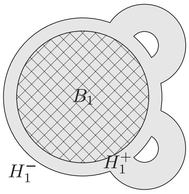

Fig. 1.

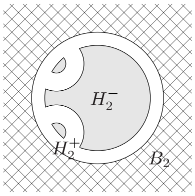

Fig. 2.

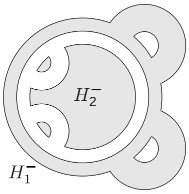

Fig. 3.

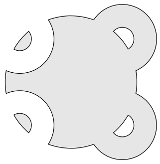

Fig. 4.

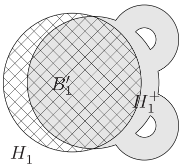

Fig. 5.

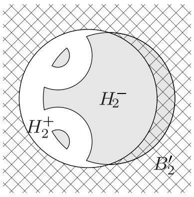

Fig. 6.

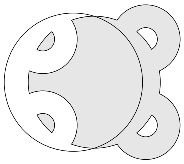

Fig. 7.

the above definitions. It is useful in our arguments to deal with amalgamations along sphere boundaries while they are no other than connected sums as the following:

Proposition 2. *Suppose* (Σ_i, H_i^-, H_i^+) *is a Heegaard splitting for a closed* 3*-manifold* M_i*, and* B_i *is an open ball in* H_i^+ *for* $i = 1, 2$*. Then the amalgamation of* $(\Sigma_1, H_1^-, H_1^+ \setminus B_1)$ *and* $(\Sigma_2, H_2^-, H_2^+ \setminus B_2)$ *is isotopic* (*in the oriented version*) *to the connected sum of* (Σ_1, H_1^-, H_1^+) *and* (Σ_2, H_2^+, H_2^-).

Proof. See above pictures. In Fig. 1, H_1^+ is regarded as a ball attached 1-handles while H_1^- as its complement. In Fig. 2, H_2^+ and H_2^- are figured similarly but inside out. The handlebodies H_1^+, H_2^- are painted gray and B_1, B_2 are patterned with meshes. The amalgamation is constructed by gluing $M_1 \setminus B_1$ and $M_2 \setminus B_2$ as Fig. 3 and collapsing the product part as Fig. 4. On the other hand, choose a ball B_i' which intersects Σ_i in a disk for each $i = 1, 2$ as Figs. 5, 6. The connected sum is constructed by gluing $M_1 \setminus B_1'$ and $M_2 \setminus B_2'$ as Fig. 7, which is equivalent to Fig. 4. □

Proposition 3. *Suppose* (Σ, H^-, H^+) *is a Heegaard splitting for a closed* 3-*manifold* M, *and* B^-, B^+ *are open balls in* H^-, H^+, *respectively. Then the boundary stabilization of* $(\Sigma, H^-, H^+ \setminus B^+)$ *is isotopic* (*in the oriented version*) *to* $(\Sigma, H^+, H^- \setminus B^-)$.

This can be proved by pushing B^+ into H^- from H^+. The details are left to the reader.

3. Sweep-outs and graphics

Rubinstein and Scharlemann [13] introduced a powerful machinery to analyze Heegaard splittings. It is called the *Rubinstein–Scharlemann graphic* or just the *graphic* for short. Roughly speaking, it is a 1-complex in $[-1, 1] \times [-1, 1]$ representing the relation between two Heegaard splittings for a 3-manifold. While their original construction was based on the Cerf theory [3], it is useful to define it in terms of *stable* maps after Kobayashi and Saeki [11].

Suppose X, Y are smooth manifolds and $\varphi, \psi \colon X \to Y$ are smooth maps. The maps φ and ψ are called *isotopic* if there are diffeomorphisms $h_X \colon X \to X$ and $h_Y \colon Y \to Y$, each isotopic to the identity map on its respective space, such that $\varphi = h_Y \circ \psi \circ h_X$. A smooth map $\varphi \colon X \to Y$ is called stable if there exists an open neighborhood U of φ in $C^\infty(X, Y)$ (under the Whitney C^∞ topology, see [6]) such that every map in U is isotopic to φ. A Morse function is a stable function from a smooth manifold to $\mathbb{R}$.

Suppose M is a compact, orientable, connected, smooth 3-manifold, and $\partial M = \partial_- M \sqcup \partial_+ M$ is a partition of boundary components of M. Let Θ^- be a finite graph in M adjacent to all components of $\partial_- M$ and let Θ^+ similarly for $\partial_+ M$. A *sweep-out* for M is a smooth function $f \colon M \to [-1,1]$ such that $f^{-1}(t)$ is a closed, connected surface parallel to $f^{-1}(0)$ for $t \in (-1,1)$, while $f^{-1}(-1) = \Theta^- \cup \partial_- M$ and $f^{-1}(1) = \Theta^+ \cup \partial_+ M$. The sets $\Theta^- \cup \partial_- M$ and $\Theta^+ \cup \partial_+ M$ are called the *spines* of f. We will say that f *represents* a Heegaard splitting (Σ, H^-, H^+) for M if f can be isotoped so that $f^{-1}(0) = \Sigma$, $f^{-1}(-1) \subset H^-$ and $f^{-1}(1) \subset H^+$.

Suppose M_i is a compact, orientable, connected, smooth, 3-dimensional submanifold of a smooth 3-manifold M, and f_i is a sweep-out for M_i for each $i = 1, 2$. Assume $M_1 \cap M_2$ is a non-empty 3-dimensional submanifold of M. We define a smooth map $f_1 \times f_2 \colon M_1 \cap M_2 \to [-1, 1] \times [-1, 1]$ by $(f_1 \times f_2)(p) = (f_1(p), f_2(p))$. In the case when $M_1 = M_2 = M$, Kobayashi and Saeki [11] showed that we can deform f_1 and f_2 by an arbitrarily small isotopy so that $f_1 \times f_2$ is stable on the complement of the spines of f_1 and f_2. An almost identical argument induces the same property in the general case. Thus, we can assume $f_1 \times f_2$ is a stable map on the complement M^* of the spines of f_1 and f_2 in $M_1 \cap M_2$.

The Rubinstein–Scharlemann graphic for f_1 and f_2 is a properly embedded 1-complex in $[-1, 1] \times [-1, 1]$ naturally extended from the discriminant set of $(f_1 \times f_2)|_{M^*}$. We mean the discriminant set as the image of the singular set $S_{f_1 \times f_2} = \{p \in$

$M^* \mid \operatorname{rank}(d(f_1 \times f_2))_p \leq 1\}$. The singular set $S_{f_1 \times f_2}$ is a 1-dimensional smooth submanifold in M^* consisting of all the points where a level surface of f_1 is tangent to a level surface of f_2. The tangent point is either a "center" or a "saddle". The discriminant set is a smooth immersion of $S_{f_1 \times f_2}$ into $(-1, 1) \times (-1, 1)$ with normal crossings except for finitely many cusps. We regard the crossings as valence-four vertices and the cusps as valence-two vertices of the graphic. They are called *crossing vertices* and *birth-death vertices*, respectively. On the boundary of $[-1, 1] \times [-1, 1]$, there are valence-one or valence-two vertices of the graphic. Each edge is monotonously increasing or decreasing as a graph in $(-1, 1) \times (-1, 1)$. See [11] or [13] for detailed descriptions.

For each $s \in (-1, 1)$, the pre-image in $f_1 \times f_2$ of the vertical arc $\{s\} \times [-1, 1]$ is the level surface $f_1^{-1}(s)$. The restriction of f_2 to the level surface has critical levels corresponding to the intersections of the vertical arc and the graphic.

DEFINITION 4. Sweep-outs f_1 and f_2 are called *generic* if $f_1 \times f_2$ is stable on M^* and every vertical or horizontal arc on $[-1, 1] \times [-1, 1]$ contains at most one vertex of the graphic.

4. Labeling the graphics

We will characterize some relations of the level surfaces of sweep-outs. It gives a "labeling" for the complementary regions of the graphic. This kind of labeling is one of the most useful techniques for reading graphics.

Suppose M is a compact, orientable, connected, smooth 3-manifold, and N is a 3-dimensional submanifold of M. Let (Σ, H^-, H^+) and (T, G^-, G^+) be Heegaard splittings for M and N, respectively. Let f and g be sweep-outs representing (Σ, H^-, H^+) and (T, G^-, G^+), respectively. We will use the notations like $\Sigma_s = f^{-1}(s)$, $H_s^- = f^{-1}([-1, s])$, $H_s^+ = f^{-1}([s, 1])$ and $T_t = g^{-1}(t)$.

DEFINITION 5. For $s, t \in (-1, 1)$, we will say that T_t is *mostly above* Σ_s if $H_s^- \cap T_t$ is contained in a disk in T_t. Similarly, T_t is *mostly below* Σ_s if $H_s^+ \cap T_t$ is contained in a disk in T_t.

DEFINITION 6. For generic sweep-outs f and g, we will say that f *spans* g if T_{t_-} is mostly below Σ_s and T_{t_+} is mostly above Σ_s for some values $s, t_-, t_+ \in (-1, 1)$. Moreover, we will say that f spans g *positively* if $t_- < t_+$, or *negatively* if $t_- > t_+$.

DEFINITION 7. For generic sweep-outs f and g, we will say that f *splits* g if there is a value $s \in (-1, 1)$ such that for every $t \in (-1, 1)$, the level surface T_t is neither mostly above nor below Σ_s.

Let R_a be the set of points $(s, t) \in (-1, 1) \times (-1, 1)$ such that T_t is mostly above Σ_s. Similarly, let R_b be the set of points such that T_t is mostly below Σ_s. Note that

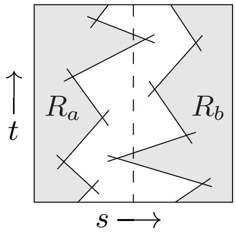

Fig. 8.

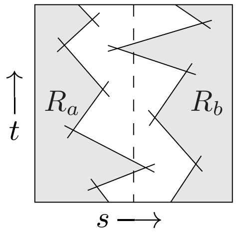

Fig. 9.

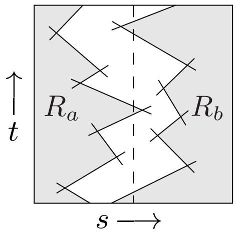

Fig. 10.

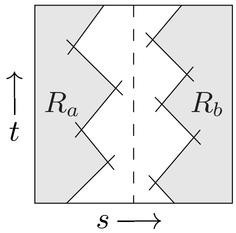

Fig. 11.

if a point (s, t) is in R_a then its left side $(-1, s] \times \{t\}$ is contained in R_a because the area $H_s^- \cap T_t$ in the surfaces T_t increase with s. Symmetrically, if $(s, t) \in R_b$ then $[s, 1) \times \{t\} \subset R_b$. The right side of R_a and the left side of R_b are bounded by edges of the graphic.

Fig. 8 illustrates the condition that f spans g positively. In Fig. 9, f spans g negatively. In Fig. 10, f spans g positively and negatively. In Fig. 11, f splits g. Note that exactly one of the conditions spanning or splitting happens for any generic pair of sweep-outs.

DEFINITION 8. We will say that (Σ, H^-, H^+) *spans* (T, G^-, G^+) positively (negatively) if (Σ, H^-, H^+) and (T, G^-, G^+) are represented by generic sweep-outs f and g, respectively, such that f spans g positively (negatively). We will also say that (Σ, H^-, H^+) *splits* (T, G^-, G^+) if (Σ, H^-, H^+) and (T, G^-, G^+) are represented by generic sweep-outs f and g such that f splits g.

Note that if (Σ, H^-, H^+) spans (T, G^-, G^+) positively, (Σ, H^+, H^-) spans (T, G^-, G^+) negatively.

5. Spanning sweep-outs

The spanning condition gives a bound for the genus of one of the Heegaard splittings. Suppose (Σ, H^-, H^+) is a Heegaard splitting for a smooth 3-manifold M, and (T, G^-, G^+) is a Heegaard splitting for a 3-dimensional submanifold N of M. Suppose

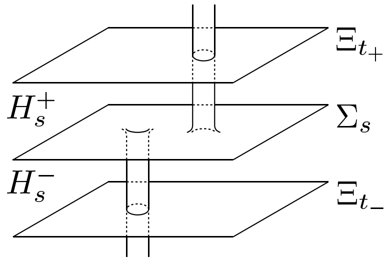

Fig. 12.

f and g are generic sweep-outs representing (Σ, H^-, H^+) and (T, G^-, G^+), respectively. Assume f spans g positively.

By the definition, there is a value $-1 < s < 1$ and values $-1 < t_- < t_+ < 1$ such that T_{t_-} is mostly below Σ_s and T_{t_+} is mostly above Σ_s. That is to say, T_{t_-} is contained in H_s^- except for some disks while T_{t_+} is contained in H_s^+ except for some disks as Fig. 12. In the product manifold $g^{-1}([t_-, t_0])$, the surface Σ_s must be "mostly separating" one boundary component from the other. The reader can notice that $\Sigma_s \cap g^{-1}([t_-, t_+])$ has genus at least the genus of T. By similar observations, we have the following:

Lemma 9. *If f spans g then $\Sigma_s \cap N$ has genus at least the genus of T for some value $s \in (-1, 1)$. If f spans g positively and negatively then $\Sigma_s \cap N$ has genus at least twice the genus of T for some value $s \in (-1, 1)$.*

Recall that we allow 3-manifolds to have sphere boundaries. Still, next four lemmas can be proved identically as those in brackets.

Lemma 10 ([8, Lemma 9]). *Every Heegaard splitting spans itself positively.*

Lemma 11 ([9, Lemma 12]). *If (Σ, H^-, H^+) spans (T, G^-, G^+) positively (negatively) then every stabilization of (Σ, H^-, H^+) spans (T, G^-, G^+) positively (negatively).*

Lemma 12 ([9, Lemma 14]). *Suppose (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+) are Heegaard splittings for compact, smooth 3-manifolds M_1 and M_2, respectively. Let (Σ, H^-, H^+) be the amalgamation of (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+). Suppose (T, G^-, G^+) is a Heegaard splitting for a 3-dimensional submanifold N of M_1. If (Σ_1, H_1^-, H_1^+) spans (T, G^-, G^+) positively (negatively) then (Σ, H^-, H^+) spans (T, G^-, G^+) positively (negatively).*

Lemma 13 ([9, Lemma 16]). *Suppose M is a smooth 3-manifold with a single boundary component and (Σ, H^-, H^+) is a Heegaard splitting for M such that $\partial_- H^+ =$*

∂M. Suppose (T, G^-, G^+) is a Heegaard splitting for a 3-dimensional submanifold N of M. Let (Σ', H'^-, H'^+) be the boundary stabilization of (Σ, H^-, H^+). If (Σ, H^-, H^+) spans (T, G^-, G^+) positively (negatively) then (Σ', H'^-, H'^+) spans (T, G^-, G^+) negatively (positively).

6. Splitting sweep-outs

The *curve complex* $C(T)$ of a closed, orientable, connected surface T is a simplicial complex defined as follows: The vertices of $C(T)$ are isotopy classes of essential loops in T. Distinct n vertices span a $(n-1)$-simplex of $C(T)$ if and only if they are represented by pairwise disjoint loops in T. There is a canonical distance d among the vertices. We mean that $d(v_1, v_2)$ is the number of edges on the shortest path between two vertices v_1 and v_2 in the 1-skeleton of $C(T)$.

Suppose (T, G^-, G^+) is a Heegaard splitting. When D^- and D^+ are essential disks in G^- and G^+, respectively, ∂D^- and ∂D^+ can be regarded as vertices of $C(T)$. Hempel [7] defined the *distance* of (T, G^-, G^+), denoted by $d(T)$, as the minimum of $d(\partial D^-, \partial D^+)$ over all pairs of essential disks $D^- \subset G^-$, $D^+ \subset G^+$. It is a numerical invariant indicating the irreducibility of Heegaard splittings (see [7]).

The goal in this section is to estimate the genus of (Σ, H^-, H^+) by $d(T)$ when a Heegaard splitting (Σ, H^-, H^+) splits another Heegaard splitting (T, G^-, G^+). We will almost trace the way of [9, Section 6] but modify it slightly to avoid arguments with the irreducibility of the manifolds.

Suppose M_1 and M_2 are irreducible, closed, smooth 3-manifolds other than S^3. Let M_i^* be the 3-manifold obtained by removing an open ball from M_i for each $i = 1, 2$. Let M be the union of M_1^* and M_2^* glued at their boundaries, namely, the connected sum of M_1 and M_2. Take either M_1^* or M_2^*, and rewrite it as N. Suppose (Σ, H^-, H^+) is a Heegaard splitting of genus k for M, and (T, G^-, G^+) is a Heegaard splitting of genus at least 2 with distance at least 2 for N. Assume (Σ, H^-, H^+) splits (T, G^-, G^+). By definition, there are generic sweep-outs f and g representing (Σ, H^-, H^+) and (T, G^-, G^+), respectively such that f splits g.

Lemma 14. *There exists a value $s_0 \in (-1, 1)$ such that:*

(1) *There are no vertices of the graphic on the vertical arc $\{s_0\} \times [-1, 1]$.*

(2) *$\Sigma_{s_0} \cap T_t$ contains an essential loop in T_t for each regular value t for $g|_{\Sigma_{s_0}}$.*

Proof. Let C be the set of values $s_0 \in (-1, 1)$ satisfying the condition (2). When the condition (2) fails, either $H^-_{s_0} \cap T_t$ or $H^+_{s_0} \cap T_t$ is contained in a disk in T_t for some value t, so T_t is mostly above or below Σ_{s_0}. Therefore C can be considered as the complement of the projections of $R_a \cup R_b$ in $[-1, 1] \times \{\mathrm{pt}\}$. Since f splits g, the set C is a non-empty closed interval.

If C is a single point $\{s_1\}$, there is a crossing vertex (s_1, t_1) of which the left quadrant is contained in R_a and the right quadrant is contained in R_b. For a small ε, the intersection $H^+_{s_1-\varepsilon} \cap T_{t_1}$ becomes $H^+_{s_1+\varepsilon} \cap T_{t_1}$ by a transformation including only two singularities. However, $H^+_{s_1-\varepsilon} \cap T_{t_1}$ is contained in a disk while $H^+_{s_1+\varepsilon} \cap T_{t_1}$ covers T_{t_1} except for some disks. This is possible only when T_{t_1} is a torus. Since we assume the genus of (T, G^-, G^+) is at least 2, the closed interval C is non-trivial.

There are finitely many vertices in the graphic, so there exists a value s_0 in C such that the vertical arc $\{s_0\} \times [-1, 1]$ passes through no vertices of the graphic. □

Similarly to H^-_s and H^+_s, we will write $G^-_t = g^{-1}([-1, t])$ and $G^+_t = g^{-1}([t, 1])$.

Lemma 15. *There exists a non-trivial closed interval* $[a, b] \subset [-1, 1]$ *such that*:
(1) *For a small* ε, *the intersection* $\Sigma_{s_0} \cap T_{a-\varepsilon}$ *has a component bounding an essential disk of* $G^-_{a-\varepsilon}$ *or* $a = -1$.
(2) *For each* $t \in (a, b)$, *the intersection* $\Sigma_{s_0} \cap T_t$ *does not have any loops bounding essential disks of* G^-_t *or* G^+_t.
(3) *For a small* ε, *the intersection* $\Sigma_{s_0} \cap T_{b+\varepsilon}$ *has a component bounding an essential disk of* $G^+_{b+\varepsilon}$ *or* $b = 1$.

Proof. Let R_- be the set of points $(s, t) \in (-1, 1) \times (-1, 1)$ such that $\Sigma_s \cap T_t$ has a component bounding an essential disk of G^-_t. Similarly, Let R_+ be the set of points such that $\Sigma_s \cap T_t$ has a component bounding an essential disk of G^+_t. They determine another labeling for the graphic.

Let a be the maximum of the closure of $R_- \cap (\{s_0\} \times [-1, 1])$ (or -1 if $R_- \cap (\{s_0\} \times [-1, 1]) = \emptyset$). Let b be the minimum of the closure of $R_+ \cap (\{s_0\} \times [a, 1])$ (or 1 if $R_+ \cap (\{s_0\} \times [a, 1]) = \emptyset$).

If there is a horizontal arc $[-1, 1] \times \{t_0\}$ which intersects both R_- and R_+, the level surface T_{t_0} has a level loop of $f|_{T_{t_0}}$ bounding an essential disk of G^-_t and a level loop bounding an essential disk of G^+_t. It contradicts that the distance of (T, G^-, G^+) is at least 2. Therefore no horizontal arcs intersect both R_- and R_+. If $a = b$ then (s_0, a) must be a crossing vertex of the graphic. Since there are no vertices on $\{s_0\} \times [-1, 1]$, the closed interval $[a, b]$ is non-trivial. □

Fig. 13 illustrates the segment $\{s_0\} \times [a, b]$. We will consider the intersection loops on this segment and construct a subcomplex of $C(T_0)$ from these loops.

Let a' be a regular value for $g|_{\Sigma_{s_0}}$ just above a and let b' be a regular value for $g|_{\Sigma_{s_0}}$ just below b. Let Δ be the union of the disks bounded by the inessential loops of $\Sigma_{s_0} \cap g^{-1}(\{a', b'\})$ in Σ_{s_0}. Let F be the union of $\Sigma_{s_0} \cap g^{-1}([a', b'])$ and Δ. Consider a projection map π from $g^{-1}([a', b'])$ onto T_0.

Lemma 16. *If two level loops of* $g|_F$ *are isotopic in* F *then their projections are isotopic in* T_0.

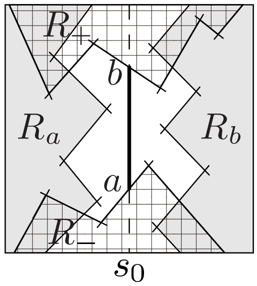

Fig. 13.

Proof. Any two level loops are disjoint in F so if two level loops are isotopic then they bound an annulus $A \subset F$. Note that A may contain some disks of Δ. By the condition (2) in Lemma 15, the boundary of a disk of Δ also bounds a disk in $T_{a'}$ or $T_{b'}$. Replacing the disks of Δ by the disks in $T_{a'}$ or $T_{b'}$, we can produce a new annulus A' contained in $g^{-1}([a', b'])$. The projection of A' into T_0 determines a homotopy from the image of one boundary of A' to the image of the other. Thus the projections of the two loops are isotopic. □

Let L be the set of isotopy classes of level loops of $g|_F$. A representative of an element $l \in L$ projects to a simple closed curve in T_0. If the projection is essential in T_0, we define $\pi_*(l)$ to be the corresponding vertex of the curve complex $C(T_0)$. If the projection is inessential, we define $\pi_*(l) = 0$. By the previous lemma, π_* is well defined as a map from L to the disjoint union $C(T_0) \sqcup \{0\}$.

Isotopy classes of essential level loops of $g|_F$ determine a pair-of-pants decomposition for F. The following can be proved identically as [9, Lemma 23].

Lemma 17. *If l_1 and l_2 are cuffs of the same pair of pants in $F \setminus L$ then their projections can be isotoped to be disjoint.*

For each regular value $t \in [a', b']$ for $g|_F$, let L^t be the set of isotopy classes of loops in $F \cap T_t$. Loops in $F \cap T_t$ are pairwise disjoint so their projections are pairwise disjoint. Moreover the projections contain at least one essential loop by the condition (2) in Lemma 14. Therefore the subcomplex L^t_C of $C(T_0)$ spanned by $\pi_*(L^t) \cap C(T_0)$ is non-empty.

If there are no critical levels for $g|_F$ between regular values t_1 and t_2 then $L^{t_1} = L^{t_2}$, so $L^{t_1}_C = L^{t_2}_C$. If there is a single critical level of center tangency between t_1 and t_2, the difference between L^{t_1} and L^{t_2} is the isotopy class of a trivial loop in F. By the condition (2) in Lemma 15, a trivial loop in F projects to a trivial loop in T_0. It implies $\pi_*(L^{t_1}) \cap T_0 = \pi_*(L^{t_2}) \cap T_0$, so $L^{t_1}_C = L^{t_2}_C$. If there is a single critical level of saddle tangency between t_1 and t_2, either one loop in $F \cap T_{t_1}$ is replaced by two loops in $F \cap T_{t_2}$ or two loops in $F \cap T_{t_1}$ is replaced by one loop in $F \cap T_{t_2}$ at the critical level.

If those three loops are essential in F, they bound a pair of pants in $F \setminus L$. By the previous lemma, their projections can be isotoped to be pairwise disjoint. Thus, there is an edge of $C(T_0)$ connecting $L_C^{t_1}$ and $L_C^{t_2}$. If one of those three loops is trivial in F then $L_C^{t_1}$ and $L_C^{t_2}$ have common vertices. Because L is the union of L^t over all regular values for $g|_F$, the subcomplex L_C of $C(T_0)$ spanned by $\pi_*(L) \cap C(T_0)$ is connected.

Consider two vertices v and v' in L_C. Suppose $v = v_0, v_1, \dots, v_n = v'$ is the shortest edge path connecting them in L_C. Let $l_i \in L$ projects to v_i for each $i = 0, 1, \dots, n$. If l_i and l_j are cuffs of the same pair of pants in $F \setminus L$ then there is an edge of L_C connecting v_i and v_j. Since the path is minimal, i and j must be consecutive. Then, we can estimate the diameter of L_C by the number of pairs of pants in $F \setminus L$. The number of pairs of pants in $F \setminus L$ is at most the negative Euler characteristic of F. Since the boundary components of F are essential in Σ_{s_0}, the Euler characteristic of F is at least that of Σ_{s_0}. We can conclude that the diameter of L_C is at most $2k - 2$. See the proof of [9, Lemma 24] for the details of this argument.

We are ready to prove the following:

Lemma 18. *If* (Σ, H^-, H^+) *splits* (T, G^-, G^+) *then* $2k \geq d(T_0)$.

Proof. Consider the case $a > -1$. By the condition (1) and (2) in Lemma 15, $\Sigma_{s_0} \cap T_{a-\varepsilon}$ has a component bounding an essential disk of $G^-_{a-\varepsilon}$ while $\Sigma_{s_0} \cap T_{a+\varepsilon}$ does not. That implies a must be a critical level for $g|_{\Sigma_{s_0}}$ containing a saddle tangency. As above, the projections of the level loops before and after this singularity can be isotoped to be pairwise disjoint. The projection of one of the level loops before this singularity bounds an essential disk of G^-_0. The projections of the level loops after this singularity are contained in L_C. Thus, the boundary of the essential disk of G^-_0 is connected to L_C by an edge in $C(T_0)$.

Consider the case $a = -1$. The compression body $G^-_{a'}$ is a small neighborhood of the spine. If $G^-_{a'}$ is a handlebody, every component of $\Sigma_{s_0} \cap G^-_{a'}$ is an essential disk of $G^-_{a'}$. It contradicts the condition (2) in Lemma 15. Therefore $\partial_- G^-_{a'} = \partial N$ and every component of $\Sigma_{s_0} \cap T_{a'}$ is parallel to $\partial_- G^-_{a'}$. The compression body $G^-_{a'}$ has essential disks disjoint from any such loop because the genus of $\partial_+ G^-_{a'}$ is at least 2. Similarly to the above argument, the boundary of an essential disk of G^-_0 is connected to L_C by an edge in $C(T_0)$.

Symmetrical arguments for b imply that the boundary of an essential disk of G^+_0 is connected to L_C by an edge in $C(T_0)$. Since the diameter of L_C is at most $2k - 2$, the distance of (T, G^-, G^+) is at most $2k$. □

7. Isotopies of sweep-outs

While we recognize Heegaard splittings up to isotopy, the spanning or splitting condition can be changed by isotopies of the sweep-outs. In this section, we need to observe the transition of the condition during an isotopy of one of the sweep-outs. Recall we defined isotopies of smooth maps in Section 3.

Suppose again M_1 and M_2 are irreducible, closed, smooth 3-manifolds other than S^3. Let M_i^* be the 3-manifold obtained by removing an open ball from M_i for each $i = 1, 2$. Let M be the union of M_1^* and M_2^* glued at their boundaries. Take either M_1^* or M_2^*, and rewrite it as N. Suppose (Σ, H^-, H^+) is a Heegaard splitting for M, and (T, G^-, G^+) is a Heegaard splitting of genus at least 2 for N.

Lemma 19. *If (Σ, H^-, H^+) spans (T, G^-, G^+) positively and negatively then either there is a pair of sweep-outs f and g representing (Σ, H^-, H^+) and (T, G^-, G^+) such that f spans g positively and negatively or (Σ, H^-, H^+) splits (T, G^-, G^+).*

Proof. Since (Σ, H^-, H^+) spans (T, G^-, G^+) positively, there are generic sweep-outs f_0 and g representing (Σ, H^-, H^+) and (T, G^-, G^+), respectively such that f_0 spans g positively. Since (Σ, H^-, H^+) also spans (T, G^-, G^+) negatively, there are generic sweep-outs f' and g' representing (Σ, H^-, H^+) and (T, G^-, G^+), respectively such that f' spans g' negatively.

The sweep-outs g and g' represent the same Heegaard splitting, so g' will be isotopic to g after an appropriate sequence of handle slides of the spines. The handle slides can be done in an arbitrarily small neighborhood of the original spines so that f' still spans g' negatively. Therefore we can assume there is an isotopy taking g' to g. By the definition, there are diffeomorphisms $h_N\colon N \to N$ and $h_I\colon [-1.1] \to [-1, 1]$ such that $g = h_I \circ g' \circ h_N$. Let $h_M\colon M \to M$ be an arbitrary extension of h_N, and define $f_1 = h_I \circ f' \circ h_M$. Then f_1 spans g negatively.

Similarly, we can assume f_0 is isotopic to f_1 because f_0 and f_1 represent the same Heegaard splitting. According to [9, Lemma 26], there is a continuous family of sweep-outs $\{f_r \mid r \in [0, 1]\}$ such that f_r and g is generic for all but finitely many $r \in [0, 1]$. At the finitely many non-generic points, there are at most two valence-two or valence-four vertices at the same level, or one valence-six vertex.

For a generic value r, the sweep-out f_r either spans g or splits g. Then we can assume that except for finitely many non-generic values, f_r spans g positively or negatively, but not both. Since f_0 spans g positively and f_1 spans g negatively, there must be some non-generic value r_0 such that $f_{r_0-\varepsilon}$ spans g positively while $f_{r_0+\varepsilon}$ spans g negatively for a small $\varepsilon > 0$. Then we may consider three cases like Figs. 14, 15 and 16. In the case Fig. 14 or 15, there are three valence-four vertices at the same level, which is a contradiction. In the case Fig. 16, if the vertex v is valence-four, T must be a torus, as explained above. Even if the vertex v is valence-six, the same argument implies T is a torus, which is a contradiction. □

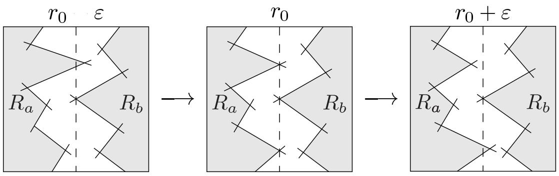

Fig. 14.

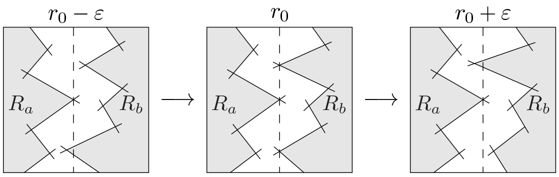

Fig. 15.

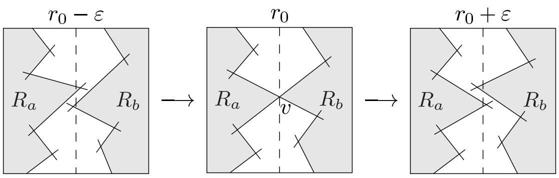

Fig. 16.

8. Planar surfaces in a product space

This section is for the final phase of the proof of the main theorem. It may possibly be easy for the reader to take this section after a view of Section 9.

Suppose Σ is a closed, orientable, connected surface of genus g. Let W be the product space $\Sigma \times [s_-, s_+]$ where $s_- < s_+$. Suppose P is a separating, planar surface with m_0 components properly embedded in W. Suppose P separates W into W_- and W_+. For each level $s \in [s_-, s_+]$, let $\Sigma^{\pm}(s)$ be the intersection of $\Sigma \times \{s\}$ with $W_{\pm}$. We will focus on $\Sigma^-(s_-)$ and $\Sigma^+(s_+)$. Let $g_-$and g_+ be the sum of the genera of all components of $\Sigma^-(s_-)$ and $\Sigma^+(s_+)$, respectively.

Lemma 20. $g \geq g_- + g_+$

Proof. We can assume P is incompressible in W because compressions of P does not change g_- or g_+.

Consider a component of P which has all its boundary components on $\Sigma \times \{s_-\}$. Such a surface is ∂-parallel, i.e. it can be isotoped onto $\Sigma \times \{s_-\}$ [18, Corollary 3.2]. Whichever it is parallel to a component of $\Sigma^-(s_-)$ or $\Sigma^+(s_-)$, the component has no

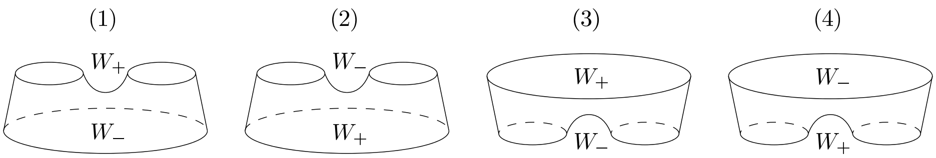

Fig. 17.

genus because P is planar. Therefore deleting the component of W_- or W_+ between these parallel surfaces does not reduce g_- or g_+. Thus, it is sufficient to prove the lemma assuming all such component has been deleted. In other words, we can assume every components of P has the boundaries both on $\Sigma \times \{s_-\}$ and $\Sigma \times \{s_+\}$.

Let $m_\pm$ be the number of components of $\Sigma^\pm(s_\pm)$ and let $p_\pm$ be the number of boundary components of $\Sigma^\pm(s_\pm)$. Then the Euler numbers of the surfaces concerned can be written as fallows:

$$\chi(\Sigma) = 2 - 2g,$$
$$\chi(\Sigma^-(s_-)) = 2m_- - 2g_- - p_-,$$
$$\chi(\Sigma^+(s_+)) = 2m_+ - 2g_+ - p_+,$$
$$\chi(P) = 2m_0 - p_- - p_+.$$

Let $f : W \to [s_-, s_+]$ be a projection. We can assume P is in general position with respect to f. Moreover, we can assume P has been isotoped so that there are no extrema because every component of P has the boundaries both on $\Sigma \times \{s_-\}$ and $\Sigma \times \{s_+\}$. Write $s_1 = s_-$, $s_{n+1} = s_+$ and let $s_2 < s_3 < \cdots < s_n$ be the regular values for $f|_P$ such that there is a single critical value for $f|_P$ between s_i and s_{i+1} for each $i = 1, 2, \ldots, n$. Write $P_i = P \cap f^{-1}([s_i, s_{i+1}])$ for each $i = 1, 2, \ldots, n$. Each P_i is a collection of annuli except for one pair of pants component of some of types in Fig. 17.

Consider the case where P_i has a component of type (1) for example. The Euler number of P_i is -1. The surface $\Sigma^+(s_{i+1})$ is homeomorphic to the union of $\Sigma^+(s_i)$ and P_i. Therefore the Euler number of $\Sigma^+(s_{i+1})$ is one less than that of $\Sigma^+(s_i)$. Considering the other cases similarly, we obtain the following:

$$\chi(P) = \sum_{i=1}^{n} \chi(P_i) = -n_1 - n_2 - n_3 - n_4,$$

$$\chi(\Sigma^+(s_+)) - \chi(\Sigma^+(s_-)) = \sum_{i=1}^{n} \{\chi(\Sigma^+(s_i)) - \chi(\Sigma^+(s_{i+1}))\} = -n_1 + n_2 - n_3 + n_4$$

where n_j is the number of critical points of type (j).

Because $\Sigma \times \{s_-\}$ is the union of $\Sigma^-(s_-)$ and $\Sigma^+(s_-)$,

$$\chi(\Sigma) = \chi(\Sigma^-(s_-)) + \chi(\Sigma^+(s_-)).$$

Applying above equations, we can arrive at a formula:

$$g = g_- + g_+ + 1 + m_0 - m_- - m_+ + n_2 + n_4.$$

Let $w_\pm$ be the number of components of $W_\pm$. Then $w_- + w_+$ is the number of components of $W \setminus P$. It implies

$$1 + m_0 \geq w_- + w_+.$$

Each of m_- components of $\Sigma^-(s_-)$ is contained in one of the w_- components of W_-. Let W_-^0 be a component of W_- which contains m_-^0 components of $\Sigma^-(s_-)$. Observe the transformation of $W_-^0 \cap \Sigma^-(s)$ during the increasing of s from s_- to s_+. Since W_-^0 is connected, there must be at least $m_-^0 - 1$ critical points for $f|_{P\cap W_-^0}$ where two components of $W_-^0 \cap \Sigma^-(s)$ come to be connected. Such critical points are type (4). Thus,

$$n_4 \geq m_- - w_-.$$

By the symmetrical argument,

$$n_2 \geq m_+ - w_+.$$

These inequalities immediately induce $g \geq g_- + g_+$. □

9. The main theorem

Johnson [9] constructed a counterexample for the stabilization conjecture by amalgamations of two Heegaard splittings with high distance along the torus boundaries. We will make the same construction changing the place of torus boundaries by sphere boundaries. By Proposition 2, an amalgamation along sphere boundaries is no other than a connected sum. In this way, we arrive at the following conclusion. Since Hempel [7] showed that there exist Heegaard splittings with arbitrarily high distance, this immediately induces Theorem 1.

Theorem 21. *Suppose $k \geq 2$ and (T_i, G_i^-, G_i^+) is a Heegaard splitting of genus k with distance at least $6k$ for a closed 3-manifold M_i for each $i = 1, 2$. Let (Σ_1, H_1^-, H_1^+) be the connected sum of (T_1, G_1^-, G_1^+) and (T_2, G_2^-, G_2^+). Let (Σ_2, H_2^-, H_2^+) be the connected sum of (T_1, G_1^-, G_1^+) and (T_2, G_2^+, G_2^-). Then the stable genus of (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+) is $3k$.*

Proof. Since the genus of a connected sum is equal to the sum of the genera of original splittings, the genus of (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+) is $2k$. As remarked in [5, Section 2], the flip genus of any Heegaard splitting is at most twice the initial genus. Therefore the Heegaard splitting (T_2, G_2^-, G_2^+) become flippable after adding k trivial handles. It implies that adding k trivial handles makes (Σ_1, H_1^-, H_1^+) isotopic to (Σ_2, H_2^-, H_2^+). Thus, the stable genus is at most $3k$. Then, we will show that the stable genus is at least $3k$.

Let B_1 and B_2 be open balls in G_1^+ and G_2^+, respectively. Write $M_i^* = M_i \setminus B_i$ and $G_i^{*+} = G_i^+ \setminus B_i$ for each $i = 1, 2$. The connected sum M of M_1 and M_2 can be obtained by gluing M_1^* and M_2^* at their sphere boundaries. (T_i, G_i^-, G_i^{*+}) is a Heegaard splitting for a 3-dimensional submanifold M_i^* of M for each $i = 1, 2$. By Proposition 2, (Σ_2, H_2^-, H_2^+) is the amalgamation of (T_1, G_1^-, G_1^{*+}) and (T_2, G_2^-, G_2^{*+}). By Propositions 2 and 3, (Σ_1, H_1^-, H_1^+) is the amalgamation of (T_1, G_1^-, G_1^{*+}) and the boundary stabilization of (T_2, G_2^-, G_2^{*+}).

By Lemma 10, (T_1, G_1^-, G_1^{*+}) spans itself positively. By Lemma 12, (Σ_2, H_2^-, H_2^+) spans (T_1, G_1^-, G_1^{*+}) positively. Similarly, (Σ_2, H_2^-, H_2^+) spans (T_2, G_2^-, G_2^{*+}) negatively and (Σ_1, H_1^-, H_1^+) spans (T_1, G_1^-, G_1^{*+}) positively. By Lemmas 10, 12 and 13, (Σ_1, H_1^-, H_1^+) spans (T_2, G_2^-, G_2^{*+}) positively.

Suppose $(\Sigma_i', H_i'^-, H_i'^+)$ is a stabilization of (Σ_i, H_i^-, H_i^+) for each $i = 1, 2$. By Lemma 11, $(\Sigma_i', H_i'^-, H_i'^+)$ spans (T_1, G_1^-, G_1^{*+}) and (T_2, G_2^-, G_2^{*+}) with the same signs as (Σ_i, H_i^-, H_i^+). If $(\Sigma_1', H_1'^-, H_1'^+)$ and $(\Sigma_2', H_2'^-, H_2'^+)$ are isotopic, the isotopy takes $H_1'^-$ to either $H_2'^-$ or $H_2'^+$.

Consider the case where the isotopy takes $H_1'^-$ to $H_2'^-$ and $H_1'^+$ to $H_2'^+$. The Heegaard splitting $(\Sigma_1', H_1'^-, H_1'^+)$ spans (T_2, G_2^-, G_2^{*+}) positively and negatively. If $(\Sigma_1', H_1'^-, H_1'^+)$ splits (T_2, G_2^-, G_2^{*+}), Lemma 18 implies that the genus of $(\Sigma_1', H_1'^-, H_1'^+)$ is at least $3k$. By Lemma 19, we can assume there is a pair of sweep-outs f and g_2 representing $(\Sigma_1', H_1'^-, H_1'^+)$ and (T_2, G_2^-, G_2^{*+}) such that f spans g_2 positively and negatively. By Lemma 9, $f^{-1}(s_2) \cap M_2^*$ has genus at least $2k$ for some value $s_2 \in (-1, 1)$. For a sweep-out g_1 representing (T_1, G_1^-, G_1^{*+}), if f splits g_1 then Lemma 18 can be applied again. Therefore we can assume f spans g_1. By Lemma 9, $f^{-1}(s_1) \cap M_1^*$ has genus at least k for some value $s_1 \in (-1, 1)$. Assume $s_1 < s_2$ without loss of generality. The intersection $M_1^* \cap M_2^* \cap f^{-1}([s_1, s_2])$ is a separating, planar surface properly embedded in a product space $f^{-1}([s_1, s_2])$. By Lemma 20, the genus of Σ_1' is at least $k + 2k = 3k$.

On the other hand, when the isotopy takes $H_1'^-$ to $H_2'^+$ and $H_1'^+$ to $H_2'^-$, the Heegaard splitting $(\Sigma_1', H_1'^-, H_1'^+)$ spans (T_1, G_1^-, G_1^{*+}) positively and negatively. The same argument implies that the genus of Σ_1' is at least $3k$. Thus, any common stabilization of (Σ_1, H_1^-, H_1^+) and (Σ_2, H_2^-, H_2^+) has genus at least $3k$. □

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