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A COMMUTATIVITY THEOREM FOR RINGS. II

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Throughout the present paper, R will represent a ring with center C , and D the commutator ideal of R . A ring R is called left (resp. right) s -unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$; R is called s -unital if R is both left and right s -unital. Given a positive integer n , we say that R has the property $Q(n)$ if for any $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$ (see [1]).

Our present objective is to generalize [2, Theorem] for left s -unital rings as follows:

Theorem. *Let $n > 0$, r, s and t be non-negative integers and let $f(X, Y) = \sum_{i=1}^r \sum_{j=2}^s f_{ij}(X, Y)$ be a polynomial in two noncommuting indeterminates X, Y with integer coefficients such that each f_{ij} is a homogeneous polynomial with degree i in X and degree j in Y and the sum of the coefficients of f_{ij} equals zero. Suppose a left s -unital ring R satisfies the polynomial identity*

$$(1) \quad X^t[X^n, Y] - f(X, Y) = 0.$$

If either $n=1$ or $r=1$ and R has the property $Q(n)$, then R is commutative.

We shall use freely the following well known result stated without proof.

Lemma. *Let x, y be elements of a ring with 1, and let k be a positive integer. If $x^k y = 0 = (x+1)^k y$ then $y = 0$.*

Proof of Theorem. Let y be an arbitrary element of R , and choose an element e of R such that $ey = y$. Then (1) gives $y - ye^n = f(e, y) \in yR$. We have thus seen that R is right s -unital, and hence s -unital. Therefore, in view of [1, Proposition 1], it suffices to prove the theorem for R with 1.

Observe that D is a nil ideal of R , by a theorem of Kezlan-Bell (see, e.g., [1, Proposition 2]), since $x = e_{11}$ and $y = e_{12}$ fail to satisfy (1).

I) We consider first the case $n=1$. Let a, b be elements of R . By Lemma, it is easy to see that if $x^t a[x, b] = 0$ for all $x \in R$ then $a[x, b] = 0$. Noting this fact, we can apply the argument employed in the proof of [2, Theorem] to see the commutativity of R .

II) Next, suppose that $n > 1$, $r = 1$ and R has the property $Q(n)$. We claim that $D \subseteq C$. In fact, if $a \in D \setminus C$ then there exists a positive integer p such that $a^p \notin C$ and $a^k \in C$ for all $k > p$. For any $y \in R$, by repeated use of (1), we have $n(1+a^p)^t [a^p, y] = (1+a^p)^t [(1+a^p)^n, y] = f(1+a^p, y) = f(1, y) + f(a^p, y) = f(a^p, y) = a^{pt} [a^{pn}, y] = 0$. Since $1+a^p$ is a unit in R , we have

$$(2) \quad n[a^p, y] = 0.$$

Hence, $[a^p, y] = 0$ by $Q(n)$, a contradiction. We have thus seen that $D \subseteq C$. We write $f_{1,j}(X, Y) = \sum_{k=0}^j \alpha_{jk} Y^k X Y^{j-k}$. Since $\sum_{k=0}^j \alpha_{jk} = 0$ by assumption, we have $f_{1,j}(x, y) = \sum_{k=0}^{j-1} \alpha_{jk} (y^k x y^{j-k} - y^j x) = \sum_{k=0}^{j-1} \alpha_{jk} y^k [x, y^{j-k}] = \sum_{k=0}^{j-1} (j-k) \alpha_{jk} y^{j-1} [x, y]$ for any $x, y \in R$. Therefore, we can write $f(X, Y) = g(Y) [X, Y]$ with some polynomial g with integer coefficients, and (1) becomes

$$(3) \quad nX^{t'} [X, Y] - g(Y) [X, Y] = 0, \text{ where } t' = n+t-1 > 0.$$

For any positive integers k, l , we denote by $h_{kl}(X, Y)$ the polynomial $(k+1)(n^{kl} - g(Y)^{kl}) [X, Y]$. By repeated use of (3), for any $x, y \in R$ we have $(k+1)n^{kl} x^{t'+k} [x, y] = (k+1)n^{kl-1} x^k [x, y] g(y) = n^{kl-1} [x^{k+1}, y] g(y) = n^{kl} x^{(k+1)t'} [x^{k+1}, y] = (k+1)n^{kl} x^{(k+1)t'+k} [x, y]$. Then, $(k+1)n^{kl} x^{t'+k} [x, y] = (k+1)n^{kl} x^{t'+k} [x, y] x^{kt'} = (k+1)n^{kl} x^{t'+k} [x, y] x^{kt'} = (k+1)x^{t'+k} g(y)^{kl} [x, y]$. Therefore, $(k+1)x^{t'+k}(n^{kl} - g(y)^{kl}) [x, y] = 0$, and hence $h_{kl}(x, y) = 0$ (Lemma). In particular, $n^2[x, (1-x^{2t'})y] = n^2(1-x^{2t'}) [x, y] = (n^2 - g(y)^2) [x, y] = h_{21}(x, y) - h_{12}(x, y) = 0$, and therefore $(1-x^{2t'}) [x, y] = 0$. Exchanging x and y , we have $[x, y] = y^{2t'} [x, y]$, which comes under the case I). This completes the proof.

As an application of our theorem, we shall prove the following which includes [3, Theorem], [4, Theorem] and [5, Theorems 1 and 2].

Corollary 1. *Let $n > 0$, m, t and s be fixed non-negative integers such that $(n, t, m, s) \neq (1, 0, 1, 0)$. Suppose a left s -unital ring R satisfies the polynomial identity*

$$(4) \quad X^t [X^n, Y] - [X, Y^n] Y^s = 0.$$

- (a) *If R has the property $Q(n)$ then R is commutative.*
- (b) *If n and m are relatively prime then R is commutative.*

Proof. Let x, y be arbitrary elements of R , and choose an element e of R such that $ex = x$ and $ey = y$. If $(m, s) \neq (1, 0)$ then (4) gives $y = ye^n + ey^{m+s} - y^m ey^s \in yR$. On the other hand, if $(m, s) = (1, 0)$ then $(n, t) \neq (1, 0)$ and (4) gives $x = xe - x^{n+t} e + x^{n+t} \in xR$. We have thus seen that R is s -unital. Therefore, by [1, Proposition 1], we may assume that R has 1.

If $m = 0$ (in the case of (a)), the assertion is clear by Theorem. Next,

we consider the case $n=1$. If $m>0$ and $(m, s) \neq (1, 0)$ then $m+s>1$, and hence the assertion is clear, again by Theorem. Also, if $(m, s)=(1, 0)$ then, exchanging the roles of X and Y , we get the assertion. Similarly, we can prove the assertion for $m=1$. Therefore, we may assume henceforth that $n>1$ and $m>1$. For the case (a), the assertion is immediate by Theorem. So, we consider the case (b). Let $a \in D$, and $y \in R$. If a is not in C then there exists a positive integer p such that $a^p \notin C$ and $a^k \in C$ for all $k>p$ and $n[a^p, y]=0$ by (2); similarly we can prove $m[a^p, y]=0$. Hence, $[a^p, y]=0$. This contradiction shows that $D \subseteq C$, and (4) becomes

$$(5) \quad nX^{n+t-1}[X, Y]=mY^{m+s-1}[X, Y].$$

If $n[x, y]=0$ ($x, y \in R$) then (5) gives $my^{m+s-1}[x, y]=nx^{n+t-1}[x, y]=0=nx^{n+t-1}[x, y+1]=m(y+1)^{m+s-1}[x, y]$, whence $m[x, y]=0$ follows by Lemma, and hence $[x, y]=0$. This prove that R has the property $Q(n)$. Hence, R is commutative by Theorem, completing the proof.

Corollary 2. *Let $n>0$ and m be fixed non-negative integers. Suppose a left s -unital ring R satisfies the polynomial identity $[XY, X^n+Y^m]=0$. If either R has the property $Q(n)$ or n and m are relatively prime, then R is commutative.*

Proof. Actually, R satisfies the polynomial identity $X[X^n, Y]-[X, Y^m]Y=0$. Hence R is commutative by Corollary 1.

REMARK. In case $n>0$ and $m=0$, Corollary 1 need not be true for right s -unital rings (see [3, Remark]).

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