Hashimoto, K
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EXPLICIT FORM OF QUATERNION MODULAR EMBEDDINGS

KI-ICHIRO HASHIMOTO

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1. Introduction

In this note, we describe some results on arithmetic of the modular embeddings of the upper half plane $\mathfrak{H}$ into the Siegel upper half space $\mathfrak{H}_2$ of degree two, with respect to the unit groups of Eichler orders of indefinite quaternion algebras over the rational number field $\boldsymbol{Q}$.

Let $\boldsymbol{B}$ be an indefinite quaternion algebra over $\boldsymbol{Q}$ with discriminant D_0, and $\mathcal{O}$ be an Eichler order of $\boldsymbol{B}$ of level $D=D_0N$, where N is a positive integer prime to D_0. Put $\Gamma=\{\gamma\in\mathcal{O}\ ;\ \mathrm{Nr}(\gamma)=1\}$, where Nr denotes the reduced norm of $\boldsymbol{B}$. Then Γ is regarded as a discrete subgroup of $\boldsymbol{B}_\infty^{(1)}\cong\mathrm{SL}_2(\boldsymbol{R})$ with finite quotient volume. As is well known, $\Gamma\backslash\mathfrak{H}$ is the $\boldsymbol{C}$-valued points of the Shimura curve S attached to $\mathcal{O}$, and interpreted as a moduli space of principally polarized abelian surface having quaternion multiplication by $\mathcal{O}$ (cf. [13], [14], [9]). More precisely, let ρ be an element of $\mathcal{O}$ such that $\rho^2=-D$, $\rho\mathcal{O}=\mathcal{O}\rho$. Then the involution of $\boldsymbol{B}$ defined by $\alpha\mapsto\alpha^\iota:=\rho^{-1}\bar{\alpha}\rho$ is positive, and satisfies $\mathcal{O}^\iota=\mathcal{O}$. The points of $S(\boldsymbol{C})$ are in one to correspondence with the set of isomorphism classes of (A,ψ,Θ), where (A,Θ) is a principally polarized abelian surface, and $\psi:\mathcal{O}\hookrightarrow\mathrm{End}(A)$ is an injective ring homomorphism such that the Rosati involution with respect to Θ coincides with ι_ρ on $\psi(\mathcal{O})$. This correspondence can be described by the quaternion modular embedding. Indeed, it is known that there is a holomorphic embedding

$$\Phi:\mathfrak{H}\to\mathfrak{H}_2,\ z\mapsto\Omega(z),$$

which is compatible with the actions of Γ, $\mathrm{Sp}(4,\boldsymbol{Z})$, through an embedding of the group

$$\varphi:\Gamma\hookrightarrow\mathrm{Sp}(4,\boldsymbol{Z})$$

Our purpose is twofold. The first is to describe φ, Φ explicitly, by constructing a concrete model of $\mathcal{O}$ and using its $\boldsymbol{Z}$-basis. This enables us to study various arithmetic properties of such embeddings, which is our second purpose. Especially, we can characterize the image of the embedding by determining all possible singular relations of Humbert's [7]. Also we can associate with it a definite binary quadratic form with discriminant $-16D$, which is an invariant of the equivalence class of the embedding. The form represents precisely those positive integers Δ which are the discriminants of orders $\boldsymbol{O}$ of real quadratic fields, such that the locus of the Shimura curve S on $\mathfrak{H}_2$ is contained in the Humbert surface H_Δ.

Most of the results of this note are not quite new except for being explicit everywhere, and are regarded as examples of general results given e.g., in [12],[13], [14], and [11]. However, we think it is convenient to have the explicit descriptions of the special cases as a step to further investigations. In fact, a motivation to the present work is [5] where we construct examples of concrete models of algebraic families over Shimura curves, the fibres of which are curves of genus two whose jacobians have quaternion multiplications by $\mathcal{O}$. There one of the key ingredients is to interprete the locus of a Shimura curve as a component of the intersection of two Humbert surfaces (cf. Corollary 5.3).

Finally we remark that all results in this note remain valid if we assume $D_0=1$, in which case we have $\Gamma\cong\Gamma_0(N)$ so that we obtain a new (non-standard) interpretation of the modular curve $X_0(N)$ as a moduli of certain abelian surfaces.

NOTATION. For an odd prime p and an integer $a\in\boldsymbol{Z}$, $\left(\frac{a}{p}\right)$ denotes the quadratic residue symbol modulo p. For a, $b\in\boldsymbol{Q}$, $(a, b)_p$ denotes the Hilbert symbol. A quaternion algebra $\boldsymbol{B}$ over a field K is a central simple algebra over K such that $[\boldsymbol{B} : K]=4$.

2. Construction of Eichler orders

Let $\boldsymbol{B}$ be as above. We construct a model of an Eichler order of $\boldsymbol{B}$. The idea is those of Ibukiyama [8], combined with a lemma of Hijilate [6] which characterize the Eichler (or split) order in $M_2(\boldsymbol{Q}_p)$.

As is well known, the isomorphism class of $\boldsymbol{B}$ is determined by the discriminant $D(\boldsymbol{B}/\boldsymbol{Q})$, i.e., the product of distinct primes at which $\boldsymbol{B}$ ramifies. We denote them by $p_1, \ldots, p_t$, and also put $D_0:=D(\boldsymbol{B}/\boldsymbol{Q})=p_1\ldots p_t$. Note that t is even, since $\boldsymbol{B}$ is assumed to be indefinite (cf.[15]).

Let $\alpha\mapsto\bar{\alpha}$ be the canonical involution on $\boldsymbol{B}$, and let $\mathrm{Nr}(\alpha):=\alpha\bar{\alpha}$ be the reduced norm. Let N be an arbitrary positive integer which is prime to D_0, and put $D:=D_0N$.

DEFINITION 2.1. *A subring $\mathcal{O}$ of $\boldsymbol{B}$ is called an order, if $\mathcal{O}$ is finitely generated $\boldsymbol{Z}$-module. $\mathcal{O}$ is called an Eichler order of level D if the following conditions are satisfied.*

1. *For each prime $p|D_0$, $\mathcal{O} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$ is the (unique) discrete valuation ring of the division algebra $\boldsymbol{B}_p$.*

2. *For each prime divisor $q^m \| N$,*

$$\mathcal{O}_q \cong R_q(m) := \begin{pmatrix} \boldsymbol{Z}_q & \boldsymbol{Z}_q \\ q^m \boldsymbol{Z}_q & \boldsymbol{Z}_q \end{pmatrix}.$$

3. *For each prime p not dividing D, $\mathcal{O}_p \cong M_2(\boldsymbol{Z}_p)$.*

Choose a prime p satisfying the following conditions, the existence of which is assured by the theorem of arithmetic progression :

1. $p \equiv 1 \pmod 4$; moreover, $p \equiv 5 \pmod 8$ if $2|D_0$, and $p \equiv 1 \pmod 8$ if $2|N$.

2. $\left(\frac{p}{p_i}\right) = -1$ for each $p_i \neq 2$.

3. $\left(\frac{p}{q}\right) = +1$ for each odd prime factor q of N.

Then one can easily prove that, for a prime l, $(-D, p)_l = -1$ if and only if $l = p_i$ $(1 \leq i \leq t)$. Hence $\boldsymbol{B}$ is expressed as

$$\begin{aligned} &\boldsymbol{B} = \boldsymbol{Q} + \boldsymbol{Q}i + \boldsymbol{Q}j + \boldsymbol{Q}ij, \\ &i^2 = -D,\ j^2 = p,\ ij = -ji. \end{aligned} \tag{1}$$

Note also that the conditions imply $\left(\frac{-D}{p}\right) = +1$, hence there exits an integer $a \in \boldsymbol{Z}$ satisfying $a^2 D + 1 \equiv 0 \pmod p$. Now put

$$e_1 = 1,\ e_2 = (1+j)/2,\ e_3 = (i + ij)/2,\ e_4 = (aDj + ij)/p. \tag{2}$$

Our first result is the following theorem, which is a slight generalization of Ibukiyama [8].

Theorem 2.2. *Notation being as above, the $\boldsymbol{Z}$-lattice*

$$\mathcal{O} = \boldsymbol{Z}e_1 + \boldsymbol{Z}e_2 + \boldsymbol{Z}e_3 + \boldsymbol{Z}e_4$$

forms an Eichler order of $\boldsymbol{B}$ of level $D = D_0 N$.

Proof. That the above $\mathcal{O}$ forms a $\boldsymbol{Z}$-order is proved as in [8] by expressing the products $e_h e_k$ as $\boldsymbol{Z}$-linear combinations. We omit the detail. Now we have

$$\det(\mathrm{Tr}(e_h, e_k)) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & \frac{p+1}{2} & 0 & aD \\ 0 & 0 & \frac{(p-1)D}{2} & D \\ 0 & aD & D & \frac{2D(a^2D+1)}{p} \end{pmatrix} = -D^2.$$

Next we note that $\mathcal{O}$ contains a subring $\boldsymbol{Z}[e_2] \cong \boldsymbol{Z}\left[\frac{1+\sqrt{p}}{2}\right]$ which splits, by q-adic completion, as

$$\boldsymbol{Z}_q[e_z] \cong \boldsymbol{Z}_q\left[\frac{1+\sqrt{p}}{2}\right] = \boldsymbol{Z}_q \oplus \boldsymbol{Z}_q$$

for any prime divisor q of N. Then a lemma of Hijikata [6] implies that $\mathcal{O}_q$ is conjugate in $\boldsymbol{B}_q \cong M_2(\boldsymbol{Q}_q)$ to a split order:

$$\mathcal{O}_q \cong R_q(n) :=\begin{pmatrix} \boldsymbol{Z}_q & \boldsymbol{Z}_q \\ q^n\boldsymbol{Z}_q & \boldsymbol{Z}_q \end{pmatrix} \ (n \in \boldsymbol{Z},\ n \geq 0).$$

Taking the standard $\boldsymbol{Z}_q$-basis of $R_q(n)$, we have immediately

$$\det(\mathrm{Tr}(e'_h, e'_k)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^n \\ 0 & 0 & q^n & 0 \end{pmatrix} = -q^{2n}.$$

Comparing the above calculation, we obtain $q^n \| N$.

□

We note that, since $\boldsymbol{B}$ is indefinite, any Eichler order of level D is obtained from $\mathcal{O}$ by an inner automorphism of $\boldsymbol{B}^\times$. Therefore we can fix $\mathcal{O}$ as above without loss of generality. Let ρ be an element of $\boldsymbol{B}$ such that $\rho^2 < 0$, and let $\iota = \iota_\rho$ be an involution of $\boldsymbol{B}$ given by $\xi^\iota = \rho^{-1}\bar{\xi}\rho$. It is positive : $\mathrm{Tr}(\xi\xi^\iota) \geq 0 (\forall \xi \in \boldsymbol{B})$. Put

$$(3) \qquad E_\rho(\xi, \eta) := \mathrm{Tr}(\rho\xi\bar{\eta}) \quad (\xi, \eta \in \boldsymbol{B})$$

Then, since $\bar{\rho} = -\rho$, we have

$$\begin{aligned} E_\rho(\eta, \xi) &= \mathrm{Tr}(\rho\eta\bar{\xi}) \\ &= \mathrm{Tr}(-\rho\xi\bar{\eta}) \\ &= -E_\rho(\xi, \eta). \end{aligned}$$

Hence E_ρ defines a skew symmetric form on $\boldsymbol{B}$ over $\boldsymbol{Q}$.

Lemma 2.3. (i) The right multiplication of $\gamma \in \boldsymbol{B}$ defines a similitude

transformation of $(\boldsymbol{B}, E_\rho)$ with multiplicator $N\mathrm{r}(\gamma)$.
(ii) *The left multiplication of* $\alpha\in\boldsymbol{Q}(\rho)$ *defines a similitude transformation of* $(\boldsymbol{B}, E_\rho)$ *with multiplicator* $\mathrm{Nr}(\alpha)$.

Proof. Both assertions follow from

$$\begin{aligned} E_\rho(\alpha\eta\gamma, \alpha\xi\gamma) &= \mathrm{Tr}(\rho\alpha\eta(\gamma\bar{\gamma})\bar{\xi}\bar{\alpha}) \\ &= \mathrm{Tr}(\bar{\alpha}\rho\alpha\eta(\gamma\bar{\gamma})\bar{\xi}) \\ &= \mathrm{Nr}(\alpha)\mathrm{Nr}(\gamma)E_\rho(\xi, \eta). \end{aligned} \tag{4}$$

□

Lemma 2.4. *E_ρ is $\boldsymbol{Z}$-valued on $\mathcal{O}$ if and only if $\rho i\in\mathcal{O}$. Moreover, E_ρ defines a non-degenerate skew symmetric pairing on $\mathcal{O}\times\mathcal{O}$ if and only if $\rho i\in\mathcal{O}^\times$.*

Proof. This is a consequence of the fact that the dual lattice of $\mathcal{O}$ with respect to the symmetric bilinear form $(x, y)\mapsto\mathrm{Tr}(xy)$ is the two-sided ideal $i^{-1}\mathcal{O}=\frac{i}{D}\mathcal{O}$. Indeed, we have for $\rho=i^{-1}$

$$E_{i^{-1}}(e_h, e_k)=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & (p-1)/2 & 1 \\ 1 & -(p-1)/2 & 0 & -aD \\ 0 & -1 & aD & 0 \end{pmatrix},$$

hence $\det(E_{i^{-1}}(e_h, e_k))=1$.

□

Throughout the following of this note, we assume that $\rho=i^{-1}$. To get a symplectic $\boldsymbol{Z}$-basis of $\mathcal{O}$ with respect to $E_{i^{-1}}$, we put

$$\eta_1=e_3-\frac{p-1}{2}e_4,\ \eta_2=-aDe_1-e_4,\ \eta_3=e_1,\ \eta_4=e_2. \tag{5}$$

Then we obtain

$$(E_{i^{-1}}(\eta_h, \eta_k))=J:=\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Now put

$$\Gamma:=\{\gamma\in\mathcal{O}\ ;\ \mathrm{Nr}(\gamma)=1\}.$$

Then Γ is a discrete subgroup of $(\boldsymbol{B}\otimes_{\boldsymbol{Q}}\boldsymbol{R})^{(1)}\cong\mathrm{SL}_2(\boldsymbol{R})$. From the above lemmas, we have the following:

Proposition 2.5. *The data $\{i^{-1}, \vec{\eta}=(\eta_1, \ldots, \eta_4)\}$ determine an embedding*

$$\varphi_{\vec{\eta}}: \Gamma \longrightarrow \mathrm{Aut}_{\boldsymbol{Z}}(\mathcal{O}, E_{i^{-1}}) \cong \mathrm{Sp}(4, \boldsymbol{Z}) \tag{6}$$

of Γ which is optimal w.r.t. Sp(4, $\boldsymbol{Z}$), *i.e. $\varphi_{\vec{\eta}}$ is not extended to an embedding of a bigger subgroup of* $\mathrm{SL}_2(\boldsymbol{R})$ *into* Sp(4, $\boldsymbol{Z}$).

We denote the natural extension of φ to $\mathrm{SL}_2(\boldsymbol{R})$ by the same letter:

$$\varphi_{\vec{\eta}}: \mathrm{SL}_2(\boldsymbol{R}) \longrightarrow \mathrm{Aut}_{\boldsymbol{R}}((\boldsymbol{B} \otimes_{\boldsymbol{Q}} \boldsymbol{R})^{(1)}, E_{i^{-1}}) \cong \mathrm{Sp}(4, \boldsymbol{R}).$$

Proof. For any $\gamma \in \Gamma$, $\vec{\eta}\gamma := (\eta_1\gamma, \ldots, \eta_4\gamma)$ is also a symplectic basis of $\boldsymbol{Z}$-basis of $\mathcal{O}$ with respect to $E_{i^{-1}}$, hence there exists $M_\gamma \in \mathrm{Sp}(4, \boldsymbol{Z})$ such that

$$\vec{\eta}\gamma = \vec{\eta}\,{}^tM_\gamma. \tag{7}$$

The map $\varphi_{\vec{\eta}}(\gamma) = M_\gamma$ gives a desired embedding, for which the last assertion is easily seen from Lemma 2.3.

□

3. Quaternion modular embeddings

Let $\boldsymbol{B}$, $\mathcal{O}$, Γ be as in §2. Γ is regarded as a discrete subgroup of $\boldsymbol{B}_\infty^{(1)} \cong \mathrm{SL}_2(\boldsymbol{R})$ with finite quotient volume. Moreover, the quotient $\Gamma\backslash\mathrm{SL}_2(\boldsymbol{R})$ is compact if $\boldsymbol{B}$ is a division algebra. As is well known, the space $\Gamma\backslash\mathfrak{H}$ is the $\boldsymbol{C}$-valued points of the Shimura curve S attached to $\mathcal{O}$, and interpreted as a moduli space of principally polarized abelian surface having quaternion multiplication by $\mathcal{O}$ (cf. [13],[14]). More precisely, we have

$$S(\boldsymbol{C}) \overset{1:1}{\longleftrightarrow} \left\{ (A, \psi, \Theta) \middle| \begin{array}{c} (A, \Theta): \textit{principally polarized abelian surface} \\ \psi: \ \mathcal{O} \hookrightarrow End(A) \\ \textit{Rosati involution w.r.t. } \Theta_{|\mathcal{O}} = \iota_\rho \end{array} \right\}.$$

The above isomorphism can be described by the quaternion modular embedding

$$\Phi: \ \mathfrak{H} \to \mathfrak{H}_2, \ z \mapsto \Omega(z),$$

which we shall describe explicitly. The following fact is well known as a special case of a result of Shimura ([13], [14]):

Proposition 3.1. *Let A be a principally polarized abelian variety of dimension two such that*

1. $\mathrm{End}(A) \supseteq \mathcal{O}$
2. *The Rosati involution coincides with the involution ι_ρ on $\mathcal{O}$.*

Then there exists an element $z \in \mathfrak{H}$ such that A is isomorphic to $A_{\Omega(z)}$ as principally polarized abelian variety.

Definition 3.2. *Notation being as above, A is said to have Quaternion*

Multiplication by $(\mathcal{O}, \iota_\rho)$. We call $\tau \in \mathfrak{H}_2$ a QM point of type $(\mathcal{O}, \iota_\rho)$ if A_τ belongs to this type.

Let $z \in \mathfrak{H}$ and let $\eta_1, \eta_2, \eta_3, \eta_4$ be the basis of $\mathcal{O}$ given in (5). We identify $\boldsymbol{B} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ with $M_2(\boldsymbol{R})$ by

$$i \mapsto \begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}, \; j \mapsto \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix},$$

and by $\boldsymbol{R}$-linearlity. For $z \in \mathfrak{H}$, we define an $\boldsymbol{R}$-linear isomorphism

$$f_z : \boldsymbol{B} \otimes_{\boldsymbol{Q}} \boldsymbol{R} \longrightarrow \boldsymbol{C}^2, \; \alpha \mapsto \alpha \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{8}$$

Using f_z, the $\boldsymbol{R}$-vector sapace $M_2(\boldsymbol{R})$ is equipped with the complex structure. The following lemma shows that this is compatible with the fact that the center of the maximal compact subgroup gives the complex structure on $\mathrm{SL}_2(\boldsymbol{R})/\mathrm{SO}(2) \cong \mathfrak{H}$. Put $i_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and for each $z \in \mathfrak{H}$, take an element $\gamma_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\boldsymbol{R})$ such that $\gamma_z(\sqrt{-1}) := \dfrac{a\sqrt{-1}+b}{c\sqrt{-1}+d} = z$.

Lemma 3.3. *We have*

$$f_z(\xi(\gamma_z i_0 \gamma_z^{-1})) = \sqrt{-1} f_z(\xi) \quad (\forall \xi \in M_2(\boldsymbol{R})). \tag{9}$$

Let $L_z = f_z(\mathcal{O})$ be the image of $\mathcal{O}$. It is easily seen that L_z is a lattice in $\boldsymbol{C}^2$. Define a pairing $E_z : \boldsymbol{C}^2 \times \boldsymbol{C}^2 \longrightarrow \boldsymbol{R}$ by

$$E_z(f_z(\xi), f_z(\eta)) := -E_\rho(\xi, \eta) = -\mathrm{Tr}(\rho \xi \bar{\eta}) \quad (\xi, \eta \in \boldsymbol{B}) \tag{10}$$

Then, by Lemma 2.4 we see that it induces a nondegenerate skew symmetric pairing

$$E_z : L_z \times L_z \longrightarrow \boldsymbol{Z}.$$

Moreover, as ι_ρ is a positive involution, we have

$$E_z(f_z(\xi), \sqrt{-1} f_z(\xi)) = \mathrm{Tr}(\eta \eta^\iota) > 0 \; (\forall \xi, \in M_2(\boldsymbol{R}), \xi \neq 0)$$

where $\eta = \xi \gamma_z \gamma_0$. Thus E_z is a Riemann form on the complex torus $\boldsymbol{C}^2/L_z$. Put

$$\begin{aligned} \omega_1 &:= f_z(\eta_1) & \omega_2 &:= f_z(\eta_2) \\ \omega_3 &:= f_z(\eta_3) & \omega_4 &:= f_z(\eta_4) \end{aligned}$$

Then $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is a symplectic basis, i.e., $E_z(\omega_i, \omega_j) = J$.
Put $(\Omega_1(z)\Omega_2(z)) = (\omega_1 \omega_2 \omega_3 \omega_4)$, and

$$\Omega_{\bar{\eta}} : = \Omega_2(z)^{-1} \Omega_1(z). \tag{11}$$

Proposition 3.4. *The map $\Phi_{\vec{\eta}}: z \mapsto \Omega_{\vec{\eta}}(z)$ is a holomorphic embedding of $\mathfrak{H}$ into $\mathfrak{H}_2$. Moreover, for each $\gamma \in \mathrm{SL}_2(\boldsymbol{R})$, the following diagram is commutative*:

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\Phi_{\vec{\eta}}} & \mathfrak{H}_2 \\ \gamma \downarrow & & \downarrow \varphi_{\vec{\eta}}(\gamma) \\ \mathfrak{H} & \xrightarrow{\Phi_{\vec{\eta}}} & \mathfrak{H}_2 \end{array}$$

where the action of γ(resp. $\varphi(\gamma)$) on $\mathfrak{H}$(resp. $\mathfrak{H}_2$) is the usual one.

Proof. Writing $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, put

$$(\Omega_1'(z)\Omega_2'(z))=(f_z(\eta_1\gamma),\ f_z(\eta_2\gamma),\ f_z(\eta_3\gamma),\ f_z(\eta_4\gamma)).$$

Then from the equality

$$\gamma\begin{pmatrix} z \\ 1 \end{pmatrix}=\begin{pmatrix} az+b \\ cz+d \end{pmatrix}=(cz+d)\begin{pmatrix} \gamma(z) \\ 1 \end{pmatrix},$$

we have

$$\Phi_{\vec{\eta}\gamma}(z)=\Phi_{\vec{\eta}}(\gamma(z))=\Omega_2'(z)^{-1}\Omega_1'(z)(=\Omega'(z),\ say).$$

On the other hand, using (7), we have the following equalities of matrices with coefficients in $M_2(\boldsymbol{C})$:

$$\begin{aligned}(\eta_1\gamma,\ \eta_2\gamma,\ \eta_3\gamma,\ \eta_4\gamma)\zeta&=(\eta_1,\ \eta_2,\ \eta_3,\ \eta_4)^t M_\gamma\zeta \\ &=(\eta_1,\ \eta_2,\ \eta_3,\ \eta_4)\zeta^t M_\gamma, \qquad \left(\zeta=\begin{pmatrix} z & \bar{z} \\ 1 & 1 \end{pmatrix}\right).\end{aligned}$$

Writing $M_\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and taking first columns of each components in $M_2(\boldsymbol{C})$, we obtain

$$\begin{aligned}(\Omega_1'(z)\Omega_2'(z))&=(\eta_1\gamma,\ \eta_2\gamma,\ \eta_3\gamma,\ \eta_4\gamma)\begin{pmatrix} z \\ 1 \end{pmatrix} \\ &=(f_z(\eta_1\gamma),\ f_z(\eta_2\gamma),\ f_z(\eta_3\gamma),\ f_z(\eta_4\gamma))\begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix} \\ &=(\Omega_1(z)\Omega_2(z))\begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix} \\ &=(\Omega_1(z)^tA+\Omega_2(z)^tB,\ \Omega_1(z)^tC+\Omega_2(z)^tD).\end{aligned}$$

Hence we have

$$\begin{aligned}\Omega'(z)&={}^t\Omega'(z) \\ &={}^t(\Omega_2'(z)^{-1}\Omega_1'(z)) \\ &=(A^t\Omega_1(z)+B^t\Omega_2(z))(C^t\Omega_1(z)+C^t\Omega_2(z))^{-1} \\ &=(A\Omega(z)+B)(C\Omega(z)+D)^{-1}.\end{aligned}$$

This proves

$$\Phi_{\vec\eta\gamma}=\Phi_{\vec\eta}\circ\gamma=\varphi_{\vec\eta}(\gamma)\circ\Phi_{\vec\eta}.$$

□

Summing up the above results with a direct calculation using $\rho=i^{-1}$ and $\vec\eta$ we have the following explicit form of a quaternion modular embedding.
Put $\varepsilon:=\frac{1+\sqrt{p}}{2}$.

Theorem 3.5. *Let $\mathcal{O}$ be an Eichler order of $\boldsymbol{B}$ of level $D=D_0N$ as given by* (2), *and let $\eta_1,\dots,\eta_4$ be a symplectic $\boldsymbol{Z}$-basis of $\mathcal{O}$. Then the following map $\Phi_{\vec\eta}(z)=\Omega(z)$ gives a modular embedding of $\mathfrak{H}$ into $\mathfrak{H}_2$ with respect to Γ and* $\mathrm{Sp}(4,\boldsymbol{Z})$:

$$\Omega(z)=\frac{1}{pz}\begin{pmatrix} -\bar\varepsilon^2+\dfrac{(p-1)aD}{2}z+D\varepsilon^2z^2, & \bar\varepsilon-(p-1)aDz-D\varepsilon z^2 \\ \bar\varepsilon-(p-1)aDz-D\varepsilon z^2, & -1-2aDz+Dz^2 \end{pmatrix}. \tag{13}$$

In particular, $\Phi_{\vec\eta}$ induces an embedding of the modular varieties:

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\Phi_{\vec\eta}} & \mathfrak{H}_2 \\ \pi\downarrow & & \pi\downarrow \\ \Gamma\backslash\mathfrak{H}\cong S(\boldsymbol{C}) & \xrightarrow{\Phi_{\vec\eta}} & \mathrm{Sp}(2,\boldsymbol{Z})\backslash\mathfrak{H}_2 \end{array}$$

4. Singular relations for period matrices

We recall some work of Humbert [7] on singular relations. A point $\tau=\begin{pmatrix}\tau_1 & \tau_{12}\\ \tau_{12} & \tau_2\end{pmatrix}$ of $\mathfrak{H}_2$ is called to have a singular relation with invariant Δ, if there exist relatively prime integers $\alpha,\beta,\gamma,\delta,\varepsilon\in\boldsymbol{Z}$ such that (c.f. [7]):

$$\alpha\tau_1+\beta\tau_{12}+\gamma\tau_2+\delta(\tau_{12}^2-\tau_1\tau_2)+\varepsilon=0, \tag{14}$$

$$\Delta=\beta^2-4\alpha\gamma-4\delta\varepsilon. \tag{15}$$

Define

$$N_\Delta=\{\tau\in\mathfrak{h}_2 \mid \tau \text{ has a singular relation with invariant } \Delta\}$$

and

$$H_\Delta:=\text{ image of } N_\Delta \text{ under the canonical map } \mathfrak{H}_2\longrightarrow\mathrm{Sp}(4,\boldsymbol{Z})\backslash\mathfrak{H}_2.$$

H_Δ is called *the Humbert surface* of invariant Δ. For $\tau\in\mathfrak{H}_2$, let L_τ be the lattice in $\boldsymbol{C}^2$ spanned by the columns of the matrix $(1_2\ \tau)=(p_1,\dots,p_4)$, and put $A_\tau:=\boldsymbol{C}^2/L_\tau$. Then A_τ, together with the standard Riemann form E on $\boldsymbol{C}^2$ defined by

$E(p_h, p_k)=J$, forms a principally polarized abelian surface. Let $\mathrm{End}(A_\tau)$ be the endomorphism algebra of A_τ. Using the rational representation, it is expressed as

$$\mathrm{End}(\mathrm{A}_\tau)=\{\phi\in M_2(\boldsymbol{C})\,|\,\exists\, M\in M_4(\boldsymbol{Z}) \text{ s.t. } \phi(\tau\ 1_2)=(\tau\ 1_2)M\cdots(*)\}.$$

Writing $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, we see that the above condition $(*)$ is equivalent to

$$\phi=\tau B+D,\ \phi\tau=\tau A+C \Longleftrightarrow \tau B\tau+D\tau-\tau A-C=0\ \cdots(**)$$

Let E be the Riemann form associated to the polarization Θ. Then E defines an involution on $\mathrm{End}(A_\tau)$, $\phi\mapsto\phi^\circ$, called the Rosati involution, which is determined by $E(\phi z, w)=E(z, \phi^\circ w)(\forall z, w\in \boldsymbol{C}^2)$. We have

$$\begin{aligned}\phi^\circ=\phi \quad &\Longleftrightarrow \quad {}^tM\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0\end{pmatrix}=\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0\end{pmatrix}M \\ &\Longleftrightarrow \quad A={}^tD, B=\begin{pmatrix} 0 & b \\ -b & 0\end{pmatrix}, C=\begin{pmatrix} 0 & c \\ -c & 0\end{pmatrix}.\end{aligned}$$

Put $A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4\end{pmatrix}$. Under the assumption $\phi^\circ=\phi$, it follows that

$$(**) \quad \Longleftrightarrow \quad a_2\tau_1+(a_4-a_1)\tau_{12}-a_3\tau_2+b(\tau_{12}^2-\tau_1\tau_2)+c=0.$$

Then we have

$$\phi=\tau B+D=\begin{pmatrix} -b\tau_{12}+a_1 & b\tau_1+a_3 \\ -b\tau_2+a_2 & b\tau_{12}+a_4\end{pmatrix}$$

and

$$\begin{aligned}&\mathrm{Tr}\phi=a_1+a_4 \\ &\det\phi=-b\{a_2\tau_1+(a_4-a_1)\tau_{12}-a_3\tau_2+b(\tau_{12}^2-\tau_1\tau_2)\}+a_1a_4-a_2a_3 \\ &\qquad=a_1a_4-a_2a_3+bc.\end{aligned}$$

So the characteristic polynomial of ϕ is

$$T^2-(a_1+a_4)T+(a_1a_4-a_2a_3+bc)$$

and its discriminant Δ is

$$\Delta := (a_1+a_4)^2-4(a_1a_4-a_2a_3+bc)=(a_4-a_1)^2-4a_2(-a_3)-4bc.$$

Let $\boldsymbol{O}_\Delta$ be the order of discriminant Δ in the real quadratic field $\boldsymbol{Q}(\sqrt{\Delta})$. The above argument gives a simple proof of the following well known fact (cf. [7], [2]):

Proposition 4.1. $\mathrm{End}(A_\tau)$ *contains* $\boldsymbol{O}_\Delta$ *optimally if and only if* $\tau\in H_\Delta$.

5. Invariants of QM points

Applying the results of §3 to those in §4, we can characterize the image of $\Phi_{\bar{\eta}}$, by determining the singular relations satisfied by its points simultaneously.

Theorem 5.1. *Let $\Phi_{\bar{\eta}} : \mathfrak{H} \longrightarrow \mathfrak{H}_2$ be the modular embedding given by* (13). *Then $\tau = \Phi_{\bar{\eta}}(z)$ satisfy simultaneously the following sinfgular relations parametrized by two independent integers x, $y \in \boldsymbol{Z}$:*

$$x\tau_1 + (x + 2aDy)\tau_{12} - \frac{p-1}{4} x\tau_2 + y(\tau_{12}^2 - \tau_1\tau_2) + (a^2D - b)Dy = 0 \tag{16}$$

where we put $a^2D + 1 = pb$. Moreover, if $z \in \mathfrak{H}$ is not a CM point, then it has no other singular relation.

Proof. From (13), we easily have

$$\begin{pmatrix} \tau_1 \\ \tau_{12} \\ \tau_2 \\ \tau_{12}^2 - \tau_1\tau_2 \\ 1 \end{pmatrix} = \frac{-1}{pz} \begin{pmatrix} \bar{\varepsilon}^2 & -\dfrac{(p-1)aD}{2} & -D\varepsilon^2 \\ -\bar{\varepsilon} & (p-1)aD & D\varepsilon \\ 1 & 2aD & -D \\ 2aD\bar{\varepsilon} & -(p-1)a^2D^2 - D & -2aD^2\varepsilon \\ 0 & -p & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \end{pmatrix}.$$

Suppose first that z is not a *CM* point. Then from the above equality, we immediately see that (14) holds with α, β, γ, δ, $\varepsilon \in \boldsymbol{Z}$ if and only if

$$\begin{cases} \bar{\varepsilon}^2\alpha - \bar{\varepsilon}\beta + \gamma + 2aD\bar{\varepsilon}\delta = 0, \\ \dfrac{(p-1)aD}{2}\alpha - (p-1)aD\beta - 2aD\gamma + ((p-1)a^2D^2 + D)\delta + p\varepsilon = 0. \end{cases}$$

Since 1, $\bar{\varepsilon}$ are linearly independent over $\boldsymbol{Q}$, we see from $\bar{\varepsilon}^2 = \bar{\varepsilon} + \frac{p-1}{4}$ that the first equality is equivalent to

$$\alpha - \beta + 2aD\delta = 0, \ \frac{p-1}{4}\alpha + \gamma = 0,$$

and then from the second equality we obtain

$$\{pa^2D^2 - D(a^2D + 1)\}\delta - p\varepsilon = 0.$$

Hence they are solved by two independent integers x, y :

$$\alpha = x, \ \beta = x + 2aDy, \ \gamma = -\frac{p-1}{4}x, \ \delta = y, \ \varepsilon = (a^2D - b)Dy.$$

This gives the relation (16), which are satisfied for any $z \in \mathfrak{H}$ by continuity.

□

The invariant $\varDelta$ of the singular relation (16) is

$$\varDelta(x, y)=px^2+4aDxy+4bDy^2, \tag{17}$$

which is an integral positive definite quadratic form in x, y, and its discriminant is

$$16a^2D^2-4pbD=-16D,$$

which is independent of the choice of $\mathcal{O}$, ρ. As a corollary, we obtain the following

Theorem 5.2. *For a positive non-square integer $\varDelta$, such that $\varDelta\equiv 1, 0$ (mod 4), the following conditions are equivalent*:
(i) *$\varDelta$ is represented by the quadratic form $\varDelta(x, y)$ with relatively prime integers $x,y\in\boldsymbol{Z}$.*
(ii) *There exists an embedding $\psi:\boldsymbol{O}_\varDelta\rightarrow\mathcal{O}$, such that $\psi(\xi)^\iota=\psi(\xi)$ $\forall\xi\in\boldsymbol{O}_\varDelta$, and that ψ is optimal*: $\psi(\boldsymbol{Q}(\sqrt{\varDelta}))\cap\mathcal{O}=\boldsymbol{O}_\varDelta$
(iii) *The image $\Phi_{\bar\eta}(S(\boldsymbol{C}))$ of the Shimura curve is contained in the Humbert surface $H_\varDelta$.*

Proof. Let x, y be integers such that $\varDelta(x, y)=px^2+4aDxy+4bDy^2=\varDelta$. To such pair (x, y) we associate

$$\xi_\varDelta:=-xe_1+2xe_2+2ye_4\in\mathcal{O}. \tag{18}$$

Then we have

$$\begin{aligned}\xi_\varDelta^2&=-\mathrm{Nr}(\xi_\varDelta)\\&=p\left(x+\frac{2aD}{p}y\right)^2+\frac{4D}{p}y^2\\&=\varDelta(x, y).\end{aligned} \tag{19}$$

Moreover, one easily sees that $\xi_\varDelta^\iota=\xi_\varDelta$. Suppose first that $\varDelta\equiv 1$(mod 4). Then x is odd, and we see that $(1+\xi_\varDelta)/2\in\mathcal{O}$. Thus, putting $\psi((1+\sqrt{\varDelta})/2)=(1+\xi_\varDelta)/2$, we see that ψ gives the embedding of the order $\boldsymbol{O}_\varDelta=\boldsymbol{Z}[(1+\sqrt{\varDelta})/2]$. The converse assertion is clear from the above equality. Also we see from (19) that $\gcd(x, y)=1$ if and only if ψ is optimal. Next suppose that $\varDelta=4\varDelta_0$, $\varDelta_0\in\boldsymbol{Z}$, hence $x=2x_0$ is even. Then we see that the element

$$\eta_{\varDelta_0}:=\frac{1}{2}\xi_\varDelta=-x_0e_1+2x_0e_2+ye_4$$

is in $\mathcal{O}$, and satisfies $\eta_{\varDelta_0}^2=\varDelta_0$. So putting $\psi(\sqrt{\varDelta_0})=\eta_{\varDelta_0}$, we get the embedding of the order $\boldsymbol{Z}[\sqrt{\varDelta_0}]$ of discriminant $\varDelta$. Again we have $\gcd(2x_0,y)=1$ if and only if ψ is optimal This proves the equivalence of (i) and (ii). The rest of the assertion follows from Proposition 4.1. □

Corollary 5.3. *The Shimura curve S attached to $(\mathcal{O}, \iota_\rho)$ is contained as a component of the intersection $H_{\Delta_1} \cap H_{\Delta_2}$ of two Humbert surfaces, if and only if Δ_1, Δ_2 are represented by $\Delta(x, y)$ with relatively prime integers x, y.*

EXAMPLE 5.4. *The first two cases of maximal orders with smallest discriminants are* :

$$D=D_0=2\cdot 3,\ (p,\ a,\ b)=(5,\ 2,\ 5):\ \Delta(x,\ y)=5x^2+48xy+120y^2$$
$$D=D_0=2\cdot 5,\ (p,\ a,\ b)=(13,\ 3,\ 7):\ \Delta(x,\ y)=13x^2+120xy+280y^2.$$

The integers represented by $\Delta(x, y)$ with relatively prime x, y are $\{5, 8, 12, 21, 29, \ldots\}$, $\{5, 8, 13, 28, 37, \ldots\}$, respectively. Thus we see that the Shimura curves for $D=D_0=6$, 10 are components of $H_5 \cap H_8$ (see [5]).
Next we give some cases for which there are more than one equivalence classes of the modular embeddings, for maximal orders of fixed discriminant $D=D_0$, which shows that $\Phi_{\bar{\eta}}(z)$ really depends on the choice of ρ.

EXAMPLE 5.5. $D=D_0=2\cdot 13$. *The following two models give nonequivalent modular embeddings* :

$$(p,\ a,\ b)=(5,\ 2,\ 21):\ \Delta(x,\ y)=5x^2+208xy+2184y^2,$$
$$(p,\ a,\ b)=(149,\ 19,\ 63):\ \Delta(x,\ y)=149x^2+1976xy+6552y^2.$$

Indeed, the integers represented by $\Delta(x, y)$ with relatively prime x, y are $\{5, 21, 24, 28, 37, \ldots\}$, $\{8, 13, 21, 45, 60, \ldots\}$, *respectively.*

EXAMPLE 5.6. $D=D_0=3\cdot 5$. *The following two models also give nonequivalent modular embeddings* :

$$(p,\ a,\ b)=(13,\ 6,\ 97):\ \Delta(x,\ y)=13x^2+840xy+13580y^2,$$
$$(p,\ a,\ b)=(73,\ 5,\ 12):\ \Delta(x,\ y)=73x^2+700xy+1680y^2.$$

The integers represented by $\Delta(x, y)$ with relatively prime x, y are $\{5, 28, 33, 48, 73, \ldots\}$, $\{12, 13, 17, 33, 45, \ldots\}$,*respectively.*

Finally we prove the following:

Proposition 5.7. *The quadratic form $\Delta(x, y)$ represents a non-zero square over $\boldsymbol{Q}$ if and only if $D_0=1$, or equivalently, $\boldsymbol{B} \cong M_2(\boldsymbol{Q})$.*

Proof. $\Delta(x, y)$ represents a non-zero square if and only if the ternary quadratic form $F(x, y, z):=x^2+4Dy^2-pz^2$ is isotropic over $\boldsymbol{Q}$. By Minkowski-Hasse principle, this is equivalent to $(-4D, p)_l=-1$ for all primes l. On the other hand, we have

$$(-4D,\ p)_l=-1 \Longleftrightarrow l=p_i (1\leq i\leq t).$$

This proves the assertion.

References

[1] M. Eichler : Zur Zahlentheorie der Quaternionen-Algebren, J.reine angew. Math., 195 (1955), 127-151.

[2] G.v.d. Geer : Hilbert modular surface. Springer-Verlag, Berlin, Heidel-berg, 1988.

[3] K. Hashimoto : Elliptic conjugacy classes of the Siegel modular group and unimodular hermitian forms over the ring of cyclotomic integers, J.Fac.Sci.Univ. of Tokyo, 33-1 (1986), 57-82.

[4] K..Hashimoto : Base change of simple algebras and symmetric maximal orders of quaternion algebras, Memoirs of Sci. & Eng. Waseda Univ., 53 (1989) 21-45.

[5] K. Hashimoto and N. Murabayashi : Shimura curves as intersections of Humbert surfaces and defining equations of QM-curves of genus two, Tohoku Math.J. 47 (1995), 271-296.

[6] H. Hijikata : Explicit formula of the traces of the Hecke operators for $\Gamma_0(N)$, J.Math.Soc.Japan 26(1974), 56-82.

[7] G. Humbert : Sur les fonctions abeliennes singulieres, (Oeuvres de G. Humbert 2, pub. par les soins de Pierre Humbert et de Gaston Julia, Paris, Gauthier-Villars, 1936, 297-401.

[8] T. Ibukiyama : On maximal orders of division quaternion algebras over the rational number field with certain optimal embeddings, Nagoya Math. J. 88 (1982), 181-195.

[9] B.W. Jordan : On the Diophantine Arithmetic of Shimura Curves, Thesis, Harvard Univ. (1981)

[10] D. Mumford : Abelian varieties. Oxford University Press, London, 1970.

[11] I. Satake : Algebraic structures of symmetric domains, Publ. Math. Soc. Japan 14 (1980)

[12] G. Shimura : On the zeta-functions of the algebraic curves uniformized by certain automorphic functions, J. Math. Soc. Japan, 13 (1961) 275-331.

[13] G. Shimura : Construction of class fields and zetafunctions of algebraic curves. Ann. of Math. 85 (1967), 58-159.

[14] G. Shimura : Introduction to the arithmetic theory of automorphic functions. Publ. Math. Soc. Japan 11 (1971)

[15] M.F. Vigneras : Arithmetique des Algebres de Quaternions, Lecture Notes in Math. 800, Springer (1980).

Department of Mathematics
Waseda University
3-4-1, Okubo Shinjuku-ku
Tokyo, 169 Japan