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HOLOMORPHIC MAPPINGS FROM THE UNIT DISK TO ALGEBRAIC VARIETIES

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Introduction

In 1925, R. Nevanlinna established the second main theorem for meromorphic functions on the complex plane \mathbf{C} and developed the value distribution theory. His theory was extended by many authors. In particular, H. Cartan proved the second main theorem for holomorphic maps from \mathbf{C} to the complex projective space $P^n(\mathbf{C})$ (cf., e.g., S. Lang [7]). And J. Noguchi [9] studied holomorphic maps from \mathbf{C} to algebraic varieties and showed a version of second main theorem for these maps. Nevanlinna's lemma on the logarithmic derivative plays a crucial role in these theory.

On the other hand R. Nevanlinna also gave the second main theorem on a disk of finite radius (cf., e.g., W. Hayman [5]).

In this paper we shall study holomorphic maps from a disk of finite radius into an algebraic variety and derive a version of the second main theorem.

Let V be a nonsingular projective algebraic variety and Σ an effective divisor of simply normal crossing. Let Ω be a Kähler form on V and $\Delta(R)$ the disk of \mathbf{C} around the origin with radius R . In this paper, we assume that R is greater than 1 for technical reasons. Let us denote by $T_f(r)$ and $\bar{N}_f(r, \Sigma)$ the characteristic function of f relative to Ω and the counting function for Σ without multiplicities (see §1) respectively. Suppose that $V - \Sigma$ satisfy condition (A) in §1; namely, there exists a system of logarithmic 1-forms $\{\omega_i\}_{i=1}^{n+1}$ along Σ such that $\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1}$ are linearly independent over \mathbf{C} , where n is the dimension of V . A holomorphic map $f: \Delta(R) \rightarrow V$ is by definition *degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$ if the image $f(\Delta(R) - f^{-1}(\Sigma))$ is contained in a subvariety

$$\{x \in V - \Sigma : \sum_{i=1}^{n+1} a_i (\omega_1 \wedge \cdots \wedge \check{\omega}_i \wedge \cdots \wedge \omega_{n+1})_x = 0\}$$

with $(a_1, \dots, a_{n+1}) \neq (0, \dots, 0)$ ($a_i \in \mathbf{C}$). Then the main theorem of this paper is stated as follows.

Theorem A. *Let V , Σ and $\{\omega_i\}_{i=1}^{n+1}$ be as above. Let $f: \Delta(R) \rightarrow V$ be a*

holomorphic map which is nondegenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$. Then

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O\left(\log \frac{1}{R-r}\right) + O(1), \quad (\text{I})$$

where κ is a constant independent of r and f . Furthermore if f is of finite order, then

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + O\left(\log \frac{1}{R-r}\right) + O(1), \quad (\text{II})$$

Throughout this paper we shall write, $\varphi(r) \leq \phi(r)$ when $\varphi(r) \leq \phi(r)$ except on an open set E with $\int_E (R-r)^{-1} dr < \infty$. As applications of Theorem A, we shall prove the following two results.

Theorem B. Under the same assumptions as in Theorem A, if

$$\int_I^R \bar{N}_f(t, \Sigma) (R-t)^{\mu-1} dt < \infty$$

for a positive number μ , then the holomorphic map f is of finite order and

$$\int_I^P T_f(t) (R-t)^{\mu-1} dt < \infty$$

holds.

Corollary C. Let $V, \Sigma, \{\omega_i\}_{i=1}^{n+1}$ and f be as in Theorem A, and let $\text{supp } f^* \Sigma = \{a_1, a_2, \dots\}$. Suppose that

$$\sum_i (R - |a_i|)^{\lambda+1} < \infty \quad (\lambda > 0).$$

Then for any effective divisor D such that $f(\Delta(R)) \not\subset \text{supp } D$, we have

$$\sum_i (R - |b_i|)^{\lambda+1} < \infty,$$

where $f^* D = b_1 + b_2 + \dots$.

We remark that when $V = P^1(\mathbf{C})$, Theorem B and Corollary C are well known (cf., e.g., [17] P. 140, [13] P. 104).

In §1 we recall some definitions and known results. §2 is devoted to the extension of Nevanlinna's lemma on the logarithmic derivative. Theorem A (resp. Theorem B and Corollary C) will be proved in §3 (resp. §4). In §5, we shall discuss holomorphic maps from $\Delta(R)$ into an algebraic variety with bounded characteristic functions.

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1. Preliminaries

(1) We call a meromorphic 1-form ω a logarithmic 1-form along Σ if any point $a \in V$ we can take a holomorphic coordinate system $U, (x_1, \dots, x_n)$ around a such that $\{x_1 \cdots x_k = 0\} = \Sigma \cap U$ ($k \leq n$) and

$$\omega = a_1(x) \frac{dx_1}{x_1} + \dots + a_k(x) \frac{dx_k}{x_k} + \eta \text{ on } U,$$

where $a_1(x), \dots, a_k(x)$ are holomorphic functions on U and η is a holomorphic 1-form on U . Let $H^0(V, \Omega_V^1(\log \Sigma))$ be the vector space of logarithmic 1-forms along Σ on V . An element of $H^0(V, \Omega_V^1(\log \Sigma))$ is d -closed on $V - \Sigma$ and

$$\dim H^0(V, \Omega_V^1) + \dim H^0(V, \Omega_V^1(\log \Sigma)) = \dim H_1(V - \Sigma, \mathbf{C})$$

where Ω_V^1 denotes the sheaf of germs of holomorphic 1-forms (see Deligne [2]). We assume the following condition (A):

(A) "There exists a system $\{\omega_i\}_{i=1}^{n+1}$ of $n+1$ logarithmic 1-forms ω_i in $H^0(V, \Omega_V^1(\log \Sigma))$ such that the n -forms

$$\omega_1 \wedge \dots \wedge \check{\omega}_i \wedge \dots \wedge \omega_{n+1} \quad (i = 1, \dots, n+1)$$

are linearly independent over \mathbf{C} , where n is the dimension of V ."

A holomorphic map $f: \Delta(R) \rightarrow V$ is by definition *degenerate* with respect to $\{\omega_i\}_{i=1}^{n+1}$ if the image $f(\Delta(R) - f^{-1}(\Sigma))$ is contained in a subvariety

$$\{x \in V - \Sigma : \sum_{i=1}^{n+1} a_i(\omega_1 \wedge \dots \wedge \check{\omega}_i \wedge \dots \wedge \omega_{n+1})_x = 0\},$$

where $(a_1, \dots, a_{n+1}) \neq (0, \dots, 0)$ ($a_i \in \mathbf{C}$). If f is degenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$, then $f(\Delta(R))$ is contained in the support of an element of the complete linear system $|K_V + \Sigma|$.

(2) We denote by $\text{supp } D$ the support of a divisor D on $\Delta(R)$ or V . For an effective divisor D on V such that $f(\Delta(R)) \not\subset \text{supp } D$ we denote by $n_f(t, D)$ the sum of orders of the divisor $f^*D \cap \Delta(t)$. We define $\bar{n}_f(t, D)$ the number of points of $\text{supp } f^*D$ in $\Delta(t)$. We denote $n_f(0, D)$ the order of f^*D at 0 and $\bar{n}_f(0, D)$ the order of $\text{supp } f^*D$ at 0. Set

$$N_f(r, D) = \int_0^r \{n_f(t, D) - n_f(0, D)\} \frac{dt}{t} + n_f(0, D) \log r,$$

$$\bar{N}_f(r, D) = \int_0^r \{\bar{n}_f(t, D) - \bar{n}_f(0, D)\} \frac{dt}{t} + \bar{n}_f(0, D) \log r.$$

(3) For a meromorphic function α in $\Delta(R)$ we write

$$m(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\alpha(re^{i\theta})| d\theta,$$

$$N(r, \alpha) = \int_0^r \{n(t, \alpha) - n(0, \alpha)\} \frac{dt}{t} + n(0, \alpha) \log r,$$

where $\log^+ |\alpha| = \max \{\log |\alpha|, 0\}$, and $n(t, \alpha)$ denotes the number of poles of α in $\Delta(t)$ with counting multiplicities and $n(0, \alpha)$ the order of α at 0. We also set

$$T(r, \alpha) = m(r, \alpha) + N(r, \alpha).$$

(4) Let f be a holomorphic map from $\Delta(R)$ to V . The characteristic function $T_f(r)$ of f relative to Ω is defined by

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* \Omega.$$

We say that f is of finite order $\lambda \in [0, \infty)$ if

$$\limsup_{r \rightarrow R} \frac{\log T_f(r)}{\log \frac{1}{R-r}} = \lambda$$

and of infinite order if

$$\limsup_{r \rightarrow R} \frac{\log T_f(r)}{\log \frac{1}{R-r}} = \infty.$$

We note that f is of finite order λ if and only if

$$\int_1^R T_f(t) (R-t)^{\mu-1} dt = \begin{cases} \infty & (\text{for } \mu < \lambda) \\ \text{finite} & (\text{for } \mu > \lambda). \end{cases}$$

(5) Let $[D] \rightarrow V$ be the line bundle over V defined by a divisor D on V . Let $\Psi \in c_1([D])$ be the first Chern form defined by a fiber metric $\|\cdot\|$ in $[D]$ and take a section $\sigma \in \Gamma(V, [D])$ such that σ defines the divisor D and $\|\sigma\| \leq 1$, then the first main theorem says that

$$T_f(r, c_1([D])) = N_f(r, D) + m_f(r, D) + O(1),$$

where $T_f(r, c_1([D])) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* \Psi$

and $m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log (\|(\sigma \circ f)(re^{i\theta})\|^{-1}) d\theta$

(cf., e.g., Shabat [16], p.61). Since Ω is positive definite and V is compact,

there exists a constant $K > 0$ such that

$$(1.1) \quad T_f(r, c_1([D])) \leq K T_f(r).$$

Let $\mathfrak{R}(V)$ be the field of rational functions over V and $\{\phi_1, \dots, \phi_l\}$ the generators of $\mathfrak{R}(V)$ such that each $f^*\phi_j$ is defined. Put

$$\tilde{T}_f(r) = \max_{1 \leq j \leq l} \{T(r, f^*\phi_j)\}.$$

Then we have

$$(1.2) \quad B' T_f(r) + O(1) \leq \tilde{T}_f(r) \leq B T_f(r) + O(1),$$

where B and B' are positive constants (cf., [12]).

(6) Let V and Σ be as in Theorem A. We denote by $\mathfrak{M}^*(\Sigma)$ the sheaf of germs of non-zero meromorphic functions whose zeros and poles are contained in Σ , and $H^0(V, \mathfrak{A}_V(\log \Sigma))$ the Z module of meromorphic closed 1-forms whose germs coincide with $d \log \zeta$ where $\zeta \in \mathfrak{M}^*(\Sigma)$. Let $\Sigma_i (i=1, 2, \dots)$ be the irreducible components of Σ and $\dot{\Sigma}$ the set of regular points of Σ . For each point of $\dot{\Sigma} \cap \Sigma_i$ we can take a neighborhood U and a holomorphic coordinate system (x_1, \dots, x_n) such that $\{x_1=0\} = \Sigma_i \cap U$. Then every section ω in $H^0(V, \mathfrak{A}_V(\log \Sigma))$ is written in U as

$$\omega = \nu_i \frac{dx_1}{x_1} + \eta,$$

where ν_i is an integer and η is a holomorphic 1-form. The integer ν_i is independent of the choice of a local coordinate system (x_1, \dots, x_n) . Since $\Sigma_i \cap \dot{\Sigma}$ is connected ν_i is constant on $\Sigma_i \cap \dot{\Sigma}$. We define the residue of ω on $\Sigma_i \cap \dot{\Sigma}$ by

$$\text{res}(\omega, \Sigma_i) = \nu_i.$$

Thus we get a divisor $D = \sum \text{res}(\omega, \Sigma_i) \Sigma_i$.

(7) **Proposition.** *There exists a basis $\{\omega_i\}$ of the vector space $H^0(V, \Omega_V^1(\log \Sigma))$ over C such that every ω_i is an element of $H^0(V, \mathfrak{A}_V(\log \Sigma))$*

Proof. See Iitaka [6] Sections 2~4.

(8) **Ochai's theorem** (cf., [14], [9]). *Suppose that there exists a system $\{\omega_i\}_{i=1}^{n+1}$ of logarithmic 1-forms on V satisfying (A). Let f be a holomorphic map from $\Delta(R)$ to V which is non-degenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$. Then for every rational function $\phi \in \mathfrak{R}(V)$ such that $f^*\phi$ is defined, the meromorphic function $f^*\phi$ is algebraic over the field generated by $\{\zeta_i^{(k)} : 0 \leq k \leq n-1, 1 \leq i \leq n+1\}$, where ζ_i is defined by $f^*\omega_i = \zeta_i dz$ and $\zeta_i^{(k)}$ denotes the k -th derivative of ζ_i .*

(9) **Proposition.** *Let F be a meromorphic function on $\Delta(R)$ and A_i ($i=0, \dots, l$) holomorphic functions on $\Delta(R)$ such that $A_0 \not\equiv 0$ and*

$$A_0 F^l + A_1 F^{l-1} + \dots + A_{l-1} F + A_l = 0 .$$

Then

$$T(r, F) \leq \sum_{j=0}^l T(r, A_j) + O(1) .$$

Proof. See Noguchi-Ochiai [12] Lemma (6.1.5).

2. A generalization of Nevanlinna’s lemma on logarithmic derivative

In this section we give a generalization of Nevanlinna’s lemma on logarithmic derivative.

Lemma 2.1. *Let $\varphi(r)$ be a positive valued C^1 function with non-negative derivatives on $[0, R)$. Then*

$$\varphi^{(1)}(r) \leq \{\varphi(r)\}^2 \frac{1}{R-r} \parallel .$$

Proof. Suppose that $\varphi^{(1)}(r)/(\varphi(r))^2 > (R-r)^{-1}$ for a subset E of $[0, R)$. Then

$$\int_{E \cap [1, R)} \frac{dr}{R-r} \leq \int_{E \cap [1, R)} \frac{\varphi^{(1)}(r)}{(\varphi(r))^2} dr \leq [-(\varphi(r)^{-1})]_1^R \leq \varphi(1)^{-1} . \quad \text{Q.E.D.}$$

Let V, Σ , and f be as in Theorem A , and let Σ_i denote the irreducible components of Σ . For an element ω of $H^0(V, \mathfrak{A}_V(\log \Sigma))$ we set $f^* \omega = \zeta(z) dz$. Then $\xi(z)$ is a meromorphic function with poles of order one and their residues are integers. Set

$$G(z) = \int_0^z f^* \omega \pmod{2\pi i}, \quad g(z) = \exp(G(z)) .$$

By the same arguments as in [9: lemma 2.2] we get the following

Lemma 2.2. *There exists constants $K (> 0)$, A and B such that*

$$T(r, g) \leq K \left\{ \left(\frac{1}{2\pi} \right)^{1/2} \left(r \frac{d}{dr} T_f(r) + A \right)^{1/2} + T_f(r) \right\} + B .$$

Next we shall prove

Main lemma 2.3. *Let f, ω and ζ be as above. Then*

$$m(r, \zeta) \leq O(\log^+ T_f(r)) + O\left(\log \frac{1}{R-r}\right) + O(1) \parallel .$$

Furthermore, if f is a map of finite order, then ζ is of finite order and

$$m(r, \zeta) \leq O\left(\log \frac{1}{R-r}\right) + O(1) .$$

Proof. Applying Lemma 2.2 and the classical Nevanlinna's lemma on the logarithmic derivative to $\zeta(z)$, we have

$$\begin{aligned} (2.3.1) \quad m(r, \zeta) &= m(r, g^{(1)}/g) \\ &\leq O(\log^+ T(r, g)) + O\left(\log \frac{1}{R-r}\right) + O(1) \parallel \\ &\leq O(\log^+ (K \{(2\pi)^{-1/2} (r \frac{d}{dr} T_f(r) + A)^{1/2} + T_f(r)\} + B)) \\ &\quad + O\left(\log \frac{1}{R-r}\right) + O(1) \parallel . \end{aligned}$$

It follows from Lemma 2.1 and (2.3.1) that

$$m(r, \zeta) \leq O(\log^+ T_f(r)) + O\left(\log \frac{1}{R-r}\right) + O(1) \parallel .$$

This is the first part of the lemma.

By Lemma 2.1 the inequality

$$\frac{d}{dr} T_f(r) \leq (T_f(r))^2 (R-r)^{-1}$$

holds except for a disjoint union $E = \cup_j I_j$ of intervals $I_j = (a_j, b_j)$ such that $\int_E (R-r)^{-1} dr = M < \infty$. We define r' by

$$\begin{aligned} r' &= r \quad \text{if } r \notin E, \\ r' &= b_j \quad \text{if } r \in I_j = (a_j, b_j). \end{aligned}$$

Then

$$\begin{aligned} r \left(\frac{d}{dr} T_f \right) (r) &\leq r' \left(\frac{d}{dr} T_f \right) (r') \leq R (T_f(r'))^2 (R-r')^{-1} \\ &\leq R (R-r')^{-2\mu-2} \leq R \left(\frac{R-r}{R-r'} \right)^{2\mu+2} \left(\frac{1}{R-r} \right)^{2\mu+2}, \end{aligned}$$

where μ is the order of f . On the other hand,

$$\int_r^{r'} \frac{dt}{R-t} \leq \int_E \frac{dt}{R-t} = M < \infty ,$$

and hence

$$\frac{R-r}{R-r'} \leq e^M .$$

Therefore we obtain

$$r \left(\frac{d}{dr} T_f \right) (r) \leq R e^{(2\mu+2)M} \left(\frac{1}{R-r} \right)^{2\mu+2} .$$

Since f is of finite order, it follows from Lemma 2.2 that g is also of finite order. Hence by the classical Nevanlinna's lemma on logarithmic derivative, we get

$$m(r, \zeta) = m(r, g^{(1)}/g) \leq O(\log \frac{1}{R-r}) + O(1) .$$

Moreover

$$N(r, \zeta) \leq \bar{N}_f(r, \Sigma) \leq N_f(r, \Sigma) = O(T_f(r)) .$$

It follows that $N(r, \zeta)$ is of finite order. Therefore ζ is of finite order. Q.E.D.

For the sake of convenience we use the following notation:

$$S_f(r) = O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \parallel$$

if f is of infinite order, and

$$S_f(r) = O(\log \frac{1}{R-r}) + O(1)$$

if f is of finite order.

Corollary 2.4. *Let ζ and ζ bs as above. Then*

$$T(r, \zeta) \leq \bar{N}_f(r, D) + S_f(r) ,$$

where $D = \Sigma \setminus \text{res}(\omega, \Sigma_i) \mid \Sigma_i$.

Proof. Since ζ has a pole at z only if $f(z)$ belongs to $\text{supp } D$ and every pole of ζ has order one, we have

$$N(r, \zeta) \leq \bar{N}_f(r, D) .$$

Hence our assertion follows from Main lemma 2.3.

Q.E.D.

Corollary 2.5. *Let ω be an element of $H^0(V, \Omega_V^1(\log \Sigma))$ and put $f^*\omega = \zeta(z) dz$. Then*

$$T(r, \zeta) \leq \bar{N}_f(r, \Sigma) + S_f(r) .$$

Proof. By the proposition of Section 1 (7) there exist

$$\omega_j \in H^0(V, \mathfrak{A}_r(\log \Sigma))$$

and $c_j \in \mathbb{C}$ ($j=1, \dots, q$) such that

$$\omega = c_1 \omega_1 + c_2 \omega_2 + \dots + c_q \omega_q.$$

We set $f^* \omega_j = \zeta_j(z) dz$. Then

$$\zeta = c_1 \zeta_1 + c_2 \zeta_2 + \dots + c_q \zeta_q.$$

By Main lemma 2.3 we obtain

$$m(r, \zeta) \leq \sum_{i=1}^q m(r, \zeta_i) + O(1) = S_f(r).$$

Since any pole of ζ is of order one and ζ has a pole at z only if $f(z)$ belongs to $\text{supp } \Sigma$, We have

$$N(r, \zeta) \leq \bar{N}_f(r, \Sigma).$$

Hence we have

$$T(r, \zeta) \leq \bar{M}_f(r, \Sigma) + S_f(r). \quad \text{Q.E.D.}$$

REMARK. Using the same idea as in Noguchi [11: p. 224, Remark (1)], we obtain from Corollary 2.5 the classical Nevanlinna's second main theorem for meromorphic functions on $\Delta(R)$.

3. Proof of Theorem A.

We keep the same notation as in Theorem A. Let ζ_i be the meromorphic function on $\Delta(R)$ defined by $f^* \omega_i = \zeta_i dz$. Applying the classical Nevanlinna's lemma on logarithmic derivative to k -th derivative of ζ_i we have

$$(3.1) \quad \begin{aligned} T(r, \zeta_i^{(k)}) &\leq (k+1) T(r, \zeta_i) + O(\log^+ T(r, \zeta_i)) \\ &\quad + O\left(\log \frac{1}{R-r}\right) + O(1). \end{aligned}$$

Moreover, combining Corollary 2.5 with the first main theorem and (1.1) we have

$$\begin{aligned} \log^+ T(r, \zeta_i) &\leq \log^+ \bar{N}_f(r, \Sigma) + \log^+ S_f(r) + O(1) \\ &\leq \log^+ T_f(r, [\Sigma]) + S_f(r) + O(1) \\ &\leq O(\log^+ T_f(r)) + S_f(r) + O(1) \\ &= O(\log^+ T_f(r)) + O\left(\log \frac{1}{R-r}\right) + O(1). \end{aligned}$$

This inequality and (3.1) imply

$$T(r, \zeta_i^{(k)}) \leq (k+1) T(r, \zeta_i) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \parallel .$$

Hence by Coroallary 2.5

$$(3.2) \quad T(r, \zeta_i^{(k)}) \leq (k+1) \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \parallel .$$

Let $\{\phi_j\}_{j=1, \dots, l}$ be a system of generators of the rational function field $\mathfrak{K}(V)$ over C such that $f^*\phi_j$ are defined. Then by the Ochiai's theorm (§1 (8)) there are algebraic relations

$$(3.3) \quad (f^*\phi_j)^{m_j} + R_{j1}(\zeta_i^{(k)}) (f^*\phi_j)^{m_j-1} + \dots + R_{jm_j}(\zeta_i^{(k)}) = 0$$

$j=1, \dots, l$ where $R_{j\nu}(\zeta_i^{(k)})$ are rational functions of $\zeta_i^{(k)}$ $k=0, \dots, n-1, i=1, \dots, n+1$. By making use of [10] we see that there is a positive constant K independent of r and f such that

$$T(r, f^*\phi_j) \leq K \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \parallel$$

for all j . Thus we have

$$(3.4) \quad \begin{aligned} \tilde{T}_f(r) &= \max_j T(r, f^*\phi_j) \\ &\leq K \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) \\ &\quad + O(\log \frac{1}{R-r}) + O(1) \parallel . \end{aligned}$$

Inequalities (1.2) and (3.4) yield the inequality (I). This completes the first part of Theorem *A*. By the assumption that f is of finite order, and by Lemma 2.3 we see that ζ_i is of finite order. Then by the classical Nevanlinna's lemma on logarithmic derivative, we have

$$T(r, \zeta_i^{(k)}) \leq (k+1) T(r, \zeta_i) + O(\log \frac{1}{R-r}) + O(1) .$$

And by the same arguments as in the first part of Theorem *A* we obtain

$$(3.5) \quad \tilde{T}_f(r) \leq K \bar{N}_f(r, \Sigma) + O(\log \frac{1}{R-r}) + O(1) .$$

Thus inequalities (1.2) and (3.5) yield the inequality (II).

Q.E.D.

4. Proof of Theorem B and Corollary C

Proof of Theorem B. Without loss of generality we may assume $T_f(r) \rightarrow \infty$

as $r \rightarrow R$. We shall prove that f is of finite order. By the assumption, we have

$$N_f(r, \Sigma) = O\left(\left(\frac{1}{R-r}\right)^{\mu+1}\right),$$

and hence by Theorem *A* we get

$$\begin{aligned} \kappa T_f(r) \leq & L_1 \left(\frac{1}{R-r}\right)^{\mu+1} + M_1 \log^+ T_f(r) \\ & + M_2 \log \frac{1}{R-r} + M_3, \end{aligned}$$

where L_1, M_1, M_2 and M_3 are constants. On the other hand, there exists a constant $\delta (>0)$ such that

$$\kappa x - M_1 \log^+ x > \delta x$$

for sufficiently large x . Therefore

$$\delta T_f(r) \leq L_1 \left(\frac{1}{R-r}\right)^{\mu+1} + M_2 \log \frac{1}{R-r} + M_3$$

except the countable disjoint union of open intervals $E = \cup_j I_j$ such that

$$\int_E \frac{dr}{R-r} = M_4 < \infty.$$

We define r' as follow

$$\begin{aligned} r' &= r & \text{if } r \notin E \\ r' &= b_j & \text{if } r \in I_j = (a_j, b_j). \end{aligned}$$

Then we see that

$$\delta T_f(r) \leq \delta T_f(r') \leq L_1 \left(\frac{1}{R-r'}\right)^{\mu+1} + M_1 \log \frac{1}{R-r'} + M_3.$$

Moreover, since

$$M_4 \geq \int_r^{r'} \frac{dt}{R-t} = \log \frac{1}{R-r'} - \log \frac{1}{R-r},$$

we obtain

$$\frac{R-r}{R-r'} \leq \exp(M_4).$$

Therefore we get

$$\delta T_f(r) \leq L_1 \exp((\mu+1)M_4) \left(\frac{1}{R-r}\right)^{\mu+1} + M_2 \log \frac{1}{R-r} + M_6,$$

where M_5 is a constant. Hence f is a map of finite order. Therefore it follows from Theorem A that

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + \bar{M}_1 \log \frac{1}{R-r} + \bar{M}_2,$$

where \bar{M}_1, \bar{M}_2 are constants. Therefore

$$\begin{aligned} \kappa \int_1^R T_f(t) (R-t)^{\mu-1} dt &\leq \int_1^R \bar{N}_f(t, \Sigma) (R-t)^{\mu-1} dt \\ &+ \bar{M}_1 \int_1^R \log \frac{1}{R-t} (R-t)^{\mu-1} dt + \bar{M}_2 \int_1^R (R-t)^{\mu-1} dt. \end{aligned}$$

On the other, hand by the assumption

$$\int_1^R \bar{N}_f(t, \Sigma) (R-t)^{\mu-1} dt < \infty,$$

we obtain

$$\int_1^R T_f(t) (R-t)^{\mu-1} dt < \infty. \quad \text{Q.E.D.}$$

Proof of Corollary C. This is an immediate consequence of Theorem B and the following

Lemma 4.1. *Let μ be a positive number and D an effective divisor on V which satisfies $f(\Delta(R)) \not\subset \text{supp } D$, Then for $\text{supp } f^*D = \{a_1, a_2, a_3, \dots\}$,*

$$\begin{aligned} &\int_1^r \bar{N}_f(t, D) (R-t)^{\mu-1} dt \\ &\int_1^r \bar{n}_f(t, D) (R-t)^\mu dt, \text{ and} \\ &\sum_{1 \leq |a_i| \leq r} (R - |a_i|)^{\mu+1} \end{aligned}$$

are convergent or divergent at the same time as $r \rightarrow R$.

Proof. The same argument as in Shimizu [17] (p.107) can be applied for this case.

REMARK If $g(z) = \exp \frac{1}{1-z}$ then

$$\overline{\lim}_{r \rightarrow 1} \log T(r, g) / \log \frac{1}{1-r} = 0,$$

and

$$\overline{\lim}_{r \rightarrow 1} \log \log M(r, g) / \log \frac{1}{1-r} = 1,$$

where $M(r, g) = \text{Max}_{|z|=r} |g(z)|$.

5. Holomorphic maps of bounded characteristic functions

In this section we study holomorphic maps from the unit disk to an algebraic variety with $T_f(r) = O(1)$. Here we shall prove some analogous result to the classical Fatou's and Blaschke's theorems concerning bounded holomorphic functions on the unit disk.

Let f be a holomorphic Map from $\Delta(R)$ to a nonsingular algebraic variety V and Ω a Kähler form on V .

Proposition 5.1. *Suppose that $r \left(\frac{d}{dr} T_f \right) (r) = O(1)$ as $r \rightarrow R$. Then the length of a curve of the image of $\{te^{i\theta}; 0 < t < R\}$ by f with respect to the Hermitian metric h determined by Ω is finite for almost all θ , i.e.*

$$\lim_{r \rightarrow R} \int_0^r (s(te^{i\theta}))^{1/2} dt \quad (s(z) dzd\bar{z} = f^*h)$$

is finite for almost all θ .

Proof. By Schwarz's inequality

$$\begin{aligned} \int_0^{2\pi} \left(\int_0^r (s(te^{i\theta}))^{1/2} dt \right)^2 d\theta &\leq \int_0^{2\pi} \left(r \int_0^r s(te^{i\theta}) dt \right) d\theta \leq R \int_0^{2\pi} \int_0^r s(te^{i\theta}) dt d\theta \\ &\leq R \int_0^{2\pi} \int_0^r s(te^{i\theta}) t dt d\theta + R \int_0^{2\pi} \int_0^1 s(te^{i\theta}) dt d\theta \\ &= r \left(\frac{d}{dr} T_f \right) (r) \cdot R + B \cdot R \\ &\quad (B = \int_1^{2\pi} \int_0^1 s(te^{i\theta}) dt d\theta). \end{aligned}$$

Hence we get using the assumption

$$\lim_{r \rightarrow R} \int_0^{2\pi} \left(\int_1^r (s(te^{i\theta}))^{1/2} dt \right)^2 d\theta < \infty .$$

Therefore we see

$$\int_0^{2\pi} \left(\lim_{r \rightarrow R} \int_0^r (s(te^{i\theta}))^{1/2} dt \right)^2 d\theta < \infty .$$

Thus

$$\lim_{r \rightarrow R} \int_0^r (s(te^{i\theta}))^{1/2} dt$$

are finite for almost all θ .

Q.E.D.

Before showing a Blaschke-type theorem we need a lemma (cf., e.g., Shimizu

[17] p.107).

Lemma 5.2. *Let D be an effective divisor on V with $f(\Delta(R)) \not\subset \text{supp } D$. Then for $f^*D = a_1 + a_2 + \dots$*

$$N_f(r, D), \int_1^r n_f(t, D) dt, \text{ and } \sum_{1 \leq i, j \leq r} (R - |a_i|)$$

are convergent or divergent at the same time as $r \rightarrow R$.

Theorem 5.3. *Suppose that $T_f(r) = O(1)$ as $r \rightarrow R$. Then for any effective divisor D on V which satisfies $f(\Delta(R)) \not\subset \text{supp } D$ we have*

$$(5.3.1) \quad N_f(r, D) = O(1) \quad (r \rightarrow R)$$

$$(5.3.2) \quad n_f(r, D) = o\left(\frac{1}{R-r}\right) \quad (r \rightarrow R)$$

$$(5.3.3) \quad \sum_{|a_i| < R} (R - |a_i|) < \infty,$$

where $f^*D = a_1 + a_2 + \dots$.

Proof. By the first main theorem and (1.1) we obtain (5.3.1). and from Lemma 5.2, we have

$$\int_1^R n_f(t, D) dt < \infty.$$

Since $n_f(t, D)$ is non-decreasing (5.3.2) holds. Moreover by Lemma 5.2 we obtain (5.3.3). Q.E.D.

Finally we shall give an example related to the above results. W. Rudin [15] gave an example of a holomorphic map from $\Delta(1)$ to $\Delta(1)$ with

$$\int_0^1 |f^{(1)}(re^{i\theta})| dr = \infty \quad \text{for almost all } \theta.$$

this map is an example such that

$$\int_1^r \frac{|f^{(1)}(re^{i\theta})|}{1 + |f(re^{i\theta})|^2} dr = \infty$$

for almost all θ . This map has the property $r \left(\frac{d}{dr} T_f\right)(r) \rightarrow \infty$ as $r \rightarrow 1$. But clearly $T_f(r) = O(1)$ as $r \rightarrow 1$.

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