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HOLOMORPHIC MAPPINGS FROM THE UNIT DISK TO ALGEBRAIC VARIETIES

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Introduction

In 1925, R. Nevalinnna established the second main theorem for meromorphis functions on the complex plane $\boldsymbol{C}$ and developed the value distribution thoery. His theory was extended by many authors. In particular, H. Cartan proved the second main theorem for holomorphic mpas from $\boldsymbol{C}$ to the complex projective space $P^n(\boldsymbol{C})$ (cf., e.g., S. Lang [7]). And J. Noguchi [9] studied holomorphic maps from $\boldsymbol{C}$ to algebraic verieties and showed a version of second main theorem for these maps. Nevanlinna's lemma on the logarithmic derivative plays a crucial role in these theory.

On the other hand R. Nevanlinna also gave the second main theorem on a disk of finite radius (cf., e.g., W. Hayman [5]).

In this paper we shall study holomorphic maps from a disk of finite radius into an algebraic variety and derive a version of the second main theorem.

Let V be a nonsingular projective algebraic variety and Σ an effective divisor of simply normal crossing. Let Ω be a Kahler form on V and $\Delta(R)$ the disk of $\boldsymbol{C}$ around the origin with radius R. In this paper, we assume that R is greater than 1 for technical reasons. Let us denote by $T_f(r)$ and $\bar{N}_f(r, \Sigma)$ the charactearistic function of f relative to Ω and the counting function for Σ without multiplicities (see §1) respectively. Suppose that V-Σ satisfy condition (A) in §1; namely, there exists a system of logarithmic 1-forms $\{\omega_i\}_{i=1}^{n+1}$ along Σ such that $\omega_1\wedge\cdots\wedge\check{\omega}_i\wedge\cdots\wedge\omega_{n+1}$ are linearly independent over $\boldsymbol{C}$, where n is the dimension of V. *A* holomorphic map f: $\Delta(R)\to V$ is by definition *degenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$ if the image $f(\Delta(R)-f^{-1}(\Sigma))$ is contained in a subvariety

$$\{x\in V-\Sigma: \sum_{i=1}^{n+1} a_i(\omega_1\wedge\cdots\wedge\check{\omega}_i\wedge\cdots\wedge\omega_{n+1})_x=0\}$$

with $(a_1, \cdots, a_{n+1})\neq(0, \cdots, 0)$ $(a_i\in\boldsymbol{C})$. Then the main theorem of this paper is stated as follows.

Theorem A. *Let V, Σ and $\{\omega_i\}_{i=1}^{n+1}$ be as above. Let f: $\Delta(R)\to V$ be a* *holomorphic map which is nondegenerate with respect to* $\{\omega_i\}_{i=1}^{n+1}$. *Then*

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \| , \tag{I}$$

where κ *is a constant independent of* r *and* f. *Furthermore if* f *is of finite order, then*

$$\kappa T_f(r) \leq \bar{N}_f(r, \Sigma) + O(\log \frac{1}{R-r}) + O(1) , \tag{II}$$

Throughout this paper we shall write, $\varphi(r) \leq \phi(r) \|$ when $\varphi(r) \leq \phi(r)$ except on an open set E with $\int_E (R-r)^{-1} dr < \infty$. As applications of Theorem A, we shall prove the following two results.

Theorem B. *Under the same assumptions as in Theorem* A, *if*

$$\int_I^R \bar{N}_f(t, \Sigma) (R-t)^{\mu-1} dt < \infty$$

for a positive number μ, *then the holomorphic map* f *is of finite order and*

$$\int_I^P T_f(t) (R-t)^{\mu-1} dt < \infty$$

holds.

Corollary C. *Let* V, Σ, $\{\omega_i\}_{i=1}^{n+1}$ *and* f *be as in Theorem* A, *and let* supp $f^*\Sigma = \{a_1, a_2, \cdots\}$. *Suppose that*

$$\sum_i (R - |a_i|)^{\lambda+1} < \infty \quad (\lambda > 0) .$$

Then for any effective divisor D *such that* $f(\Delta(R)) \not\subset \operatorname{supp} D$, *we have*

$$\sum_i (R - |b_i|)^{\lambda+1} < \infty ,$$

where $f^*D = b_1 + b_2 + \cdots$.

We remark that when $V = P^1(\boldsymbol{C})$, Theorem B and Corollary C are well known (cf., e.g., [17] P. 140, [13] P. 104).

In §1 we recall some definitions and known results. §2 is devoted to the extension of Nevanlinna's lemma on the logarithmic derivative. Theorem A (resp. Theorem B and Corollary C) will be proved in §3 (resp. §4). In §5, we shall discuss holomorphic maps from $\Delta(R)$ into an algebraic variety with bounded charactaristic functions.

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1. Preliminaries

(1) We call a meromorphic 1-form ω a logarithmic 1-form along Σ if any point $a\in V$ we can take a holomorphic coordinate system U, $(x_1, \cdots, x_n)$ around a such that $\{x_1\cdots x_k=0\}=\Sigma\cap U$ $(k\leq n)$ and

$$\omega = a_1(x)\frac{dx_1}{x_1}+\cdots+a_k(x)\frac{dx_k}{x_k}+\eta \text{ on } U,$$

where $a_1(x), \cdots, a_k(x)$ are holomorphic functions on U and η is a holomorphic 1-form on U. Let $H^0(V, \Omega_V^1(\log\Sigma))$ be the vector space of logarithmic 1-forms along Σ on V. An element of $H^0(V, \Omega_V^1(\log\Sigma))$ is d-closed on V-Σ and

$$\dim H^0(V, \Omega_V^1)+\dim H^0(V, \Omega_V^1(\log\Sigma)) = \dim H_1(V-\Sigma, \boldsymbol{C})$$

where Ω_V^1 denotes the sheaf of germs of holomorphic 1-forms (see Delgine [2]). We assume the following condition (A):

(A) "Three exists a system $\{\omega_i\}_{i=1}^{n+1}$ of $n+1$ logarithmic 1-forms ω_i in H^0 $(V, \Omega_V^1(\log\Sigma))$ such that the n-forms

$$\omega_1\wedge\cdots\wedge\check{\omega}_i\wedge\cdots\wedge\omega_{n+1} \quad (i=1, \cdots, n+1)$$

are linearly independent over $\boldsymbol{C}$, where n is the dimension of V."

A holomorphic map f: $\Delta(R)\rightarrow V$ is by definition *degenerate* with respect to $\{\omega_i\}_{i=1}^{n+1}$ if the image $f(\Delta(R)-f^{-1}(\Sigma))$ is contained in a subvariety

$$\{x\in V-\Sigma: \sum_{i=1}^{n+1} a_i(\omega_1\wedge\cdots\wedge\check{\omega}_i\wedge\cdots\wedge\omega_{n+1})_x = 0\},$$

where $(a_1, \cdots, a_{n+1})\neq(0, \cdots, 0)$ $(a_i\in\boldsymbol{C})$. If f is degenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$, then $f(\Delta(R))$ is cotained in the support of an element of the complete linear system $|K_V+\Sigma|$.

(2) We denote by supp D the support of a divisor D on $\Delta(R)$ or V. For an effect,ve divisor D on V such that $f(\Delta(R))\not\subset\text{supp } D$ we denote by $n_f(t, D)$ the sum of orders of the divisor $f^*D\cap\Delta(t)$. We define $\bar{n}_f(t, D)$ the number of points of supp f^*D in $\Delta(t)$. We denote $n_f(0, D)$ the order of f^*D at 0 and $\bar{n}_f(0, D)$ the order of supp f^*D at 0. Set

$$N_f(r, D) = \int_0^r \{n_f(t, D)-n_f(0, D)\}\frac{dt}{t}+n_f(0, D)\log r,$$

$$\bar{N}_f(r, D) = \int_0^r \{\bar{n}_f(t, D)-\bar{n}_f(0, D)\}\frac{dt}{t}+\bar{n}_f(0, D)\log r.$$

(3) For a meromorphic function α in $\Delta(R)$ we write

$$m(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\alpha(re^{i\theta})| \, d\theta \,,$$

$$N(r, \alpha) = \int_0^r \{n(r, \alpha) - n(0, \alpha)\} \frac{dt}{t} + n(0, \alpha) \log r \,,$$

where $\log^+|\alpha| = \max\{\log|\alpha|, 0\}$, and $n(t, \alpha)$ denotes the number of poles of α in $\Delta(t)$ with counting multiplicities and $n(0, \alpha)$ the order of α at 0. We also set

$$T(r, \alpha) = m(r, \alpha) + N(r, \alpha) \,.$$

(4) Let f be a holomorphic map from $\Delta(R)$ to V. The characteristic function $T_f(r)$ of f relative to Ω is defined by

$$T_f(r) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* \Omega \,.$$

We say that f is of finite order $\lambda \in [0, \infty)$ if

$$\limsup_{r \to R} \frac{\log T_f(r)}{\log \frac{1}{R-r}} = \lambda$$

and of infinite order if

$$\limsup_{r \to R} \frac{\log T_f(r)}{\log \frac{1}{R-r}} = \infty \,.$$

We note that f is of finite order λ if and only if

$$\int_1^R T_f(t) (R-t)^{\mu-1} dt = \begin{cases} \infty & (\text{for } \mu < \lambda) \\ \text{finite} & (\text{for } \mu > \lambda) \,. \end{cases}$$

(5) Let $[D] \to V$ be the line bundle over V defined by a divisor D on V. Let $\Psi \in c_1([D])$ be the first Chern form defined by a fiber metric $\|\cdot\|$ in $[D]$ and take a section $\sigma \in \Gamma(V, [D])$ such that σ defines the divisor D and $\|\sigma\| \leq 1$, then the first main theorem says that

$$T_f(r, c_1([D])) = N_f(r, D) + m_f(r, D) + O(1) \,,$$

where $T_f(r, c_1([D])) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* \Psi$

and $$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log (\|(\sigma \circ f)(re^{i\theta})\|^{-1}) \, d\theta$$

(cf., e.g., Shabat [16], p.61). Since Ω is positive definite and V is compact, there exists a constant $K>0$ such that

(1.1) $$T_f(r, c_1([D]))\leq K\, T_f(r).$$

Let $\mathfrak{R}(V)$ be the field of rational functions over V and $\{\phi_1, \cdots, \phi_l\}$ the generators of $\mathfrak{R}(V)$ such that each $f^*\phi_j$ is defined. Put

$$\tilde{T}_f(r) = \max_{1\leq j\leq l} \{T(r, f^*\phi_j)\}.$$

Then we have

(1.2) $$B'\, T_f(r)+O(1)\leq \tilde{T}_f(r)\leq B\, T_f(r)+O(1),$$

where B and B' are positive constants (cf., [12]).

(6) Let V and Σ be as in Theorem A. We denote by $\mathfrak{M}_V^*(\Sigma)$ the sheaf of germs of non-zero meromorphic functions whose zeros and poles are contained in Σ, and $H^0(V, \mathfrak{A}_V(\log \Sigma))$ the Z module of meromorphic closed 1-forms whose germs concide with $d \log \zeta$ where $\zeta\in\mathfrak{M}_V^*(\Sigma)$. Let $\Sigma_i(i=1, 2, \cdots)$ be the irreducible components of Σ and $\mathring{\Sigma}$ the seto of regular points of Σ. For each point of $\mathring{\Sigma}\cap\Sigma_i$ we can take a neighborhood U and a holomrorphic coordinate system $(x_1, \cdots, x_n)$ such that $\{x_1=0\}=\Sigma_i\cap U$. Then every seation ω in $H^0(V, \mathfrak{A}_V(\log \Sigma))$ is written in U as

$$\omega = \nu_i \frac{dx_1}{x_1}+\eta,$$

where ν_i is an integer and η is a holomorphic 1-form. The integer ν_i is independent of the choice of a local coordinate system $(x_1, \cdots, x_n)$. Since $\Sigma_i\cap\mathring{\Sigma}$ is connected ν_i is constant on $\Sigma_i\cap\mathring{\Sigma}$. We define the residue of ω on $\Sigma_i\cap\mathring{\Sigma}$ by

$$\operatorname{res}(\omega, \Sigma_i) = \nu_i.$$

Thus we get a divisor $D=\Sigma \operatorname{res}(\omega, \Sigma_i)\, \Sigma_i$.

(7) **Proposition.** *There exists a basis $\{\omega_i\}$ of the vector space $H^0(V, \Omega_V^1$ $(\log \Sigma))$ over **C** such that every ω_i is an element of $H^0(V, \mathfrak{A}_V(\log \Sigma))$*

Proof. See Iitaka [6] Sections 2~4.

(8) **Ochai's theorem** (cf., [14], [9].). *Suppose that there exists a system $\{\omega_i\}_{i=1}^{n+1}$ of logarithmic 1-forms on V satisfying (A). Let f be a holomorphic map from $\Delta(R)$ to V which is non-degenerate with respect to $\{\omega_i\}_{i=1}^{n+1}$. Then for every rational function $\phi\in\mathfrak{R}(V)$ such that $f^*\phi$ is defined, the meromorphic function $f^*\phi$ is algebraic over the field generated by $\{\zeta_i^{(k)}: 0\leq k\leq n-1, 1\leq i\leq n+1\}$, where ζ_i is defined by $f^*\omega_i=\zeta_i\, dz$ and $\zeta_i^{(k)}$ denotes tne k-the derivative of ζ_i.*

(9) **Proposition.** *Let F be a meromorphic function on $\Delta(R)$ and A_i $(i=0, \cdots, l)$ holomorphic functions on $\Delta(R)$ such that $A_0 \not\equiv 0$ and*

$$A_0 F^l + A_1 F^{l-1} + \cdots + A_{l-1} F + A_l = 0 .$$

Then

$$T(r, F) \leq \sum_{j=0}^{l} T(r, A_i) + O(1) .$$

Proof. See Noguchi-Ochiai [12] Lemma (6.1.5).

2. A generalization of Nevanlinna's lemma on logarithmic derivative

In this section we give a generalization of Nevanlinna's lemma on logarithmic derivative.

Lemma 2.1. *Let $\varphi(r)$ be a positive valued C^1 function with non-negative derivatives on $[0, R)$. Then*

$$\varphi^{(1)}(r) \leq \{\varphi(r)\}^2 \frac{1}{R-r} \,||\, .$$

Proof. Suppose that $\varphi^{(1)}(r)/(\varphi(r))^2 > (R-r)^{-1}$ for a subset E of $[0, R)$. Then

$$\int_{E \cap [1,R)} \frac{dr}{R-r} \leq \int_{E \cap [1,R)} \frac{\varphi^{(1)}(r)}{(\varphi(r))^2} dr$$
$$\leq [-(\varphi(r)^{-1})]_1^R \leq \varphi(1)^{-1} . \qquad \text{Q.E.D.}$$

Let V, Σ, and f be as in Theorem A, and let Σ_i denote the irreducible components of Σ. For an element ω of $H^0(V, \mathfrak{A}_V(\log \Sigma))$ we set $f^*\omega = \zeta(z)\, dz$. Then $\xi(z)$ is a meromorphic function with poles of order one and their residues are integers. Set

$$G(z) = \int_0^n f^*\omega \quad (\text{mod. } 2\pi i) , \quad g(z) = \exp(G(z)) .$$

By the same arguments as in [9: lemma 2.2] we get the following

Lemma 2.2. *There exists constants $K(>0)$, A and B such that*

$$T(r, g) \leq K \left\{ \left(\frac{1}{2\pi}\right)^{1/2} \left(r \frac{d}{dr} T_f(r) + A\right)^{1/2} + T_f(r) \right\} + B .$$

Next we shall prove

Main lemma 2.3. *Let f, ω and ζ be as above. Then*

$$m(r, \zeta)\leq O(\log^+ T_f(r))+O(\log \frac{1}{R-r})+O(1) \,||\,.$$

Furthermore, if f is a map of finite order, then ζ is of finite order and

$$m(r, \zeta)\leq O(\log \frac{1}{R-r})+O(1)\,.$$

Proof. Applying Lemma 2.2 and the classical Nevanlinna's lemma on the logarithmic derivative to $\zeta(z)$, we have

$$\begin{aligned} m(r, \zeta) &= m(r, g^{(1)}/g) \\ &\leq O(\log^+ T(r, g))+O(\log \frac{1}{R-r})+O(1) \,|| \\ &\leq O(\log^+ (K \{(2\pi)^{-1/2} (r \frac{d}{dr} T_f(r)+A)^{1/2}+T_f(r)\} +B)) \\ &\quad +O(\log \frac{1}{R-r})+O(1) \,||\,. \end{aligned} \tag{2.3.1}$$

It follows from Lemma 2.1 and (2.3.1) that

$$m(r, \zeta)\leq O(\log^+ T_f(r))+O(\log \frac{1}{R-r})+O(1) \,||\,.$$

This is the first part of the lemma.

By Lemma 2.1 the inequality

$$\frac{d}{dr} T_f(r)\leq (T_f(r))^2 (R-r)^{-1}$$

holds except for a disjoint union $E=\cup_j I_j$ of intervals $I_j=(a_j, b_j)$ such that $\int_E (R-r)^{-1}\, dr=M<\infty$. We define r' by

$$\begin{aligned} r' &= r \quad \text{if} \quad r\notin E\,, \\ r' &= b_j \quad \text{if} \quad r\in I_j = (a_j, b_j)\,. \end{aligned}$$

Then

$$\begin{aligned} r\left(\frac{d}{dr} T_f\right)(r)&\leq r'\left(\frac{d}{dr} T_f\right)(r')\leq R(T_f(r'))^2 (R-r')^{-1} \\ &\leq R(R-r')^{-2\mu-2}\leq R\left(\frac{R-r}{R-r'}\right)^{2\mu+2}\left(\frac{1}{R-r}\right)^{2\mu+2}, \end{aligned}$$

where μ is the order of f. On the other hand,

$$\int_r^{r'} \frac{dt}{R-t}\leq \int_E \frac{dt}{R-t} = M<\infty\,,$$

and hence

$$\frac{R-r}{R-r'} \leq e^M .$$

Therefore we obtain

$$r\left(\frac{d}{dr} T_f\right)(r) \leq R\, e^{(2\mu+2)M} \left(\frac{1}{R-r}\right)^{2\mu+2} .$$

Since f is of finite order, it follows from Lemma 2.2 that g is also of finite order. Hence by the classical Nevanlinna's lemma on logarithmic derivative, we get

$$m(r, \zeta) = m(r, g^{(1)}/g) \leq O(\log \frac{1}{R-r}) + O(1) .$$

Moreover

$$N(r, \zeta) \leq \bar{N}_f(r, \Sigma) \leq N_f(r, \Sigma) = O(T_f(r)) .$$

It follows that $N(r, \zeta)$ is of finite order. Therefore ζ is of finite order. Q.E.D.

For the sake of convenience we use the following notation:

$$S_f(r) = O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) \;||$$

if f is of infinite order, and

$$S_f(r) = O(\log \frac{1}{R-r}) + O(1)$$

if f is of finite order.

Corollary 2.4. *Let ζ and ζ bs as above. Then*

$$T(r, \zeta) \leq \bar{N}_f(r, D) + S_f(r) ,$$

where $D = \Sigma\, |\text{res}(\omega, \Sigma_i)|\, \Sigma_i$.

Proof. Sinse ζ has a pole at z only if $f(z)$ belongs to supp D and every pole of ζ has order one, we have

$$N(r, \zeta) \leq \bar{N}_f(r, D) .$$

Hence our assertion follows from Main lemma 2.3. Q.E.D.

Corollary 2.5. *Let ω be an element of $H^0(V, \Omega^1_V(\log \Sigma))$ and put $f^*\omega = \zeta(z)\, dz$. Then*

$$T(r, \zeta) \leq \bar{N}_f(r, \Sigma) + S_f(r) .$$

Proof. By the proposition of Section 1 (7) there exist

$$\omega_j \in H^0(V, \mathfrak{A}_V(\log \Sigma))$$

and $c_j \in \boldsymbol{C}$ ($j=1, \cdots, q$) such that

$$\omega = c_1 \omega_1 + c_2 \omega_2 + \cdots + c_q \omega_q .$$

We set $f^*\omega_j = \zeta_j(z)\, dz$. Then

$$\zeta = c_1 \zeta_1 + c_2 \zeta_2 + \cdots + c_q \zeta_q .$$

By Main lemma 2.3 we obtain

$$m(r, \zeta) \leq \sum_{i=1}^{q} m(r, \zeta_j) + O(1) = S_f(r) .$$

Since any pole of ζ is of order one and ζ has a pole at z only if $f(z)$ belongs to supp Σ, We have

$$N(r, \zeta) \leq \bar{N}_f(r, \Sigma) .$$

Hence we ha have

$$T(r, \zeta) \leq \bar{M}_f(r, \Sigma) + S_f(r) . \qquad \text{Q.E.D.}$$

REMARK. Using the same idea as in Noguchi [11: p. 224, Remark (1)], we obtain from Corollary 2.5 the classical Nevanlinna's second main theorem for meromorphic functions on $\Delta(R)$.

3. Proof of Thorem A.

We keep the same notation as in Theorem A. Let ζ_i be the meromorphic function on $\Delta(R)$ defined by $f^*\omega_i = \zeta_i\, dz$. Applying the classical Nevanlinna's lemma on logarithmic derivative to k-th derivative of ζ_i we have

$$\text{(3.1)} \qquad \begin{aligned} T(r, \zeta_i^{(k)}) \leq (k+1)\, T(r, \zeta_i) + O(\log^+ T(r, \zeta_i)) \\ + O(\log \frac{1}{R-r}) + O(1) \, \| . \end{aligned}$$

Moreover, combining Corollary 2.5 with the first main theorem and (1.1) we have

$$\begin{aligned} \log^+ T(r, \zeta_i) &\leq \log^+ \bar{N}_f(r, \Sigma) + \log^+ S_f(r) + O(1) \\ &\leq \log^+ T_f(r, [\Sigma]) + S_f(r) + O(1) \\ &\leq O(\log^+ T_f(r)) + S_f(r) + O(1) \\ &= O(\log^+ T_f(r)) + O(\log \frac{1}{R-r}) + O(1) . \end{aligned}$$

This inequality and (3.1) imply

$$T(r, \zeta_i^{(k)}) \leq (k+1)\, T(r, \zeta_i) + O(\log^+ T_f(r)) \\ + O(\log \frac{1}{R-r}) + O(1) \; || \, .$$

Hence by Coroallary 2.5

$$T(r, \zeta_i^{(k)}) \leq (k+1)\, \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) \\ + O(\log \frac{1}{R-r}) + O(1) \; || \, . \tag{3.2}$$

Let $\{\phi_j\}_{j=1,\cdots,l}$ be a system of generators of the rational function field $\mathfrak{R}(V)$ over $\boldsymbol{C}$ such that $f^*\phi_j$ are defined. Then by the Ochiai's theorrm (§1 (8)) there are algebraic relations

$$(f^*\phi_j)^{m_j} + R_{j1}(\zeta_i^{(k)})\, (f^*\phi_j)^{m_j-1} + \cdots + R_{jm_j}(\zeta_i^{(k)}) = 0 \tag{3.3}$$

$j=1, \cdots, l$ where $R_{j\nu}(\zeta_i^{(p)})$ are rational functions of $\zeta_i^{(k)}$ $k=0, \cdots, n-1, i=1, \cdots, n+1$. By making use of [10] we see that there is a positive constant K independent of r and f such that

$$T(r, f^*\phi_j) \leq K\, \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) \\ + O(\log \frac{1}{R-r}) + O(1) \; ||$$

for all j. Thus we have

$$\begin{aligned} \tilde{T}_f(r) &= \max_j T(r, f^*\phi_j) \\ &\leq K\, \bar{N}_f(r, \Sigma) + O(\log^+ T_f(r)) \\ &\quad + O(\log \frac{1}{R-r}) + O(1) \; || \, . \end{aligned} \tag{3.4}$$

Inequalities (1.2) and (3.4) yield the inequality (I). This completes the first part of Theorem A. By the assumption that f is of finite order, and by Lemma 2.3 we see that ζ_i is of finite order. Then by the classical Nevanlinna's lemma on logarithmic derivative, we have

$$T(r, \zeta_i^{(k)}) \leq (k+1)\, T(r, \zeta_i) + O(\log \frac{1}{R-r}) + O(1) \, .$$

And by the same arguments as in the first part of Theorem A we obtain

$$\tilde{T}_f(r) \leq K\, \bar{N}_f(r, \Sigma) + O(\log \frac{1}{R-r}) + O(1) \, . \tag{3.5}$$

Thus inequalities (1.2) and (3.5) yield the inequality (II). Q.E.D.

4. Proof of Theorem B and Corollary C

Proof of Theorem B. Without loss of generality we may assume $T_f(r) \to \infty$

as $r \to R$. We shall prove that f is of finite order. By the assumption, we have

$$N_f(r, \Sigma) = O\left(\left(\frac{1}{R-r}\right)^{\mu+1}\right),$$

and hence by Theorem A we get

$$\kappa T_f(r) \leq L_1 \left(\frac{1}{R-r}\right)^{\mu+1} + M_1 \log^+ T_f(r) + M_2 \log \frac{1}{R-r} + M_3 \,||\,,$$

where L_1, M_1, M_2 and M_3 are constants. On the other hand, there exists a constant δ (>0) such that

$$\kappa x - M_1 \log^+ x > \delta x$$

for sufficiently large x. Therefore

$$\delta\, T_f(r) \leq L_1 \left(\frac{1}{R-r}\right)^{\mu+1} + M_2 \log \frac{1}{R-r} + M_3$$

except the countable disjoint union of open intervals $E = \cup_j I_j$ such that

$$\int_E \frac{dr}{R-r} = M_4 < \infty\,.$$

We define r' as follow

$$r' = r \quad \text{if} \quad r \notin E$$
$$r' = b_j \quad \text{if} \quad r \in I_j = (a_j, b_j)\,.$$

Then we see that

$$\delta\, T_f(r) \leq \delta\, T_f(r') \leq L_1 \left(\frac{1}{R-r'}\right)^{\mu+1} + M_1 \log \frac{1}{R-r'} + M_3\,.$$

Moreover, since

$$M_4 \geq \int_r^{r'} \frac{dt}{R-t} = \log \frac{1}{R-r'} - \log \frac{1}{R-r},$$

we obtain

$$\frac{R-r}{R-r'} \leq \exp(M_4)\,.$$

Therefore we get

$$\delta\, T_f(r) \leq L_1 \exp((\mu+1) M_4) \left(\frac{1}{R-r}\right)^{\mu+1} + M_2 \log \frac{1}{R-r} + M_6\,,$$

where M_5 is a constant. Hence f is a map of finite order. Therefore it follows from Theorem A that

$$\kappa\, T_f(r) \leq \bar{N}_f(r, \Sigma) + \bar{M}_1 \log \frac{1}{R-r} + \bar{M}_2\,,$$

where $\bar{M}_1$, $\bar{M}_2$ are constants. Therefore

$$\kappa \int_1^R T_f(t)(R-t)^{\mu-1}\,dt \leq \int_1^R \bar{N}_f(t, \Sigma)(R-t)^{\mu-1}\,dt$$
$$+\bar{M}_1 \int_1^R \log \frac{1}{R-t}(R-t)^{\mu-1}\,dt + \bar{M}_2 \int_1^R (R-t)^{\mu-1}\,dt\,.$$

On the other, hand by the assumption

$$\int_1^R \bar{N}_f(t, \Sigma)(R-t)^{\mu-1}\,dt < \infty\,,$$

we obtain

$$\int_1^R T_f(t)(R-t)^{\mu-1}\,dt < \infty\,. \qquad \text{Q.E.D.}$$

Proof of Corollary C. This is an immediate consequence of Thoerem B and the following

Lemma 4.1. *Let μ be a positive number and D an effective divisor on V which satisfies $f(\Delta(R)) \not\subset \operatorname{supp} D$, Then for* $\operatorname{supp} f^*D = \{a_1, a_2, a_3, \cdots\}$,

$$\int_1^r \bar{N}_f(t, D)(R-t)^{\mu-1}\,dt$$
$$\int_1^r \bar{n}_f(t, D)(R-t)^{\mu}\,dt\,, \quad \text{and}$$
$$\sum_{1\leq|a_i|\leq r} (R-|a_i|)^{\mu+1}$$

are convergent or divergent at the same time as $r \to R$.

Proof. The same argument as in Shimizu [17] (p.107) can be applied for this case.

REMARK If $g(z) = \exp \dfrac{1}{1-z}$ then

$$\overline{\lim_{r\to 1}} \log T(r, g)/\log \frac{1}{1-r} = 0\,,$$

and

$$\overline{\lim_{r\to 1}} \log\log M(r, g)/\log \frac{1}{1-r} = 1\,,$$

where $M(r, g)=\operatorname*{Max}_{|z|=r} |g(z)|$.

5. Holomorphic maps of bounded characteristic functions

In this section we study holomorphic maps from the unit disk to an algebraic variety with $T_f(r)=O(1)$. Here we shall prove some analogous result to the classical Fatou's and Blaschke's theorems concerning bounded holomorphic functions on the unit disk.

Let f be a holomorphic Map from $\Delta(R)$ to a nonsinglar algebraic variety V and Ω a Kähler form on V.

Proposition 5.1. *Suppose that* $r\left(\frac{d}{dr}T_f\right)(r)=O(1)$ *as* $r\to R$. *Then the length of a curve of the image of* $\{te^{i\theta};\ 0<t<R\}$ *by* f *with respect to the Hermitian metric* h *determined by* Ω *is finite for almost all* θ, *i.e.*

$$\lim_{r\to R}\int_0^r (s(te^{i\theta}))^{1/2}\,dt \quad (s(z)\,dzd\bar{z}=f^*h)$$

is finite for almost all θ.

Proof. By Schwarz's inequality

$$\int_0^{2\pi}\left(\int_0^r (s(te^{i\theta}))^{1/2}\,dt\right)^2 d\theta\le\int_0^{2\pi}\left(r\int_0^r s(te^{i\theta})\,dt\right)d\theta\le R\int_0^{2\pi}\int_0^r s(te^{i\theta})\,dtd\theta$$
$$\le R\int_0^{2\pi}\int_0^r s(te^{i\theta})\,tdtd\theta+R\int_0^{2\pi}\int_0^1 s(te^{i\theta})\,dtd\theta$$
$$=r\left(\frac{d}{dr}T_f\right)(r)\cdot R+B\cdot R$$
$$\left(B=\int_1^{2\pi}\int_0^1 s(te^{i\theta})\,dtd\theta\right).$$

Hence we get using the assumption

$$\lim_{r\to R}\int_0^{2\pi}\left(\int_1^r (s(te^{i\theta}))^{1/2}\,dt\right)^2 d\theta<\infty\,.$$

Therefore we see

$$\int_0^{2\pi}\left(\lim_{r\to R}\int_0^r (s(te^{i\theta}))^{1/2}\,dt\right)^2 d\theta<\infty\,.$$

Thus

$$\lim_{r\to R}\int_0^r (s(te^{i\theta}))^{1/2}\,dt$$

are finite for almost all θ. Q.E.D.

Before showing a Blaschke-type theorem we need a lemma (cf.,e.g., Shimizu

[17] p.107).

Lemma 5.2. *Let D betan effective divisor on V with* $f(\Delta(R))\not\subset \mathrm{supp}\ D$. *Then for* $f^*D=a_1+a_2+\cdots$

$$N_f(r, D), \int_1^r n_f(t, D)\, dt, \quad and \quad \sum_{1\leq|a_i|\leq r}(R-|a_i|)$$

are convergent or divergent at the same time as $r\to R$.

Theorem 5.3. *Suppose that* $T_f(r)=O(1)$ *as* $r\to R$ *Then for any effective divisor D on V which satisfies* $f(\Delta(R))\not\subset \mathrm{supp}\ D$ *we have*

(5.3.1) $$N_f(r, D)=O(1) \quad (r\to R)$$

(5.3.2) $$n_f(r, D)=o\left(\frac{1}{R-r}\right)(r\to R)$$

(5.3.3) $$\sum_{|a_i|<R}(R-|a_i|)<\infty,$$

where $f^*D=a_1+a_2+\cdots$.

Proof. By the first main theorem and (1.1) we obtain (5.3.1). and from Lemma 5.2, we have

$$\int_1^R n_f(t, D)\, dt<\infty .$$

Since $n_f(t, D)$ is non-decreasing (5.3.2) holds. Moreover by Lemma 5.2 we obtain (5.3.3). Q.E.D.

Finally we shall given an example related to the above results. W. Rudin [15] gave an example of a holomorphic map from $\Delta(1)$ to $\Delta(1)$ with

$$\int_0^1 |f^{(1)}(re^{i\theta})|\, dr=\infty \qquad \text{for almost all } \theta .$$

this map is an example such that

$$\int_1^1 \frac{|f^{(1)}(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\, dr=\infty$$

for almost all θ. This map has the property $r\left(\frac{d}{dr}T_f\right)(r)\to\infty$ as $r\to 1$. But clearly $T_f(r)=O(1)$ as $r\to 1$.

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