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SEQUENCES OF DERIVABLE TRANSLATION PLANES

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1. Introduction

In [6], Hiramine, Matsumoto, and Oyama study translation planes of order q^4 and kernel $\supseteq K \cong GF(q^2)$ that admit an affine elation group E of order at least q^2 and a Baer group $\mathscr{B}$ of order $q+1$ such that $[E, \mathscr{B}] \neq 1$. It is shown in [6] that for q odd, such translation planes always have solvable translation complements. Further, a construction is given by which translation planes of the above type may be obtained from arbitrary translation planes of order h^2 and kernel $\supseteq GF(h)$.

In this article, we extend the construction method of Hiramine, Matsumoto, and Oyama and show how to obtain infinite sequences of potentially new derivable translation planes. This method allows the identification of certain other recently constructed translation planes.

In [2], Boerner-Lantz constructs a new class of semifield planes of order q^4. We show that these planes may be obtained by the construction methods under consideration from the Desarguesian planes. Furthermore, there are other similar but nonisomorphic semifield planes which may be constructed from Desarguesian planes.

We also complete the study of the groups of the translation planes of order q^4, kernel $GF(q^2)$ admitting elation and Baer groups as described above for q even.

2. The collineation groups

In this section, we extend the results of Hiramine, Matsumoto, and Oyama to include the even order case.

Theorem 2.1 (See Hiramine, et al. [6] for q odd.). *Let π denote a translation plane of order q^4 and kernel $\geq K \cong GF(q^2)$ which admits an elation group E of order $\geq q^2$ and a Baer group $\mathscr{B}$ of order $q+1$ such that $[\mathscr{B}, E] \neq 1$. Then the full collineation group is solvable.*

Proof. We give the proof as a series of lemmas.

Let the translation plane $\pi = \{(x_1, x_1, y_1, y_2) \mid x_i, y_i \in K, i = 1, 2,\}$. Choose the axis of E as $(x_1, x_2) = (0, 0)$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\mathcal{O} = (0, 0)$.

In [6], it is shown that the spread for π may be represented as follows:

$x = \mathcal{O}$, $y = x \begin{bmatrix} u & v \\ f(v) & u^q \end{bmatrix}$ where $u, v \in K$ and f is a function $K \to K$ such that $u^{q+1} \neq (v-s)(f(v)-f(s))$ for any u, v, s; $v \neq s$ in K. Further,

$$E \cong \left\{ \begin{bmatrix} I & \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix} \\ 0 & I \end{bmatrix} \;\middle|\; u \in K \right\} = E_0 \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 1 & & \\ & e & \\ & & 1 \end{bmatrix} \;\middle|\; e^{q+1} = 1, e \in K \right\}.$$

We assume that E is the full elation group with axis $\mathcal{L} \equiv (x = \mathcal{O})$. We first observe

Lemma 2.2. *$\mathcal{B}$ fixes each E orbit of components.*

Proof. $E_0 \trianglelefteq E$ so that each E orbit is a union of E_0 orbits. Let

$\tau_u = \begin{bmatrix} I & \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix} \\ 0 & I \end{bmatrix} \in E_0$. Then $y = x \begin{bmatrix} 0 & v \\ f(v) & 0 \end{bmatrix} \xrightarrow{\tau_u} y = x \begin{bmatrix} u & v \\ f(v) & u^q \end{bmatrix}$. Let $\sigma_e =$

$\begin{bmatrix} 1 & & \\ & e & \\ & & 1 \end{bmatrix} \in \mathcal{B}$. $y = x \begin{bmatrix} u & v \\ f(v) & u^q \end{bmatrix} \xrightarrow{\sigma_e} y = x \begin{bmatrix} 1 & & \\ & e^{-1} & \end{bmatrix} \begin{bmatrix} u & v \\ f(v) & u^q \end{bmatrix} \begin{bmatrix} e & \\ & 1 \end{bmatrix}$

$= \left[y = x \begin{bmatrix} ue & v \\ f(v) & u^q e^{-1} \end{bmatrix} = x \begin{bmatrix} ue & v \\ f(v) & (ue)^q \end{bmatrix} \right]$. Hence, $\mathcal{B}$ fixes each E_0 orbit and thus fixes each E orbit of components.

Let $\mathcal{F}$ denote the full translation complement of π.

Lemma 2.3. *The axis $\mathcal{L}$ of E is $\mathcal{F}$-invariant.*

Proof. If the axis is not $\mathcal{F}$-invariant then the elations generate $SL(2, q^2)$, $S_z(q)$ or $S_z(q^2)$. The proof of [3] shows that the group cannot be $S_z(\sqrt[4]{q^4})$. Furthermore, let $S_z(2^m)$ be a Suzuki group. Then $m \equiv 1 \bmod 2$ so that the group cannot be $S_z(\sqrt[2]{q^4})$. Hence, the elations generate $SL(2, q^2)$ which implies that the plane is Desarguesian (see [3]). However, as the group $\mathcal{B}$ is Baer, we obtain a contradiction.

We recall the following theorem of Jha-Johnson-Wilke [8].

Lemma 2.4. *Let Σ be a translation plane of even order m^2 where the kernel $\geq GF(m)$. Assume Σ admits a nonsolvable reducible group $\mathcal{G}$ in the linear translation complement.*

(0) *If the involutions of $\mathcal{G}$ are Baer then there is a derivable net $\mathfrak{N}$ which contains all of the Baer subplanes which are pointwise fixed by involutions in $\mathcal{G}$.*

(1) *If Σ admits affine elations with axes in $\mathfrak{N}$ then the elations leave $\mathfrak{N}$ invariant. Furthermore, any affine elation group must have order $<m$.*

Lemma 2.5. *If $\mathcal{F}$ is nonsolvable then $\mathcal{F}\,|\,\mathcal{L}$ is nonsolvable.*

Proof. Assume that $\mathcal{F}\,|\,\mathcal{L}$ is solvable. Let $\hat{\mathcal{F}}$ denote the subgroup of $\mathcal{F}$ which fixes $\mathcal{L}$ pointwise. So $E \trianglelefteq \hat{\mathcal{F}}$ where E is the full elation subgroup with axis $\mathcal{L}$ of order $\geq q^2$.

Let $|E| = q^2 \cdot 2^t$, $q = 2^r$ so that $q^2 \cdot 2^t = 2^{2r+t}$ for $0 \leq t \leq 2r$. There are $\dfrac{q^4}{q^2 \cdot 2^t} = 2^{4r-(2r+t)} = 2^{2r-t}$ E-orbits on the line at infinity $l_\infty - (\infty)$ where $\mathcal{L} \cap l_\infty = (\infty)$.

Since E is solvable and $\mathcal{F}\,|\,\mathcal{L}$ is assumed solvable, it must be that there exists a homology h in $\hat{\mathcal{F}}$. Let the cocenter Q of h be in the E-orbit Γ on $l_\infty - (\infty)$. Then by Andre's result [1], Γ is $\mathcal{F}$-invariant. And, $(\mathcal{F} \cap GL(4, q^2))_Q$ has *Baer* Sylow 2-subgroups. Note that $E \trianglelefteq \mathcal{F}$ since $\mathcal{L}$ is invariant. Hence, $(\mathcal{F} \cap GL(4, q^2)) = (\mathcal{F} \cap GL(4, q^2))_Q \cdot E$. Thus, $(\mathcal{F} \cap GL(4, q^2))_Q \cong \mathcal{F}/E$ so that $(\mathcal{F} \cap GL(4, q^2))_Q$ must be nonsolvable. However, this is a contradiction from (2.4) (1). This proves (2.5).

Lemma 2.6. *Let $\mathcal{F} \cap GL(4, q^2) = \mathcal{F}^G$. Let $\hat{\mathcal{F}} \trianglelefteq \mathcal{F}^G$ denote the subgroup which fixes $\mathcal{L}$ pointwise so that $\mathcal{F}^G\,|\,\mathcal{L} \cong \mathcal{F}^G/\hat{\mathcal{F}}$. We may assume that $SL(2, 2^s) \leq \mathcal{F}^G/\hat{\mathcal{F}}$ for some $s \geq 2$.*

Proof. $\mathcal{F}^G/\hat{\mathcal{F}}$ is a nonsolvable subgroup of $GL(2, q^2)$. Furthermore, $\mathcal{F}^G$ contains the kernel homology group K^* of order $q^2 - 1$. Hence, $\mathcal{F}^G/\hat{\mathcal{F}}$ contains the subgroup $K^*/\hat{\mathcal{F}} \cap K^* \cong K^*\hat{\mathcal{F}}/\hat{\mathcal{F}} \cong K^*$. Since q is even, $GL(2, q^2) = Z(GL(2, q^2)) \times SL(2, q^2)$ where Z denotes the center. However, $K^* \cong Z$.

Let $\mathcal{F}^G/\hat{\mathcal{F}} = \mathcal{H}$. Recall that Fix $\mathcal{B}$, $\mathcal{B}$ the Baer group of order $q+1$, must be a K-space by Foulser [5] and hence $\mathcal{B}$ must fix a 1-space over K on $\mathcal{L}$ pointwise. That is, considering $\mathcal{L}$ as the Desarguesian affine plane of order q^2 then $\mathcal{B}$ acts as a homology group of order $q+1$ in $GL(2, q^2)$.

Furthermore, $\mathcal{B}\hat{\mathcal{F}}/\hat{\mathcal{F}} \cong \mathcal{B} \leq \mathcal{H}$. So $\mathcal{H}$ is a nonsolvable subgroup of $GL(2, q^2)$ and $(q+1)(q^2-1)\,|\,|\mathcal{H}|$ (since $K^*\mathcal{B} \leq \mathcal{H}$).

Let $\mathcal{H} \cap SL(2, q^2) = \hat{\mathcal{H}}$. Then $\hat{\mathcal{H}}$ is a nonsolvable subgroup of $SL(2, q^2)$ and so is isomorphic to $SL(2, 2^s)$ for some s. This proves (2.6).

Lemma 2.7. *The subgroup $\hat{\mathcal{F}}$ which fixes $\mathcal{L}$ pointwise is E.*

Proof. If $\hat{\mathcal{F}}$ contains a homology then there is an $\mathcal{F}$-invariant E-orbit Γ of components by André's result [1]. However, if $Q \in \Gamma$ then $\mathcal{F}^G_Q E = \mathcal{F}^G$. However, the involutions in $\mathcal{F}^G_Q$ are all Baer. Since $\mathcal{F}^G_Q$ is nonsolvable, we obtain a contradiction by (2.4) (1).

Lemma 2.8. *Let M denote the preimage of $\hat{\mathscr{A}}$. Let N denote the kernel of the representation $\mathscr{F}^G$ induces on the set of E-orbits on the line at infinity $l_\infty-(\infty)$. Then $M(\hat{\mathscr{A}})$ acts faithfully or trivially on this set.*

Proof. $M/N \cong M/E \big/ N/E$ as $E \subseteq N$. Since $E = \hat{\mathscr{F}}$ then $M/E = \hat{\mathscr{A}} \cong SL(2, 2^s)$ for $s \geq 2$ by (2.6).

Hence N/E is $M/E = \hat{\mathscr{A}}$ or E. This proves (2.8).

We now complete the proof to (2.1).

By (2.2), $\mathscr{B}$ fixes each E-orbit of components and so does the kernel homology group K^*. Note that $\mathscr{B}|\mathscr{L} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \Big| |a| = q+1 \right\}$ and $K^*|\mathscr{L} = \left\{ \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \Big| b \right.$ $\left. \in GF(q^2) \right\}$. Then for $a = b^{-2}$, $\begin{bmatrix} 1 & 0 \\ 0 & b^{-2} \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \in \hat{\mathscr{A}}$. Since $|a| = |b^{-2}| = |b|$, $q+1 | |\hat{\mathscr{A}}|$ and it follows that $\hat{\mathscr{A}}$ cannot act faithfully on the set of E-orbits on $l_\infty - (\infty)$. So by (2.8), $\hat{\mathscr{A}}$ acts trivially on this set. Let Γ be any such E-orbit. Then for $Q \in \Gamma$, $M = M_Q E$ and $N/E = M_Q E/E \cong M_Q \cong \hat{\mathscr{A}} \cong SL(2, 2^s)$ for $s \geq 2$ by (2.6). However, this is a contradiction by (2.4) (1) since the involutions in M_Q are Baer and the plane of order q^4 admits an affine elation group of order $\geq q^2$.

3. A construction method

In this section, we extend slightly the procedures of Hiramine, Matsumoto, and Oyama who also characterize the planes considered in (2.1) in terms of matrix spread sets.

Choose the axis of E as $x = \mathcal{O}$ and the Baer subplane pointwise fixed by $\mathscr{B}$ as $\{(x_1, 0, 0, y_2) | x_1, y_2 \in K \cong GF(q^2)\}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $\pi = \{(x_1, x_2, y_1, y_2) | x_i, y_i \in K \cong GF(q^2), i = 1, 2\}$. Note that $\mathscr{B}$ must fix the axis of E in order that π be non Desarguesian and π cannot be Desarguesian as $\mathscr{B}$ is Baer of order $q+1$.

Furthermore, it is shown in [6] that the spread may be represented as follows:

$$x = 0, \quad y = x \begin{bmatrix} u, & v \\ f(v), & u^q \end{bmatrix},$$

where

$$E = \left\{ \begin{bmatrix} I & \begin{bmatrix} u & 0 \\ 0 & u^q \end{bmatrix} \\ \mathcal{O} & I \end{bmatrix} \Big| u \in K \cong GF(q^2) \right\}$$

and

$$\mathscr{B} = \left\{ \begin{bmatrix} 1 & & & \\ & e & & \\ & & e & \\ & & & 1 \end{bmatrix} \Big| e^{1+q} = 1, \ e \in K \right\},$$

where f is a function $K \to K$ such that $u^{q+1} \neq (v-s)(f(v)-f(s))$ for any $u, v, s, v \neq s$ in K. Hence, $(v-s)(f(v)-f(s)) \notin GF(q)$ for any $v, s, v \neq s$ of K.

Now suppose that $t \in GF(q^2) - GF(q)$ where $t^2 = t\theta + \rho$, θ, ρ constants in $GF(q)$. Then write

$$f(\alpha + \beta t) = g(\alpha, \beta) - h(\alpha, \beta)t$$

for functions $g, h: GF(q) \times GF(q) \to GF(q)$. Then $(v-s)(f(v)-f(s)) \in GF(q)$

$$\Leftrightarrow ((\alpha + \beta t) - (\delta + \gamma t))((g(\alpha, \beta) - g(\delta, \gamma))$$
$$- (h(\alpha, \beta) - h(\delta, \gamma))t) \in GF(q)$$

$\Leftrightarrow t$-component in the previous equation is 0

$$\Leftrightarrow (\beta - \gamma)(g(\alpha, \beta) - g(\delta, \gamma)) - (\alpha - \delta)(h(\alpha, \beta) - h(\delta, \gamma))$$
$$- (\beta - \gamma)(h(\alpha, \beta) - h(\delta, \gamma))\theta = 0$$

$$\Leftrightarrow (\beta - \gamma)[(g(\alpha, \beta) - g(\delta, \gamma)) - \theta(h(\alpha, \beta) - h(\delta, \gamma))]$$
$$- (\alpha - \delta)(h(\alpha, \beta) - h(\delta, \gamma)) = 0$$

$$\Leftrightarrow (\beta - \gamma)[(g(\alpha, \beta) - \theta h(\alpha, \beta) - (g(\delta, \gamma) - \theta h(\delta, \gamma))]$$
$$- (\alpha - \delta)(h(\alpha, \beta) - h(\delta, \gamma) = 0$$

$$\Leftrightarrow \det \left[\begin{bmatrix} \alpha & \beta \\ g(\alpha, \beta) - \theta h(\alpha, \beta) & h(\alpha, \beta) \end{bmatrix} - \begin{bmatrix} \delta & \gamma \\ g(\delta, \gamma) - \theta h(\delta, \gamma) & h(\delta, \gamma) \end{bmatrix} \right]$$
$$= 0.$$

Hence, we have (see also H-M-0 [6]):

Theorem 3.1.

$$y = x \begin{bmatrix} u, & v \\ f(v), & u^q \end{bmatrix}, \quad x = \mathcal{O}, \quad u, v, \in GF(q^2)$$

represents a translation plane π of order q^4 kernel $GF(q^2)$ if and only if

$$y = x \begin{bmatrix} \alpha, & \beta \\ g(\alpha, \beta) - \theta h(\alpha, \beta), & h(\alpha, \beta) \end{bmatrix}, \quad x = \mathcal{O}$$

for $\alpha, \beta \in GF(q)$ represents a translation plane π^c of order q^2 and kernel $GF(q)$ where f, g, h are functions interconnected by $f(\alpha + \beta t) = g(\alpha, \beta) - h(\alpha, \beta)t$, $f: GF(q^2) \to GF(q^2)$, $g, h: GF(q) \times GF(q) \to GF(q)$ and $GF(q^2) = GF(q)[t]$ where $t^2 = t\theta + \rho$ for $\theta, \rho \in GF(q)$.

The plane π^c will be called a *contraction* of π and π an *extension* of π^c.

4. Examples

EXAMPLE 4.1. Let π_1 be the translation plane of odd order q^2 and kernel

$GF(q)$ given by $x=\mathcal{O}$, $y=x\begin{bmatrix} \alpha, & \beta \\ \gamma\beta^\rho, & \alpha^\sigma \end{bmatrix}$ for $\alpha, \beta\in GF(q)$, $\sigma, \rho\in\text{Aut } GF(q)$ and γ a nonsquare in $GF(q)$.

Let $GF(q)[t]\cong GF(q^2)$ for $t^2=t\theta+\rho$, $\theta, \rho\in GF(q)$. Construct the translation plane π_1^t as in (3.1) of order q^4 and kernel $GF(q^2)$: $x=\mathcal{O}$, $y=x\begin{bmatrix} u, & v \\ f(v), & u^q \end{bmatrix}$ where

$$f(v) = f(\alpha+\beta t) = g(\alpha, \beta)-h(\alpha, \beta)t$$

and

$$\gamma\beta^\rho = g(\alpha, \beta)-\theta h(\alpha, \beta), \quad \alpha^\sigma = h(\alpha, \beta).$$

Hence, $\gamma\beta^\rho+\theta\alpha^\sigma=g(\alpha, \beta)$. So $f(\alpha+\beta t)=(\gamma\beta^\rho+\theta\alpha^\sigma)-\alpha^\sigma t$ and π_1^t is $x=\mathcal{O}$,

$$y=x\begin{bmatrix} \delta+\gamma t, & \alpha+\beta t \\ \left\{\begin{matrix}\gamma\beta^\rho+\theta\alpha^\sigma\\ -\alpha^\sigma t\end{matrix}\right\}, & (\delta+\gamma t)^q \end{bmatrix}$$ for all $\delta, \gamma, \alpha, \beta\in GF(q)$, θ a constant.

If s is chosen so that $s^2\in GF(q)$ then a plane π_1^s may be similarly constructed of order q^4 with matrix spread set $x=\mathcal{O}$, $y=x\begin{bmatrix} \delta+\gamma s, & \alpha+\beta s \\ \gamma\beta^\rho-\alpha^\sigma s, & (\delta+\gamma s)^q \end{bmatrix}$. It is clearly possible that π_1^s is not isomorphic to π_1^t.

Reversing the technique, consider a plane defined by

$$x = \mathcal{O}, \quad y = x\begin{bmatrix} u, & s \\ \gamma s^\rho, & u^q \end{bmatrix}$$

of order q^4, kernel $GF(q^2)$ for $u, s\in GF(q^2)$, $\rho\in\text{Aut } GF(q^2)$, γ a nonsquare in $GF(q^2)$. Then assume that $GF(q^2)=GF(q)[t]$ for $t^2=t\theta+\chi$ for $\theta, \chi\in GF(q)$. Then $f(s)=\gamma s^\rho$. Writing $s=\alpha+\beta t$ then $f(\alpha+\beta t)=\gamma(\alpha+\beta t)^\rho=\gamma(\alpha^\rho+\beta^\rho t^\rho)$. Letting $t^\rho=t\delta_1+\delta_2$ for $\delta_1, \delta_2\in GF(q)$, we obtain:

$$\begin{aligned} f(\alpha+\beta t) &= g(\alpha, \beta)-h(\alpha, \beta)t \\ &= (\gamma(\alpha^\rho+\beta^\rho\delta_2))+(\gamma\beta^\rho\delta_1)t. \end{aligned}$$

We want to obtain the plane of order q^2,

$$x = \mathcal{O}, \quad y = x\begin{bmatrix} \alpha, & \beta \\ g(\alpha, \beta)-\theta h(\alpha, \beta), & h(\alpha, \beta) \end{bmatrix}.$$

However γ a nonsquare cannot be in $GF(q)$ so $\gamma=t\gamma_1+\gamma_2$ for $\gamma_1\neq0$. Then

$$\begin{aligned} g(\alpha, \beta)-h(\alpha, \beta)t &= (t\gamma_1+\gamma_2)(\alpha^\rho+\beta^\rho\delta_2)+(t\gamma_1+\gamma_2)(\beta^\rho\delta_1 t) \\ &= (\gamma_2(\alpha^\rho+\beta^\rho\delta_2)+\chi\gamma_1\beta^\rho\delta_1) \\ &\quad +(\gamma_1(\alpha^\rho+\beta^\rho\delta_2)+\gamma_2\beta^\rho\delta_1+\theta\gamma_1\beta^\rho\delta_1)t. \end{aligned}$$

So

$$g(\alpha, \beta) = \gamma_2(\alpha^\rho + \beta^\rho \delta_2) + \mathcal{X}\gamma_1\beta^\rho\delta_1,$$
$$h(\alpha, \beta) = -[(\gamma_1(\alpha^\rho + \beta^\rho \delta_2) + \gamma_2\beta^\rho\delta_1 + \theta\gamma_1\beta^\rho\delta_1)].$$

So the contraction plane has the form $x = \mathcal{O}$,

$$y = x\begin{bmatrix} \alpha & , & \beta \\ c_1\alpha^\rho + c_2\beta^\rho, & d_1\alpha^\rho + d_2\beta^\rho \end{bmatrix}$$

for certain constants c_1, c_2, d_1, d_2.

4.2. The semifield of Boerner-Lantz.

In [2], Boerner-Lantz constructs a class of semifields as follows:

Theorem ((3.2) [2], p. 115). Let $q = p^r$ with $p > 3$. Choose $\sigma \in GF(q)$ such that $x^2 - \sigma$ is irreducible over $GF(q)$ and $1 + 4\sigma$ is a nonsquare. Identify $GF(q^2)$ as $\{\alpha_1 + \alpha_2 a \mid \alpha_1, \alpha_2 \in GF(q)\}$, where $a \notin GF(q)$ is a root of $x^2 = \sigma$, and S as $\{\alpha + \beta s \mid \alpha, \beta \in GF(q^2)\}$ where $s \notin GF(q^2)$. Define addition $+$ on S to be the usual vector addition. If multiplication is defined on S by

$$(\alpha + \beta s) \cdot (\gamma + \delta s) = \alpha\gamma + \beta(\delta^q a - \delta_1) + (\alpha\delta + \beta\gamma^q)s$$

where $\delta = \delta_1 + \delta_2 a$; δ_1, $\delta_2 \in GF(q)$, then the triple $(S, +, \cdot)$ is a semifield of dimension two over $GF(q^2)$.

In terms of a matrix spread set, the spread for the associated semifield plane π of order q^4 and kernel $GF(q^2)$ is easily seen to be:

$$x = \mathcal{O}, \quad y = x\begin{bmatrix} \gamma & , & \delta \\ \delta^q a - \delta_1, & \gamma^q \end{bmatrix}$$

where $\delta = \delta_1 + \delta_2 a$ taking $x_1 + x_2 s = (x_1, x_2) = x$. Note that $a^q = -a$ so that

$$\delta^q a - \delta_1 = (\delta_1 - \delta_2 a)a - \delta_1 = -(\delta_1 + \delta_2\sigma) + \delta_1 a$$

so the matrix forms become

$$y = x\begin{bmatrix} \gamma & , & \delta = \delta_1 + \delta_2 a \\ -(\delta_1 + \delta_2\sigma) + \delta_1 a, & \gamma^q \end{bmatrix}.$$

If we take, in the notation of (3.1), $t = a$ then $f(\delta_1 + \delta_2 a) = -(\delta_1 + \delta_2\sigma) + \delta_1 a$ and $g(\delta_1, \delta_2) = -(\delta_1 + \delta_2\sigma)$ so $h(\delta_1, \delta_2) = -\delta_1$. The contraction plane π^c of order q^2 and kernel $GF(q)$ is

$$x = \mathcal{O}, \quad y = x\begin{bmatrix} \delta_1 & , & \delta_2 \\ -(\delta_1 + \delta_2\sigma), & -\delta_1 \end{bmatrix}$$

for all δ_1, $\delta_2 \in GF(q)$. Now we change bases in π^c as follows: First change bases by

$$\begin{bmatrix} I & \\ & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix},$$

then by

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} & \\ & I \end{bmatrix},$$

and finally by

$$\begin{bmatrix} I & \\ & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}$$

to obtain the spread

$$x = \mathcal{O}, \quad y = x \begin{bmatrix} \delta_1, & \delta_2 \\ \delta_2\sigma, & \delta_1 - \delta_2 \end{bmatrix}.$$

Since $x^2 - x - \sigma$ is irreducible (see [2], p. 115), we clearly obtain a field of matrices so that π is Desarguesian. That is,

Theorem 4.3. *The semifield planes of Boerner-Lantz may be obtained from the Desarguesian planes by the extension procedure given in* (3.1).

We now consider semifield planes obtained more or less *directly* from the Desarguesian planes using (3.1).

Lemma 4.4 (see [7], (3.2)). *If $x^2 + fx - g$ is irreducible over $P \cong GF(q)$ for fixed f, g in P then*

$$\left\{ \begin{bmatrix} \alpha, & \beta \\ \beta g, & \alpha + \beta f \end{bmatrix} \middle| \alpha, \beta \in P \right\}$$

is a field of order q^2.

Now let π_1 be a Desarguesian affine plane with points $\{(x_1, x_2, y_1, y_2) \mid x_i, y_i \in P, i = 1, 2\}$ and spread set $y = x \begin{bmatrix} \alpha, & \beta \\ \beta g, & \alpha + \beta f \end{bmatrix}$, $x = \mathcal{O}$ for $y = (y_1, y_2)$, $x = (x_1, x_2)$ and $x^2 + fx - g$ irreducible over P.

We may use the construction method of (3.1) directly from this spread set to obtain the following semifield plane: Let $GF(q^2) = P[t]$ for $t^2 = t\theta + \rho$ for fixed $\theta, \rho \in P$. Letting $h(\alpha, \beta) = \alpha + \beta f$, $g(\alpha, \beta) - \theta h(\alpha, \beta) = \beta g$, then

$$g(\alpha, \beta) = \beta g + \theta(\alpha + \beta f) = \theta \alpha + \beta(g + f\theta) .$$

Hence,

$$f(\alpha + \beta t) = g(\alpha, \beta) - h(\alpha, \beta)t$$

so that

$$f(\alpha+\beta t) = \theta\alpha+\beta(g+f\theta)-(\alpha+\beta f)t .$$

4.5.   The semifield plane of order q^4 and kernel $GF(q^2)$ has the following form:

$$x = \mathcal{O}, \left[y = x\begin{bmatrix} u & , & v \\ f(v), & u^q \end{bmatrix}\right] \equiv \left[y = x\begin{bmatrix} \alpha+\beta t & , & \delta+\gamma t \\ \theta\delta+\gamma(g+f\theta)-(\delta+\gamma f)t, & (\alpha+\beta t)^q \end{bmatrix}\right].$$

It is probable that this class of semifield planes is not isomorphic to the class of (4.2) even though they are constructed from the same Desarguesian plane. Notice that this also shows that a basis change in a base plane may lead to distinct planes in this extension process.

4.6.   Other examples—the H-M-O example.

In Hiramine, et al., an example which is potentially new, of a translation plane of order q^4 and kernel $GF(q^2)$ is given which admits collineation groups of the type we are considering.

We note here that this example may be constructed from the regular nearfield plane (see [6], Example (2.5)): Assume p is a prime >2 and $e \in GF(q^2)-GF(q)$ such that $e^2 \in GF(q)$. Define $f(v)=e^{v\left[\frac{q^2+2q-1}{2}\right]}$, then $x=\mathcal{O}$, $y=x\begin{bmatrix} u & , & v \\ f(v), & u^q \end{bmatrix}$ for all u, $v \in GF(q^2)$ defines a "new" translation plane of order q^4 and kernel $GF(q^2)$.

Note that $f(v)=\begin{cases} ev^q; v \text{ a square} \\ -ev^q; v \text{ a nonsquare} \end{cases}$. Now represent $v=\alpha+\beta e$ for $\alpha, \beta \in GF(q)$ and note that $(\alpha+\beta e)^q=\alpha-\beta e$. Hence, if $f(\alpha+\beta e)=g(\alpha, \beta)-h(\alpha, \beta)e$ for $e^2=e\theta+\rho$, we have $\theta=0$ and

$$e(\alpha+\beta e)^q = e(\alpha-\beta e) = -\beta\rho+\alpha e$$

for $\alpha+\beta e$ a square and,

$$-e(\alpha+\beta e)^q = -e(\alpha-\beta e) = \beta\rho-\alpha e$$

for $\alpha+\beta e$ a nonsquare.   Thus,

$$f(\alpha+\beta e) = \begin{cases} -\beta\rho+\alpha e, \text{ for } \alpha+\beta e \text{ a square} \\ \beta\rho-\alpha e, \text{ for } \alpha+\beta e \text{ a nonsquare} \end{cases}.$$

The contraction to the translation plane of order q^2 and kernel $GF(q)$ gives $g(\alpha, \beta)=\pm\beta\rho$ where $\pm$ is $+$ or $-$ depending on where $\alpha+\beta e$ is a nonsquare or square and $h(\alpha, \beta)=\pm\alpha$.   The contraction plane is given by the spread:

$$x = \mathcal{O}, y = x\begin{bmatrix} \alpha & , & \beta \\ \pm\beta\rho, & \pm\alpha \end{bmatrix}$$

where $x^2-\rho$ is irreducible.

To see that this is the regular nearfield plane, we start from the André construction.

Consider $(GF(q^2), +, \cdot)$, q odd, and define a multiplication "$\circ$" as follows: $x \circ y = x^q \cdot y$ if y^{1+q} is a nonsquare and $x \circ y = x \cdot y$ if y^{1+q} is a square for all $x, y \in GF(q^2)$. Then it follows that $(GF(q^2), +, \circ)$ is a regular nearfield of order q^2 and kernel $GF(q)$. Let $t \in GF(q^2) - GF(q)$ so that $t^2 = \gamma$ is a nonsquare in $GF(q)$. $x^q = (tx_1 + x_2)^q = -tx_1 + x_2$ for $x_1, x_2 \in GF(q)$. Hence, $x \circ y = (-tx_1 + x_2) \cdot (ty_1 + y_2)$ for $ty_1 + y_2$ a nonsquare. Thus, $x \circ y = t(x_2 y_1 - x_1 y_2) - \gamma x_1 y_1 + x_2 y_2$ for $t_1 y + y_2$ a nonsquare and

$$= t(x_2 y_1 + x_1 y_2) + \gamma x_1 y_1 + x_2 y_2$$

for $ty_1 + y_2$ a square.

The matrix spread set is, for $x = tx_1 + x_2 = (x_1, x_2)$,

$$x = \mathcal{O}, \left[y = x \begin{bmatrix} -y_2, & -\gamma y_1 \\ y_1, & y_2 \end{bmatrix} \right]$$

for $ty_1 + y_2$ a nonsquare and

$$\left[y = x \begin{bmatrix} y_2, & \gamma y_1 \\ -y_1, & y_2 \end{bmatrix} \right]$$

for $ty_1 + y_2$ a square. It is now clear that by making the proper identifications the Hiramine, Matsumoto, Oyama plane of order q^4 may be constructed (extended) from the regular nearfield plane of order q^2. However, the nearfield plane can conceivably construct (extendt $\circ$) a number of distinct nonisomorphic planes.

4.7. Non semifield planes from semifield planes.

By looking at the form of (3.1), it follows that $g(\alpha, \beta)$ and $h(\alpha, \beta)$ are (bi)additive if and only if $f(\alpha + \beta t)$ is also additive.

Thus, starting from a non Desarguesian semifield plane of order q^2 and kernel $GF(q)$ then, by choosing $x = \mathcal{O}$ to represent the shears axis, the extension plane of order q^4 is also a semifield plane. However, by choosing a non-shears axis to represent $x = \mathcal{O}$, the associated functions $g(\alpha, \beta)$, $h(\alpha, \beta)$ will not be additive so that the matrices of the plane of order q^4 will not be additive. Since this plane also admits a shears axis with group of order q^2 in this representation, it follows that we now do not have a semifield plane.

An example will suffice to illustrate this idea.

Consider the semifield plane of odd order q^2 with spread:

$$x = \mathcal{O}, \quad y = x \begin{bmatrix} \alpha, & \beta \\ \gamma \beta^w, & \alpha^\sigma \end{bmatrix}$$

for γ a nonsquare in $GF(q)$, w, $\sigma \in \mathrm{Aut}\ GF(q)$ (see (4.1)). First let $\alpha \equiv \rho^{\sigma^{-1}}$ so $\alpha^\sigma = \rho$ and $\beta = -\mathcal{X}$ so that the spread has the form

$$x = \mathcal{O},\ y = x \begin{bmatrix} \rho^{\sigma^{-1}}, & -\mathcal{X} \\ -\gamma \mathcal{X}^w, & \rho \end{bmatrix}.$$

Change bases by $(x, y) \leftrightarrow (y, x)$ to obtain the spread in the form

$$x = \mathcal{O},\ y = x \begin{bmatrix} \rho^{\sigma^{-1}}, & -\mathcal{X} \\ -\gamma \mathcal{X}^w, & \rho \end{bmatrix}^{-1} \equiv \left(y = x \begin{pmatrix} \dfrac{\rho}{\Delta \rho,\ \gamma}, & \dfrac{\mathcal{X}}{\Delta \rho,\ \gamma} \\ \dfrac{\gamma \mathcal{X}^w}{\Delta \rho,\ \gamma}, & \dfrac{\rho^{\sigma^{-1}}}{\Delta \rho,\ \gamma} \end{pmatrix} \right)$$

where

$$\Delta_{\rho, \gamma} = \mathrm{Det}\begin{bmatrix} \rho^{\sigma^{-1}}, & -\mathcal{X} \\ -\gamma \mathcal{X}^w, & \rho \end{bmatrix} = \rho^{\sigma^{-1}+1} - \gamma \mathcal{X}^{w+1}.$$

We may now follow the example of (4.1) to obtain a "new" non semifield translation plane.

5. Sequences of derivable planes

Let π_1 be any translation plane of order q^2 and kernel $GF(q)$. Write π_1 in the form $x = \mathcal{O}$, $y = x \begin{bmatrix} \alpha, & \beta \\ g_1(\alpha, \beta), & h(\alpha, \beta) \end{bmatrix}$. Extend $GF(q)$ to $GF(q^2)$ writing $GF(q^2) \simeq GF(q)[t]$. If $t^2 = t\theta + \mathcal{X}$, let $g(\alpha, \beta) = g_1(\alpha, \beta) - \theta h(\alpha, \beta)$. Construct the plane π_1^t of order q^4 and kernel $GF(q^2)$: $x = \mathcal{O}$,

$$y = x \begin{bmatrix} u, & v \\ f(v), & u^q \end{bmatrix}$$

where $f(v) = f(\alpha + \beta t) = g(\alpha, \beta) - h(\alpha, \beta)t$. Now extend $GF(q^2)$ to $GF(q^4)$ by writing $GF(q^4) = GF(q^2)[t_1]$ for $t_1^2 = t_1 \theta_1 + \mathcal{X}_1$.

Let $f_1(v) = f(v) + \theta_1 u^q$ so that the components of the plane of order q^4 have the form $x = \mathcal{O}$, $y = x \begin{bmatrix} u, & v \\ f_1(v) - \theta_1 u^q, & u^q \end{bmatrix}$. Construct the plane $(\pi_1^t)^{t_1}$ of order q^8 and kernel $GF(q^4)$ with components $\begin{bmatrix} u, & m \\ f_2(m), & u^q \end{bmatrix}$ where $f_2(m) = f_2(u + vt_1)$ $= (f_1(v) - u^q t_1)$. In this way, we obtain a sequence of derivable translation planes from any translation plane of order q^2 and kernel $GF(q)$.

$$\begin{array}{ccccccc} \pi_1 & \to & \pi_1^t & \to & (\pi_1^t)^{t_1} & \to & ((\pi_1^t)^{t_1})^{t_2} \to \cdots \\ \text{order } q^2 & & \text{order } q^4 & & \text{order } q^8 & & \text{order } q^{16} \end{array}$$

Clearly, there are many other ways to vary the sequence. The isomorphism questions are completely open.

The preceding shows that there are a vast number of new translation planes of order q^4 and kernel $GF(q^2)$ which may be constructed by (3.1) from translation planes of order q^2 and kernel $GF(q)$. The point of view might be therefore to identify the planes obtained from one another by (3.1).

Also, each of the constructed planes of order q^4 is derivable where the derivable net is not a regulus over the field in question. Note the derivable net is $x=\mathcal{O}$, $y=x\begin{bmatrix} v, & 0 \\ 0, & v^q \end{bmatrix}\forall v \in GF(q^2)$.

The translation plane derived from replacing this net has order q^4 and kernel $GF(q)$.

References

[1] T. André: *Über Perspektivitäten in endlichen projektiven Ebenen,* Arch. Math. **6** (1954), 29–32.

[2] V. Boerner-Lantz: *A class of semifields of order q^4,* J. Geom. **27** (1986), 112–118.

[3] J. Fink, N.L. Johnson and F.W. Wilke: *Transitive affine planes admitting elations,* J. Geom. **21** (1983), 59–65.

[4] D. Foulser: *Baer p-elements in translation planes,* J. Algebra **31** (1974), 354–366.

[5] D. Foulser: *Subplanes of partial spreads in translation planes,* Bull. London Math. Soc. **4** (1972), 32–38.

[6] Y. Hiramine, M. Matsumoto and T. Oyama: *On some extension of 1 spread sets,* Osaka J. Math. **24** (1987), 123–137.

[7] N.L. Johnson: *The translation planes of Dempwolff,* Canad. J. Math. **33** (1981), 1060–1073.

[8] V. Jha, N.L. Johnson and F.W. Wilke: *Translation planes of order q^2 that admit a group of order $q^2(q-1)$; Bartolone's Theorem,* Rend. Circ. Mat. Palermo **33** (1984), 407–424.

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