Higuchi, Y.
Osaka J. Math.
5 (1968), 103-135

A PRIORI ESTIMATES AND EXISTENCE THEOREM ON ELLIPTIC BOUNDARY VALUE PROBLEMS FOR UNBOUNDED DOMAINS

Yoshiki HIGUCHI

(Received February 7, 1968)

1. Introduction

Elliptic boundary value problems for bounded domains have been studied by many authors ([2], [7], [8], [9], etc.). A powerful tool was the so-called **coercive inequalities** introduced by N. Aronszajn. Under the Lopatinski condition, Schechter obtained L_2-estimates of solutions of general elliptic boundary value problems by using Fourier transforms. Agmon, Douglis and Nirenberg got the Schauder estimates and L_p-estimates by means of singular integral operators. We should mention among these, the work of Agranovich and Vishik [2], which treated a vast class of elliptic boundary value problems. However there are few literatures in the case of unbounded domains. Among these, we mention Browder's paper [4], in which he introduced the concept of uniformly regular domains and investigated Dirichlet problems for such domains. In this paper, extending Aronszajn's coercive inequalities to unbounded domains, we study general elliptic boundary value problems for fairly general unbounded domains.

Let Ω be an open set of n-dimensional Euclidean space E^n and let us denote its closure and boundary by $\bar{\Omega}$ and Γ respectively. Let us consider a linear partial differential operator of order $2m$:

$$A=\sum_{|\mu|\leq 2m} a_\mu(x)D^\mu ,$$

where the coefficients $a_\mu(x)$ are complex-valued functions defined in $\bar{\Omega}$ and a system of linear boundary operators of order less than $2m$:

$$B_j=\sum_{|\mu|\leq m_j} b_{j\mu}(x)D^\mu \qquad j=1, 2, \cdots, m ,$$

where the coefficients $b_{j\mu}(x)$ are complex-valued functions defined in some neighborhood of Γ.

We make the following assumptions on Ω and $(A, \{B_j\}_{j=1}^m)$ (precise definitions and notations will be given in §2).

(A.1) Ω is uniformly regular of class C^{2m+s+1}.

(A.2) There exist constants θ $(0\leqslant\theta<2\pi)$ and $\gamma_1>0$ (independent of x, ξ, and λ) such that

$$|P(x, \xi)-\lambda^{2m}|\geqslant\gamma_1(|\xi|^2+|\lambda|^2)^m$$

where $\lambda=re^{i(\theta/2m)}$, and $P(x, \xi)$ is the characteristic polynomial of A.

Theroughout this paper λ will mean complex numbers with argument $\theta/2m$

(A.3) For every real, tangential vector τ at $x\in\Gamma$ and the unit inner normal vector ν_0 at $x\in\Gamma$, the polynomial $P(x, \tau+z\nu_0)-\lambda^{2m}$ in z has exactly m roots $z_k^+(x, \tau, \lambda)$ $(k=1, 2, \cdots, m)$ with positive imaginary part for $|\lambda|+|\tau|\neq 0$.

(A.4) Set $P_+(z)=\prod_{k=1}^{m}(z-z_k^+(x, \tau, \lambda))$ and

$$L_{jk}(x, \tau, \lambda)=\frac{1}{2\pi i}\oint\frac{Q_j(x, \tau+z\nu_0)}{P_+(z)}z^{k-1}dz,$$

where $Q_j(x, \xi)$ is the characteristic polynomial of B_j and the integration is taken along a closed curve in the complex z-plane enclosing all the roots $z_k^+(x, \tau, \lambda)$ $(k=1, 2, \cdots, m)$. We assume that there exists a constant $\gamma_2>0$ (independent of x, τ and λ) such that

$$|\det(L_{jk}(x, \tau, \lambda))|\geqslant\gamma_2(|\tau|^2+|\lambda|^2)^{N/2},$$

where $N=\sum_{j=1}^{m}(m_j-j+1)$.

(A.5) $\{B_j\}_{j=1}^m$ forms a normal system, that is,

(i) $m_j\neq m_k$ if $j\neq k$, $m_j<2m$.

(ii) $Q_j(x, \nu_0)\neq 0$ for $x\in\Gamma$.

(A.6) (Assumptions on the smoothness of coefficients)

(i) $a_\mu(x)\in\mathscr{B}^s(\Omega)$. In addition, if $s=0$, the coefficients $a_\mu(x)$ for $|\mu|=2m$ are uniformly continuous in $\bar{\Omega}$.

(ii) $b_{j\mu}(x)\in\mathscr{B}^{2m+s-m_j}(\Gamma)$.

Then we have

Theorem 1. *Under the above assumptions* $(A.1)$–$(A.6)$, *there exist constants* c *and* $r_1>0$ *such that for* $u\in H^{2m+s}(\Omega)$,

$$(1.1)\qquad |u|_{2m+s}^2+|\lambda|^{2(2m+s)}|u|_0^2\leqslant c[|(A-\lambda^{2m})u|_s^2+|\lambda|^{2s}|(A-\lambda^{2m})u|_0^2 +\sum_{j=1}^{m}(\langle B_ju\rangle_{2m+s-m_j-(1/2)}^2+|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_ju\rangle_0)^2]$$

if $|\lambda|\geqslant r_1$, *where* $\lambda=re^{i(\theta/2m)}$.

REMARK 1.1. Theorem 1 is equivalent to the following

Theorem 1′. *Under the above assumptions* $(A.\ 1)$–$(A.\ 6)$, *there esist constants* c' *and* $r_1'>0$ *such that for* $u\in H^{2m+s}(\Omega)$,

$$(1.1)' \quad \|u\|_{2m+s}^2+|\lambda|^{2(2m+s)}\|u\|_0^2\leqslant c[\|(A-\lambda^{2m})u\|_s^2+\|(A-\lambda^{2m})u\|_0^2 + \sum_{j=1}^{m}(\langle B_ju\rangle_{2m+s-m_j-(1/2)}^2+|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_ju\rangle_0^2)],$$

if $|\lambda|\geqslant r_1'$.

Next we have to make stronger assumptions to obtain the existence theorem, that is,

(A*. 1) Ω is uniformly regular of class C^{6m+s+1}.

(A*.2), (A*.3), (A*.4) and (A*.5) are identical with (A.2), (A.3), (A.4) and (A.5) respectively (remark that (A.4) and (A.5) imply that there exists a constant $\gamma_3>0$ such that

$$|Q_j(x,\nu)|\geqslant\gamma_3|\nu|^{m_j}, \qquad j=1,2,\cdots,m,$$

where $x\in\Gamma$ and ν is any normal vector at x).

(A*.6) (Assumptions on the smoothness of coefficients)
 (i) $a_\mu(x)\in\mathscr{B}^{4m+s}(\Omega)$.
 (ii) $b_{j\mu}(x)\in\mathscr{B}^{6m+s}(\Gamma)$.
 Then we have

Theorem 2. *Under the above assumptions, let* $|\lambda|\geqslant r_0$. *Then, for any* $f(x)\in H^s(\Omega)$, *there exists a unique solution* $u(x)\in H^{2m+s}(\Omega)$ *satisfying*

$$(A-\lambda^{2m})u=f \text{ in } \Omega,$$
$$B_ju=0 \text{ on } \Gamma, \qquad j=1,2,\cdots,m.$$

Theorem 1 will be proved in §6 and Theorem 2 in §8. Notations and precise definitions of terminologies mentioned above will be given in §2. In §3 we give the definition of the boundary norms. §§4–5 are devoted to the study of linear partial differential operators in a half space. We consider the adjoint boundary system in §7.

Finally, I would like to express my gratitude to Professor S. Mizohata for various suggestions and corrections. I would also like to thank Mr. K. Asano for helpful discussions.

2. Notations and definitions

Let Ω be an open set in E^n and let us denote its closure and boundary by $\bar{\Omega}$ and Γ respectively. Let $x=(x_1,x_2,\cdots,x_n)$ be a point of E^n, $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ be any real vector and $\mu=(\mu_1,\mu_2,\cdots,\mu_n)$ be any n-tuple of non-negative integers. We shall use the following notations:

$$|\mu| = \mu_1 + \mu_2 + \cdots + \mu_n ,$$
$$\xi^\mu = \xi_1^{\mu_1} \xi_2^{\mu_2} \cdots \xi_n^{\mu_n} ,$$
$$D^\mu = D_1^{\mu_1} D_2^{\mu_2} \cdots D_n^{\mu_n} ,$$

where $D_k = \partial / i\partial x_k$, $k=1, 2, \cdots, n$.

For operators A and B_j introduced in §1, we denote their characteristic polynomials by

$$P(x, \xi) = \sum_{|\mu|=2m} a_\mu(x) \xi^\mu$$

and

$$Q_j(x, \xi) = \sum_{|\mu|=m_j} b_{j\mu}(x) \xi^\mu$$

respectively.

Furthermore, the formally adjoint operator A^* of A will be defined, if it has a meaning, by

$$A^* v = \sum_{|\mu| \leqslant 2m} D^\mu \overline{(a_\mu(x) v)}$$

and the characteristic polynomial of A^* is

$$P^*(x, \xi) = \sum_{|\mu|=2m} \overline{a_\mu(x)} \xi^\mu .$$

Now, we give the definitions of domains considered here and the function spaces on them (We use the notations and terminologies of Browder [4] on the definition of domains).

DEFINITION 2.1. Let m and s be two integers such that $m>0$ and $s \geqslant 0$. Ω is said to be *uniformly regular of class* C^{2m+s+1} if there exist an open covering $\{N_k\}$ of Γ, a family of homeomorphisms $\{\Phi_k\}$ of N_k upon the unit ball $B_1 = \{y;\ |y| < 1\}$, and an integer R such that the following conditions are satisfied:

(1) Let $N_k' = \Phi_k^{-1}(B_{1/2})$, where $B_{1/2} = \{y; |y| < \frac{1}{2}\}$. Then $\bigcup_k N_k'$ contains the R^{-1}-neighborhood of Γ.

(2) For each k, $\Phi_k(N_k \cap \Gamma) = B_1^+ = \{y; |y| < 1, y_n > 0\}$, while $\Phi_k(N_k \cap \Gamma) = \Sigma_1 = \{y; |y| < 1, y_n = 0\}$.

(3) Any $(R+1)$ distinct open sets $\{N_k\}$ have an empty intersection.

(4) Let Ψ_k be the mapping of B_1 on N_k which is inverse to Φ_k. Then Φ_k and Ψ_k are mappings of class C^{2m+s}, and if Φ_{jk} and Ψ_{jk} are the j-th components of Φ_k and Ψ_k respectively, there exists a constant M (independent of x, y and k) such that $|D^\beta \Phi_{jk}(x)| \leqslant M$, $|D^\beta \Psi_{jk}(y)| \leqslant M$, $|\Phi_{nk}(x)| \leqslant M \cdot \mathrm{dist}\,(x, \Gamma)$ for $|\beta| \leqslant 2m+s$, $x \in N_k$ and $y \in B_1$.

(5) To the normal direction at $x \in \Gamma$ corresponds the normal direction at $y_n = 0$.

In Browder's definition the condition (5) is not explicitly stated, but we make this assumption to avoid the ambiguity in our reasoning. Of course, our domains defined above are uniformly regular of class C^{2m+s} in Browder's sense.

The following lemma due to Browder [4] makes clear the structure of uniformly regular domains of class C^{2m+s+1}.

Lemma 2.1. *Let $\Omega\subset E^n$ be uniformly regular of class C^{2m+s+1}. Then there exists a constant $\delta_0>0$ such that given δ with $0<\delta<\delta_0$, there exist a countable open covering $\{N_k\}$ of $\bar{\Omega}$ with the diameter less than δ, a family of functions $\{\eta_k\}$ with $\eta_k(x)\in C_0^{2m+s}(N_k)$, and a family of homeomorphisms $\{\Phi_k\}$ of N_k into E^n which satisfy the following conditions*:

(1) *There exists an integer R (independent of δ) such that at most R distinct numbers of $\{N_k\}$ have a non-empty intersection.*

(2) *For every k, the image N_k under Φ_k is the ball B_δ of radius δ about the origin. If $N_k\cap\Gamma\neq\phi$, $\Phi_k(N_k\cap\Omega)=B_\delta^+=\{y;\ |y|<\delta,\ y_n>0\}$, $\Phi_k(N_k\cap\Gamma)=\sum_\delta=\{y;|y|<\delta,\ y_n=0\}$. Furthermore, let $\Phi_k=\{\Phi_{jk}\}$ and $\Psi_k=\{\Psi_{jk}\}$. Then Φ_k and Ψ_k are mappings of class C^{2m+s}, and there exists a constant K (independent of x, y, k and δ) such that $|D^\beta\Phi_{jk}(x)|\leqslant K$, $|D^\beta\Psi_{jk}(y)|\leqslant K$ for all β with $|\beta|\leqslant 2m+s$, $x\in N_k$ and $y\in B_\delta$.*

(3) *There exists a constant K (independent of x but depending on δ) such that for every α and β with $|\alpha|\leqslant 2m+s$, $|\beta|\leqslant 2m+s$,*

$$\sum_{k=1}^{\infty}|D^\alpha\eta_k(x)|^2|D^\beta\eta_k(x)|^2\leqslant K_\delta\,.$$

(4) *For $x\in\bar{\Omega}$, $0\leqslant\eta_k(x)\leqslant 1$ and $\sum_{k=1}^{\infty}\eta_k(x)^2=1$.*

(5) *There exists a constant ρ with $0<\rho<1$ (depending only on n) such that $\Omega\subseteq\bigcup_k N_k'$, where $N_k'=\Psi_k(B_{\rho\delta})$, with $B_{\rho\delta}=\{y;\ |y|<\rho\delta\}$.*

REMARK 2.1. If $N_k\cap\Gamma=\phi$, we can take as N_k a n-dimensional ball of radius less than δ with some point in Ω as a center and if $N_k\cap\Gamma\neq\phi$, the image of a n-dimensional ball contained in B_1 under Ψ_k of Definition 2.1. If $N_k\cap\Gamma\neq\phi$, Φ_k satisfies the condition (5) of Definition 2.1.

Next we give the definitions of some function spaces.

DEFINITION 2.2. $H^j(\Omega)$. Let Ω be an open set in E^n. $u\in H^j(\Omega)$ if $D^\alpha u\in L^2(\Omega)$ for $|\alpha|\leqslant j$. $H^j(\Omega)$ is a Hilbert space equipped with the inner product

$$(u,\ v)_j=\sum_{|\alpha|\leqslant j}(D^\alpha u,\ D^\alpha v)_{L^2(\Omega)}\,.$$

We set

$$||u||_j^2=(u,\ u)_j\quad\text{for}\quad u\in H^j(\Omega)\,.$$

We also use the semi-norm

$$|u|_j^2=\sum_{|\alpha|=j}\|D^\alpha u\|_{L^2(\Omega)}^2 .$$

DEFINITION 2.3. $\mathcal{B}^j(\Omega)$, $\mathcal{B}^j(\Gamma)$. Let Ω be a domain of E^n whose boundary Γ is a hypersurface of class C^{2m+s} in the ordinary sense. We say that $a(x)\in\mathcal{B}^j(\Omega)$, $0\leqslant j\leqslant m$, if $a(x)\in C^j(\Omega)$ and $D^\alpha a(x)$, $|\alpha|\leqslant j$, are bounded in $\bar{\Omega}$. $\mathcal{B}^j(\Omega)$ is equipped with the norm

$$|a(x)|_{j,\mathcal{B}(\Omega)}^2=\sum_{|\alpha|\leqslant j}\sup_{x\in\bar{\Omega}}|D^\alpha a(x)|^2 .$$

Let $b(x)$ be a function defined in a neighborhood of Γ. We say that $b(x)$ belongs to $\mathcal{B}^j(\Gamma)$, $0\leqslant j\leqslant 2m+s$, if $b(x)\in C^j$ in the domain of definition and if for all $|\alpha|\leqslant j$, $D^\alpha b(x)|_\Gamma$ are bounded on Γ. We give the following norm

$$|b(x)|_{j,\mathcal{B}(\Gamma)}^2=\sum_{|\alpha|\leqslant j}\sup_{x\in\Gamma}|D^\alpha b(x)|^2 .$$

REMARK 2.2. The neighborhood of Γ mentioned above may depend on each $b(x)$. Moreover, in the above definition, we can require the domain of definition only to be a part of $\bar{\Omega}$ containing a neighborhood of Γ. In fact, such a function can be extended as a C^j-class function into a neighborhood of Γ.

The following lemmas due to Browder [4] are used in a later section in proving a priori estimates and the regularity of the solutions of our boundary value problems.

Lemma 2.2. *Let Ω be uniformly regular of class C^{2m+s+1} and $\{\eta_k(x)\}$ be a family of functions satisfying the conditions of Lemma 2.1. Then, there exist constants K_j and $K_{j,\delta}>0$, $0\leqslant j\leqslant 2m+s$, such that for $u\in H^{2m+s}(\Omega)$,*

$$\|u\|_j^2\leqslant K_j\sum_{k=1}^{\infty}\|\eta_k^2 u\|_j^2 ,$$

$$\sum_{k=1}^{\infty}\|\eta_k^2 u\|_j^2\leqslant K_{j,\delta}\|u\|_j^2 .$$

Lemma 2.3. *Let Ω be uniformly regular of class C^{2m+s+1}. Then, there exist constants C_j and $C_{j,\varepsilon}>0$ such that for $u\in H^{2m+s}(\Omega)$, we have, for $0\leqslant j<2m+s$,*

$$\|u\|_j^2\leqslant\varepsilon\|u\|_{2m+s}^2+C_{j,\varepsilon}\|u\|_0^2 ,$$

$$\|u\|_j^2\leqslant\varepsilon|u|_{2m+s}^2+C_{j,\varepsilon}|u|_0^2 ,$$

$$\lambda^{2(2m+s-j)}\|u\|_j^2\leqslant\|u\|_{2m+s}^2+C_j\lambda^{2(2m+s)}\|u\|_0^2 ,$$

where λ and ε are arbitrary positive numbers.

3. Boundary norms

In this section we give the definition of the boundary norms employed in

Theorem 1. To begin with the simplest case of a half space, we shall use the following notations: $y'=(y_1,\cdots,y_{n-1})$, $\xi'=(\xi_1,\cdots,\xi_{n-1})$, $E_+^n=\{y=(y',y_n), y_n>0\}$. Let $v(y)=v(y',y_n)\in H^j(E_+^n)$. We know that the trace $v(y',+0)$ belongs to $H^{j-(1/2)}(E^{n-1})$. Let us introduce a new notation for boundary norms. Let $v_1, v_2\in H^j(E_+^n)$, then

$$(3.1)\qquad \langle v_1, v_2\rangle_{j-(1/2)}=\int_{E^{n-1}}|\xi'|^{2j-1}\hat{v}_1(\xi',+0)\overline{\hat{v}_2(\xi',+0)}d\xi',$$

where

$$(3.2)\qquad \hat{v}_i(\xi',+0)=(2\pi)^{-(n-1)/2}\int_{E^{n-1}}e^{-i\xi'y'}v_i(y',+0)dy'.$$

In particular denote

$$(3.3)\qquad \langle v\rangle_{j-(1/2)}^2=\langle v,v\rangle_{j-(1/2)}.$$

We know that

$$(3.4)\qquad \langle v\rangle_{j-(1/2)}^2\leqslant c_j|v|_j^2 \qquad \text{for } v\in H^j(E_+^n),$$

where c_j is a constant depending only on j (see, for example, Schechter [8]).

The following lemma is also useful for our reasoning.

Lemma 3.1. *For $v\in H^1(E_+^n)$, we have*

$$(3.5)\qquad \langle v\rangle_0^2\leqslant\lambda^{-1}|v|_1^2+\lambda|v|_0^2,$$

where λ is an arbitrary positife number.

Proof. $$|v(y',+0)|^2=-\int_0^\infty\frac{\partial}{\partial y_n}|v(y',y_n)|^2dy_n=-\int_0^\infty\left(\frac{\partial v}{\partial y_n}\bar{v}+v\frac{\partial\bar{v}}{\partial y_n}\right)dy_n$$
$$\leqslant 2\int_0^\infty|v|\left|\frac{\partial v}{\partial y_n}\right|dy_n\leqslant\lambda^{-1}\int_0^\infty\left|\frac{\partial v}{\partial y_n}\right|^2dy_n+\lambda\int_0^\infty|v|^2dy_n.$$

The integration in y' gives (3.5).

Now let $b(y')\in\mathscr{B}^j(E^{n-1})$, $v\in H^j(E_+^n)$. Applying (3.4), we get

$$\langle b(y')v\rangle_{j-(1/2)}^2\leqslant c_j|b(y')v|_j^2.$$

Hence

$$(3.6)\qquad \langle b(y')v\rangle_{j-(1/2)}^2\leqslant c_j'|b(y')|_{j,\mathscr{B}(E^{n-1})}^2\|v\|_j^2,$$

more precisely,

$$(3.7)\qquad \langle b(y')v\rangle_{j-(1/2)}^2\leqslant c_j''(|b|_0^2|v|_j^2+|b|_j^2\|v\|_{j-1}^2).$$

Let Ω be a uniformly regular of class C^{2m+s+1} and take $\{N_k\}$, $\{\Phi_k\}$ and $\{\eta_k\}$ corresponding to a fixed δ $(0<\delta<\delta_0)$ in Lemma 2.1. Let $u(x)\in H^j(\Omega)$ $(0\leqslant j\leqslant 2m+s)$. Then $(\eta_k^2u)(\Psi_k(y))\in H^j(B_\delta^+)$. Since this function vanishes in a neighborhood of $|y|=|\delta|$, we can consider this as an element of $H^j(E^n_+)$, hence its trace to the hyperplane $y_n=0$ belongs to $H^{j-(1/2)}(E^{n-1})$. Now we define the boundary norm on Γ as follows:

DEFINITION 3.1. For $u\in H^j(\Omega)$ $(0<j\leqslant 2m+s)$, let us denote

$$\langle u\rangle^2_{j-(1/2),\Gamma}=\sum_{N_k\cap\Gamma\neq\phi}\langle(\eta_k^2u)(\Psi_k(y))\rangle^2_{j-(1/2),E^{n-1}}. \tag{3.8}$$

The convergence of the right-hand side is shown as follows: By (3.4),

$$\langle(\eta_k^2u)(\Psi_k(y))\rangle^2_{j-(1/2)}\leqslant c_j|(\eta_k^2u)(\Psi_k(y))|^2_{j,E^n_+}\leqslant c_jc_{j\delta}\|\eta_k^2(x)u(x)\|^2_j,$$

where $c_{j\delta}$ is a constant independent of k. Taking account of Lemma 2.2, we get

$$\langle u\rangle^2_{j-(1/2),\Gamma}\leqslant c_j\|u\|^2_j. \tag{3.9}$$

It should be remarked that the left-hand side of (3.8) depends on the partition of the unity. However as we shall show in Appendix, the norms $\langle u\rangle^2_{j-(1/2),\Gamma}+\langle u\rangle^2_{0,\Gamma}$ are equivalent to each other. Hence we can define the space $H^{j-(1/2)}(\Gamma)$ independently of the choice of the partition of the unity.

Finally, let $b(x)\in\mathscr{B}^j(\Gamma)$, $(0<j\leqslant 2m+s)$ (see Definition 2.3). For $u(x)\in H^j(\Omega)$,

$$\begin{aligned}&\langle b(\Psi_k(y',0))(\eta_k^2u)(\Psi_k(y',0)\rangle^2_{j-(1/2),E^{n-1}}\\&\quad\leqslant C'_j|b(\Psi_k(y'\ 0))|^2_{j,E^{n-1}}\|(\eta_k^2u)(\Psi_k(y))\|^2_{j,B^+_\delta},\end{aligned}$$

where we used (3.6) by putting $b(y')=b(\Psi_k(y',0))$ and taking its j-norm on $E^{n-1}\cap B_\delta$. Now from the expression,

$$b(\Psi_k(y',0))=b(\Psi_{1k}(y',0),\cdots,\Psi_{nk}(y',0)),$$

taking account of $(\Psi_{1k}(y',0),\cdots,\Psi_{nk}(y',0))\in\Gamma$, we see that

$$|b(\Psi_k(y',0))|_{j,E^{n-1}}\leqslant c|b(x)|_{j,\mathscr{B}(\Gamma)}.$$

Thus we have

$$\langle b(x)u(x)\rangle^2_{j-(1/2),\Gamma}\leqslant\text{const.}|b(x)|^2_{j,\mathscr{B}(\Gamma)}\|u(x)\|^2_j. \tag{3.10}$$

Finally let us remark the following estimate:

$$\langle b(x)u(x)\rangle^2_{0,\Gamma}\leqslant\text{const.}|b(x)|^2_{1,\mathscr{B}(\Gamma)}\|u(x)\|^2_1. \tag{3.11}$$

4. A priori estimates for operators with constant coefficients in a half space

In this section we consider operators with constant (complex) coefficients in a half-space E^n_+. Modifying the notations of preceding sections, we denote the points of E_+^n by $x=(x', t)$, where $x'=(x_1, \cdots, x_{n-1})$, and the points in the dual space by $\xi=(\xi', \tau)$.

Let s be a non-negative integer, and $u\in H^s(E_+^n)$. In this case the norm $||u||_s$ and the semi-norm $|u|_s$ can be expressed by means of the Fourier transform in x'. In fact, let $\hat{u}(\xi, t)$ be the Fourier transform of u in x', then

$$||u||_s^2 \simeq \sum_{k=1}^{s} \int_{E^{n-1}} \int_0^\infty (1+|\xi'|^2)^{s-k} |D_t^k \hat{u}(\xi, t)|^2 d\xi' dt\,,$$

$$|u|_s^2 \simeq \sum_{k=1}^{s} \int_{E^{n-1}} \int_0^\infty |\xi'|^{2(s-k)} |D_t^k \hat{u}(\xi', t)|^2 d\xi' dt\,.$$

We need the following interpolation lemma, whose proof we refer, for example, to Mizohata [6].

Lemma 4.1. *There exist constants $C_j>0$ and $C_{j\varepsilon}>0$ (depending only on s, j and ε) such that for $u\in H^s(E_+^n)$ and $0\leqslant j<s$,*

(4.1) $$\lambda^{2(s-j)}|u|_j^2 \leqslant C_j(|u|_s^2+\lambda^{2s}|u|_0^2)\,, \qquad \lambda>0\,.$$

(4.2) $$||u||_j^2 \leqslant \varepsilon |u|_s^2 + C_{j\varepsilon}|u|_0^2$$

and if $\lambda>1$,

(4.3) $$\lambda^{2(s-j)}||u||_j^2 \leqslant C_j(|u|_0^2+\lambda^{2s}|u|_0^2)\,.$$

Now let us consider linear differential operators with constant coefficients in the half space E_+^n:

$$A(D_{x'}, D_t) = \sum_{|\mu|+k=2m} a_{\mu k} D_{x'}^{\mu} D_t^k\,,$$

$$B_j(D_{x'}, D_t) = \sum_{|\mu|+k=m_j} b_{j\mu} D_{x'}^{\mu} D_t^k\,,$$

where $m_j \neq m_k$, $m_j<2m$, $j=1, 2, \cdots, m$, and we denote the characteristic polynomials by

$$A(\xi', \tau) = \sum_{|\mu|+k=2m} a_\mu \xi'^\mu \tau^k\,,$$

$$B_j(\xi', \tau) = \sum_{|\mu|+k=m_j} b_{j\mu} \xi'^\mu \tau^k\,.$$

We assume now that (A, B_j) satisfies the following conditions:

(L.1) $|a_\mu|\leqslant\gamma_1\,,\quad |b_{j\mu}|<\gamma_1\,.$

(L.2) $|A(\xi', \tau)-\lambda^{2m}| \geqslant \gamma_2(|\xi'|^2+|\lambda|^2+\tau^2)^m$ for $(\xi', \tau)\in E^n$ and $\lambda = r\, e^{i(\theta/2m)}$.

(L.3) For every $(\xi', \tau)\neq(0, 0)$ the polynomial in τ $A(\xi', \tau)-\lambda^{2m}$ has just m roots $\tau_k^+(\xi', \lambda)$ with positive imaginary part.

(L.4) Set $A_+(\xi', \lambda, \tau) = \prod_{k=1}^{m} (\tau - \tau_k^+(\xi', \lambda))$ and

$$d_{jk}(\xi', \lambda) = \frac{1}{2\pi i} \oint \frac{B_j(\xi', \tau)}{A_+(\xi', \lambda, \tau)} \tau^{k-1} d\tau ,$$

where the integration is taken along a closed curve in the complex τ-plane enclosing all the roots $\tau_k^+(\xi', \lambda)$. We assume that

$$|\det (d_{jk}(\xi', \lambda))| \geqslant \gamma_3(|\xi_{\prime}|^2 + |\lambda|^2)^{N/2} \qquad (N = \sum_{j=1}^{m} (m_j - j + 1)) ,$$

where γ_1, γ_2 and γ_3 are positive constants.

Let us remark that, by (L.3), the normal direction at the boundary is not characteristic for any B_j. In fact, if B_{j_0} is so, then $B_{j_0}(0, \tau) = 0$, hence we have $d_{j_0 k}(0, \lambda) = 0$, contrary to the condition (L.4).

At first we prove the following

Proposition 4.1. *Under the above assumptions, given any* $f(x', t) \in H^s(E_+^n)$ *and* $(g_1(x'), \cdots, g_m(x')) \in \prod_{j=1}^{m} H^{2m+s-m_j-(1/2)}(E^{n-1})$, *there exists a unique solution* $u(x', t; \lambda) \in H^{2m+s}(E_+^n)$ *satisfying*

$$(A(D_{x'}, D_t) - \lambda^{2m}) u(x', t; \lambda) = f(x', t), \ t > 0 , \tag{4.4}$$

$$B_j(D_{x'}, D_t) u(x', 0; \lambda) = g_j(x'), \qquad j = 1, 2, \cdots, m , \tag{4.5}$$

where $\lambda = re^{i(\theta/2m)}$. $u(x', t; \lambda)$ *can be expressed in the form*

$$u(x', t; \lambda) = w(x', t; \lambda) + v(x', t; \lambda) , \tag{4.6}$$

where

$$\hat{w}(\xi', t; \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \frac{e^{it\tau}}{A(\xi', \tau) - \lambda^{2m}} \widetilde{F}(\xi', \tau) d\tau , \tag{4.7}$$

$$\begin{aligned} \hat{v}(\xi', t; \lambda) = \sum_{j=1}^{m} (g_j(\xi') - B_j(\xi', D_t) \hat{w}(\xi', 0; \lambda)) \\ \times \sum_{k=1}^{m} \frac{D_{jk}(\xi', \lambda)}{D(\xi', \lambda)} I_k(t; \xi', \lambda) , \end{aligned} \tag{4.8}$$

where

$$I_k(t; \xi', \lambda) = \frac{1}{2\pi i} \oint \frac{e^{it\tau} \tau^{k-1}}{A_+(\xi', \lambda, \tau)} d\tau . \tag{4.9}$$

In the formula (4.7), $\widetilde{F}(\xi', \tau)$ represents the Fourier transform of $F(x', t) \in H^s(E^n)$——an extension of $f(x', t)$ to the whole space E^n——and in (4.8), $D_{jk}(\xi', \lambda)$ represents the (j, k)-cofactor of the matrix $[d_{jk}(\xi', \lambda)]$.

Proof. Since the existence and the uniqueness of the solution $u(x', t; \lambda)$

is well-known, we restrict ourselves to explain how to deduce the expression of u. By the well-known method we can define the extension $F(x', t)\in H^s(E^n)$ of $f(x', t)\in H^s(E_+^n)$ (linear operator) such that

$$|F|_s \leqslant C|f|_s, \tag{4.10}$$

where C is a constant depending only on s. It is evident that the function $w(x', t; \lambda)$, inverse transform of $\hat{w}(\xi', t; \lambda)$, belongs to $H^{2m+s}(E^n)$, and it satisfies

$$(A(D_{x'}, D_t)-\lambda^{2m})w(x', t; \lambda) = F(x', t)\,.$$

Put $u(x', t, \lambda)=v(x', t; \lambda)+w(x', t; \lambda)$. Taking the Fourier transform in x', we see that

$$\begin{aligned}&(A(\xi', D_t)-\lambda^{2m})\hat{v}(\xi', t; \lambda) = \hat{f}(\xi', t; \lambda)\,, \qquad t>0\,,\\ &B_j(\xi', D_t)\hat{v}(\xi', 0; \lambda) = \hat{g}_j(\xi')-B_j(\xi', D_t)\hat{w}(\xi', 0; \lambda)\,.\end{aligned}$$

Namely $\hat{v}(\xi', t; \lambda)$ is a solution of an ordinary differential equation in t containing the parameter (ξ', λ), and satisfying the m boundary conditions at $t=0$. Then it is also well-known that the solution is expressed by (4.8) (see also L. Hörmander [5] and J. Peetre [7]).

In the second place, we give the a priori estimate for the solutions given in Proposition 4.1.

Proposition 4.2. *Under the above assumptions, there exists a constant* $c(\gamma_1, \gamma_2, \gamma_3)>0$ *such that for* $u\in H^{2m+s}(E_+^n)$,

$$\begin{aligned}|u|_{2m+s}^2+|\lambda|^{2(2m+s)}|u|_0^2\leqslant c(\gamma_1, \gamma_2, \gamma_3)[&|(A-\lambda^{2m})u|_s^2+|\lambda|^{2s}|(A-\lambda^{2m})u|_0^2\\ &+\sum_{j=1}^{m}(\langle B_ju\rangle_{2m+s-m_j-(1/2)}^2+|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_ju\rangle_0)]\,.\end{aligned} \tag{4.11}$$

Proof. At first let us consider the estimate of w. For this purpose we introduce the quantity α and (η, λ') as follows:

$$\alpha = \{|\xi'|^2+|\lambda|^2\}^{1/2}, \ (\eta, \lambda') = \left(\frac{\xi'}{\alpha}, \frac{\lambda}{\alpha}\right). \tag{4.12}$$

Let us remark that

$$|\eta|^2+|\lambda'|^2 = 1\,. \tag{4.13}$$

Then, by virtue of (L.2),

$$|A(\xi', \tau)-\lambda^{2m}|\geqslant \gamma_2\alpha^{2m}(1+|\alpha^{-1}\tau|^2)^m\,. \tag{4.14}$$

Now, using the Plancherel formula in (4.7),

$$\int_{-\infty}^{+\infty}(|D_t^{2m+s}\hat{w}(\xi', t; \lambda)|^2+\alpha^{2(2m+s)}|\hat{w}(\xi', t; \lambda)|^2)dt$$
$$\leqslant c_1\gamma_2^{-2}\int_{-\infty}^{+\infty}(|D_t^s\hat{F}(\xi', t)|^2+\alpha^{2s}|\hat{F}(\xi', t)|^2)dt\,.$$

The integration in ξ' gives

$$(4.15)\quad |w(x', t; \lambda)|_{2m+s}^2+|\lambda|^{2(2m+s)}|w|_0^2\leqslant c_2\gamma_2^{-2}(|F(x', t)|_s^2+|\lambda|^{2s}|F|_0^2)\,,$$

where the semi-norms are taken in E^n, and c_2 depends only on m, s.

In the second step, we estimate $|B_j(\xi', D_t)\hat{w}(\xi', 0; \lambda)|$ appearing in (4.8). At first, it is evident that

$$B_j(\xi', D_t)\hat{w}(\xi', 0; \lambda)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\frac{B_j(\xi', \tau)}{A(\xi', \tau)-\lambda^{2m}}\tilde{F}(\xi', \tau)d\tau\,.$$

By the Schwarz inequality,

$$|B_j(\xi', D_t)\hat{w}(\xi', 0; \lambda)|^2\leqslant(2\pi)^{-1}\int(|\tau|^2+|\xi'|^2+|\lambda|^2)^{-s}\left|\frac{B_j}{A-\lambda^{2m}}\right|^2 d\tau\times$$
$$\int(|\tau|^2+|\xi|^2+|\lambda|^2)^s|\tilde{F}(\xi', \tau)|^2 d\tau\,.$$

Take $\tau'=\alpha^{-1}\tau$ as the new variable in the first integral, then

$$(4.16)\qquad \begin{aligned}&(|\tau|^2+|\xi'|^2+|\lambda|^2)^{-s}|B_j(\xi', \tau)(A(\xi', \tau)-\lambda^{2m})^{-1}|^2\\ &=\alpha^{2(m_j-2m-s)}(1+|\tau'|^2)^{-s}|B_j(\eta, \tau')(A(\eta, \tau')-\tau'^{2m})^{-1}|^2\,.\end{aligned}$$

Taking account of (L.1) and (4.14), we see that there exists a positive constant $c_1(\gamma_1, \gamma_2)$ such that (4.16) is estimated by $c_1(\gamma_1, \gamma_2)\alpha^{2(m_j-2m-s)}(1+|\tau'|^2)^{-1}$. Thus,

$$(4.17)\quad |B_j(\xi', D_t)\hat{w}(\xi', 0; \lambda)|^2\leqslant c_1(\gamma_1, \gamma_2)\alpha^{2(m_j-2m-s)+1}\times$$
$$\int(|\tau|^2+|\xi'|^2+|\lambda|^2)^s||\tilde{F}(\xi', \tau)|^2 d\tau\leqslant c(n, s)c_1(\gamma_1, \gamma_2)\alpha^{2(m_j-2m-s)+1}\times$$
$$\int(|D_t^s\hat{F}(\xi', t)|^2+\alpha^{2s}|\hat{F}(\xi', t)|^2)dt\,.$$

Finally, we estimate $v(x', t; \lambda)$. Let us remark that $D_{jk}(\xi', \lambda)/D(\xi', \lambda)$ is homogeneous of degree $(m-m_j-k)$ in (ξ', λ). Next, the conditions (L.1)–(L.4) imply that there exist three constants depending only on $(\gamma_1, \gamma_2, \gamma_3)$:

$$(4.18)\qquad \begin{aligned}&\operatorname{Im}\tau_j^+(\eta, \lambda')\geqslant\delta_0(\gamma_1, \gamma_2)\,,\quad |\tau_j^+(\eta, \lambda')|\leqslant M(\gamma_1, \gamma_2)\,,\\ &|D_{jk}(\eta, \lambda')/D(\eta, \lambda')|\leqslant c_2(\gamma_1, \gamma_2, \gamma_3)\,.\end{aligned}$$

On the other hand,

$$(4.19)\qquad I_k(t; \xi', \lambda)=\alpha^{k-m}I_k(\alpha t; \eta, \lambda')\,,\qquad k=1, 2, \cdots,$$

$$(4.20)\qquad D_t^{2m+s}I_k(t; \xi', \lambda)=I_{k+2m+s}(t; \xi', \lambda)\,.$$

Now, by (4.18), we see that there exists a constant $c_3(\gamma_1, \gamma_2, \gamma_3)$ such that

$$|I_k(\alpha t; \eta, \lambda')| \leqslant c_3(\gamma_1, \gamma_2, \gamma_3) \exp\left(-\frac{\delta_0}{2}\alpha t\right). \tag{4.21}$$

In fact, in the definition of (4.9), we can take a fixed closed curve of integration independent of (η, λ').

Now

$$\sum_{k=1}^{m} |D_{jk}(\xi', \lambda)/D(\xi', \lambda)|^2 \int_0^\infty |I_k(t; \xi', \lambda)|^2 dt \leqslant c_2(\gamma_1, \gamma_2, \gamma_3) c_3(\gamma_1, \gamma_2, \gamma_3) \times$$
$$\sum_{k=1}^{m} \alpha^{2(m-m_j-k)} \frac{\alpha^{2(k-m)-1}}{\delta_0} \leqslant c_4(\gamma_1, \gamma_2, \gamma_3) \alpha^{-2m_j-1}.$$

In view of (4.20),

$$\sum_{k=1}^{m} |D_{jk}(\xi', \lambda)/D(\xi', \lambda)|^2 \int_0^\infty |D_t^{2m+s} I_k(t; \xi', \lambda)|^2 dt \leqslant c_4(\gamma_1, \gamma_2, \gamma_3) \alpha^{2(2m+s-m_j)-1}.$$

Thus, from (4.8)

$$\alpha^{2(2m+s)} \int_0^\infty |\hat{v}(\xi', t; \lambda)|^2 dt + \int_0^\infty |D_t^{2m+s} \hat{v}(\xi', t; \lambda)|^2 dt$$
$$\leqslant c \sum_{j=1}^{m} [|g_j(\xi')|^2 + |B_j(\xi', D_t)\hat{w}(\xi', 0; \lambda)|^2] c_5(\gamma_1, \gamma_2, \gamma_3) \alpha^{2(m+s-m_j-(1/2))},$$

in view of (4.17), this is again estimated by

$$c_5(\gamma_1, \gamma_2, \gamma_3) \left(\sum_{j=1}^{m} \alpha^{2(2m+s-m_j-(1/2))} |g_j(\xi')|^2 + \int_0^\infty |D_t^s \hat{F}(\xi', t)|^2 dt + \alpha^{2s} \int_0^\infty |\hat{F}(\xi', t)|^2 dt\right).$$

Taking account of (4.10), this estimate and (4.15) prove the estimate (4.11)

5. A priori estimates for operators with variable coefficients in a half space

In this section we use the notations of §4 and consider linear partial differential operators with variable coefficients in a half space;

$$A(x, D_{x'}, D_t) = A_0(D_{x'}, D_t) + \sum_{|\mu|+k \leqslant 2m} c_{\mu k}(x) D_{x'}^{\mu} D_t^k,$$
$$B_j(x', D_{x'}, D_t) = B_{j0}(D_{x'}, D_t) + \sum_{|\mu|+k \leqslant m_j} b_{j\mu k}(x') D_{x'}^{\mu} D_t^k,$$

where $m_j < 2m$, $m_j \neq m_k$, $j=1, 2, \cdots, m$, $c_{\mu k}(0)=0$ for $|\mu|+k=2m$, $b_{j\mu k}(0)=0$ for $|\mu|+k=m_j$. A_0 and B_{j0} are homogeneous differential operators of degree $2m$ and m_j respectively.

We assume that (A_0, B_{j0}) satisfies the assumptions (L.1)–(L.4) of §4, and $c_{\mu k}(x) \in \mathscr{B}^s(E^n)$ and $b_{j\mu k}(x') \in \mathscr{B}^{2m+s-m_j}(E^{n-1})$.

Let us introduce the two quantities;

$$\zeta=\sum_{|\mu|+k=2m}\sup_{x\in E_+^n}|c_{\mu k}(x)|^2+\sum_{\substack{|\mu|+k=m_j\\ 1\leqslant j\leqslant m}}\sup_{x'\in E^{n-1}}|b_{j\mu k}(x')|^2,$$

$$M=\sum_{\mu,k}|c_{\mu k}(x)|^2_{s,\mathcal{B}(E_+^n)}+\sum_{j,\mu,k}|b_{j\mu k}(x')|^2_{2m+s-m_j,\mathcal{B}(E^{n-1})},$$

where each supremum is taken in the whole space. Then we have

Proposition 5.1. *Under the above assumptions, there exists a positive constant* ζ_0 *depending only on* $c(\gamma_1, \gamma_2, \gamma_3)$ (*introduced in Proposition* 4.2) *such that, if* $\zeta\leqslant\zeta_0$, *the following inequality holds*

$$(5.1)\qquad |u|^2_{2m+s}+|\lambda|^{2(2m+s)}|u|^2_0\leqslant 4c(\gamma_1, \gamma_2, \gamma_3)[|(A-\lambda^{2m})u|^2_s+|\lambda|^{2s}|(A-\lambda^{2m})u|^2_0+\sum_{j=1}^m(\langle B_ju\rangle^2_{2m+s-m_j-(1/2)}+|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_ju\rangle^2_0)],\qquad u\in H^{2m+s}(E_+^n),$$

for $|\lambda|\geqslant r_0$ ($\lambda=re^{i(\theta/2m)}$), *where* r_0 *is determined by* $c(\gamma_1, \gamma_2, \gamma_3)$ *and* M. *More precisely,* r_0 *can be considered as an increasing function of* M.

Proof. From Proposition 4.2.

$$(5.2)\quad |u|^2_{2m+s}+|\lambda|^{2(2m+s)}|u|^2_0\leqslant c(\gamma_1, \gamma_2, \gamma_3)[|(A_0-\lambda^{2m})u|^2_s+|\lambda|^{2s}|(A_0-\lambda^{2m})u|^2_0+\sum_{j=1}^m(\langle B_{j0}u\rangle^2_{2m+s-m_j-(1/2)}+|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_{j0}u\rangle^2_0)].$$

From now on, we assume that $|\lambda|>1$ and that ε be a positive number less than 1. From (5.2) we have

$$(5.3)\qquad |u|^2_{2m+s}+|\lambda|^{2(2m+s)}|u|^2_0-2c(\gamma_1, \gamma_2, \gamma_3)J\leqslant\frac{1}{2}\ \text{(right-hand side of (5.1))},$$

where

$$J=|(A-A_0)u|^2_s+|\lambda|^{2s}|(A-A_0)u|^2_0+\sum_{j=1}^m\langle(B_j-B_{j0})u\rangle^2_{2m+s-m_j-(1/2)}+\sum_{j=1}^m|\lambda|^{2(2m+s-m_j-(1/2))}\langle(B_j-B_{j0})u\rangle^2_0.$$

The decomposition

$$(A-A_0)u=\sum_{|\mu|+k<2m}c_{\mu k}(x)D^{\mu}_{x'}D^k_tu+\sum_{|\mu|+k=2m}c_{\mu k}(x)D^{\mu}_{x'}D^k_tu$$

gives

$$|(A-A_0)u|^2_s\leqslant c_1(\zeta|u|^2_{2m+s}+M||u||^2_{2m+s-1}).$$

Applying Lemma 4.1,

$$(5.4)\qquad \begin{aligned}&|(A-A_0)u|^2_s\leqslant{}_1c(\zeta+\varepsilon M)|u|^2_{2m+s}+c_1c_\varepsilon M|u|^2_0,\\&|\lambda|^{2s}|(A-A_0)u|^2_0\leqslant c_2|\lambda|^{2s}(\zeta|u|^2_{2m}+M||u||^2_{2m-1}).\end{aligned}$$

Applying Lemma 4.1, we have

(5.5) $$|\lambda|^{2s}|(A-A_0)u|_0^2\leqslant c_2(\zeta+\varepsilon M)|u|_{2m+s}^2+|\lambda|^{2(2m+s)}(c_2\zeta+C'_\varepsilon M|\lambda|^{-2})|u|_0^2 .$$

The decomposition

$$(B_j-B_{j0})=\sum_{|\mu|+k=m_j} b_{j\mu k}(x')D_{x'}^{\mu}D_t^k u+\sum_{|\mu|+k<m_j} b_{j\mu k}(x')D_{x'}^{\mu}D_t^k u$$

gives, applying (3.7),

$$\langle(B_j-B_{j0})u\rangle_{2m+s-m_j-(1/2)}^2\leqslant c_3(\zeta|u|_{2m+s}^2+M\|u\|_{2m+s-1}^2) .$$

Applying Lemma 4.1,

(5.6) $$\langle(B_j-B_{j0})u\rangle_{2m+s-m_j-(1/2)}^2\leqslant(c_3\zeta+c_3\varepsilon M)|u|_{2m+s}^2+c_3C''M|u|_0^2 .$$

Finally,

$$\begin{aligned}|\lambda|^{2(2m+s-m_j-(1/2))}\langle(B_j-B_{j0})u\rangle_0^2\leqslant c'|\lambda|^{2(2m+s-m_j-1/2)}[&|\lambda|^{-1}\zeta|u|_{m_j+1}^2\\ &+|\lambda|^{-1}M|u|_{m_j}^2+|\lambda|\zeta|u|_{m_j}^2+|\lambda|M\|u\|_{m_j-1}^2] .\end{aligned}$$

In fact, consider a term $\langle b_{j\mu k}(x')D_{x'}^{\mu}D_t^k u\rangle_0^2$, where $|\mu|+k=m_j$. Making use of Lemma 3.1., this term is estimated by

$$\begin{aligned}&c\{|\lambda|^{-1}|b_{j\mu k}(x')D_{x'}^{\mu}D_t^k u|_1^2+|\lambda|\,|b_{j\mu k}(x')D_{x'}^{\mu}D_t^k u|_0\}\\ &\leqslant c'\{|\lambda|^{-1}(\zeta|u|_{m_j+1}^2+M|u|_{m_j}^2)+|\lambda|\zeta|u|_{m_j}^2\} .\end{aligned}$$

In the case where $|\mu|+k<m_j$, we have similarly

$$\langle b_{j\mu k}D_{x'}^{\mu}D_t^k u\rangle_0\leqslant c'(|\lambda|^{-1}M|u|_{m_j}^2+|\lambda|M|u|_{m_j-1}^2) .$$

Therefore, we have

(5.7) $$\begin{aligned}|\lambda|^{2(2m+s-m_j-(1/2))}\langle(B_j-B_{j0})u\rangle_0^2&\leqslant c_4(\zeta+M|\lambda|^{-2})|u|_{2m+s}^2\\ &+|\lambda|^{2(2m+s)}c_4(\zeta+M|\lambda|^{-2})|u|_0^2 .\end{aligned}$$

Adding the right-hand sides of (5.4)–(5.7), we have

$$\begin{aligned}J\leqslant&\{(c_1+c_2+c_3+c_4)\zeta+\varepsilon(c_1+c_2+c_3)M+M|\lambda|^{-2}\}|u|_{2m+s}^2\\ &+|\lambda|^{2(2m+s)}\{(c_1C+c_3C'')M|\lambda|^{-2(2m+s)}+(c_2+c_4)\zeta+(C'_\varepsilon+c_4)M|\lambda|^{-2}\}|u|_0^2 .\end{aligned}$$

Now we take ζ_0 and ε as follows:

(5.8) $$\begin{aligned}2c(\gamma_1,\gamma_2,\gamma_3)(c_1+c_2+c_3+c_4)\zeta_0&=\frac{1}{6},\\ 2c(\gamma_1,\gamma_2,\gamma_3)(c_1+c_2+c_3)M\varepsilon&=\frac{1}{6},\end{aligned}$$

then we see that, if we take $|\lambda|^2$ in such a way that

$$2c(\gamma_1, \gamma_2, \gamma_3)(1+c_1C_\varepsilon+C'_\varepsilon+c_3C''_\varepsilon+c_4)M \leqslant \frac{1}{6}|\lambda|^2,$$

we have

$$2c(\gamma_1, \gamma_2, \gamma_3)J \leqslant \frac{1}{2}(|u|^2_{2m+s}+|\lambda|^{2(2m+s)}|u|^2_0).$$

Thus we have (5.1). Finally, we remark that ε is a decreasing function of M. Since the constants C_ε, C'_ε, C''_ε are all decreasing functions of ε, they are increasing functions of M. Therefore, the quantity defined by

$$r_0=\{12c(\gamma_1, \gamma_2, \gamma_3)(1+c_1C_\varepsilon+\cdots+c_4)M\}^{1/2}$$

is an increasing function of M.

REMARK. The proof shows that ζ can be replaced by

$$\zeta'=\sum_{|\mu|+k=2m}\sup_{x\in\mathrm{supp}[u]}|c_{\mu k}(x)|^2+\sum_{\substack{|\mu|+k=m_j\\ 1\leqslant j\leqslant m}}\sup_{x\in\mathrm{supp}[u]}|b_{j\mu k}(x')|^2,$$

where supp $[u]$ means the support of u. It is the same with M. Thus, if we consider only the functions whose support are contained in B^+_δ, then, in the definition of ζ and M, each supremum can be replaced by the one taken over $\bar{B}^+_\delta$.

6. Proof of Theorem 1

In this section we give the proof of Theorem 1. Our method is to reduce the proof to the results of §5 by using $\{N_k, \Phi_k, \eta_k, \delta\}$ mentioned in Lemma 2.1. Let $A^{(k)}(y, D)$ and $B^{(k)}_j(y, D)$ be the transformed operators of $A(x, D)$ and $B_j(x, D)$ and denote their principal parts by $A^{(k)}_0(y, D)$ and $B^{(k)}_{j0}(y, D)$. The main point of the proof is to show that $(A^{(k)}_0(y, D), B^{(k)}_{j0}(y, D))$ satisfies the conditions (L.1)–(L.4) of §4 with constants γ_1, γ_2, γ_3 independent of k and δ.

Denote $u(\Psi_k(y))$ by $v_k(y)$ and $\eta_k(\Psi_k(y))$ by $\eta_k(y)$. Let us begin with an elementary lemma.

Lemma 6.1. *Let Ω be uniformly regular of class C^{2m+s+1}. Then*

1) *For $u\in H^j(\Omega)$, $j\leqslant 2m+s$, and $\lambda\geqslant 1$,*

$$\sum_{k=1}^{\infty}\{|\eta_k^2v_k|^2_j+\lambda^{2j}|\eta_k^2v_k|^2_0\}\geqslant c_1(j)(|u|^2_{j,\Omega}+\lambda^{2j}|u|^2_{0,\Omega}),$$

where $c_1(j)$ is independent of δ.

2) *Suppose that $u\in H^j(\Omega)$, $j\leqslant 2m+s$, and that the support of u is contained in N_k. Then for $\lambda\geqslant 1$,*

$$\begin{aligned}c_2(j)^{-1}(|u|^2_{j,\Omega}+\lambda^{2j}|u|^2_{0,\Omega})&\leqslant|v_k|^2_j+\lambda^{2j}|v_k|^2_0\\ &\leqslant c_2(j)(|u|^2_{j,\Omega}+\lambda^{2j}|u|^2_{0,\Omega}),\end{aligned}$$

where $c_2(j)$ is a positive constant independent of k and δ.

3) *For $u\in H^{j+|\gamma|}(\Omega)$, $j+|\gamma|\leqslant 2m+s$, there exists a constant $c(\alpha, \beta, \delta)>0$ such that*

$$\sum_{k=1}^{\infty} |D_y^{\alpha}\eta_k(y)D_y^{\beta}\eta_k(y)D_y^{\gamma}v_k(y)|_j^2\leqslant c(\alpha, \beta, \delta)\|u\|_{j+|\gamma|}^2 .$$

Proof.

2) Since $v_k(y)=u(\Psi_k(y))$, in view of the condition $|D^{\alpha}\Psi_k(y)|\leqslant K$ (see Lemma 2.1.), we see easily

$$|v_k|_j^2+|v_k|_0^2\leqslant c_j\|u\|_{j,\Omega}^2 .$$

Now in view of Lemma 2.3, $\|u\|_{j,\Omega}^2\leqslant c_j'(|u|_{j,\Omega}^2+|u|_{0,\Omega}^2)$. Hence

$$|v_k|_j^2+|v_k|_0^2\leqslant c_jc_j'(|u|_{j,\Omega}^2+|u|_{0,\Omega}^2) .$$

Adding to this the evident inequality $|v_k|_0^2\leqslant C|u|_0^2$, multiplied by $(\lambda^{2j}-1)$, we have

$$\begin{aligned}|v_k|_j^2+\lambda^{2j}|v_k|_0^2&\leqslant c_jc_j'(|u|_{j,\Omega}^2+|u|_{0,\Omega}^2)+(\lambda^{2j}-1)C|u|_{0,\Omega}^2\\ &= c_jc_j'|u|_{j,\Omega}^2+\{c_jc_j'+(\lambda^{2j}-1)C\}|u|_{0,\Omega}^2 ,\end{aligned}$$

setting $c_2(j)=\max(c_jc_j', C)$,

$$\sum_{k=1}^{\infty}\{|v_k|_j^2+\lambda^{2j}|v_k|_0^2\}\leqslant c_2(j)(|u|_{j,\Omega}^2+\lambda^{2j}|u|_{0,\Omega}^2) .$$

Another inequality is proved in the same way as above by considering $u(x)=v_k(\Phi_k(x))$.

1) We apply 2) to the function $\eta_k^2(x)u(x)$. Then

$$|\eta_k^2v_k|_j^2+\lambda^{2j}|\eta_k^2v_k|_0\geqslant c_2(j)^{-1}(|\eta_k^2u|_{j,\Omega}^2+\lambda^{2j}|\eta_k^2u|_{0,\Omega}^2) .$$

On the other hand, by the interpolation lemma on Ω (see Lemma 2.3), we have

$$|\eta_k^2u|_{j,\Omega}^2+|\eta_k^2u|_{0,\Omega}^2\geqslant c_0\|\eta_k^2u\|_{j,\Omega}^2 ,$$

where c_0 is a small constant independent of k and δ. So using Lemma 2.2, we have

$$\begin{aligned}\sum_k(|\eta_k^2u|_{j,\Omega}^2+\lambda^{2j}|\eta_k^2u|_{0,\Omega}^2)&\geqslant c_0K_j^{-1}\|u\|_{j,\Omega}^2+(\lambda^{2j}-1)\sum_k|\eta_k^2u|_{0,\Omega}^2\\ &\geqslant c(j)(|u|_{j,\Omega}^2+\lambda^{2j}|u|_{0,\Omega}^2) , \qquad \lambda>1 .\end{aligned}$$

Thus

$$\sum_k(|\eta_k^2v_k|_{j,\Omega}^2+\lambda^{2j}|\eta_k^2v_k|_{0,\Omega}^2)\geqslant c_2(j)^{-1}c(j)(|u|_{j,\Omega}^2+\lambda^{2j}|u|_{0,\Omega}^2) .$$

3) Since $\eta_k(y)=\eta_k(\Psi_k(y))$, $v_k(y)=u(\Psi_k(y))$, representing the derivative $D_y^{\nu}(D_y^{\alpha}\eta_k(y)D_y^{\beta}\eta_k(y)D_y^{\gamma}v_k(y))$, $|\nu|=j$, as a linear combination of

$\{D_y^\alpha u(\Psi_k(y))\}_{|\alpha|\leqslant|\gamma|+|\nu|}$ with coefficients in the polynomials of $D_y^\mu \eta_k(\Psi_k(y))$, $D_y^\mu \Psi_k(y)$, we see easily the desired inequality.

Lemma 6.2. *We suppose* $(A.1)$–$(A.6)$. *Further, suppose that there exists a partition of unity* $\{\Phi_k, N_k, \eta_k, \delta\}$ *in Lemma* 2.1 *such that, for* $u\in H^{2m+s}(\Omega)$, *the inequalities*

$$(6.1)\qquad \begin{aligned}|\eta_k^2 v_k|_{2m+s}^2+|\lambda|^{2(2m+s)}|\eta_k^2 v_k|_0^2 &\leqslant C[|(A^{(k)}-\lambda^{2m})(\eta_k^2 v_k)|_s^2\\ &+\lambda^{2s}|(A^{(k)}-\lambda^{2m})(\eta_k^2 v_k)|_0^2+\sum_{j=1}^m(\langle B_j^{(k)}(\eta_k^2 v_k)\rangle_{2m+s-m_j-(1/2)}^2\\ &+|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_j^{(k)}(\eta_k^2 v_k)\rangle_0^2)]\end{aligned}$$

hold for $k=1, 2, \cdots$, *with some fixed constant* C *and* $|\lambda|\geqslant r_0$ *(independent of* k*). Then there exists* r_1 *such that the inequality* (1.1) *holds for* $|\lambda|\geqslant r_1$.

Proof. In view of Lemma 6.1, 2),

$$(6.2)\qquad \begin{aligned}&|(A^{(k)}-\lambda^{2m})(\eta_k^2 v_k)|_s^2+|\lambda|^{2s}|(A^{(k)}-\lambda^{2m})(\eta_k^2 v_k)|_0^2\\ &\leqslant c_2(s)[|(A-\lambda^{2m})(\eta_k^2 u)|_s^2+|\lambda|^{2s}|(A-\lambda^{2m})(\eta_k^2 u)|_0^2].\end{aligned}$$

Now

$$(6.3)\qquad (A-\lambda^{2m})(\eta_k^2 u)=\eta_k^2(A-\lambda^{2m})u+\varphi_k(x),$$

where

$$\varphi_k(x)=\sum_{\substack{|\alpha|+|\beta|+|\gamma|\leqslant 2m\\ |\gamma|\leqslant 2m-1}} c_{\alpha\beta}(x)D^\alpha\eta_k D^\beta\eta_k D^\gamma u.$$

Since $c_{\alpha\beta}(x)\in\mathscr{B}^s(\Omega)$;

$$(6.4)\qquad \sum_k(|\varphi_k|_0^2+|\lambda|^{2s}|\varphi_k|_0^2)\leqslant c_8 M\{||u||_{2m+s-1}^2+|\lambda|^{2s}||u||_{2m-1}^2\}.$$

Next, put

$$B_j^{(k)}(\eta_k^2 v_k)=\eta_k^2(B_j^{(k)}v_k)+\psi_{jk}(y),$$

where $\qquad \psi_{jk}(y)=\sum c_{jk\nu\beta}(y)D^\alpha\eta_k D^\beta\eta_k\gamma_k, \qquad |\gamma|\leqslant m_j-1.$

Then

$$(6.5)\qquad \begin{aligned}&\langle B_j^{(k)}(\eta_k^2 v_k)\rangle_{2m+s-m_j-(1/2)}^2+|\lambda|^{2(2m+s-m_j-(1/2))}\langle\eta_k^2(B_j^{(k)}v_k)\rangle_0^2\}\\ &\leqslant 2\{\langle\eta_k^2(B_j^{(k)}v_k)\rangle_{2m+s-m_j-(1/2)}^2+|\lambda|^{2(2m+s-m_j-(1/2))}\langle\eta_k^2(B_j^{(k)}v_k)\rangle_0^2\}\\ &+2\langle\psi_{jk}\rangle_{2m+s-m_j-(1/2)}^2+2|\lambda|^{2(2m+s-m_j-(1/2))}\langle\psi_{jk}\rangle_0^2.\end{aligned}$$

Now, by (3.4),

$$2\langle\psi_{jk}\rangle_{2m+s-m_j-(1/2)}^2\leqslant c|\psi_{jk}|_{2m+s-m_j}^2\leqslant c'M'\sum_{\substack{|\alpha|+|\beta|+|\gamma|\leqslant m_j\\ |\gamma|\leqslant m_j-1}}|D^\alpha\eta_k D^\beta\eta_k D^\gamma v_k|_{2m+s-m_j}^2$$

and making use of Lemma 3.1,

$$2|\lambda|^{2(2m+s-m_j-(1/2))}\langle\psi_{jk}\rangle_0^2\leqslant 2|\lambda|^{2(2m+s-m_j-1)}|\psi_{jk}|_1^2+2|\lambda|^{2(2m+s-m_j)}|\psi_{jk}|_0^2.$$

Adding in k, and making use of Lemma 6.1, 3),

$$(6.6)\quad 2\langle\psi_{jk}\rangle^2_{2m+s-m_j-(1/2)}+2|\lambda|^{2(2m+s-m_j-(1/2))}\langle\psi_{jk}\rangle^2_0 \\ \leqslant c''\{\|u\|^2_{2m+s-1}+|\lambda|^{2(2m+s-m_j-1)}\|u\|^2_{m_j}+|\lambda|^{2(2m+s-m_j)}\|u\|_{m_j-1}\}\,.$$

In view of Lemma 2.3, this is estimated again by

$$\varepsilon|u|^2_{2m+s}+c''C|\lambda|^{2(2m+s-1)}|u|^2_0\,,\qquad \text{for}\quad |\lambda|>r_1\,.$$

(6.2)–(6.6) and Lemma 5.1, 1) prove (1.1).

Proof of Theorem 1. Let $(A^{(k)}(y, D), B_j^{(k)}(y, D))$ be the transformed operators of $(A(x, D), B_j(x, D))$ by the transformation $\Phi_k(x)$. From Lemma 6.2, it remains to prove that $(A^{(k)}, B_j^{(k)})$ satisfies the assumptions (L.1)–(L.4) of §4 with constants $\gamma_1, \gamma_2, \gamma_3$ independent of k and δ, and that the quantities ζ_k become uniformly small as δ is taken small.

$$(6.7)\qquad A^{(k)}(y, D)=\sum_{|\mu|\leqslant 2m} a_\mu(\psi_k(y))\prod_{i=1}^n\Big(\sum_{j=1}^n\frac{\partial\Phi_{jk}}{\partial x_i}(\psi_k(y))D_j\Big)^{\mu_i}\,,$$

$$(6.8)\qquad B_j^{(k)}(y, D)=\sum_{|\mu|\leqslant m_j} b_{j\mu}(\psi_k(y))\prod_{i=1}^n\Big(\sum_{j=1}^n\frac{\partial\Phi_{jk}}{\partial x_i}(\Psi_k(y))D_j\Big)^{\mu_i}\,,$$

where $D_j=D_{y_j}$.

Let us introduce the linear transformation:

$$(6.9)\qquad \xi=T^{(k)}\eta\,,\qquad \text{where}\quad T^{(k)}=\left[\frac{\partial\Phi_{jk}}{\partial x_i}\right].$$

We interpret this as follows: $T^{(k)}$, more precisely, $T_y^{(k)}$ maps the vectors η at the point y to the vectors ξ at $x=\Psi_k(y)$. This mapping is isomorphism. We can say more:

1) $T^{(k)}\nu_y=c_k(y)\nu_x$, $x=\Psi_k(y)$, where $\nu_x(\nu_y)$ represents the normal of unit length at $x\in\Gamma$ (at $y\in S$, where S: $y_n=0$).

2) For any tangent vector η' at $y\in S$, $T^{(k)}\eta'$ is a tangent vector at $x=\Psi_k(y)$.

Evidently

$$(6.10)\qquad c^{-1}|\eta|\leqslant|T^{(k)}\eta|\leqslant c|\eta|\,,$$

where c is a positive constant independent of k and δ.
In particular

$$(6.11)\qquad c^{-1}\leqslant c_k(y)\leqslant c\,.$$

Hereafter we change slightly the notations. We dentoe the tangent vectors at $x\in\Gamma$ by ξ'_x or simply by ξ', and that of $y\in S$ by η'.

Let $A_0^{(k)}(y, D)$, $B_{j0}^{(k)}(y, D)$ be the principal part of $A^{(k)}(y, D)$ and $B_j^{(k)}(y, D)$ respectively. We have

$$A_0^{(k)}(y, \eta) = A_0(\Psi_k(y), T^{(k)}\eta), \; B_{j0}^{(k)}(y, \eta) = B_{j0}(\Psi_k(y), T^{(k)}\eta) .$$

Then it is easy to see that we can take γ_1 and γ_2 in (L.1) and (L.2) independently of k and δ, in view of (6.7), (6.8), and that the quantities ζ_k become uniformly small (in k) if δ is taken small. Consider now (L.3).

Let $z_i^{+(k)}(y,\eta', \lambda)$ be the roots with positive imaginary part of $A_0^{(k)}(y, \eta'+z\nu_y) -\lambda^{2m}=0$. Now, $A_0^{(k)}(y, \eta'+z\nu_y)=A_0(\Psi_k(y), T^{(k)}(\eta'+z\nu_y))=A_0(\Psi_k(y), T^{(k)}\eta' +c_k(y)z\nu_x)$, where $x=\Psi_k(y)$. Hence

$$z_i^{+(k)}(y, \eta', \lambda) = z_i^+(\Psi_k(y), T^{(k)}\eta', \lambda)c_k(y)^{-1} .$$

Hence (L.3) is satisfied. Consider

$$L_{ij}^{(k)}(y, \eta', \lambda) = \frac{1}{2\pi_i} \oint \frac{B_{i0}(y, \eta'+z\nu_y)z^{j-1}}{\prod_{i=1}^{m} (z- z_i^{+(k)}(y, \eta', \lambda))} dz .$$

Since $B_{i0}^{(k)}(y, \eta'+z\nu_y)=B_{i0}(\Psi_k(y), T^{(k)}\eta'+c_k(y)z\nu_x)$, putting $c_k(y)z=z'$ in the integral, we have

$$L_{ij}^{(k)}(y, \eta', \lambda) = c_k(y)^{m-j}L_{ij}(\Psi_k(y), T^{(k)}\eta', \lambda) .$$

Hence,
$$\begin{aligned} |\det L_{ij}^{(k)}(y, \eta', \lambda)| &= c_k(y)^{N'}| \det L_{ij}(\Psi_k(y), T^{(k)}\eta', \lambda)| \\ &\geqslant c_k(y)^{N'}\gamma_3(|T^{(k)}\eta'|^2+|\lambda|^2)^{N/2} . \end{aligned}$$

Now, in view of (6.10) and (6.11), we see that (L.4) is verified with a constant independe of k and δ. Finally, by (6.7) and (6.8), we see that the quantities M_k (in $\Phi_k(N_k)$) introduced just before Proposition 5.1 are bounded sequences (for each fixed δ).

In conclusion, to apply Lemma 6.2, we choose a partition of unity $\{\Phi_k(x), N_k, \delta\}$ as follows: Since γ_1, γ_2 and γ_3 can be considered independent of k and δ, the ζ_0 in Propositon 5.1 can be considered independent of k and δ. Thus we choose δ in such a way that $\zeta_{\delta,k}$ introduced before Proposition 5.1 (see the remark at the end of that proposition) be less than ζ_0 for all k.

7. A priori estimates for adjoint systems

Let Ω be a bounded domain of class C^{2m+s+1} (which is necessarily uniformly regular of class C^{2m+s+1}) and assume that (A, B_j) satisfies the assumptions of Theorem 2. Then we know that there exists another boundary system $\{B_j'\}$, $j=1, 2, \cdots, m$, which is called an adjoint system, such that, for all u, $v\in H^{2m}(\Omega)$, the identity

$$(7.1) \quad ((A-\lambda^{2m})u, v)-(u, (A^*-\bar{\lambda}^{2m})v) = \sum_{j=1}^{m} (B_ju, C_jv)_\Gamma+ \sum_{j=1}^{m} (C_j'u, B_j'v)_\Gamma$$

holds, where $C_j(x, D)$ and $C_j'(x, D)$ are also differential operators. More precisely, $(\{B_j\}_{j=1,\cdots,m}, \{C'_j\}_{j=1,\cdots,m})$ and $(\{B'_j\}_{j=1,\cdots,m}, \{C_j\}_{j=1,\cdots,m})$ form two Dirichlet systems (in the sense of Aronszajn-Milgram [3]). The form $(u, v)_\Gamma$ means $\int_\Gamma u\bar{v}\,dS$.

We know that $(A^*-\bar{\lambda}^{2m}, \{B'_j\}_{j=1,2,\cdots,m})$ satisfies also the assumptions of Theorem 1 (see N. Aronszajn [3] or M. Schechter [9]). In this section we shall show that the above fact remains true in our case and derive an a priori estimate for such an adjoint system, which is used in proving Theorem 2. Our statement is as follows:

Proposition 7.1. *Under the assumptions of Theorem* 2, *there exists another system* $\{B'_j\}$ (*which is called an adjoint system*) *satisfying* (7.1), *and that* $(A^*-\bar{\lambda}^{2m}, B'_j)$ *satisfies the assumptions of Theorem* 1 (*the constants appearing there may be different*). B'_j *are all independent of* λ.

Before proving this proposition, we begin with some remarks.

Let $A(D_t)$ be a differential operator of order $2m$ such that $A(\tau)\neq 0$ for $\tau\in E^1$. We assume that $A(\tau)=0$ has m roots τ_j^+ (counting the multiplicities) with positive imaginary part (therefore m roots τ_j^- with negative imaginary part). Given $\{B_j(D_t)\}_{j=1,2,\cdots,m}$ satisfying the following condition: $B_j(\tau)$ are lineary independent modulo $A_+(\tau)=\prod_{j=1}^{m}(\tau-\tau_j^+)$, and assume that the order m_j of $B_j(\tau)$ is less than $2m$. Then

Lemma 7.1. *Suppose that there exists another m differential operators* $B'_j(D_t)$ *such that the identity*

$$(7.2)\qquad (A(D_t)u, v)-(u, A^*(D_t)v) = \sum_{j=1}^{m} B_j(D_t)u(0)C_j(D_t)v(0) + \sum_{j=1}^{m} C'_j(D_t)u(0)B'_j(D_t)v(0)$$

holds for all $u(t), v(t)\in H^\infty(E^1_+)$. *Then* $\{B'_j(\tau)\}_{j=1,\cdots,m}$ *are linearliy independent modulo* $\bar{A}_-(\tau)=\prod_{j=1}^{m}(\tau-\overline{\tau_j^-})$.

Proof. Suppose that $B'_j(\tau)$ are linearly dependent modulo $\bar{A}_-(\tau)$. Then it is easy to see that there exists a function $v_0(t)\in H^\infty(E^1_+)$, $v_0(t)\not\equiv 0$ such that $A^*(D_t)v_0(t)=0$, $B'_j(D_t)v_0(0)=0$. Now we know that, under the assumptions, there exists a (unique) function $u(t)\in H^\infty(E^1_+)$ such that $A(D_t)u(t)=v_0(t)$, $B_j(D_t)u(0)=0$. This contradicts with (7.2).

Now we consider the simplest case in E^n. The boundary is $x_n=0$, and $\Omega=E^n_+=\{x_n>0\}$. Let $A(D)$, $\{B_j(D)\}_{j=1,2,\cdots,m}$ be differential operators of homogeneous order $2m$, $m_j(<2m)$, respectively. More precisely,

$$(7.3)\qquad A(D)=\sum_{|\mu|=2m}a_\mu D^\mu = a_0D_n^{2m}+a_1(D')D_n^{2m-1}+\cdots+a_{2m}(D')$$

$$B_j(D)=\sum_{|\mu|=m_j}b_\mu D^\mu = b_{j0}D_n^{m_j}+b_{j1}(D')D_n^{m_j-1}+\cdots+b_{j,m_j}(D'),$$

where $D'=(D_1,\cdots,D_{n-1})$. We assume the same conditions as in Theorem 1:

$$(7.4)\qquad |A(\xi)-\lambda^{2m}|\geqq\gamma_1(|\xi|^2+|\lambda|^2)^m,\quad \lambda=re^{i(\theta/2m)}.$$

Let $\tau_j^+(\xi',\lambda)$ be the m roots with positive imaginary part of $A(\xi',z)-\lambda^{2m}=0$. $B_j(\xi',z)$ are assumed lineary independent modulo $A_+(z)=\prod_{j=1}^m(z-\tau_j^+(\xi',\lambda))$. More precisely, we assume that the Lopatinski's determinant satisfies

$$(7.5)\qquad |\det L_{jk}(\xi',\lambda;A-\lambda^{2m},B_j)|\geqq\gamma_2(|\xi'|^2+|\lambda|^2)^{N/2}.$$

Now we construct $\{B_j'\}$ in the following way (see Schechter [9]). Take $u,v\in H^{2m}(E_+^n)$. The integration by parts gives

$$(7.6)\qquad (A(D)u,v)-(u,A^*(D)v)=i\sum_{k=0}^{2m-1}(D_n^ku,N_{2m-1-k}v)_{E^{n-1}},$$

where

$$(7.7)\qquad N_{2m-1-k}(D)=\sum_{j=0}^{2m-1-k}\bar a_{2m-1-k-j}(D')D_n^j.$$

Now we add the differential operators $\{D_n^{m_i'}\}_{i=1,2,\cdots,m}$ $(m_i'<2m)$ to the given system $\{B_j(D)\}$, where $\{m_1,\cdots,m_m\}\cap\{m_1',\cdots,m_m'\}=\phi$. By rearrangement, we may assume that the order of B_j is j. We denote the original set by $\{B_{m_j}\}_{j=1,\cdots,m}$ and the added set by $\{B_{m_j}\}_{j=m+1,\cdots,2m}$. This rearrangement made, let

$$B_j=b_{j0}(D')+b_{j1}(D')D_n+\cdots+b_{j,j-1}(D')D_n+b_jD_n^j\qquad (j=0,\cdots,2m-1).$$

Then we have

$$(7.8)\qquad D_n^j=d_{j0}(D')B_0+d_{j1}(D')B_1+\cdots+d_jB_j,$$

where $d_{ij}(\xi')$ are uniquely determined. More precisely they are polynomials of b_j^{-1}, $b_{ij}(\xi')$ $(i,j=0,1,\cdots,2m-1)$. If we put

$$(7.9)\qquad C_j(D)=-i\sum_{k=j}^{2m-1}\bar d_{kj}(D')N_{2m-1-k}(D',D_n),$$

we have

$$(7.10)\quad (Au,v)-(u,A^*v)=\sum_{j=0}^{2m-1}(B_ju,C_jv)=\sum_{j=1}^m(B_{m_j}u,C_{m_j}v)+\sum_{j=m+1}^{2m}(B_{m_j}u,C_{m_j}v).$$

If we put $B_j'=C_{m_{j+m}}$, $C_j'=B_{m_{j+m}}$, $C_j=C_{m_j}$, $(j=1,2,\cdots,m)$ and using the original index, we can write (7.10) under the form

$$(7.11)\qquad (Au, v)-(u, A^*v) = \sum_{j=1}^{m} (B_j u, C_j v)+ \sum_{j=1}^{m} (C'_j u, B'_j v) .$$

Clearly, $(B_j, C'_j)_{j=1,2,\cdots,m}$ and $(B'_j, C_j)_{j=1,2,\cdots,m}$ form two Dirichlet systems.

Now we claim

Lemma 7.2. *Let us assume* (7.4) *and* (7.5), *and that*

$$|a_\mu|,\ |b_{j\mu}|,\ |b_j^{-1}|\leqslant M .$$

Then there exists a positive constant γ'_2 depending only on M, γ_1 and γ_2 such that, for all $\{A, B_j\}$ satisfying those conditions, the adjoint system $\{A^*-\bar\lambda^{2m}, B'_j\}$ thus constructed satisfies

$$(7.12)\qquad |\det L_{jk}(\xi', \lambda;\ A^*-\bar\lambda^{2m}, B'_j)| \geqslant \gamma'_2(|\xi'|^2+|\gamma|^2)^{N/2} .$$

Proof. The above construction shows that, if we put $A(D_t)=A(\xi', D_n) -\lambda^{2m}$, $B_j(D_t)=B_j(\xi', D_t)$, then the identity (7.2) holds for all ξ', λ. This means that the left-hand side of (7.12) does not vanish for all $(\xi', \lambda)\neq 0$. Since this is a continuous function of (ξ', λ), and homogeneous in (ξ', λ), we conclude that, there exists a positive minimum when $|\xi'|^2+|\lambda|^2=1$, for each fixed (A, B'_j).

Now we consider the space of the points $P=\{a_\mu, b_{j\mu}\}$, in the cartesian coordinates. The set satisfying the conditions mentioned above forms a compact set F in that space. Next, let us remark that the polynomials $B'_j(\xi', \xi_n)$ are all continuous functions of P. Thus, the mapping defined for $|\xi'|^2+|\lambda|^2=1$, and $P\in F$,

$$(\xi', \lambda, P) \rightarrow |\det L_{jk}(\xi', \lambda;\ A^*-\bar\lambda^{2m}, B'_j)|$$

is continuous. We can see that there exists $\gamma'_2>0$ satisfying (7.12).

Proof of Proposition 7.1. Since Γ is supposed to be unifromly regular, we consider at first the local construction of adjoint systems by using the mappings $\Phi_k(x)$. We shall construct the adjoint system $B'^{(k)}_j(x, D)$ $(j=1, 2, \cdots, m)$ in each N_k. However, we shall construct them in such a way that their principal parts $B'^{(k)}_{j0}(x, D)$ coincide with each other. Namely, if $x\in N_k\cap N_l\cap\Gamma$, then we have $B'^{(k)}_{j0}(x, D)=B'^{(l)}_{j0}(x, D)(j=1, 2, \cdots, m)$. For that purpose, we consider an open set N_k, and the mappying $y=\Phi_k(x)$ $(x=\Psi_k(y))$. Hereafter, we omit the suffix k.

By assumptions, $\Phi(x)$ satisfies the conditions;

1) $x\in\Gamma\Leftrightarrow\Phi_n(x)=0$ (that is $y_n=0$),

2) To the normal directions at x corresponds the normal direction at S: $y_n=0$.

Now

$$(7.13)\qquad \frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial\Phi_j}{\partial x_i}\frac{\partial}{\partial y_j} = \sum_{j=1}^{n} \Phi_{j,i}\frac{\partial}{\partial y_j} .$$

Attached to this, we consider the linear mapping

$$\xi_i = \sum_j \Phi_{j,i}\eta_j \,, \tag{7.14}$$

or more simply

$$\xi = T\eta \,. \tag{7.15}$$

We interpret this in the following way: For the vector η at the point y, $T\eta$ represents the vector at the point $x=\Psi(y)$.

We see easily the following facts;

1) For $y\in S$, let ν_y be the normal of unit length at the point y, then $T\nu_y = c(x, T)\nu_x$, where ν_x is the normal of unit length at the point $x=\Psi(y)$ and $c(x, T)$ is a scalar.

2) Let η'_y be a tangent vector at $y\in S$, then $T\eta'_y=\xi'_x$, where ξ'_x is a tangent vector at $x\in\Gamma$.

3) Let $|T|=\det\Phi_{j,i}$. We may assume it to be positive. Then $J=\dfrac{dx}{dy} =|T|^{-1}$. We see that $dS=c(x, T)Jdy'$, where $c(x, T)$ is the quantity introduced in 1), and dS is the surface element of Γ.

Now let $A(x, D)$ be a differential operator of order $2m$. Let us denote its principal part by $A^0(x, D)$ and its transformed operator by $\mathcal{A}(y, D)$. We know that the principal part $\mathcal{A}^0(y, D)$ is defined by

$$\mathcal{A}^0(y, \eta) = A^0(x, T\eta). \tag{7.16}$$

More precisely,

$$\mathcal{A}^0(y, \eta) = A^0(\Psi(y), T\eta)\,. \tag{7.17}$$

Moreover, if $x\in\Gamma$, then

$$A^0(x, T(\eta_n\nu_y+\eta')) = A^0(x, \eta_n T\nu_y+T\eta')\,.$$

In view of 1) and 2), we see that, if

$$A^0(x, z\nu_x+\xi') = a_0(x)z^m+a_1(x, \xi')z^{m-1}+\cdots+a_m(x, \xi')\,, \tag{7.18}$$

then

$$\mathcal{A}^0(y, \eta) = a_0(y)(c(x, T)\eta_n)^m+a_1(y, T\eta')(c\eta_n)^{m-1}+\cdots+a_m(y, T\eta'), \quad y\in S. \tag{7.19}$$

Conversely, given a homogeneous differential operator $a(y, D')$ where $D'=(D_{y_1}, \cdots, D_{y_{n-1}})$, if $a(y, \eta')$ can be written as

$$a(y, \eta') = h(\Psi(y), T\eta') \tag{7.20}$$

with a homogeneous polynomial $h(x, \xi)$, then the operator $a(y, D')$ is equal, neglecting the lower terms, to the transformed operator $h(x, D'_\xi)$. In this sense, we shall call the polynomial $a(y, \eta')$ of the form (7.20) *quasi-invariant*. Let us remark that the product of two quasi-invariant polynomials is quasi-invariant. Finally, the transformed operator D_ν is $c(\Phi(x), T)D_n$.

Now we construct the adjoint system. We proceed just in the same way as in the simplest case explained as above. Let us write

$$\mathcal{A}(y, D', D_n) = \sum_{k=0}^{2m} \mathcal{A}_{2m-k}(y, D')(cD_n)^k .$$

We observe that

$$\mathcal{A}^0_{2m-k}(y, \eta') = a_{2m-k}(\Psi(y), T\eta') , \qquad y \in S .$$

$\mathcal{A}^0_{2m-k}(y, \eta')$ is quasi-invariant.

$$(Au, v)-(u, A^*v) = (\mathcal{A}(y, D)u, v)_J-(u, \mathcal{A}^*(y, D)v)_J ,$$

where $(\,,\,)_J$ means the scalar product with the measure Jdy. By integration by parts,

$$= i \sum_{k=0}^{2m-1} ((cD_n)^k u, N_{2m-1-k}v)_{S,cJ} ,$$

where $(\,,\,)_{cJ}$ means the integration on S with the surface measure $cJdy'$ and

$$N_{2m-k-1}v = \sum_{j=0}^{2m-1-k} J^{-1}(cD_n)^j \mathcal{A}^*_{2u-1-k-j}(y, D)(Jv) .$$

Let us remark that

$$N^0_{2m-1-k}(y, \eta', cD_n) = \sum_{j=0}^{2m-1-k} \mathcal{A}^0_{2m-1-k-j}(y, \eta')(cD_n)^j . \tag{7.21}$$

Next, we add to the system $\{B_j\}$, the m differential poerators $\{D_\nu^{m'_i}\}_{i=1,2,\cdots,m}$ $(m'_i<2m)$, where $\{m_2, \cdots, m_m\} \cap \{m'_1, \cdots, m'_m\} = \phi$. By rearrangement, we denote this by $\{B_j(x, D)\}_{j=0,\cdots,2m-1}$. We may assume that the order of B_j is j. Let $\mathcal{B}_j(y, D)$ be the transformed operator of B_j. Then

$$\mathcal{B}_j = \mathcal{B}_{j0}(y, D')+\mathcal{B}_{j1}(y, D)(cD_n)+ \cdots +b_j(y)(cD_n)^j . \tag{7.22}$$

Hence

$$(cD_n)^j = \sum_{i=0}^{j} \Lambda_{ji}(y, D')\mathcal{B}_i(y, D', cD_n) , \tag{7.23}$$
$$\Lambda_{jj}(y, D') = b_j(y)^{-1}, \ (j=0, 1, \cdots, 2m-1) ,$$

where $\Lambda^0_{ij}(y, \eta)$ are also quasi-invariant, for $\mathcal{B}^0_{ij}(y, \eta')$ are all so. Put

$$C_j(v) = -i(cJ)^{-1} \sum_{k=j}^{2m-1} \Lambda^*_{kj}(y, D')(cJN_{2m-1-k}v) .$$

Then

(7.24) $$(Au, v)-(u, A^*v) = \sum_{j=0}^{2m-1} (\mathcal{B}_j u, C_j v)_{S,cJ}\,.$$

Let us remark that

(7.25) $$C_j^0(y, \eta', cD_n) = -i\sum_{k=j}^{2m-1} \Lambda_{kj}^0(y, \eta')N_{2m-1-k}^0(y, \eta', cD_n)\,.$$

Denote by C_j the transformed operator of C_j. We observe that the principal part of C_j is invariant with respect to Φ.

This remarked, write (7.24) in the scalar product in the x-space:

$$\sum_{j=0}^{2m-1} (B_j u, C_j v)_\Gamma = \sum_{j=1}^{m} (B_{m_j} u, v) + \sum_{j=m+1}^{2m} (B_{m_j} u, C_{m_j} v)\,.$$

Up to now, we omitted the suffix k. Now we put the suffix k. Put

$$B_j'^{(k)} = C_{m_{j+m}}\,,\quad C_j' = B_{m_{j+m}}\,,\quad C_j^{(k)} = C_{m_j} \qquad (j=1, 2, \cdots, m),$$

then observing $B_j = B_{m_j}$ $(j=1, 2, \cdots, m)$ in the original notation, we can write

(7.26) $$(Au, v)-(u, A^*v) = \sum_{j=1}^{m} (B_j u, C_j^{(k)} v) + \sum_{j=1}^{m} (C_j' u, B_j'^{(k)} v)\,,$$

for v having its support in N_k.

From the above construction, we observe the following fact: In order to know the principal parts of $B_j'^{(k)}(x, D)$ $(j=1, \cdots, m)$ at the point $x_0 \in N_k$, it suffices to construct the adjoint system by the process explained just before Lemma 7.2, replacing there $A(x, D)$, $B_j(x, D)$ by $A^0(x_0, D)$, $B_j^0(x_0, D)$ and Γ by the tangent plane at x_0. For this reason, we dentoe the principal part of $B_j'^{(k)}(x, D)$ simply by $B_{j0}'(x, D)$. This is the same with $C_j^{(k)}$.

Now

$$\begin{aligned}(Au, v)-(u, A^*v) &= \sum_k (Au, \eta_k^2 v) - \sum_k (u, A^*(\eta_k^2 v)) \\ &= \sum_{j=1}^{m} (B_j u, \sum_k C_j^{(k)}(\eta_k^2 u)) + \sum_{j=1}^{m} (C_j' u, \sum_k B_k'^{(k)}(\eta_k^2 v)\,.\end{aligned}$$

Put $B_j' v = \sum_k B_j'^{(k)}(\eta_k^2 v)$, $C_j v = \sum_k C^{(k)}(\eta_k^2 v)$. Then we see that (7.1) holds. We see also that the principal part of B_j' is nothing but $B_{j0}'(x, D)$. Then, by virtue of Lemma 7.2, we see that the condition (A.4) is satisfied for $(A^* - \bar{\lambda}^{2m}, B_j')$. Since it is easy to check the other conditions, we omit it.

From Proposition 7.1, we have the following

Theorem 1*. *Under the assumptions* $(A^*, 1)$-$(A^*, 6)$, *there exist constants* $c^* > 0$ *and* $r_1^* > 0$ *such that for* $u \in H^{2m+s}(\Omega)$,

$$\begin{aligned}|u|_{2m+s}^2 + |\lambda|^{2(2m+s)}|u|_0^2 \leqslant c[&|(A-\bar{\lambda}^{2m})u|_s^2 + |\lambda|^{2s}|(A-\bar{\lambda}^{2m})u|_0^2 \\ &+ \sum_{j=1}^{m} (\langle B_j' u\rangle_{2m+s-m_j-(1/2)}^2 + |\lambda|^{2(2m+s-m_j'-(1/2))}\langle B_j' u\rangle_0^2)]\end{aligned}$$

if $|\lambda| \geqslant r_1^*$, *where* $\lambda = re^{i(\theta/2m)}$ *and* m_j' *is the order of* B_j'.

8. Existence theorem (Proof of Theorem 2)

In this section we give the proof of Theorem 2. We use the method of the proof employed by M. Schechter [9]. But we need one lemma on the global regularity of solutions of our boundary value problems.

Lemma 8.1. *Suppose that* $(A-\lambda^{2m}, \{B_j\}_{j=1}^m)$ *satisfies the assumptions of Theorem* 1. *Then, if* $(A-\lambda^{2m})u=f\in H^s(\Omega)$, $u\in L^2(\Omega)\cap H^{2m+s}(\Omega\cap N_k)$ *for each* N_k *and* $B_ju=0$ $(j=1, 2, \cdots, m)$, *it follows that* $u\in H^{2m+s}(\Omega)$.
This lemma is an extension of Theorem 3 in Browder [4], in the case of Dirichlet boundary value problem. The proof of Lemma 8.1 will be given at the end of this section.

Proof of Theorem 2. Consider the quadratic form in $H^{2m}(\Omega)$;

$$[u, v] = ((A^*-\bar{\lambda}^{2m})u, (A^*-\bar{\lambda}^{2m})v)+\sum_{j=1}^m (\langle B_j'u, B_j'v\rangle_{2m-m_j'-(1/2)} + |\lambda|^{2(2m-m_j'-(1/2))}\langle B_j'u, B_j'v\rangle_0)\,.$$

From Theorem 1* for $s=0$, we have

$$C^{-1}\|u\|_{2m}^2 \leqslant [u, u] \leqslant C\|u\|_{2m}^2\,, \tag{8.1}$$

and it follows from (8.1) that $[u, u]^{1/2}$ defines a norm equivalent to the usual one of $H^{2m}(\Omega)$. We denote by $\mathfrak{H}$ the space $H^{2m}(\Omega)$ equipped with the positive Hermitian form $[u, v]$. $\mathfrak{H}$ is a Hilbert space and its elements and topology coincide with those of $H^{2m}(\Omega)$. On the other hand, for $f\in H^s(\Omega)$, $v\to(f, v)$ is a bounded anti-linear functional on $\mathfrak{H}$. Hence, by the Theorem of Riesz, there exists a unique $w\in\mathfrak{H}(=H^{2m}(\Omega))$ such that

$$[w, v] = (f, v) \qquad \text{for all} \quad v\in H^{2m}(\Omega)\,. \tag{8.2}$$

On the other hand, if we consider the problem in each N_k, it is known that $w\in H^{4m+s}(\Omega\cap N_k)$ (see, M. Schechter [9]. Theorem 6.1). Then, put

$$u = (A^*-\bar{\lambda}^{2m})w\in L^2(\Omega)\cap H^{2m+s}(\Omega\cap N_k)\,.$$

Take a function $v\in C_0^{2m}(N_k)$ which satisfies $B_j'v|_\Gamma=0$. Then (8.2) becomes

$$(u, (A^*-\bar{\lambda}^{2m})v) = (f, v)\,.$$

Since $u\in H^{2m+s}(\Omega\cap N_k)$, we can apply the identity (7.1). Then taking account of $B_j'v|_\Gamma=0$ $(j=1, 2, \cdots, m)$, the left-hand side is equal to

$$((A-\lambda^{2m})u, v)-\sum_{j=1}^m (B_ju, C_jv)_\Gamma\,.$$

Supposing in particular that $v\in C_0^{2m}(\Omega\cap N_k)$, we conclude at first that

$$(A-\lambda^{2m})u=f, \qquad x\in\Omega\cap N_k \text{ (hence } x\in\Omega).$$

Therefore,

$$\sum_{j=1}^{m}(B_j u, C_j v)_\Gamma=0 \qquad \text{for all} \quad v\in C_0^{2m}(N_k)$$

satisfying $B'_j v|_\Gamma=0$.

Since $\{B'_j, C_j\}_{j=1}^m$ forms a Dirichlet system, we see that $B_j u|_\Gamma=0$ ($j=1, 2, \cdots, m$) for all $x\in\Gamma\cap N_k$, and hence for all $x\in\Gamma$. Finally, applying Lemma 8.1, we see that $u\in H^{2m+s}(\Omega)$.

We conclude this section with the

Proof of Lemma 8.1. From the assumptions of the lemma, it follows that $u\in H^{2m+s}(K)$ for any compact set $K\subset\Omega$. From Lemma 2.1, $\bigcup_k N'_k\supset\Omega$, where $N'_k=\Psi_k(B_{\rho\delta})$. Let $N_0=\{y;\ |y_i|<1,\ i=1, 2, \cdots, n\}$, $N_0^+=\{y;\ |y_i|<1,\ i=1, 2, \cdots, n-1,\ 1>y_n>0\}$ and $B_0=\{y;\ |y|<n^{1/2}+1\}$.
Then, there exists a homeomorphism Ψ_0 of class C^{2m+s} of B_0 on B_δ carrying $B_{1/2}$ onto $B_{\rho\delta}$ with an inverse Φ_0 of class C^{2m+s}. Furthermore, if we define

$$\zeta(y)=\begin{cases}\prod_{j=1}^{n}(1-y_j^2) & y\in N_0\\ 0 & y\notin N_0\end{cases}$$

and if we set

$$\zeta_k(x)=\begin{cases}\zeta(\Phi_0(\Phi_k(x)) & x\in N_k\\ 0 & x\notin N_k,\end{cases}$$

it follows that $\zeta_k^{2m+s}u\in H^{2m+s}(\Omega\cap N_k)$. Then the following estimate holds (see, Browder [4]).

$$\sum_{|\beta|=j}\|\zeta_k^j D^\beta u\|_0^2\leqslant\varepsilon\sum_{|\beta|\leqslant 2m+s}\|\zeta_k^\beta D^\beta u\|_0^2+K_\varepsilon\|u\|_{L^2(N_k\cap\Omega)}^2, \tag{8.3}$$

where ε is an arbitrary positive number, and K_ε is independent of k. From Theorem 1, we have

$$\begin{aligned}\|\zeta_k^{2m+s}u\|_{2m+s}^2+|\lambda|^{2(2m+s)}\|\zeta_k^{2m+s}u\|_0^2\leqslant C[\|(A-\lambda^{2m})(\zeta_k^{2m+s}u)\|_s^2\\ +|\lambda|^{2s}\|(A-\lambda^{2m})(\zeta_k^{2m+s}u)\|_0^2+\sum_{j=1}^{m}(\langle B_j(\zeta_k^{2m+s}u)\rangle_{2m+s-m_j-(1/2)}^2\\ +|\lambda|^{2(2m+s-m_j-(1/2))}\langle B_j(\zeta_k^{2m+s}u)\rangle_0^2)].\end{aligned} \tag{8.4}$$

Since $\quad D^\alpha(\zeta_k^{2m+s}u)=\zeta_k^{2m+s}D^\alpha u+\sum_{|\beta|\leqslant|\alpha|-1}C_{\alpha\beta}(x)\zeta^{|\beta|}D^\beta u, \qquad |\alpha|\leqslant 2m+s,$

we have

$$\sum_{|\alpha|\leqslant 2m+s}\|D^\alpha(\zeta_k^{2m+s}u)\|_0^2\geqslant 2\sum_{|\alpha|\leqslant 2m+s}\|\zeta_k^\alpha D^\alpha u\|_0^2-K_1\sum_{|\beta|\leqslant 2m+s-1}\|\zeta_k^{|\beta|}D^\beta u\|_0^2,$$

where K_1 is a constant independent of k.

Similarly,

$$\sum_{|\alpha|\leqslant s}\|D^{\alpha}(A-\lambda^{2m})(\zeta_k^{2m+s}u)\|_0^2\leqslant 2\sum_{|\alpha|\leqslant s}\|\zeta_k^{2m+s}D^{\alpha}(A-\lambda^{2m})u\|_0^2+K_2\sum_{|\beta|\leqslant 2m+s-1}\|\zeta_k^{|\beta|}D^{\beta}u\|_0^2.$$

On the other hand,

$$B_j(\zeta_k^{2m+s}u)=\zeta_k^{2m+s}B_ju+\sum_{|\beta|\leqslant m_j-1}d_{j\beta}(x)\zeta_k^{2m+s-m_j+|\beta|}D^{\beta}u.$$

Since $B_ju|_{\Gamma}=0$, and taking account of (3.10), we have

$$\begin{aligned}\langle B_j(\zeta_k^{2m+s}u)\rangle^2_{2m+s-m_j-(1/2)}&\leqslant C\sum_{|\beta|\leqslant m_j-1}\|d_{j\beta}\zeta_k^{2m+s-m_j+|\beta|}D^{\beta}u\|^2_{2m+s-m_j}\\&\leqslant K_2\sum_{|\beta|\leqslant 2m+s-1}\|\zeta_k^{\beta}D^{\beta}u\|_0^2.\end{aligned}$$

Since $\langle B_j(\zeta_k^{2m+s}u)\rangle_0^2$ is estimated in the same way taking account of Lemma 3.1, we have finally

$$\sum_{|\alpha|\leqslant 2m+s}\|\zeta_k^{|\alpha|}D^{\alpha}u\|_0^2\leqslant C_1\sum_{|\alpha|\leqslant s}\|\zeta_k^{2m+s}D^{\alpha}(A-\lambda^{2m})u\|_0^2+C_2\sum_{|\beta|\leqslant 2m+s-1}\|\zeta_k^{|\beta|}D^{\beta}u\|_0^2.$$

Since $0\leqslant\zeta_k(x)\leqslant 1$, by using (8.3), we obtain

$$\sum_{|\alpha|\leqslant 2m+s}\|\zeta_k^{|\alpha|}D^{\alpha}u\|_0^2\leqslant C(\|(A-\lambda^{2m})u\|^2_{H^s(\Omega\cap N_k)}+\|u\|^2_{L^2(\Omega\cap N_k)}).\tag{8.4}$$

Since $\zeta_k(x)\geqslant\kappa>0$ for $x\in N'_k$, where κ is a constant independent of k, adding in k, we have $\|u\|_{2m+s}<+\infty$.

Appendix

To justify Definition 3.1, more precisely to define the space $H^{j-(1/2)}(\Gamma)$, we want to prove a theorem which is known when Γ is compact.

Given any two partitions of untiy $\{N_k, \Phi_k, \eta_k\}$ and $\{N'_k, \Phi'_k, \zeta'_k\}$ satisfying the following conditions:

1) There exists an integer R_1 (resp. R_2) such that at most R_1 (resp. R_2) distinct numbers of N_k (resp. of N'_k) have a non-empty intersection.

2) Φ'_k satisfies the same conditions as (2) of Lemma 2.1 with δ and K. Φ'_k satisfies also the same conditions with δ', K'.

3) There exists $\theta_1<1$ (resp. $\theta_2<1$) such that $\eta_k(\Psi_k(y))$ (resp. $\zeta_k(\Psi'_k(y))$ has its support in $B_{\theta_1\delta}$ (resp in $B_{\theta_2\delta'}$) ($k=1, 2, \cdots$).

4) $\sum_k|D^{\alpha}\eta_k(x)|^2|D^{\beta}\eta_k(x)|^2\leqslant K_1$, $\sum_k|D^{\alpha}\zeta_k(x)|^2|D^{\beta}\zeta_k(x)|^2\leqslant K_2$, for $|\alpha|+|\beta|\leqslant 2m+l$.

5) $0\leqslant\eta_k(x)$, $\zeta_k(x)\leqslant 1$, $\sum\eta_k^2(x)=1$, $\sum\zeta_k^2(x)=1$, for $x\in\bar{\Omega}$.

Then we can define two different semi-norms $\langle u\rangle_{j-(1/2)}$ following Definition 3.1. Let us denote them by $\langle u\rangle_{j-(1/2),1}$ and $\langle u\rangle_{j-(1/2),2}$ respectively. Now we claim

Proposition. *Two norms* $\langle u\rangle^2_{j-(1/2),1}+\sum_{k=1}^{\infty}|\eta_k^2u|_0^2$ *and* $\langle u\rangle^2_{j-(1/2),2}+\sum_{k=1}|\zeta_k^2u|_0^2$ *are equivalent to each other.*

To prove this, we prepare several lemmas.

Lemma A.1. *Let $u(x)$ be a function defined on Γ such that* $\operatorname{supp}[u]\subset N_i\cap N'_k\cap\Gamma$, *then*

$$c\,|u(\Psi'_k(y))|_s\leqslant|u(\Psi_i(y))|_s\leqslant c'\,|u(\Psi'_k(y))|_s\,,$$

where $0<s<1$ and c, c' are two positive constants independent of i, k.

Proof. We change slightly the notations. Let $v(y)=u(\Psi_i(y))$, $z=\Phi'_k(x)$, and $y_{ik}(z)=\Phi_i\circ\Phi_k'^{-1}=\Phi_i\,(\Phi'_k(z))$ and let us denote $\tilde{v}(z)=v(y_{ik}(z))$. Now we know that in E^{n-1} the s-norm $(0<s<1)$ has the relation

$$\int|\eta|^{2s}|\hat{v}(\eta)|^2d\eta=c_{n-1,s}\int_{E^{n-1}}\int_{E^{n-1}}\frac{|v(y_1)-v(y_2)|^2}{|y_1-y_2|^{n-1+s}}dy_1dy_2\,,$$

which can be seen by taking the Fourier transform of $v(y_2)$ after changing y_1 by $y=y_1-y_2$. Denote $y_{ik}(z_1)=y_1$, $y_{ik}(z_2)=y_2$, $dy=J_{ik}(z)dz$. Then the right-hand side can be written

$$\iint\frac{|\tilde{v}(z_1)-\tilde{v}(z_2)|^2}{|y_{ik}(z_1)-y_{ik}(z_2)|^{n-+s}}|J_{ik}(z_1)J_{ik}(z_2)|\,dz_1dz_2\,.$$

Now $\qquad c_0^{-1}|z_1-z_2|\leqslant|y_1-y_2|\leqslant c_0|z_1-z_2|\,,\quad c_1^{-1}\leqslant|J_{ik}(z)|\leqslant c_1\,,$

where c_0, c_1 are constants independent of i, k. Thus the lemma is proved.

Lemma A.2. *Let $v(y)\in H^s(E^{n-1})$, $0<s<1$, and $a(y)\in\mathscr{B}^1(E^{n-1})$. Let $\gamma_s(y)$ be the temperate distribution whose transform of Fourier is $|\eta|^s$. Then there exists a constant $c(s,n)$ such that denoting*

$$Cv=[\gamma_s*,\,a]v=\gamma_s*(a(y)v(y))-a(y)(\gamma_s*v(y))\,,$$

we have

$$\|Cv\|_{L^2}\leqslant c(s,n)\,|a(y)|_{1,\mathscr{B}(E^{n-1})}\|v\|_{L^2}\,.$$

Proof. Let $\alpha(\eta)\in C_0^\infty$ be a function which takes the value 1 in a neighborhood of the origin and $0\leqslant\alpha(\eta)\leqslant1$. By the decomposition $|\eta|^s=\alpha(\eta)|\eta|^s+(1-\alpha(\eta))|\eta|^s$, and taking the inverse transform of Fourier, we have the corresponding decomposition $\gamma_s=\gamma_0+\gamma_1$, where $\hat{\gamma}_0=\alpha(\eta)|\eta|^s$. Then

$$[\gamma_s*,\,a]v=[\gamma_0*,\,a]v+[\gamma_1*,\,a]v\,.$$

Since $\|[\gamma_0*,\,a]v\|\leqslant c|a|_0\|c\|$, it suffices to consider the second term. At first, let us show that $y_i\gamma_1\in L^1(E^{n-1})$.

$$|y|^{2p}\gamma_1=c\int e^{iy\eta}\Delta^p[(1-\alpha(\eta))|\eta|^s]d\eta$$

shows that, if we take $2p<n+1+s$, $|y|^{2p}\gamma_1(y)$ is continuous and bounded. Next, $y_i\gamma_1=y_i\gamma_s-y_i\gamma_0$ shows, since $y_1\gamma_s$ is a function of homogeneous of degree $-(n-1+s-1)$, that $y_i\gamma_1$ is locally summable. Hence $y_i\gamma_s\in L^1$ ($i=1, 2, \cdots, n-1$).

Now

$$[\gamma_1*, a]v = \int \gamma_1(x-y)[a(y)-a(x)]v(y)dy ,$$

using $a(y)-a(x)=\sum a(x, y)(y_i-x_i)$, the Hausdorff-Young inequality gives

$$||[\gamma_1*, a]v||_{L^2}\leqslant c|a|_1\sum_i ||y_i\gamma_2||_{L^1}||v||_{L^2} .$$

Lemma A.3. *Let* $a(y)\in\mathscr{B}^1(E^{n-1})$, $v(y)\in H^s(E^{n-1})$, $0<s<1$. Then

$$|a(y)v(y)|_s\leqslant\sup|a(y)|\,|v|_s+c(s, n)|a(y)|_1||v||_0 .$$

Proof.

$$\gamma_s*(av) = a(\gamma_s*v)+[\gamma_s*, a]v .$$

Taking the L^2-norm of the both sides and in view of the previous Lemma, we get the above inequality.

Lemma A.4. *Under the same assumption as in Lemma A.1, for* $0\leqslant j<2m+l$, $0\leqslant s<1$,

$$\begin{aligned} C_{j+s}[|u(\Psi_k'(y))|_{j+s}^2+|u(\Psi_k'(y))|_0^2]&\leqslant |u(\Psi_i(y))|_{j+s}^2+|u(\Psi_i(y))|_0^2 \\ &\leqslant C_{j+s}'[|u(\Psi_k'(y))|_{j+s}^2+|u(\Psi_k'(y))|_0^2] .\end{aligned}$$

Proof. For $|\alpha|=j$,

$D^\alpha\tilde{v}(z)=D^\alpha v(y_{ik}(z))=\sum_{|\beta|\leqslant|\alpha|} c_{\alpha\beta}(z)D^\beta v(y_{ik}(z))$, where $|c_{\alpha\beta}(z)|\leqslant K$ (independent of i, k). Thus, for $|\alpha|=j$,

$$|D^\alpha\tilde{v}(z)|_s\leqslant\sum_{|\beta|=j}|c_{\alpha\beta}(z)D^\beta v(y_{ik}(z))|_s+\sum_{|\beta|\leqslant j-1}|c_{\alpha\beta}D^\beta v(y_{ik}(z))|_s .$$

By virtue of Lemma A.1 and A.3, this is estimated by

$$c(|v(y)|_{j+s}+||v(y)||_{H^j}) .$$

This is estimated again by $c''[|v(y)|_{j+s}+||v(y)||_{L^2}]$. The converse inequality is obtained changing the role of y and z.

Now we prove Proposition in the following form.

Theorem. *Two norms* $\langle u\rangle_{j+s,1}^2+\sum_k|\eta_k^2u|_0^2$ *and* $\langle u\rangle_{j+s,2}^2+\sum_k|\eta_k^2u|_0^2$ *are equivalent, where* $0\leqslant j\leqslant 2m+l-1$, $0\leqslant s<1$.

REMARK. $|\eta_k^2 u|_0^2$ means $\|\eta_k^2(\Psi_k(y))u(\Psi_k(y))\|_{L^2(E^{n-1})}^2$.

Proof. Put

$$\begin{aligned}
I_1(u) &= \sum_i |\eta_i^2 u|_{j+s}^2 + \sum_i |\eta_i^2 u|_0^2 \\
I_2(u) &= \sum_k |\zeta_k^2 u|_{j+s}^2 + \sum_k |\zeta_k^2 u|_0^2 \\
I_{12}(u) &= \sum_{ik} |\zeta_k^2 \eta_i^2 u|_{j+s}^2 + \sum_{ik} |\zeta_k^2 \eta_i^2 u|_0^2 \\
I_{21}(u) &= \sum_{ik} |\eta_i^2 \zeta_k^2 u|_{j+s}^2 + \sum_{ik} |\eta_i^2 \zeta_k^2 u|_0^2 ,
\end{aligned}$$

where in $I_{12}(u)$ the semi-norm $|\zeta_k^2\eta_i^2 u|_{j+s}^2$ is calculated by the transform $\Phi_i(x)$, whereas in $I_{21}(u)$ $|\eta_i^2\zeta_k^2 u|_{j+s}^2$ is calculated by $\Phi_k'(x)$.

At first we remark that $I_{12}(u)$ and $I_{21}(u)$ are equivalent by virtue of Lemma A.4. Our purpose is then to prove that $I_1(u)$ and $I_{12}(u)$ are equivalent, because the equivalency of $I_2(u)$ and $I_{21}(u)$ is proved by the same reasoning.

Let $v_i(y)=u(\Psi_i(y))$. For $|\alpha|=j$,

$$(*)\qquad D^\alpha(\zeta_k^2\eta_i^2 v_i) = \zeta_k^2 D^\alpha(\eta_i^2 v_i) + \sum_{|\beta|\leqslant|\alpha|-1} c_\beta^\alpha D^{\alpha-\beta}(\zeta_k^2) D^\beta(\eta_i^2 v_i)$$

shows, taking account of Lemma A.3,

$$\begin{aligned}
|D^\alpha(\zeta_k^2\eta_i^2 v_i)|_s &\leqslant |\zeta_k^2 D^\alpha(\eta_i v_i)|_s + K_0\|\eta_i^2 v_i\|_{H^j} \\
&\leqslant \|\zeta_k^2\cdot\gamma_s * D^\alpha(\eta_i^2 v_i)\|_{L^2} + K_0'\|\eta_i^2 v_i\|_{H^j} ,
\end{aligned}$$

where K_0' is independent of i, k.
From the assumption, we see that there exists an integer R' such that each $\operatorname{supp}[\eta_i]$ has at most R' numbers of $\operatorname{supp}[\zeta_k]$ having common points (R' is independent of i). Hence

$$\sum_k |D^\alpha(\zeta_k^2\eta_i^2 v_i)|_s^2 \leqslant 2\sum_k \|\zeta_k^2\cdot\gamma_s * D^\alpha(\eta_i^2 v_i)\|_{L^2}^2 + 2K_0'^2 R'\|\eta_i^2 v_i\|_{H^j}^2$$

implies that, since $\sum\zeta_k^4(x)\leqslant 1$, the right-hand side is estimated by

$$2\|\gamma_s * D^\alpha(\eta_i^2 v_i)\|_{L^2}^2 + 2R'K_0'^2\|\eta_i^2 v_i\|_{H^j}^2 .$$

Therefore

$$\sum_k |\zeta_k^2\eta_i^2 v_i|_{j+s}^2 \leqslant 2|\eta_i^2 v_i|_{j+s}^2 + C\|\eta_i^2 v_i\|_{H^j}^2 \leqslant C'(|\eta_i^2 v_i|_{j+s}^2 + \|\eta_i^2 v_i\|_{L^2}^2) ,$$

where C' is a constant independent of i. Thus we have proved

$$I_{12}(u) \leqslant \text{const. } I_1(u) .$$

Conversely, $(*)$ gives the decomposition

$$\begin{aligned}
\gamma_s * D^\alpha(\zeta_k^2\eta_i^2 v_i) &= \zeta_k^2\cdot\gamma_s * D^\alpha(\eta_i^2 v_i) + [\gamma_s *, \zeta_k^2] D^\alpha(\eta_i^2 v_i) \\
&\quad + \sum C_\beta^\alpha D^{\alpha-\beta}(\zeta_k^2)\, \gamma_s * D^\beta(\eta_i^2 v_i) + \sum C_\beta^\alpha [\gamma_s *, D^{\alpha-\beta}(\zeta_k^2)] D^\beta(\eta_i^2 v_i) .
\end{aligned}$$

Taking account of Lemma A.3,

$$|D^\alpha(\zeta_k^2\eta_i^2 v_i)|_s \geqslant \|\zeta_k^2 \cdot \gamma_s * D^\alpha(\eta_i^2 v_i)\| - K\|\eta_i^2 v_i\|_{H^j} .$$

Then, since $\sum \zeta_k^4(x) \geqslant \delta$ for all $x \in \Gamma$,

$$\sum_{k,\alpha;\,|\alpha|=j} |D^\alpha(\zeta_k^2\eta_i^2 v_i)|_s^2 \geqslant \frac{\delta}{2}|\eta_i^2 v_i|_{j+s}^2 - 2R'K^2\|\eta_i v_i\|_{H^j}^2$$

$$\geqslant \frac{\delta}{4}|\eta_i^2 v_i|_{j+s}^2 - C\|\eta_i^2 v_i\|_{L^2}^2 .$$

Summing up in i, we have

$$\sum_{i,k} |\zeta_k^2\eta_i^2 v_i|_{j+s}^2 \geqslant \frac{\delta}{4}\sum |\eta_i^2 v_i|_{j+s}^2 - C\sum \|\eta_i^2 v_i\|_{L^2}^2 .$$

Thus we have proved $I_{12}(u) \geqslant$ const. $I_1(u)$.

OSAKA UNIVERSITY

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