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THE GROUP OF NORMALIZED UNITS OF A GROUP RING

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Introduction

Let RG be the group ring of a group G over a commutative ring R with identity and $\Delta_R(G)$ its augmentation ideal. For a normal subgroup N of G , the kernel of the natural homomorphism $RG \rightarrow R(G/N)$ will be denoted by $\Delta_R(G, N)$. It is equal to $\Delta_R(N)RG$. Also, we shall denote by $V(RG)$ the group of normalized units of RG , that is, $V(RG) = U(RG) \cap (1 + \Delta_R(G))$ where $U(RG)$ is the unit group of RG .

The aim of this paper is to prove the following theorem which generalizes [1, Proposition 1.3].

Theorem 1.3. *Let G be an arbitrary group and R an integral domain of characteristic 0. Let I be an ideal of RG and set $J = \bigcap_{n=1}^{\infty} (I + \Delta_R(G)^n)$. Then the factor group*

$$V(RG) \cap (1 + J) / V(RG) \cap (1 + \Delta_R(G, G \cap (1 + J)))$$

is torsion-free.

As an immediate consequence of this result we can weaken the condition on R in [1, Proposition 2.4]. To be more precise, let $D_{n,R}(G)$ be the n -th dimension subgroup of G over R . Then, for two groups G and H with isomorphic group algebras over an integral domain R of characteristic 0, we can show that $D_{n,R}(G) = \{1\}$ if and only if $D_{n,R}(H) = \{1\}$.

Let A be a ring and 0A the group of all quasi-regular elements in A . Here we say that A is residually nilpotent if $\bigcap_{n=1}^{\infty} A^n = 0$. As another application of Theorem 1.3, we show that if A is a residually nilpotent algebra over an integral domain R of characteristic 0, then the group $G = {}^0A$ has a torsion-free normal complement in $V(RG)$. This is proved by D.S. Passman and P.F. Smith [3, Theorem 1.4] for the case where A is a finite nilpotent ring and R is the ring of rational integers.

1. The group of normalized units

We start by making two simple observations. Let G be a group, R a

commutative ring with identity. The n -th dimension subgroup $D_{n,R}(G)$ ($n=1, 2, \dots$) of G over R is defined by $D_{n,R}(G) = G \cap (1 + \Delta_R(G)^n)$, where $\Delta_R(G)$ denotes the augmentation ideal of RG . The series $\{D_{n,R}(G)\}_{n \geq 1}$ forms a descending central series of G .

Lemma 1.1. *Suppose $D_{n,R}(G) = \{1\}$ for some n . Then no element $g \neq 1$ of G has order invertible in R .*

Proof. It suffices to verify that whenever a rational prime p is a unit in R , G is p -torsion-free. It is well-known that the map $f: D_{i,R}(G) \rightarrow \Delta_R(G)^i / \Delta_R(G)^{i+1}$ defined by $f(g) = g - 1 + \Delta_R(G)^{i+1}$ induces a monomorphism $D_{i,R}(G) / D_{i+1,R}(G) \rightarrow \Delta_R(G)^i / \Delta_R(G)^{i+1}$ of abelian groups. Therefore, if a rational prime p is a unit in R , then each additive group $\Delta_R(G)^i / \Delta_R(G)^{i+1}$ is clearly p -torsion-free, so is each $D_{i,R}(G) / D_{i+1,R}(G)$. Since $D_{n,R}(G) = \{1\}$ it follows that G is p -torsion-free and thus the lemma is proved.

Lemma 1.2. *Let $H_1 \supseteq H_2 \supseteq \dots \supseteq H_i \supseteq \dots$ be a decreasing series of subgroups of G . Then*

$$\bigcap_{i=1}^{\infty} \{\Delta_R(H_i)RG\} = \Delta_R\left(\bigcap_{i=1}^{\infty} H_i\right)RG.$$

Proof. It is trivial that the right-hand side is contained in the left-hand side. To show the reverse inclusion, let $\alpha \in \bigcap_{i=1}^{\infty} \{\Delta_R(H_i)RG\}$ and set $H = \bigcap_{i=1}^{\infty} H_i$. Then, choosing a right transversal T for H in G , we may express α , uniquely, as $\alpha = \sum_{j=1}^n \alpha_j t_j$, $\alpha_j \in RH$, $t_j \in T$. We first show that $\alpha_v \in \Delta_R(H)$ for a fixed integer ν with $1 \leq \nu \leq n$. Since the set $\{t_j t_\nu^{-1} \mid 1 \leq j \leq n\}$ is finite, we can pick some H_k with $H_k \cap \{t_j t_\nu^{-1} \mid 1 \leq j \leq n\} = \{1\}$. Then, under the natural projection map $\pi: RG \rightarrow RH_k$, we have $\pi(\alpha t_\nu^{-1}) = \alpha_v$ since π is a left RH_k -homomorphism (see [2, p. 6]). On the other hand, we have

$$\pi(\alpha t_\nu^{-1}) \in \pi(\Delta_R(H_k)RG) = \Delta_R(H_k),$$

so $\alpha_v \in \Delta_R(H_k) \cap RH = \Delta_R(H)$. Thus we see that all α_j 's are in $\Delta_R(H)$ so that $\alpha \in \Delta_R(H)RG$. This completes the proof of the lemma.

We are now in a position to prove our main theorem which is a generalization of [1, Proposition 1.3]. Recall that for any (two-sided) ideal I of RG , $V(RG) \cap (1 + I) = \{u \in V(RG) \mid u - 1 \in I\}$ forms a normal subgroup of $V(RG)$.

Theorem 1.3. *Let G be an arbitrary group and R an integral domain of characteristic 0. Let I be an ideal of RG and set $J = \bigcap_{n=1}^{\infty} (I + \Delta_R(G)^n)$. Then the factor group*

$$V(RG) \cap (1 + J) / V(RG) \cap (1 + \Delta_R(G, G \cap (1 + J)))$$

is torsion-free.

Proof. For simplicity of notation, the normal subgroup $G \cap (1+K)$ of G determined by an ideal K of RG will be denoted by $D(K)$. Let $I_n = I + \Delta_R(G)^n$. Then, $I_m I_n + I_n I_m \subseteq I_{m+n}$ for all $m, n \geq 1$, so we obtain a descending central series $\{D(I_n)\}_{n \geq 1}$ of G with $D(I_1) = G$. Note that $D_{n,R}(G) \subseteq D(I_n)$ because $\Delta_R(G)^n \subseteq I_n$.

We first prove the following:

(*) If $D(I_n) = \{1\}$, then $V(RG) \cap (1+I_n)$ is torsion-free.

We proceed by induction on n , the case $n=1$ being trivial. Let $n \geq 2$ and assume that (*) holds for $n-1$. Set $\bar{G} = G/D(I_{n-1})$ and let $\bar{\cdot} : RG \rightarrow R\bar{G}$ be the natural homomorphism. Then, since $D(\bar{I}_{n-1}) = \bar{G} \cap (1+\bar{I}_{n-1}) = \{1\}$, $V(R\bar{G}) \cap (1+\bar{I}_{n-1})$ is torsion-free by induction hypothesis. Let $u \in V(RG) \cap (1+I_n)$ have finite order. Then $\bar{u} \in V(R\bar{G}) \cap (1+\bar{I}_{n-1})$, and \bar{u} still has finite order, so $\bar{u} = 1$, that is, $u-1 \in \Delta_R(G, D(I_{n-1}))$. Note here that since $D(I_n) = \{1\}$, G is nilpotent and $D(I_{n-1})$ is central in G . Moreover, as $D_{n,R}(G) = \{1\}$, we know from Lemma 1.1 that no element $g \neq 1$ of G has order invertible in R . Thus, by [1, Lemma 1.2], $u = x$ for some $x \in D(I_{n-1})$. This implies that $x \in D(I_n) = \{1\}$ because $u-1 \in I_n$. Hence we have $u=1$, so $V(RG) \cap (1+I_n)$ is torsion-free.

Turning the proof of the theorem, let $u \in V(RG) \cap (1+J)$ and suppose $u^l \in V(RG) \cap (1+\Delta_R(G, D(J)))$ for some integer l . If $\bar{G} = G/D(I_n)$, then $D(\bar{I}_n) = \bar{G} \cap (1+\bar{I}_n) = \{1\}$ under the natural homomorphism $\bar{\cdot} : RG \rightarrow R\bar{G}$, and so (*) shows that each factor group

$$V(RG) \cap (1+I_n) / V(RG) \cap (1+\Delta_R(G, D(I_n)))$$

is torsion-free. Since $u-1 \in I_n$ and $u^l-1 \in \Delta_R(G, D(I_n))$ for all $n \geq 1$, it follows that $u-1 \in \bigcap_{n=1}^{\infty} \Delta_R(G, D(I_n))$. Furthermore, by Lemma 1.2,

$$\bigcap_{n=1}^{\infty} \Delta_R(G, D(I_n)) = \Delta_R(G, \bigcap_{n=1}^{\infty} D(I_n)) = \Delta_R(G, D(J)),$$

and hence we conclude that $u \in V(RG) \cap (1+\Delta_R(G, D(J)))$. This completes the proof.

A ring A is said to be residually nilpotent if $\bigcap_{n=1}^{\infty} A^n = 0$. In the context of the preceding theorem, the factor ring $(\Delta_R(G) + I)/I = \Delta_R(G)/(\Delta_R(G) \cap I)$ is residually nilpotent if and only if $J=I$, so we note the following

Corollary 1.4. *Let G, R and I be as in Theorem 1.3. If $\Delta_R(G)/(\Delta_R(G) \cap I)$ is residually nilpotent, then*

$$V(RG) \cap (1+I) / V(RG) \cap (1+\Delta_R(G, G \cap (1+I)))$$

is torsion-free.

By taking $I = \mathcal{A}_R(G)^*$ in this corollary we see that if $D_{n,R}(G) = \{1\}$, then $V(RG) \cap (1 + \mathcal{A}_R(G)^*)$ is torsion-free. Thus the same argument as in Proposition 2.4 of [1] gives us the following result, whose proof will be omitted.

Proposition 1.5. *Let G and H be two groups with $RG \cong RH$ as R -algebras, where R is an integral domain of characteristic 0. Then*

$$D_{n,R}(G) = \{1\} \quad \text{if and only if} \quad D_{n,R}(H) = \{1\}.$$

2. Quasi-regular groups

Let A be a ring, and let 0A denote the group of all quasi-regular elements of A , that is, 0A is the set of those elements of A which are invertible under the circle operation $a \circ b = a + b + ab$. In case ${}^0A = A$, A is a Jacobson radical ring and 0A is called the circle group of A . It has been shown in [3] that if G is the circle group of a finite nilpotent ring, then G has a torsion-free normal complement in $V(\mathbb{Z}G)$. We shall extend this result as follows.

Proposition 2.1. *Let R be an integral domain of characteristic 0 and let A be a residually nilpotent R -algebra. Then the group $G = {}^0A$ has a torsion-free normal complement in $V(RG)$.*

Proof. Since any R -algebra can be embedded in an R -algebra with identity, we can regard A as an ideal of some R -algebra A_1 with identity. Then $G = {}^0A$ is isomorphic to $U(A_1) \cap (1 + A)$ where $U(A_1)$ is the unit group of A_1 , so we may suppose that $G = U(A_1) \cap (1 + A)$. Then the inclusion map $G \rightarrow A_1$ can be extended to the R -algebra homomorphism $RG \rightarrow A_1$, which is denoted by f , and we set $F = V(RG) \cap (1 + \text{Ker } f)$ so that $F \triangleleft V(RG)$. Since $f(g - 1) = g - 1$ for $g \in G$, we have $f(\mathcal{A}_R(G)) \subseteq A$, and hence $f(V(RG)) \subseteq U(A_1) \cap (1 + A) = f(G)$ which implies that $V(RG) = GF$. Observe that the factor ring $\mathcal{A}_R(G)/(\mathcal{A}_R(G) \cap \text{Ker } f)$ is isomorphic to a subring of A and so is residually nilpotent. Since $G \cap F = G \cap (1 + \text{Ker } f) = \{1\}$, we conclude by Corollary 1.4 that F is a torsion-free normal complement for G in $V(RG)$.

REMARK. As seen in the above proof, the group $G = {}^0A$ of any R -algebra A has a normal complement in $V(RG)$.

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