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THE GROUP OF NORMALIZED UNITS OF A GROUP RING

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Introduction

Let RG be the group ring of a group G over a commutative ring R with identity and $\Delta_R(G)$ its augmentation ideal. For a normal subgroup N of G, the kernel of the natural homomorphism $RG \rightarrow R(G/N)$ will be denoted by $\Delta_R(G, N)$. It is equal to $\Delta_R(N)RG$. Also, we shall denote by V(RG) the group of normalized units of RG, that is, $V(RG) = U(RG) \cap (1 + \Delta_R(G))$ where U(RG)is the unit group of RG.

The aim of this paper is to prove the following theorem which generalizes [1, Proposition 1.3].

Theorem 1.3. Let G be an arbitrary group and R an integral domain of characteristic 0. Let I be an ideal of RG and set $J = \bigcap_{n=1}^{\infty} (I + \Delta_R(G)^n)$. Then the factor group

$$V(RG) \cap (1+J)/V(RG) \cap (1+\Delta_R(G, G \cap (1+J)))$$

is torsion-free.

As an immediate consequence of this result we can weaken the condition on R in [1, Proposition 2.4]. To be more precise, let $D_{n,R}(G)$ be the *n*-th dimension subgroup of G over R. Then, for two groups G and H with isomorphic group algebras over an integral domain R of characteristic 0, we can show that $D_{n,R}(G) = \{1\}$ if and only if $D_{n,R}(H) = \{1\}$.

Let A be a ring and ${}^{0}A$ the group of all quasi-regular elements in A. Here we say that A is residually nilpotent if $\bigcap_{n=1}^{\infty} A^{n} = 0$. As another application of Theorem 1.3, we show that if A is a residually nilpotent algebra over an integral domain R of characteristic 0, then the group $G={}^{0}A$ has a torsion-free normal complement in V(RG). This is proved by D.S. Passman and P.F. Smith [3, Theorem 1.4] for the case where A is a finite nilpotent ring and Ris the ring of rational integers.

1. The group of normalized units

We start by making two simple observations. Let G be a group, R a

commutative ring with identity. The *n*-th dimension subgroup $D_{n,R}(G)$ $(n=1, 2, \cdots)$ of G over R is defined by $D_{n,R}(G)=G\cap(1+\Delta_R(G)^n)$, where $\Delta_R(G)$ denotes the augmentation ideal of RG. The series $\{D_{n,R}(G)\}_{n\geq 1}$ forms a descending central series of G.

Lemma 1.1. Suppose $D_{n,R}(G) = \{1\}$ for some n. Then no element $g \neq 1$ of G has order invertible in R.

Proof. It suffices to verify that whenever a rational prime p is a unit in R, G is p-torsion-free. It is well-known that the map $f: D_{i,R}(G) \rightarrow \Delta_R(G)^i / \Delta_R(G)^{i+1}$ defined by $f(g) = g - 1 + \Delta_R(G)^{i+1}$ induces a monomorphism $D_{i,R}(G) / D_{i+1,R}(G) \rightarrow \Delta_R(G)^i / \Delta_R(G)^{i+1}$ of abelian groups. Therefore, if a rational prime p is a unit in R, then each additive group $\Delta_R(G)^i / \Delta_R(G)^{i+1}$ is clearly p-torsion-free, so is each $D_{i,R}(G) / D_{i+1,R}(G)$. Since $D_{n,R}(G) = \{1\}$ it follows that G is p-torsion-free and thus the lemma is proved.

Lemma 1.2. Let $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_i \supseteq \cdots$ be a decreasing series of subgroups of G. Then

$$\bigcap_{i=1}^{n} \{ \mathcal{A}_{\mathbb{R}}(H_i) \mathbb{R} G \} = \mathcal{A}_{\mathbb{R}}(\bigcap_{i=1}^{n} H_i) \mathbb{R} G .$$

Proof. It is trivial that the right-hand side is contained in the left-hand side. To show the reverse inclusion, let $\alpha \in \bigcap_{i=1}^{\infty} \{ \Delta_R(H_i) RG \}$ and set $H = \bigcap_{i=1}^{\infty} H_i$. Then, choosing a right transversal T for H in G, we may express α , uniquely, as $\alpha = \sum_{j=1}^{n} \alpha_j t_j, \alpha_j \in RH, t_j \in T$. We first show that $\alpha_v \in \Delta_R(H)$ for a fixed integer ν with $1 \leq \nu \leq n$. Since the set $\{t_j t_v^{-1} | 1 \leq j \leq n\}$ is finite, we can pick some H_k with $H_k \cap \{t_j t_v^{-1} | 1 \leq j \leq n\} = \{1\}$. Then, under the natural projection map $\pi: RG \rightarrow RH_k$, we have $\pi(\alpha t_v^{-1}) = \alpha_v$ since π is a left RH_k -homomorphism (see [2, p. 6]). On the other hand, we have

$$\pi(\alpha t_{\nu}^{-1}) \in \pi(\mathcal{A}_{\mathbb{R}}(H_k) \mathbb{R} G) = \mathcal{A}_{\mathbb{R}}(H_k),$$

so $\alpha_{\nu} \in \mathcal{A}_{\mathbb{R}}(H_k) \cap \mathbb{R}H = \mathcal{A}_{\mathbb{R}}(H)$. Thus we see that all α_j 's are in $\mathcal{A}_{\mathbb{R}}(H)$ so that $\alpha \in \mathcal{A}_{\mathbb{R}}(H)\mathbb{R}G$. This completes the proof of the lemma.

We are now in a position to prove our main theorem which is a generalization of [1, Proposition 1.3]. Recall that for any (two-sided) ideal I of RG, $V(RG) \cap (1+I) = \{u \in V(RG) | u-1 \in I\}$ forms a normal subgroup of V(RG).

Theorem 1.3. Let G be an arbitrary group and R an integral domain of characteristic 0. Let I be an ideal of RG and set $J = \bigcap_{n=1}^{\infty} (I + \Delta_R(G)^n)$. Then the factor group

$$V(RG) \cap (1+J)/V(RG) \cap (1+\Delta_{\mathbb{R}}(G, G \cap (1+J)))$$

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is torsion-free.

Proof. For simplicity of notation, the normal subgroup $G \cap (1+K)$ of G determined by an ideal K of RG will be denoted by D(K). Let $I_n = I + \Delta_R(G)^n$. Then, $I_m I_n + I_n I_m \subseteq I_{m+n}$ for all $m, n \ge 1$, so we obtain a descending central series $\{D(I_n)\}_{n\ge 1}$ of G with $D(I_1) = G$. Note that $D_{n,R}(G) \subseteq D(I_n)$ because $\Delta_R(G)^n \subseteq I_n$. We first prove the following:

(*) If $D(I_n) = \{1\}$, then $V(RG) \cap (1+I_n)$ is torsion-free.

We proceed by induction on *n*, the case n=1 being trivial. Let $n \ge 2$ and assume that (*) holds for n-1. Set $\overline{G} = G/D(I_{n-1})$ and let $\overline{}: RG \to R\overline{G}$ be the natural homomorphism. Then, since $D(\overline{I}_{n-1}) = \overline{G} \cap (1 + \overline{I}_{n-1}) = \{1\}, V(R\overline{G}) \cap (1 + \overline{I}_{n-1})$ is torsion-free by induction hypothesis. Let $u \in V(RG) \cap (1 + I_n)$ have finite order. Then $\overline{u} \in V(R\overline{G}) \cap (1 + \overline{I}_{n-1})$, and \overline{u} still has finite order, so $\overline{u} = 1$, that is, $u-1 \in \mathcal{A}_R(G, D(I_{n-1}))$. Note here that since $D(I_n) = \{1\}, G$ is nilpotent and $D(I_{n-1})$ is central in G. Moreover, as $D_{n,R}(G) = \{1\}$, we know from Lemma 1.1 that no element $g \neq 1$ of G has order invertible in R. Thus, by [1, Lemma 1.2], u=x for some $x \in D(I_{n-1})$. This implies that $x \in D(I_n) = \{1\}$ because $u-1 \in I_n$. Hence we have u=1, so $V(RG) \cap (1+I_n)$ is torsion-free.

Turning the proof of the theorem, let $u \in V(RG) \cap (1+J)$ and suppose $u' \in V(RG) \cap (1+\Delta_R(G, D(J)))$ for some integer *l*. If $\overline{G} = G/D(I_n)$, then $D(\overline{I}_n) = \overline{G} \cap (1+\overline{I}_n) = \{1\}$ under the natural homomorphism $\overline{}: RG \to R\overline{G}$, and so (*) shows that each factor group

$$V(RG) \cap (1+I_n)/V(RG) \cap (1+\Delta_R(G, D(I_n)))$$

is torsion-free. Since $u-1 \in I_n$ and $u^l-1 \in \mathcal{A}_R(G, D(I_n))$ for all $n \ge 1$, it follows that $u-1 \in \bigcap_{n=1}^{\infty} \mathcal{A}_R(G, D(I_n))$. Furthermore, by Lemma 1.2,

$$\bigcap_{n=1}^{\infty} \Delta_{\mathbb{R}}(G, D(I_n)) = \Delta_{\mathbb{R}}(G, \bigcap_{n=1}^{\infty} D(I_n)) = \Delta_{\mathbb{R}}(G, D(J)),$$

and hence we conclude that $u \in V(RG) \cap (1 + \mathcal{A}_R(G, D(J)))$. This completes the proof.

A ring A is said to be residually nilpotent if $\bigcap_{n=1}^{\infty} A^n = 0$. In the context of the preceding theorem, the factor ring $(\Delta_R(G)+I)/I = \Delta_R(G)/(\Delta_R(G) \cap I)$ is residually nilpotent if and only if J=I, so we note the following

Corollary 1.4. Let G, R and I be as in Theorem 1.3. If $\Delta_R(G)/(\Delta_R(G) \cap I)$ is residually nilpotent, then

$$V(RG) \cap (1+I)/V(RG) \cap (1+\Delta_R(G, G \cap (1+I)))$$

is torsion-free.

By taking $I=\Delta_R(G)^n$ in this corollary we see that if $D_{n,R}(G)=\{1\}$, then $V(RG)\cap (1+\Delta_R(G)^n)$ is torsion-free. Thus the same argument as in Proposition 2.4 of [1] gives us the following result, whose proof will be omitted.

Proposition 1.5. Let G and H be two groups with $RG \approx RH$ as R-algebras, where R is an integral domain of characteristic 0. Then

 $D_{n,R}(G) = \{1\}$ if and only if $D_{n,R}(H) = \{1\}$.

2. Quasi-regular groups

Let A be a ring, and let ${}^{0}A$ denote the group of all quasi-regular elements of A, that is, ${}^{0}A$ is the set of those elements of A which are invertible under the circle operation $a \circ b = a + b + ab$. In case ${}^{0}A = A$, A is a Jacobson radical ring and ${}^{0}A$ is called the circle group of A. It has been shown in [3] that if G is the circle group of a finite nilpotent ring, then G has a torsion-free normal complement in $V(\mathbb{Z}G)$. We shall extend this result as follows.

Proposition 2.1. Let R be an integral domain of characteristic 0 and let A be a residually nilpotent R-algebra. Then the group $G={}^{0}A$ has a torsion-free normal complement in V(RG).

Proof. Since any *R*-algebra can be embedded in an *R*-algebra with identity, we can regard *A* as an ideal of some *R*-algebra A_1 with identity. Then $G = {}^{0}A$ is isomorphic to $U(A_1) \cap (1+A)$ where $U(A_1)$ is the unit group of A_1 , so we may suppose that $G = U(A_1) \cap (1+A)$. Then the inclusion map $G \rightarrow A_1$ can be extended to the *R*-algebra homomorphism $RG \rightarrow A_1$, which is denoted by *f*, and we set $F = V(RG) \cap (1+Ker f)$ so that $F \triangleleft V(RG)$. Since f(g-1) = g-1for $g \in G$, we have $f(A_R(G)) \subseteq A$, and hence $f(V(RG)) \subseteq U(A_1) \cap (1+A) = f(G)$ which implies that V(RG) = GF. Observe that the factor ring $A_R(G)/(A_R(G) \cap$ Ker f) is isomorphic to a subring of *A* and so is residually nilpotent. Since $G \cap$ $F = G \cap (1+Ker f) = \{1\}$, we conclude by Corollary 1.4 that *F* is a torsion-free normal complement for *G* in V(RG).

REMARK. As seen in the above proof, the group $G={}^{0}A$ of any R-algebra A has a normal complement in V(RG).

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