SMOOTH ACTIONS OF SPECIAL UNITARY GROUPS ON COHOMOLOGY COMPLEX PROJECTIVE SPACES

FUICHI UCHIDA

(Received July 24, 1974)

0. Introduction

The purpose of this paper is to study smooth $SU(n)$-actions on a compact orientable $2m$-manifold whose rational cohomology ring is isomorphic to $H^*(P_m(C); Q)$.   First we show the following result.

Theorem 2.1.   *Let* $n \geqslant 7$ *and* $0 \leqslant k < n-4$.   *Let M be a compact orientable smooth* $2(n+k)$*-manifold with*

$$H^*(M; Q) = H^*(P_{n+k}(C); Q).$$

Then for any **non-trivial** *smooth* $SU(n)$*-action on M, the stationary point set* $F = F(SU(n), M)$ *is an orientable* $2k$*-manifold with*

$$H^*(F; Q) = H^*(P_k(C); Q)$$

and there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1.$$

Here X is a compact connected orientable $(2k+2)$*-manifold which is acyclic over rationals, X admits a smooth* S^1*-action which is free on dX, the* $SU(n)$*-action is standard on* D^{2n} *and trivial on X, and*

$$\pi_1(X) = \pi_1(M).$$

Furthermore, if

$$H^*(M; Z) = H^*(P_{n+k}(C); Z),$$

then X is acyclic over integers, the S^1*-action on X is semi-free, and*

$$H^*(F; Z) = H^*(P_k(C); Z).$$

Corollary 2.2.   *Let* $n \geqslant 7$ *and* $0 \leqslant k < n-4$.   *Let M be a compact connected smooth* $2(n+k)$*-manifold which is homotopy equivalent to* $P_{n+k}(C)$. *If M admits a* **non-trivial** *smooth* $SU(n)$*-action, then M is diffeomorphic to* $P_{n+k}(C)$.

Examples of $SU(n)$-actions on cohomology complex projective spaces are constructed in section 3. And we have the following results.

Theorem 3.1. *Let $n \geqslant 2$, $k \geqslant 1$ and $p \geqslant 1$. Then there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = \mathbf{Z}/p\mathbf{Z} \quad and \quad H^*(M; \mathbf{Q}) = H^*(P_{n+k}(\mathbf{C}); \mathbf{Q})$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(\mathbf{C}).$$

Theorem 3.2. *Let $n \geqslant 2$ and $k \geqslant 3$. Let G be a finitely presentable group with $H_1(G; \mathbf{Z}) = H_2(G; \mathbf{Z}) = 0$. Then*

(a)   *there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = G \quad and \quad H^*(M; Z) = H^*(P_{n+k}(\mathbf{C}), Z)$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(\mathbf{C}),$$

(b)   *there is a smooth $SU(n)$-action on $P_{n+k}(\mathbf{C})$ such that*

$$\pi_1(F) = G \quad and \quad H^*(F; Z) = H^*(P_k(\mathbf{C}); \mathbf{Z}),$$

where $F = F(SU(n), P_{n+k}(\mathbf{C}))$.

Next, in section 4, we study a signature of closed orientable manifold which admits a smooth G-action with isotropy groups of uniform dimension, and we have a result which is a generalization of the fact that $\text{Sign}(M) = 0$ if M admits a smooth circle action without stationary points.

Next we study smooth $SU(3)$-actions on orientable 8-manifolds in section 5, and as an application we show a similar result as Theorem 2.1 for non-trivial smooth $SU(3)$-action on a cohomology complex projective 4-space. We construct examples of stationary point free $SU(3)$-actions on orientable 8-manifolds with non-zero signature in section 6.

As a concluding remark, classification of smooth $SU(n)$-actions on orientable $2n$-manifolds is done in the final section.

1. $SU(n)$-actions with certain isotropy types

Let E be a manifold with smooth $SU(n)$-action ($n \geqslant 3$). Assume that the identity component of each isotropy group is conjugate to $SU(n-1)$ or $NSU(n-1)$, the normalizer of $SU(n-1)$ in $SU(n)$. Then $S^1 = NSU(n-1)/SU(n-1)$ acts naturally on

$$X = F(SU(n-1), E),$$

the stationary point set of $SU(n-1)$.   It is easily seen that

(1.1) $$SU(n)/SU(n-1) \underset{S^1}{\times} X \to E, \quad [gSU(n-1), x] \to gx$$

is an equivariant diffeomorphism as $SU(n)$-manifolds, since $g \in SU(n)$ and $g^{-1}SU(n-1)g \subset NSU(n-1)$ imply $g \in NSU(n-1)$.

Lemma 1.2.   *Let V be a real vector space **with** linear $SU(n)$-action $(n \geqslant 3)$. Assume that the identity component of each isotropy group on the **invariant unit** sphere $S(V)$ is conjugate to $SU(n-1)$ or $NSU(n-l)$.   Then $S(V) = SU(n)/SU(n-1)$ as $SU(n)$-spaces.*

Proof.   By (1.1), there is an equivariant diffeomorphism

$$S(V) = SU(n)/SU(n-1) \underset{S^1}{\times} F(SU(n-1)S(V)),$$

where $F(SU(n-1), S(V))$ is a sphere.   Then it is easily seen that

$$F(SU(n-1), S(V)) = S^1$$

by the homotopy exact sequence of the fibre bundle

$$F(SU(n-1), S(V)) \to S(V) \to P_{n-1}(\boldsymbol{C}).$$

Considering S^1-actions on S^1, we have

$$S(V) = SU(n)/SU(n-1)$$

as $SU(n)$-spaces.   q.e.d.

Lemma 1.3.   *Let V be a real vector space **with** linear $SU(n)$-action such that $S(V) = SU(n)/SU(n-1)$ is $SU(n)$-spaces $(n \geqslant 3)$.   Then the $SU(n)$-action on $V = \boldsymbol{R}^{2n}$ is equivalent to the standard action.*

Proof.   This is a known result (see [8], Theorem I), but we give an elementary proof for the completeness.   It is well-known that a real irreducible $SU(n)$-vector space $\boldsymbol{R}^{2n}$ with an invariant complex structure is equivalent to $\boldsymbol{R}^{2n}$ with the standard $SU(n)$-action.   So we prove the existence of an invariant complex structure on V.   Denote by $\boldsymbol{Z}_n$, the center of $SU(n)$.   Then $\boldsymbol{Z}_n$ is a cyclic group of order n, and the $\boldsymbol{Z}_n$-action on $S(V)$ is free, since

$$\boldsymbol{Z}_n \cap SU(n-1) = \{1\} .$$

Consider a direct sum decomposition

$$V = V_1 \oplus \cdots \oplus V_k$$

as $\mathbf{Z}_n$-vector space, where $V_i(i=1, \cdots, k)$ are irreducible.   Leaving a non-zero vector $v_1 \in V_1$ fixed, we have an element $g_i \in SU(n)$ such that

$$v_i = g_i v_1 \in V_i \qquad (i = 1, \cdots, k)$$

by the transitivity of the $SU(n)$-action on $S(V)$.   Then

$$V_i = g_i V_1 \qquad (i = 1, \cdots, k).$$

Since the $\mathbf{Z}_n$-action on $S(V_1)$ is free, there is a complex structure J_1 on V_1 such that

$$\sigma J_1 = J_1 \sigma, \qquad \sigma v_1 = a v_1 + b J_1 v_1$$

for some $a, b \in \mathbf{R}$, $b \neq 0$, where σ is a generator of $\mathbf{Z}_n$, moreover the real vector space V_1 is spanned by $\{v_1, J_1 v_1\}$.   Therefore there is a complex structure J on V such that

$$J v_1 - J_1 v_1, \quad J g_i v_1 - g_i J_1 v_1 \quad \text{and} \quad \sigma v = a v + b J v$$

for each $v \in V$.   Then

$$g \sigma v = a g v + b g J v,$$
$$\sigma g v = a g v + b J g v$$

for any $g \in SU(n)$.   Therefore the complex structure J is $SU(n)$-invariant, since $g\sigma = \sigma g$ and $b \neq 0$.   q.e.d.

Let M be a closed connected manifold with smooth $SU(n)$-action $(n \geqslant 3)$. Assume that the identity component of each isotropy group is conjugate to one of the following

$$SU(n), \ SU(n-1) \ \text{and} \ NSU(n-1)$$

Assume that the stationary point set $F = F(SU(n), M)$ is non-empty.   Let U be an invariant closed tubular neighborhood of F in M.   Then there is an equivariant decomposition

$$M = \ U \cup (SU(n)/SU(n-1) \underset{S^1}{\times} X) = U \cup (S^{2n-1} \underset{S^1}{\times} X),$$

where $X = F(SU(n-1), M - \text{int } U)$ with the natural S^1-action.   Since

$$dU = \ SU(n)/SU(n-1) \underset{S^1}{\times} \partial X = S^{2n-1} \underset{S^1}{\times} \partial X$$

as $SU(n)$-maifolds, the S^1-action on ∂X is free, $F = \partial X/S^1$, and the disk bundle $U \to F$ with $SU(n)$-action is equivariantly isomorphic to the disk bundle

$$D^{2n} \underset{S^1}{\times} \partial X \to \partial X/S^1,$$

where the $SU(n)$-action on D^{2n} is standard by Lemma 1.2 and Lemma 1.3.

Therefore the codimension of F in M is $2n$, X is connected, and there is an equivariant diffeomorphism

$$(1.4) \qquad M = \partial(D^{2n} \times X)/S^1 = D^{2n} \underset{S^1}{\times} \partial X \cup S^{2n-1} \underset{S^1}{\times} X$$

as $SU(n)$-manifolds.

Lemma 1.5. *Let G be a closed connected proper subgroup of $SU(n), (n \geqslant 7)$. If*

$$\dim G > n^2 - 4n + 7 = \dim N(SU(n-2), SU(n)),$$

then G is conjugate to $SU(n-1)$ or $NSU(n-1)$ in $SU(n)$.

Proof. The inclusion $\rho: G \subset SU(n)$ gives an n-dimensional complex representation of G. First we show that the representation p is reducible. Suppose that p is irreducible. Then G is semi-simple from the Shur's lemma. If G is not simple, then there are integers $p \geqslant q \geqslant 2$ with $n = pq$, such that G is conjugate to a subgroup of the tensor product

$$SU(p) \otimes SU(q)$$

in $SU(pq)$, by considering the induced representation of the universal covering group of G. Therefore

$$\dim G \leqslant p^2 + q^2 - 2 \leqslant \left(\frac{n}{2}\right)^2 + 2 \leqslant \frac{n(n+1)}{2}.$$

If G is simple but not one of the type

$$A_k, D_{2k+1} \quad \text{and} \quad E_6,$$

then G is conjugate to a subgroup of $SO(n)$ or $Sp(n/2)$, (see [6], p. 336, Theorem 0.20). But

$$\dim SO(n) = \frac{n(n-1)}{2}, \quad \dim Sp\left(\frac{n}{2}\right) = \frac{n(n+1)}{2}$$

and hence

$$\dim G \leqslant \frac{n(n+1)}{2}.$$

If G is of type D_{2k+1} $(k \geqslant 2)$, then the lowest dimensional non-trivial irreducible complex representation is $(4k+2)$-dimensional (see [6], p. 378, Table 30). Therefore $4k+2 \leqslant n$ and hence

$$\dim G = \dim SO(4k+2) = (2k+1)(4k+1) \leqslant \frac{n(n-1)}{2}.$$

If G is of type E_6, then $n \geqslant 27$ (see [6], p. 378, Table 30). Therefore

F. UCHIDA

$$\dim G = 78 \leqslant 3n \leqslant \frac{n(n+1)}{2} .$$

Finally, if G is of type A_{k-1} $(k < n)$, then

$$\frac{k(k-1)}{2} \leqslant n ,$$

by the Weyl's formula (see [14], Theorem 7.5).   Therefore

$$\dim G = \dim SU(k) = k^2 - 1 \leqslant 3n - 2 \leqslant \frac{n(n+1)}{2} .$$

Consequently

$$\dim G \leqslant \frac{n(n+1)}{2} ,$$

if p: $G \subset SU(n)$ is irreducible $(n \geqslant 4)$. Therefore p is reducible, if

$$\dim G > n^2 - 4n + 7 \quad \text{and} \quad n \geqslant 7 .$$

Since p is reducible, G is conjugate to a subgroup of

$$N(SU(n\text{-}p), \quad SU(n)), \left(1 \leqslant p \leqslant \frac{n}{2}\right)$$

the normalizer of $SU(n-p)$ in $SU(n)$.   But

$$\dim \ N(SU(n\text{-}p), \ SU(n)) \leqslant n^2 - 4n + 7$$

for $2 \leqslant p \leqslant \frac{n}{2}$.   Therefore G is conjugate to a subgroup G' of $NSU(n-1)$. If $G' \neq NSU(n-1)$, then

$$\dim G' \leqslant \dim G'' + 1$$

where $G'' = G' \ \Pi \ SU(n-1)$, by the isomorphism

$$NSU(n-1)/SU(n-1) \cong \ S^1 .$$

If $G'' = \ SU(n-1)$ then $G' = G'' = SU(n-1)$. If $G'' \neq SU(n-1)$, then

$$\dim G'' \leqslant (n-2)^2 \ = \ \dim N(SU(n-2), SU(n-1)),$$

by making use of the first part of the proof of this lemma for $SU(n-1)$ instead of $SU(n)$, and hence

$$\dim G' \leqslant (n-2)^2 + 1 < n^2 - 4n + 7 .$$

Consequently we see that G is conjugate to $SU(n-1)$ or $NSU(n-1)$ in $SU(n)$.

<div align="right">q.e.d.</div>

Lemma 1.6.   *Let M be a manifold with smooth $SU(n)$-action.   If* $\dim M <$
$4n-8$, then

$$\dim SU(n)_x > n^2 - 4n + 7$$

for each $x \in M$.

Proof.   Since $SU(n)/SU(n)_x$ is equivariantly embedded in M,

$$\dim SU(n) - \dim SU(n)_x \leqslant \dim M < 4n-8 .$$

Hence $\dim SU(n)_x > \dim SU(n) - (4n-8) = n^2 - 4n + 7$.   q.e.d.

2. $SU(n)$-actions on cohomology complex projective spaces

In this section we prove the following results.

Theorem 2.1.   *Let $n \geqslant 7$ and $0 \leqslant k < n-4$.   Let M be a compact connected
orientable smooth $2(n+k)$-manifold with*

$$H^*(M; \boldsymbol{Q}) = H^*(P_{n+k}(\boldsymbol{C}); \boldsymbol{Q}) .$$

*Then for any non-trivial smooth $SU(n)$-action on M, the stationary point set
$F = F(SU(n), M)$ is an orientable $2k$-manifold with*

$$H^*(F; \boldsymbol{Q}) = H^*(P_k(\boldsymbol{C}); \boldsymbol{Q})$$

and there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1 .$$

*Here X is a compact connected orientable $(2k+2)$-manifold which is acyclic over
rationals, X admits a smooth S^1-action which is free on ∂X, the $SU(n)$-action is
standard on D^{2n} and trivial on X, and*

$$\pi_1(X) = \pi_1(M) .$$

Furthermore, if

$$H^*(M; \boldsymbol{Z}) = H^*(P_{n+k}(\boldsymbol{C}); \boldsymbol{Z}) ,$$

then X is acyclic over integers, the S^1-action on X is semi-free, and

$$H^*(F; \boldsymbol{Z}) = H^*(P_k(\boldsymbol{C}); \boldsymbol{Z}) .$$

Corollary 2.2,   *Let $n \geqslant 7$ and $0 \leqslant k < n-4$.   Let M be a compact connected
smooth $2(n+k)$-manifold which is homotopy equivalent to $P_{n+k}(\boldsymbol{C})$.   If M admits a
non-trivial smooth $SU(n)$-action, then M is diffeomorphic to $P_{n+k}(\boldsymbol{C})$.*

Proof of Theorem 2.1.   By Lemma 1.5, Lemma 1.6 and the assumption
$n \geqslant 7$ and $0 \leqslant k < n-4$, the identity component of each isotropy group of the

given $SU(n)$-action on M is conjugate to one of the following

$$SU(n),\ SU(n\text{-}l)\quad \text{and}\quad NSU(n-1).$$

(i)   First we show that the stationary point set $F=F(SU(n),M)$ is non-empty.   Assume $F=\emptyset$, then by (1.1) there is a smooth fibre bundle

$$F(SU(n-1),M)\ \to M\to P_{n-1}(C).$$

Thus

$$\chi(M)\ =\ \chi(P_{n-1}(C))\cdot\chi(F(SU(n-1)M))$$

and hence

$$k+1\equiv 0\,(\mathrm{mod}\ n)\,.$$

This is impossible by the assumption $0\leqslant k<n-4$.   Thus $F\neq\emptyset$.   Then by (1.4) there is an equivariant diffeomorphism

$$M=\partial(D^{2n}\underset{S^1}{\times}X)/S^1 = D^{2n}\underset{S^1}{\times}\partial X\cup S^{2n-1}\underset{S^1}{\times}X$$

as $SU(n)$-manifolds.   Here X is a compact connected orientable $(2k+2)$-manifold with smooth S^1-action which is free on ∂X.

(ii)   Next we show that X is acyclic over rationals.   Since

$$D^{2n}\underset{S^1}{\times}\partial X\to\partial X/S^1= F$$

is a $2n$-disk bundle, there is an isomorphism

$$H^i(M,\ S^{2n-1}\underset{S^1}{\times}X;Q) = H^{i-2n}(F;Q)\,.$$

Thus

(2.3) $H^i(M;Q) = H^i(S^{2n-1}\underset{S^1}{\times}X;Q)\qquad\text{for }i\leqslant 2n-2\,.$

Now we show that the euler class $e(p)$ of the principal S^1-bundle

$$p:\quad \partial(D^{2n}\times X)\to M$$

is non-zero in $H^2(M;Q)$.   Assume $e(p)=0$, then the euler class of the principal S^1-bundle

$$S^{2n-1}\times X\to S^{2n-1}\underset{S^1}{\times}X$$

is zero in $H^2(S^{2n-1}\underset{S^1}{\times}X;Q)$, and hence there is an isomorphism

$$H^*(S^{2n-1};Q)\otimes H^*(X;Q) = H^*(S^1;Q)\otimes H^*(S^{2n-1}\underset{S^1}{\times}X;Q)\,.$$

Therefore

$$H^i(X;Q) = Q\qquad\text{for }0\leqslant i\leqslant 2n-2$$

by (2.3) and the assumption

$$H^*(M; \mathbf{Q}) = H^*(P_{n+k}(C); \mathbf{Q}) .$$

But

$$\dim X = 2k+2 \leqslant 2n-2 .$$

Thus $H^{2k+2}(X; \mathbf{Q}) = \mathbf{Q}$ and this is a contradiction, since the connected manifold X has a non-empty boundary. Therefore $e(p) \neq 0$ and hence

(2.4) $$H^*(\partial(D^{2n} \times X)\mathbf{Q}) = H^*(S^{2n+2k+1}; \mathbf{Q}) .$$

There is an isomorphism

$$H^i(D^{2n} \times X; \mathbf{Q}) = H_{2n+2k+2-i}(D^{2n} \times X, \partial(D^{2n} \times X); Q)$$

by the Poincaré-Lefschetz duality, and the homomorphism

$$H_{2n+2k+2-i}(D^{2n} \times X; Q) \to H_{2n+2k+2-i}(D^{2n} \times X, \partial(D^{2n} \times X); Q)$$

is onto for $0 < i < 2n+2k+2$ by (2.4). Since X is a connected $(2k+2)$-manifold with a non-empty boundary,

$$H_{2n+2k+2-i}(D^{2n} \times X; \mathbf{Q}) = 0 \quad \text{for } i \leqslant 2n,$$

and hence

$$H^i(X; \mathbf{Q}) = 0 \quad \text{for } 0 < i \leqslant 2n .$$

Therefore X is acyclic over rationals. Then

$$H^*(\partial X; \mathbf{Q}) = H^*(S^{2k+1}; \mathbf{Q}) ,$$

by the Poincaré-Lefschetz duality, and hence

$$H^*(F; Q) = H^*(P_k(C); Q) .$$

Furthermore $F(S^1, X)$ consists just one point by the P.A. Smith theory (see [2], chapter IV) from the fact that X is acyclic over rationals and the S^1-action is free on ∂X.

(iii) Next we show $\pi_1(X) = \pi_1(M)$. Since $F(S^1, X) = \{x_0\}$, there is an S^1-map

$$s: \tilde{}^{-1} \to \partial(D^{2n} \times X)$$

given by $s(y) = (y, x_0)$. Then we have an isomorphism

$$\pi_1(M) = \pi_1(\partial(D^{2n} \times X))$$

from the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(S^1) & \longrightarrow & \pi_1(S^{2n-1}) \\
\downarrow id & & \downarrow s_* \\
\pi_1(S^1) & \dashrightarrow \pi_1(\partial(D^{2n} \times X)) \xrightarrow{\ p_*\ } & \pi_1(M) .
\end{array}$$

Applying the van Kampen theorem (see [5], p. 63) to the decomposition

$$\partial(D^{2n} \times X) = D^{2n} \times \partial X \cup S^{2n-1} \times X ,$$

we have

$$\pi_1(X) = \pi_1(\partial(D^{2n} \times X)) ,$$

and hence

$$\pi_1(X) = \pi_1(M) .$$

(iv)   Finally we show that the assumption

$$H^*(M; Z) = H^*(P_{n+k}(C)\ Z)$$

implies $H^*(X, x_0; Z) = 0$.   There is a commutative diagram:

$$\begin{array}{ccc}
S^{2n-1} & \xrightarrow{\ s\ } & \partial(D^{2n} \times X) \\
\downarrow p_0 & \quad t \quad & \downarrow p \\
P_{n-1}(C) & \xrightarrow{\quad} & M .
\end{array}$$

Since $t^*e(p) = e(p_0)$ is a generator of $H^*(P_{n-1}(C); Z)$, $e(p)$ is a generator of $H^*(M; Z)$.   Therefore

$$H^*(\partial(D^{2n} \times X); Z) = H^*(S^{2n+2k+1}; Z)$$

by the Gysin seqence for the principal S^1-bundle

$$p : \partial(D^{2n} \times X) \to M ,$$

and hence X is acyclic over integers and

$$H^*(F; Z) = H^*(P_k(C); Z)$$

by the same argument as in (ii).   Then the S^1-action on X is semi-free by the P.A. Smith theory from the fact that X is acyclic over integers and the S^1-action is free on ∂X.   This completes the proof of Theorem 2.1.

Proof of Corollary 2.2.   If M admits a non-trivial smooth $SU(n)$-action, then by Theorem 2.1, there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1$$

as $SU(n)$-manifolds.   Here X is a compact contractible $(2k+2)$-manifold with smooth semi-free S^1-action with just one stationary point x_0.   Therefore the

S^1-action on $D^{2n} \times X$ is semi-free and its stationary point is only $(0, x_0)$. Let U be an invariant closed disk around the point $(0, x_0)$. One may assume that the S^1-action on U is linear. Put

$$W = (D^{2n} \times X - \text{int } U)/S^1 .$$

Then

$$\partial W = dU/S^1 \cup \partial(D^{2n} \times X)/S^1 = P_{n+k}(C) \cup M .$$

Since

$$\pi_1(M) = \pi_1(W) = 0 ,$$
$$H_*(W, M \ \ Z) = 0$$

and

$$\dim W = 2n + 2k + 1 \geqslant 6 ,$$

we have

$$M = P_{n+k}(C)$$

by applying the h-cobordism theorem (see [10], Theorem 9.1) to the triad $(W; M, P_{n+k}(C))$. This completes the proof of Corollary 2.2.

3. Construction of $SU(n)$-actions

In this section we construct $SU(n)$-actions on cohomology complex projective spaces, and we have the following results.

Theorem 3.1. *Let $n \geqslant 2$, $k \geqslant 1$ and $p \geqslant 1$. Then there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = Z/pZ \quad and \quad H^*(M; Q) = H^*(P_{n+k}(C); Q)$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(C) .$$

Theorem 3.2. *Let $n \geqslant 2$ and $k \geqslant 3$. Let G be a finitely presentable group with $H_1(G; Z) = H_2(G; Z) = 0$. Then*

(a) *there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = G \quad and \quad H^*(M; Z) = H^*(P_{n+k}(C); Z)$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(C) ,$$

(b) *there is a smooth $SU(n)$-action on $P_{n+k}(C)$ such that*

$$\pi_1(F) = G \quad and \quad H^*(F; Z) = H^*(P_k(C); Z) ,$$

where $F = F(SU(n), P_{n+k}(C))$.

First we prepare the following lemma. It is proved by a similar argument as in the proof of Theorem 2.1 and Corollary 2.2, so we omit the proof.

Lemma 3.3. *Let X be a compact orientable $(2k+2)$-manifold which is acyclic over Z (resp. Q). Assume that X admits a smooth S^1-action which is free on ∂X. If $n \geqslant 2$, then*

(a) $M = \partial(D^{2n} \times X)/S^1$ is a cohomology $P_{n+k}(C)$ over Z (resp. Q),

(b) $\pi_1(M) = \pi_1(X)$.

Moreover if $n+k \geqslant 3$ and X is contractible, then $M = P_{n+k}(C)$.

Now we construct an acyclic S^1-manifold. Let W be a closed orientable smooth homology $(2k+1)$-sphere over Z (resp. Q) and let

(3.4) $Y = P_k(C) \times [0,1] \# W,$ $(k \geqslant 1)$.

Then F is a compact connected orientable smooth $(2k+1)$-manifold with boundary

$$\partial Y = P_k(C) \times 0 \cup P_k(C) \times 1 .$$

It is easily seen that

(3.5) $\pi_1(Y) = \pi_1(W)$,

(3.6) $H^i(Y; Z) = H^i(P_k(C); Z) \oplus H^i(W; Z)$, $(0 < i \leqslant 2k)$.

Furthermore there is a smooth principal S^1-bundle

$$p: E \to Y$$

such that $\partial_i E \to P_k(C) \times i$, $(i=0, 1)$ is equivalent to the Hopf bundle $S^{2k+1} \to P_k(C)$, where $\partial_i E = p^{-1}(P_k(C) \times i)$.   Then

(3.7) $\pi_1(E) = \pi_1(Y)$,

(3.8) $H^*(E, \partial_1 E; A) = 0$

where $A = Z$ (resp. Q), by (3.6) and the Gysin sequence for S^1-bundles. Furthermore

$$X = E \underset{\partial_1 E}{\cup} D^{2k+2}$$

is a compact orientable manifold with a semi-free smooth S^1-action which is linear and free on $\partial X = \partial_0 E = S^{2k+1}$.   It is easily seen that

(3.9) $\pi_1(X) = \pi_1(W)$,   by (3.5) and (3.7),

(3.10) X is acyclic over Z (resp. Q),   by (3.8).

Proof of Theorem 3.1.   Put $W = S^{2k+1}/Z_p$, a lens space, in (3.4).   Then there is a compact orientable $(2k+2)$-manifold X with a semi-free smooth S^1-action which is linear and free on $\partial X = S^{2k+1}$, such that $\pi_1(X) = Z_p$ and X is acyclic over Q.   Then by Lemma 3.3, the $SU(n)$-manifold

$$M = \partial(D^{2n} \times X)/S^1$$

is a compact orientable $2(n+k)$-manifold such that

$$\pi_1(M) = Z_p, \quad H^*(M; Q) = H^*(P_{n+k}(C); Q)$$

and

$$F(SU(n), M) = \partial X/S^1 = P_k(C) .\qquad\qquad \text{q.e.d.}$$

REMARK 3.11.   It is known that if G is a finitely presentable group with $H_1(G; Z) = H_2(GZ) = 0$, then for each $m \geqslant 7$, there is a compact contractible smooth n-manifold P such that

$$\pi_1(\partial P) = G \qquad (\text{see } [12]).$$

It is known that there are infinitely many groups satisfying the above condition.

Proof of Theorem 3.2 (a).   Let $k \geqslant 3$.   Put $W = \partial P$, a smooth homology $(2k+1)$-sphere over Z with $\pi_1(\partial P) = G$, in (3.4).   Then there is a compact orientable $(2k+2)$-manifold X with a semi-free smooth S^1-action which is linear and free on $\partial X = S^{2k+1}$, such that $\pi_1(X) = G$ and X is acyclic over Z.   Then by Lemma 3.3, the $SU(n)$-manifold

$$M = \partial(D^{2n} \times X)/S^1$$

is a compact orientable $2(n+k)$-manifold such that

$$\pi_1(M) = G, \quad \text{ff}^*(M; Z) = H^*(P_{n+k}(C); Z)$$

and

$$F(SU(n), M) = P_k(C) .\qquad\qquad \text{q.e.d.}$$

Proof of Theorem 3.2 (b).   Let $k \geqslant 3$.   For a given group G satisfying the hypothesis, there is a compact contractible smooth $(2k+1)$-manifold p such that

$$\pi_1(\partial P) = G$$

by Remark 3.11.   Let

$$Y = P_k(C) \times [0, 1] \# P ,$$

a boundary connected sum with boundary

$$\partial Y = P_k(C) \# \partial P \cup P_k(C) \times 1 .$$

Then $P_k(C) \times 1$ is a deformation retract of Y, and hence there is a smooth principal S^1-bundle

$$p : E \to Y,$$

such that $\partial_1 E \to P_k(C) \times 1$ is equivalent to the Hopf bundle $S^{2k+1} \to P_k(C)$, where $\partial_1 E = p^{-1}(P_k(C) \times 1)$.   Then

$$X = E \underset{\partial_1 E}{\cup} D^{2k+2}$$

is a compact contractible $(2k+2)$-manifold with a semi-free smooth S^1-action. Then by Lemma 3.3, the $SU(n)$-manifold

$$M = \partial(D^{2n} \times X)/S^1$$

is diffeomorphic to $P_{n+k}(C)$ for $n \geqslant 2$, and

$$F(SU(n), M) = \partial X / S^1 = P_k(C) \# \partial P.$$

Therefore there is a smooth $SU(n)$-action on $P_{n+k}(C)$ such that

$$\pi_1(F) = G \quad \text{and} \quad H^*(F; Z) = H^*(P_k(C); Z),$$

where $F = F(SU(n), P_{n+k}(C))$.   q.e.d.

4.   Signature of certain smooth G-manifolds

The purpose of this section is to study a signature of closed orientable manifold which admits a smooth G-action with isotropy groups of uniform dimension.   We have the following result.

Theorem 4.1.   *Let G be a compact Lie group and H a closed connected subgroup.   Let M be a compact orientable manifold without boundary.   Assume that M admits a smooth G-action such that the identity component of an isotropy group G_x is conjugate to H in G for each point x of M.   Then $F(H, M)$, the stationary point set with respect to the H-action, is orientable, and*
  (a)   *if $\dim N(H) \neq \dim H$,   then $\mathrm{Sign}(M) = 0$,*
  (b)   *if $\dim N(H) = \dim H$,   then*

$$|N(H)/H| \, \mathrm{Sign}(M) = \mathrm{Sign}(G/H) \, \mathrm{Sign}(F(H, M)) .$$

Here $N(H)$ is the normalizer of H in G, $|N(H)/H|$ is the order of the finite group $N(H)/H$.

The result is a generalization of the fact that $\mathrm{Sign}(M) = 0$ if M admits a smooth circle action without stationary points.

Lemma 4.2.   *Let G be a compact Lie group and H a closed connected subgroup.   Let M be a smooth G-manifold such that the identity component of G_x is*

conjugate to H in G for each **point** *x of M.* *Then*

(a) *the W(H)-action on F(H,M) is almost free (i.e. all isotropy groups are* discrete), *where* $W(H)=N(H)/H$,

(b) *there is an equívariant diffeomorphism*

$$M = G \underset{N(H)}{\times} F(H, M) = G/H \underset{W(H)}{\times} F(H, M),$$

(c) *if M is orientable, then F(H, M) is orientable.*

Proof. By the assumption, the identity component of G_x is equal to H for each point x of $F(H, M)$, and the mapping

$$f: \quad G \times F(H, M) \to M$$

given by $f(g, x)=g \chi$ is surjective. Moreover $f(g, x)$ is in $F(H, M)$ if and only if $g \in N(H)$, thus $W(H)$ acts on $F(H, M)$ naturally and (b) is proved. Next, if an isotropy group $W(H)_x$ is not discrete for a point x of $F(H, M)$, then

$$\dim G_x \neq \dim H .$$

This contradicts our assumption, and (a) is proved. By (b), the product manifold $G/H \times F(H,M)$ is a total space of a principal $W(H)$-bundle over M. Therefore $G/H \times F(H,M)$ is orientable, if M is orinetable, and hence $F(H, M)$ is orientable.   q.e.d.

Lemma 4.3. *Let G be a compact Lie group which is not discrete. Let M be a compact orientable smooth manifold without boundary. Then,* $\operatorname{Sign}(M)=0$ *if M admits an almost free smooth G-action.*

Proof. *G* contains a circle subgroup and the circle action on *M* has no stationary points. Therefore $\operatorname{Sign}(M)=0$.   q.e.d.

Proof of Theorem 4.1. Denote by $W(H)^0$, the identity component of $W(H)$. Then

$$G/H \underset{W(H)^0}{\times} F(H, M)$$

is a total space of a principal $W(H)/W(H)^0$-bundle over M by Lemma 4.2. (b). Therefore

$$| W(H)/W(H)^0| \cdot \operatorname{Sign}(M) = \operatorname{Sign}(G/H \underset{W(H)^0}{\times} F(H, M)) .$$

Next, $G/H \underset{W(H)^0}{\times} F(H, M)$ is a total space of a smooth fibre bundle over an orientable manifold $(G/H)/W(H)^0$ with a fibre $F(H, M)$ and a structure group $W(H)^0$ which is connected. Therefore

$$\text{Sign}(G/H \underset{W(H)^0}{\times} F(H, M)) = \text{Sign}((G/H)/W(H)^0) \cdot \text{Sign}(F(M))$$

for a certain orientation of $F(H, M)$ by [4]. By the above equations,

$$| W(H)/W(H)^0 | \text{Sign}(M) = \text{Sign}((G/H)/W(H)^0) \ \text{Sign}(F(H,M)) .$$

Now, if dim $W(H) \neq 0$ then $\text{Sign}(F(H,M))=0$ by Lemma 4.2 (a) and Lemma 4.3. If dim $W(H)=0$, then

$$| W(H) | \text{Sign}(M) = \text{Sign}(G/H) \ \text{Sign}(F(H, M)) .$$

This completes the proof.

REMARK 4.4. Let G be an arbitrary compact connected Lie group and T be a maximal torus. Then $\text{Sign}(G/T)=0$, since G/T is stably parallelizable (see [3], section 5.4).

REMARK 4.5. Let G be a compact connected Lie group and H a closed connected subgroup. Then $\text{Sign}(G/H)=0$ if

$$\text{rank } G \neq \text{rank } H \qquad (\text{see } [7]).$$

Because the left translation on G/H of a maximal torus of G has no stationaly points.

5. $SU(3)$-actions on orientable 8-manifolds

The purpose of this section is to prove the following result.

Theorem 5.1. *Let M be a closed connected orientable* 8-*manifold. Assume that M admits a non-trivial smooth $SU(3)$-action with a principal isotropy type (H). Then*
  (a) $H^4(M; \boldsymbol{Q})=0$, *if* dim $H=0$,
  (b) $\text{Sign}(M)=0$, *if* dim $H=1$ *and M has not isotropy types* $(NSU(2))$ *and* $(T_{(2)})$,
  (c) $\text{Sign}(M)=0$, *if* dim $H=2$,
  (d) $H^4(M \ \boldsymbol{Q})=0$, *if* dim $H=3$ *and M has not an isotropy type* $(NSU(2))$,
  (e) $M=P_2(C) \times F(NSU(2),M)$, *if* dim $H=4$.
Here NSU(2) is the normalizer of SU(2) in SU(3), the identity component of $T_{(2)}$ is a maximal torus of SU(3) and $T_{(2)}$ has 2-components.

First we recall an additivity property of the signature due to S.P. Novikov (see [1], p. 588). Suppose that Y is a compact oriented $4n$-manifold with boundary ∂Y. Let $\hat{H}^{2n}(Y; \boldsymbol{Q})$ denote the image of the natural homomorphism

$$j^*: H^{2n}(Y, \partial Y; Q) \rightarrow H^{2n}(Y; \boldsymbol{Q}) .$$

Then the bilinear form B on $\hat{H}^{2n}(Y; Q)$ defined by

$$B(j^*(a), j^*(b)) = ab[Y]$$

is symmetric and non-degenerate by Poincare-Lefschetz duality. We can now define $\mathrm{Sign}(Y)$ as the signature of B. Suppose now that Y' is another compact oriented $4n$-manifold with boundary $\partial Y' = -\partial Y$. Then $X = Y \underset{\partial Y}{\cup} Y'$ is a closed orineted $4n$-manifold and

(5.2) $\mathrm{Sign}(X) = \mathrm{Sign}(Y) + \mathrm{Sign}(Y')$.

REMARK 5.3.   Let ξ be an orientable k-plane bundle over a closed orientable manifold X. Denote by $t(\xi)$, $e(\xi)$ and $D(\xi)$, the Thom class, the Euler class and the disk bundle of ξ, respectively. Then $D(\xi)$ is a compact orientable manifold and there is a commutative diagram:

$$
\begin{array}{ccc}
H^*(D(\xi), \partial D(\xi)) & \overset{j^*}{\longrightarrow} & H^*(D(\xi)) \\
\cong \uparrow \psi & & \cong \uparrow \pi^* \\
H^*(X) & \overset{\cdot e(\xi)}{\longrightarrow} & H^*(X).
\end{array}
$$

Here ψ is the Thom isomorphism defined by

$$\psi(a) = \pi^*(a) \cdot t(\xi).$$

There is an equation

$$\psi(a) \cdot \psi(b) = (-1)^{kp}\psi(ab \cdot e(\xi)) \qquad \text{for } b \in H^p(X).$$

Therefore we can calculate $\mathrm{Sign}(D(\xi))$ from the information about the cohomology ring $H^*(X)$ and the Euler class $e(\xi)$.

Now we prepare the following results.

Lemma 5.4.
(a) $H^*(SU(3); \mathbf{Z}) = \wedge_{\mathbf{Z}}(x_3, x_5)$, $\deg x_i = i$, $(i = 3, 5)$.
(b) $H^*(SU(3)/SU(2)\mathbf{Z}) = H^*(S^5; \mathbf{Z})$ and the right translation of $NSU(2)/SU(2) = S^1$ induces a trivial action on $H^*(SU(3)/SU(2); Z)$.
(c) $H^*(SU(3)/SO(3)\mathbf{Q}) = H^*(S^5; \mathbf{Q})$, and the right translation of $NSO(3)/SO(3) = \mathbf{Z}_3$ induces a trivial action on $H^*(SU(3)/SO(3);\mathbf{Q})$.
(d) $H^*(SU(3)/T;\mathbf{Z}) = \mathbf{Z}[u_1, u_2, u_3]/(s_1s_2, s_3)$,
where T is a maximal torus of $SU(3)$ consists of all diagonal matrices, s_k is the k-th elementary symmetric polynomials, and $\deg u_i = 2$, $(i = 1, 2, 3)$. Furthermore the induced action of $N(T)/T = S_3$, the symmetric group on 3-elements, is given by

$$a^*(u_i) = u_{a(i)}, \qquad a \in S_3.$$

(e) $H^*(SU(3)/D(m,n); \mathbf{Q}) = \wedge_{\mathbf{Q}}(x_2, x_5)$, $\deg x_i = i$, $(i = 2, 5)$. Here $D(m, n)$ is a closed one-dimensional subgroup defined by

$$D(m, n) = \left\{ \begin{pmatrix} z^m & & \\ & z^n & \\ & & z^{-(m+n)} \end{pmatrix}; z \in \boldsymbol{C}, \ |z|=1 \right\}$$

for any pair of integers $(m, n) \neq (0, 0)$.

Proof.   Since $SU(3)/SU(2) = S^5$ (b) is true.   (a) is proved by making use of the Gysin sequence for

$$SU(2) \to SU(3) \to S^5 .$$

(c) is proved from

$$\pi_1(SU(3)/SO(3)) = 0 \text{ and } \pi_2(SU(3)/SO(3)) = \boldsymbol{Z}_2 .$$

(d) is a classical result (see [9]).   In fact $u_i = p_i^*(u)$, where u is a generator of $H^2(P_2(\boldsymbol{C}); Z)$ and p_i ; $SU(3)/T \to P_2(\boldsymbol{C})$ is defined by

$$p_i((x_{ab}) \cdot T) = (x_{1i} : x_{2i} : x_{3i}) .$$

Finally (e) is proved from the fact that the Euler class of principal S^1-bundle π: $SU(3)/D(m, n) \to SU(3)/T$ is

$$e(\pi) = nu_1 + mu_2 ,$$

and hence the homomorphism

$$H^2(SU(3)/T; \ Q) \xrightarrow{\cdot e(\pi)} H^4(SU(3)/T; Q)$$

is an isomorphism.   q.e.d.

Lemma 5.5.
   (a)   *Let* φ *be an* 8-*dimensional non-trivial real representation of* $SU(3)$. *Let* (H_φ) *be the principal isotropy type of the linear action given by* φ. *Then there are only the following cases:*
   (i) $\varphi = Ad_{SU(3)}, H_\varphi = T$: *a maximal torus of* $SU(3)$,
   (ii) $\varphi = \rho_3 +$ *trivial summand*, $H_\varphi = SU(2)$,
where ρ_3: $SU(3) \to O(6)$ *is the standard representation.*
   (b)   *Let* ψ *be a* 4-*dimensional non-trivial real representation of* $NSU(2)$. *Let* (H_ψ) *be the principal isotropy type of the linear action given by* ψ. *Then there are only the following cases:*
   (i) $\psi = Ad_{NSU(2)}, H_\psi = T$: *a maximal torus of* $NSU(2)$,
   (ii) $\psi = \sigma_k, H_\psi = D(k-1, -k), (k \in \boldsymbol{Z})$,
where the representation $\sigma_k: NSU(2) \to U(2) \subset O(4)$ *is given by*

$$\sigma_k \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} y^k x_{11} & y^k x_{12} \\ y^k x_{21} & y^k x_{22} \end{pmatrix}.$$

(iii) ψ is induced from a *non-trivial* real representation of S^1, via the natural projection $NSU(2) \to NSU(2)/SU(2) \cong S^1$, and $H_\psi^0 = SU(2)$, where H_ψ^0 is the identity component of H_ψ.

We omit the proof (see [8], Theorem I).

From now on we assume that M is a closed connected orientable smooth 8-manifold and M admits a non-trivial smooth $SU(3)$-action with a principal isotropy type (H). Then $SU(3)/H$ is orientable by the differentiable slice theorem (see [11], Lemma 3.1).

We will prove Theorem 5.1 by the following many propositions.

Proposition 5.6. *Assume that $SU(3)_x^0$ is conjugate to H^0 in $SU(3)$ for each $x \in M$. Here G^0 is the identity component of G and $SU(3)_x$ is the isotropy group at x. Then,*

(a) $\mathrm{Sign}(M) = 0$, if $\dim H = 1$ or 2,
(b) $H^4(M; Q) = 0$, if $\dim H = 0$ or 3,
(c) $M = P_2(C) \times F(NSU(2), M)$, if $\dim H = 4$.

Proof.   If $\dim H = 1$ or 2, then $\mathrm{Sign}(M) = 0$ by Theorem 4.1 and Remarks 4.4, 4.5.   If $\dim H = 0$, then $M = SU(3)/H$ and hence $H^4(M; Q) = 0$ by Lemma 5.4 (a).   By Lemma 4.2, there is an equivariant diffeomorphism

$$M = SU(3)/H^0 \underset{K}{\times} F,\ K = N(H^0)/H^0,\ F = F(H^0, M).$$

If $\dim H = 4$, then H^0 is conjugate to $NSU(2)$ in $SU(3)$ and $N(NSU(2)) = NSU(2)$.   Therefore

$$M = P_2(C) \times F(NSU(2), M).$$

Finally if $\dim H = 3$, then H^0 is conjugate to $SO(3)$ or $SU(2)$ in $SU(3)$.   If $H^0 = SO(3)$, then $\dim F = 3$ and

$$H^4(M; Q) = H^4(SU(3)/SO(3) \times F, Q) = 0$$

by Lemma 5.4 (c).   Next if $H^0 = SU(2)$, then $\dim F = 4$, F admits a smooth S^1-action without stationary points and there is an equivariant diffeomorphism

$$M = S^5 \underset{S^1}{\times} F.$$

There is a sufficiently large integer n such that the S^1/Z_n-action on the orbit space F/Z_n is free.   Then there is an isomorphism

$$H^*(M; \boldsymbol{Q}) = H^*(M'; \boldsymbol{Q}),$$

where

$$M' = (S^5/\boldsymbol{Z}_n \times F/\boldsymbol{Z}_n)/(S^1/\boldsymbol{Z}_n),$$

and there is a fibre bundle

$$S^5/\boldsymbol{Z}_n \to M' \to F/S^1$$

with a structure group $S^1/\boldsymbol{Z}_n$. Here $F/S^1 = (F/\boldsymbol{Z}_n)/(S^1/\boldsymbol{Z}_n)$ is a 3-dimensional rational cohomology manifold. Therefore

$$H^4(M; \boldsymbol{Q}) = H^4(M'; \boldsymbol{Q}) = 0.\qquad\qquad\qquad \text{q.e.d.}$$

REMARK 5.7. Now Theorem 5.1 is proved for dim $H = 0$ or 4. Moreover, Theorem 5.1 is proved for the case $H^0 = SO(3)$, since $SO(3)$ is not conjugate to any subgroup of $NSU(2)$ in $SU(3)$ and H with $H^0 = SO(3)$ is not a principal isotropy group of any 8-dimensional real representation of $SU(3)$ by Lemma 5.5.

Proposition 5.8. *Suppose* dim $H = 1$. *Then* $\mathrm{Sign}(M) = 0$, *if M has not isotropy types* $(NSU(2))$ *and* $(T_{(2)})$.

Proof. By Proposition 5.6 (a), one may assume that there is an isotropy type (K_1) with dim $K_1 > 1$. Then by making use of the differentiable slice theorem, there is an isotropy type (K_2) and there is an equivariant decomposition

$$M = D(\nu_1) \cup D(\nu_2),$$

where $D(\nu_i)$ is an equivariant normal disk bundle of an embedding $SU(3)/K_i$ CM, and

$$\partial D(\nu_1) = -\partial D(\nu_2) = SU(3)/H.$$

Thus

$$\mathrm{Sign}(M) = \mathrm{Sign}(D(\nu_1)) + \mathrm{Sign}(D(\nu_2)).$$

Since

$$H^4(SU(3)/K; \boldsymbol{Q}) = 0 \qquad \text{for } \dim K \neq 2, 4,$$

by Lemma 5.4, $\mathrm{Sign}(D(\nu_i)) = 0$ for dim $K_i \neq 2$, 4. Let K be a 2-dimensional closed subgroup of $SU(3)$. Then K is conjugate to one of the following

$$T, \ T_{(2)}, \ T_{(3)} \ \text{ and } \ N(T) = T_{(6)}.$$

Here $T^0_{(i)} = T$ and $T_{(i)}$ has i-components. By Lemma 5.4 (d),

$$H^4(SU(3)/T; \boldsymbol{Q}) = \boldsymbol{Q} \oplus \boldsymbol{Q},$$
$$H^4(SU(3)/T_{(2)}; \boldsymbol{Q}) = \boldsymbol{Q},$$
$$H^4(SU(3)/T_{(3)}; \boldsymbol{Q}) = H^4(SU(3)/N(T); \boldsymbol{Q}) = 0.$$

Thus $\mathrm{Sign}(D(\nu_i))=0$, if $K_i=T_{(3)}$ or $N(T)$.   If $K_i=T$, then $\mathrm{Sign}(D(\nu_i))=0$ from Lemma 5.4 (d) and Remark 5.3.   q.e.d.

REMARK 5.9.   If $\dim H=2$ in Theorem 5.1, then $H=T$ or $T_{(3)}$, since $SU(3)/T_{(2)}$ and $SU(3)/N(T)$ are non-orientable by Lemma 5.4(d).   Theorem 5.1 is proved for $H=T_{(3)}$ by Proposition 5.6, since $T_{(3)}$ is not conjugate to any subgroup of $NSU(2)$ in $SU(3)$ and $T_{(3)}$ is not a principal isotropy group of any 8-dimensional real representation of $SU(3)$ by Lemma 5.5. Therefore, it remains to prove Theorem 5.1 for the cases $H=T$ and $H^0=SU(2)$.

Proposition 5.10.   *Suppose* $H=T$.   *Then* $\mathrm{Sign}(M)=0$.

Proof.   If $F(NSU(2), M)$ is empty, then $\mathrm{Sign}(M)=0$ by Proposition 5.6. Now we assume that $F(NSU(2),M)$ is not empty.   Then

$$\dim F(NSU(2), M) = 1$$

by Lemma 5.5, and any stationary point (if exists) of $SU(3)$ is isolated by Lemma 5.5 (a).   Let

$$F(SU(3), M) =\{x_1, \cdots, x_k\}, \qquad (k\geqslant 0)$$

and let D_i be an invariant closed disk around x_i, such that

$$D_i \cap D_j = 0 \qquad \text{for} \quad i\neq j.$$

Let $D=D_1 \cup \quad \cup D_k$ and $E=M-\mathrm{int}\,D$.   Then

$$D_i \cap F(NSU(2), E)\neq\emptyset, \qquad (i = 1, \cdots, k)$$

by Lemma 5.5 (a).   Let

$$E_0 = \{x\in E \,|\,(SU(3)_x) = (NSU(2))\},$$

let U_0 be an invariant closed tubular neighborhood of E_0 in E, and let $U=U_0\cup D$. Then $M-\mathrm{int}\,U$ is connected and

$$(SU(3)^0_x)= (T), \qquad \text{for} \quad x\in M-\mathrm{int}\,U.$$

Therefore, there is an equivariant diffeomorphism

$$M-\mathrm{int}\,U = SU(3)/T \underset{N(T)/T}{\times} F, \quad F = F(T, M-\mathrm{int}\,U)$$

by Lemma 4.2, and there is a commutative diagram:

$$
\begin{array}{ccc}
H^4(M-\mathrm{int}\,U;\,\boldsymbol{Q}) & \xrightarrow{\;\;i^*\;\;} & H^4(\partial(M-\mathrm{int}\,U);\,\boldsymbol{Q}) \\[2pt]
\cong\Big\downarrow p^* & & \cong\Big\downarrow p^* \\[2pt]
H^4(SU(3)/T\times F;\,\boldsymbol{Q})^{N(T)/T} & \xrightarrow[i_0^*]{} & H^4(SU(3)/T\times \partial F;\,\boldsymbol{Q})^{N(T)/T}
\end{array}
$$

Here i_0^* is injective, since $H^{odd}(SU(3)/T\boldsymbol{Q})=0$ by Lemma 5.4 (d), dim $F=2$, and each connected component of F has non-empty boundary from the connectedness of $M-\text{int } U$.   Thus

$$\hat{H}^4(M-\text{int } U; \boldsymbol{Q}) = 0 ,$$

and hence $\text{Sign}(M-\text{int } U)=0$.   Next, let $U_1, \cdots, U_n$ be connected components of U.   Then we can prove that

$$\hat{H}^4(U_i; \boldsymbol{Q}) = 0 , \qquad \text{if } U \cap D = \emptyset ,$$
$$H^4(U_i; \boldsymbol{Q}) = 0 , \qquad \text{if } U_i \cap D \neq \emptyset ,$$

and hence

$$\text{Sign}(U) = \text{Sign}(U_1) + \cdots + \text{Sign}(U_n) = 0 .$$

Therefore

$$\text{Sign}(M) = \text{Sign}(M - \text{int } U) + \text{Sign}(U) = 0 . \qquad\qquad \text{q.e.d.}$$

We recall the following result which is essentially proved in the proof of Proposition 5.6 (b).

Lemma 5.11.   *Let X be a compact connected orientable smooth n-manifold (∂X is empty or not).   Let $n=7$ or 8.   Assume that X admits a smooth $SU(3)$-action with*

$$(SU(3)_x^0) = (SU(2)) \qquad \text{for} \quad x \in X.$$

Then

$$H^{n-4}(X; Q) = 0 .$$

Proposition 5.12.   *Assume that $H^0=SU(2)$ and M has not an isotropy type $(NSU(2))$.   Then $H^4(M; \boldsymbol{Q})=0$.*

Proof.   If $F(SU(3),M)=\emptyset$, then $H^4(M; \boldsymbol{Q})=0$ by Lemma 5.11.   Next if $F(SU(3),M)\neq\emptyset$, then dim $F(SU(3),M)=2$ by Lemma 5.5 (a).   Let U be an invariant closed tubular neighborhood of $F(SU(3),M)$ in M.   Then there is an exact sequence:

$$H^3(\partial U; Q) \to H^4(M; Q) \to H^4(U; \boldsymbol{Q}) \oplus H^4(M - \text{int} U; \boldsymbol{Q}).$$

Here

$$H^3(\partial U; Q) = H^4(M - \text{int } U; \boldsymbol{Q}) = 0$$

by Lemma 5.11, and

$$H^4(U; Q) = H^4(F(SU(3),M); Q) = 0 .$$

Therefore

$$H^4(M; Q) = 0 . \qquad\qquad\qquad \text{q.e.d.}$$

This completes the proof of Theorem 5.1.

6. SZ7(3)-actions on cohomology $P_4(C)$

In the previous paper [13] we have considered smooth $SU(3)$-actions on homotopy $P_3(C)$.   In this section, first we prove the following result as an application of Theorem 5.1.

Theorem 6.1.   *Let M be a compact connected orientable* 8-*manifold such that*

$$H^*(M; \pmb{Q}) = H^*(P_4(C); \pmb{Q}) .$$

*Then for any **non-trivial** smooth SU(3)-action on M, the stationary point set is a* 2-*sphere and the principal isotropy type is* $(SU(2))$. *Furthermore there is an equivariant   diffeomorphism*

$$M = \partial(D^6 \times X)/S^1 .$$

Here X is a compact connected orientable 4-*manifold which is acyclic over **rationals**, X admits a smooth S^1-**action** which is free on ∂X, the SU(3)-action is standard on D^6 and trivial on X.*

Proof.   Denote by (H), the principal isotropy type of the given $SU(3)$-action on M.   Since $\mathrm{Sign}(M) \neq 0$, the following are the only possible cases from Theorem 5.1,

  (a)   dim $H=l$ and M has an isotropy type $(NSU(2))$ or $(T_{(2)})$,
  (b) $H^0=SU(2)$ and M has an isotropy type $(NSU(2))$,
  (c) $H=NSU(2)$ and $M=P_2(C) \times F(NSU(2),M)$.
  If $H=NSU(2)$, then $\chi(M)=5$ is divisible by $\chi(P_2(C))=3$, and this is a contradiction.   Next if dim $H=1$, then there is a decomposition

$$M = D(\nu_1) \cup D(\nu_2)$$

as in the proof of Proposition 5.8, where $D(\nu_i)$ is a normal disk bundle over $SU(3)/K_i$.   One may assume $K_1= NSU(2)$ or $T_{(2)}$, and hence

$$\chi(SU(3)/K_1) = 3$$

by Lemma 5.4.   On the other hand,

$$5 = \chi(M) = \chi(SU(3)/K_1) + \chi(SU(3)/K_2)$$

Thus $\chi(SU(3)/K_2)=2$,   and   hence $K_2 = T_{(3)}$ by   Lemma 5.4.   Since $H^2(SU(3)/T_{(3)}; \pmb{Q})=0$, there is a contradiction in the following exact sequence of rational cohomology groups:

$$H^1(\partial D(\nu_1)) \to H^2(M) \ \ \to \ \ H^2(SU(3)/K_1) \oplus H^2(SU(3)/K_2)$$
$$\to H^2(\partial D(\nu_1)) \to H^3(M) .$$

Therefore we obtain $H^0 = SU(2)$. If $F(SU(3), M) = \emptyset$, then there is a fibre bundle

$$F(SU(2), M) \to M \to P_2(\mathbf{C}) \ .$$

Thus $\chi(M) = 5$ is divisible by $\chi(P_2(\mathbf{C})) = 3$, and this is a contradiction. Hence $F(SU(3), M) \neq \emptyset$ and this implies $H = SU(2)$ by Lemma 5.5 (a). Let U be an invariant tubular neighborhood of $F(SU(3), M)$ in M. Then

$$X = F(SU(2), M - \mathrm{int}\ U)$$

is a compact connected orientable 4-manifold with the natural action of $NSU(2)/SU(2) = S^1$ which is free on ∂X. Furthermore there is an equivariant diffeomorphism

$$M = \partial(D^6 \times X)/S^1 \ ,$$

and X is acyclic over rationals by the same argument as in the proof of Theorem 2.1. Finally,

$$F(SU(3),\ \mathrm{M}) = \partial X/S^1 = S^2 \ . \qquad\qquad \text{q.e.d.}$$

Next, as a complementary part of Theorem 5.1, we give examples of certain $SU(3)$-actions on 8-manifolds with non-zero signature.

Let $\psi : NSU(2) \to U(3)$ be a unitary representation of $NSU(2)$. Then ψ induces a smooth $NSU(2)$-action ψ_* on $P_2(\mathbf{C})$. Denote by $M(\psi)$, the orbit manifold of the free smooth action of $NSU(2)$ on $SU(3) \times P_2(\mathbf{C})$ given by

$$h \cdot (g,\ x) = (gh^{-1},\ \psi_*(h,\ x)),\ g \in SU(3),\ h \in NSU(2),\ x \in P_2(\mathbf{C}) \ .$$

Then the compact connected orientable 8-manifold $M(\psi)$ admits a natural smooth $SU(3)$-action without stationary points and

$$\mathrm{Sign}(M(\psi)) = 1 \ .$$

EXAMPLE 6.2.   Let $\alpha_k : NSU(2) \to U(3)$ be a unitary representation given by

$$\alpha_k \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^k \end{pmatrix} .$$

Then $M(\alpha_k)$ has just two isotropy types

$$(SU(2)_{(k)}) \quad \text{and} \quad (NSU(2)) \ ,$$

where $SU(2)_{(k)}$ has k-components and its identity component is $SU(2)$. (see Theorem 5.1 (d))

EXAMPLE 6.3.   Let $\beta_k : NSU(2) \to U(3)$ be a unitary representation given by

$$\beta_k \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & y^k \end{pmatrix}.$$

Then $M(\beta_k)$ has just three isotropy types

$$(D(k, -k-1)), (T) \quad \text{and} \quad (NSU(2)),$$

where $D(k, -k-1)$ is a closed one-dimensional subgroup defined in Lemma 5.4. (see Theorem 5.1 (b))

EXAMPLE 6.4.   Let $\gamma \colon NSU(2) \to U(3)$ be a unitary representation given by

$$\gamma \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} a^2 & \sqrt{2}\,ab & b^2 \\ \sqrt{2}\,ac & ad+bc & \sqrt{2}\,bd \\ c^2 & \sqrt{2}\,cd & d^2 \end{pmatrix}.$$

Then $M(\gamma)$ has just three isotropy types

$$(D(1, 1)_{(2)}), (T) \quad \text{and} \quad (T_{(2)}),$$

where $G_{(2)}$ is a subgroup of $SU(3)$ such that $G_{(2)}$ has 2-components and its identity component is G. (see Theorem 5.1 (b))

7. Classification of smooth $SU(n)$-actions on orientable $2n$-manifolds

Let M be a compact connected $2n$-manifold with non-trivial smooth $SU(n)$-action, then the identity component of each isotropy group is conjugate to one of the following

$$SU(n), \ SU(n-l) \quad \text{and} \quad NSU(n-1),$$

for $n \geqslant 5$. This is proved similarly as Lemma 1.5. Therefore there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1$$

as $SU(n)$-manifolds by (1.1) and (1.4). Here X is a compact connected 2-dimensional S^1-manifold and the S^1-action on dX is free if dX is non-empty. Furthermore if M is orientable, then X is also orientable.   Next we remark that for orientable 2-dimensional S^1-manifold X, if the isotropy group $S^1_x \neq S^1$ for $x \in X$, then S^1_x is a principal isotropy group by the differentiable slice theorem, and hence the S^1-space $X - F(S^1, X)$ has just one isotropy type.
   (i)   If X has just one isotropy type (S^1), then $\partial X = \emptyset$ and

$$M = P_{n-1}(C) \times X.$$

(ii)   If X has just one isotropy type $(\mathbf{Z}_k)$, then

$$M = S^{2n} \qquad\qquad \text{if } \partial X \neq \emptyset,$$
$$M = L^{2n-1}(k) \times S^1 \qquad \text{if } \partial X = \emptyset.$$

Here $L^{2n-1}(k) = S^{2n-1}/\mathbf{Z}_k$ is a standard lens space.

(iii)   If X has just two isotropy types $(\mathbf{Z}_k)$ and (S^1), then

$$M = P_n(\mathbf{C}) \qquad\qquad \text{if } \partial X \neq \emptyset,$$
$$M = S^{2n-1} \underset{S^1}{\times} S^2_{(k)} \qquad \text{if } \partial X = \emptyset.$$

Here $S^2_{(k)}$ is a 2-sphere with the S^1-action given by

$$e^{i\theta}(x_0, x_1, x_2) = (x_0, x_1 \cos k\theta + x_2 \sin k\theta, -x_1 \sin k\theta + x_2 \cos k\theta).$$

This completes the classification.

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