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SMOOTH ACTIONS OF SPECIAL UNITARY GROUPS ON COHOMOLOGY COMPLEX PROJECTIVE SPACES

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0. Introduction

The purpose of this paper is to study smooth $SU(n)$-actions on a compact orientable $2m$-manifold whose rational cohomology ring is isomorphic to $H^*(P_m(C);Q)$. First we show the following result.

Theorem 2.1. *Let $n \geqslant 7$ and $0 \leqslant k < n-4$. Let M be a compact orientable smooth $2(n+k)$-manifold with*

$$H^*(M; Q) = H^*(P_{n+k}(C); Q).$$

Then for any non-trivial smooth $SU(n)$-action on M, the stationary point set $F=F(SU(n), M)$ is an orientable $2k$-manifold with

$$H^*(F; Q) = H^*(P_k(C); Q)$$

and there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1.$$

Here X is a compact connected orientable $(2k+2)$-manifold which is acyclic over rationals, X admits a smooth S^1-action which is free on ∂X, the $SU(n)$-action is standard on D^{2n} and trivial on X, and

$$\pi_1(X) = \pi_1(M).$$

Furthermore, if

$$H^*(M; Z) = H^*(P_{n+k}(C); Z),$$

then X is acyclic over integers, the S^1-action on X is semi-free, and

$$H^*(F; Z) = H^*(P_k(C); Z).$$

Corollary 2.2. *Let $n \geqslant 7$ and $0 \leqslant k < n-4$. Let M be a compact connected smooth $2(n+k)$-manifold which is homotopy equivalent to $P_{n+k}(C)$. If M admits a non-trivial smooth $SU(n)$-action, then M is diffeomorphic to $P_{n+k}(C)$.*

Examples of $SU(n)$-actions on cohomology complex projective spaces are constructed in section 3. And we have the following results.

Theorem 3.1. *Let $n \geqslant 2$, $k \geqslant 1$ and $p \geqslant 1$. Then there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = \boldsymbol{Z}/p\boldsymbol{Z} \text{ and } H^*(M; \boldsymbol{Q}) = H^*(P_{n+k}(\boldsymbol{C}); \boldsymbol{Q})$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(\boldsymbol{C}).$$

Theorem 3.2. *Let $n \geqslant 2$ and $k \geqslant 3$. Let G be a finitely presentable group with $H_1(G; \boldsymbol{Z}) = H_2(G; \boldsymbol{Z}) = 0$. Then*

(a) *there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = G \text{ and } H^*(M; Z) = H^*(P_{n+k}(\boldsymbol{C}), Z)$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(\boldsymbol{C}),$$

(b) *there is a smooth $SU(n)$-action on $P_{n+k}(\boldsymbol{C})$ such that*

$$\pi_1(F) = G \text{ and } H^*(F; Z) = H^*(P_k(\boldsymbol{C}); \boldsymbol{Z}),$$

where $F = F(SU(n), P_{n+k}(\boldsymbol{C}))$.

Next, in section 4, we study a signature of closed orientable manifold which admits a smooth G-action with isotropy groups of uniform dimension, and we have a result which is a generalization of the fact that $\mathrm{Sign}(M) = 0$ if M admits a smooth circle action without stationary points.

Next we study smooth $SU(3)$-actions on orientable 8-manifolds in section 5, and as an application we show a similar result as Theorem 2.1 for non-trivial smooth $SU(3)$-action on a cohomology complex projective 4-space. We construct examples of stationary point free $SU(3)$-actions on orientable 8-manifolds with non-zero signature in section 6.

As a concluding remark, classification of smooth $SU(n)$-actions on orientable $2n$-manifolds is done in the final section.

1. $SU(n)$-actions with certain isotropy types

Let E be a manifold with smooth $SU(n)$-action ($n \geqslant 3$). Assume that the identity component of each isotropy group is conjugate to $SU(n-1)$ or $NSU(n-1)$, the normalizer of $SU(n-1)$ in $SU(n)$. Then $S^1 = NSU(n-1)/SU(n-1)$ acts naturally on

$$X = F(SU(n-1), E),$$

the stationary point set of $SU(n-1)$. It is easily seen that

$$(1.1) \qquad SU(n)/SU(n-1) \times_{S^1} X \to E, \quad [gSU(n-1), x] \to gx$$

is an equivariant diffeomorphism as $SU(n)$-manifolds, since $g \in SU(n)$ and $g^{-1}SU(n-1)g \subset NSU(n-1)$ imply $g \in NSU(n-1)$.

Lemma 1.2. *Let V be a real vector space with linear $SU(n)$-action ($n \geqslant 3$). Assume that the identity component of each isotropy group on the invariant unit sphere $S(V)$ is conjugate to $SU(n-1)$ or $NSU(n-1)$. Then $S(V) = SU(n)/SU(n-1)$ as $SU(n)$-spaces.*

Proof. By (1.1), there is an equivariant diffeomorphism

$$S(V) = SU(n)/SU(n-1) \times_{S^1} F(SU(n-1) S(V)),$$

where $F(SU(n-1), S(V))$ is a sphere. Then it is easily seen that

$$F(SU(n-1), S(V)) = S^1$$

by the homotopy exact sequence of the fibre bundle

$$F(SU(n-1), S(V)) \to S(V) \to P_{n-1}(C).$$

Considering S^1-actions on S^1, we have

$$S(V) = SU(n)/SU(n-1)$$

as $SU(n)$-spaces. q.e.d.

Lemma 1.3. *Let V be a real vector space with linear $SU(n)$-action such that $S(V) = SU(n)/SU(n-1)$ as $SU(n)$-spaces ($n \geqslant 3$). Then the $SU(n)$-action on $V = \boldsymbol{R}^{2n}$ is equivalent to the standard action.*

Proof. This is a known result (see [8], Theorem I), but we give an elementary proof for the completeness. It is well-known that a real irreducible $SU(n)$-vector space $\boldsymbol{R}^{2n}$ with an invariant complex structure is equivalent to $\boldsymbol{R}^{2n}$ with the standard $SU(n)$-action. So we prove the existence of an invariant complex structure on V. Denote by $\boldsymbol{Z}_n$, the center of $SU(n)$. Then $\boldsymbol{Z}_n$ is a cyclic group of order n, and the $\boldsymbol{Z}_n$-action on $S(V)$ is free, since

$$\boldsymbol{Z}_n \cap SU(n-1) = \{1\}.$$

Consider a direct sum decomposition

$$V = V_1 \oplus \cdots \oplus V_k$$

as Z_n-vector space, where $V_i(i=1, \cdots, k)$ are irreducible. Leaving a non-zero vector $v_1 \in V_1$ fixed, we have an element $g_i \in SU(n)$ such that

$$v_i = g_i v_1 \in V_i \qquad (i=1, \cdots, k)$$

by the transitivity of the $SU(n)$-action on $S(V)$. Then

$$V_i = g_i V_1 \qquad (i=1, \cdots, k).$$

Since the Z_n-action on $S(V_1)$ is free, there is a complex structure J_1 on V_1 such that

$$\sigma J_1 = J_1 \sigma, \quad \sigma v_1 = a v_1 + b J_1 v_1$$

for some $a, b \in R$, $b \neq 0$, where σ is a generator of Z_n, moreover the real vector space V_1 is spanned by $\{v_1, J_1 v_1\}$. Therefore there is a complex structure J on V such that

$$J v_1 - J_1 v_1, \quad J g_i v_1 - g_i J_1 v_1 \quad \text{and} \quad \sigma v = a v + b J v$$

for each $v \in V$. Then

$$g \sigma v = a g v + b g J v,$$
$$\sigma g v = a g v + b J g v$$

for any $g \in SU(n)$. Therefore the complex structure J is $SU(n)$-invariant, since $g\sigma = \sigma g$ and $b \neq 0$. q.e.d.

Let M be a closed connected manifold with smooth $SU(n)$-action ($n \geqq 3$). Assume that the identity component of each isotropy group is conjugate to one of the following

$$SU(n), \; SU(n-1) \text{ and } NSU(n-1).$$

Assume that the stationary point set $F = F(SU(n), M)$ is non-empty. Let U be an invariant closed tubular neighborhood of F in M. Then there is an equivariant decomposition

$$M = U \cup (SU(n)/SU(n-1) \times_{S^1} X) = U \cup (S^{2n-1} \times_{S^1} X),$$

where $X = F(SU(n-1), M - \text{int } U)$ with the natural S^1-action. Since

$$dU = SU(n)/SU(n-1) \times_{S^1} \partial X = S^{2n-1} \times_{S^1} \partial X$$

as $SU(n)$-maifolds, the S^1-action on ∂X is free, $F = \partial X / S^1$, and the disk bundle $U \to F$ with $SU(n)$-action is equivariantly isomorphic to the disk bundle

$$D^{2n} \times_{S^1} \partial X \to \partial X / S^1,$$

where the $SU(n)$-action on D^{2n} is standard by Lemma 1.2 and Lemma 1.3.

Therefore the codimension of F in M is $2n$, X is connected, and there is an equivariant diffeomorphism

$$M = \partial(D^{2n}\times X)/S^1 = D^{2n}\underset{S^1}{\times}\partial X \cup S^{2n-1}\underset{S^1}{\times} X \tag{1.4}$$

as $SU(n)$-manifolds.

Lemma 1.5. *Let G be a closed connected proper subgroup of $SU(n)$, $(n\geqslant 7)$. If*

$$\dim G > n^2-4n+7 = \dim N(SU(n-2), SU(n)),$$

then G is conjugate to $SU(n-1)$ or $NSU(n-1)$ in $SU(n)$.

Proof. The inclusion $\rho\colon G\subset SU(n)$ gives an n-dimensional complex representation of G. First we show that the representation p is reducible. Suppose that p is irreducible. Then G is semi-simple from the Shur's lemma. If G is not simple, then there are integers $p\geqslant q\geqslant 2$ with $n=pq$, such that G is conjugate to a subgroup of the tensor product

$$SU(p)\otimes SU(q)$$

in $SU(pq)$, by considering the induced representation of the universal covering group of G. Therefore

$$\dim G\leqslant p^2+q^2-2\leqslant\left(\frac{n}{2}\right)^2+2\leqslant\frac{n(n+1)}{2}.$$

If G is simple but not one of the type

$$A_k,\ D_{2k+1}\ \text{ and }\ E_6,$$

then G is conjugate to a subgroup of $SO(n)$ or $Sp(n/2)$, (see [6], p. 336, Theorem 0.20). But

$$\dim SO(n) = \frac{n(n-1)}{2},\quad \dim Sp\left(\frac{n}{2}\right) = \frac{n(n+1)}{2}$$

and hence

$$\dim G\leqslant\frac{n(n+1)}{2}.$$

If G is of type D_{2k+1} $(k\geqslant 2)$, then the lowest dimensional non-trivial irreducible complex representation is $(4k+2)$-dimensional (see [6], p. 378, Table 30). Therefore $4k+2\leqslant n$ and hence

$$\dim \mathrm{G} = \dim SO(4k+2) = (2k+1)(4k+1)\leqslant\frac{n(n-1)}{2}.$$

If G is of type E_6, then $n\geqslant 27$ (see [6], p. 378, Table 30). Therefore

$$\dim G = 78 \leqslant 3n \leqslant \frac{n(n+1)}{2}.$$

Finally, if G is of type A_{k-1} $(k<n)$, then

$$\frac{k(k-1)}{2} \leqslant n,$$

by the Weyl's formula (see [14], Theorem 7.5). Therefore

$$\dim G = \dim SU(k) = k^2 - 1 \leqslant 3n-2 \leqslant \frac{n(n+1)}{2}.$$

Consequently

$$\dim G \leqslant \frac{n(n+1)}{2},$$

if p: $G \subset SU(n)$ is irreducible $(n \geqslant 4)$. Therefore p is reducible, if

$$\dim G > n^2 - 4n + 7 \quad \text{and} \quad n \geqslant 7.$$

Since p is reducible, G is conjugate to a subgroup of

$$N(SU(n-p), \quad SU(n)), \left(1 \leqslant p \leqslant \frac{n}{2}\right)$$

the normalizer of $SU(n-p)$ in $SU(n)$. But

$$\dim \ N(SU(n-p), SU(n)) \leqslant n^2 - 4n + 7$$

for $2 \leqslant p \leqslant \frac{n}{2}$. Therefore G is conjugate to a subgroup G' of $NSU(n-1)$. If $G' \neq NSU(n-1)$, then

$$\dim G' \leqslant \dim G'' + 1$$

where $G'' = G' \cap SU(n-1)$, by the isomorphism

$$NSU(n-1)/SU(n-1) \cong S^1.$$

If $G'' = SU(n-1)$ then $G' = G'' = SU(n-1)$. If $G'' \neq SU(n-1)$, then

$$\dim G'' \leqslant (n-2)^2 = \dim N(SU(n-2), SU(n-1)),$$

by making use of the first part of the proof of this lemma for $SU(n-1)$ instead of $SU(n)$, and hence

$$\dim G' \leqslant (n-2)^2 + 1 < n^2 - 4n + 7.$$

Consequently we see that G is conjugate to $SU(n-1)$ or $NSU(n-1)$ in $SU(n)$.

q.e.d.

Lemma 1.6. *Let M be a manifold with smooth $SU(n)$-action. If* $\dim M < 4n-8$, *then*

$$\dim SU(n)_x > n^2-4n+7$$

for each $x \in M$.

Proof. Since $SU(n)/SU(n)_x$ is equivariantly embedded in M,

$$\dim SU(n) - \dim SU(n)_x \leqslant \dim M < 4n-8 .$$

Hence $\dim SU(n)_x > \dim SU(n) - (4n-8) = n^2-4n+7$. q.e.d.

2. $SU(n)$-actions on cohomology complex projective spaces

In this section we prove the following results.

Theorem 2.1. *Let* $n \geqslant 7$ *and* $0 \leqslant k < n-4$. *Let M be a compact connected orientable smooth* $2(n+k)$-*manifold with*

$$H^*(M; \boldsymbol{Q}) = H^*(P_{n+k}(\boldsymbol{C}); \boldsymbol{Q}) .$$

Then for any non-trivial smooth $SU(n)$-*action on* M, *the stationary point set* $F = F(SU(n), M)$ *is an orientable* $2k$-*manifold with*

$$H^*(F; \boldsymbol{Q}) = H^*(P_k(\boldsymbol{C}); \boldsymbol{Q})$$

and there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1 .$$

Here X is a compact connected orientable $(2k+2)$-*manifold which is acyclic over rationals, X admits a smooth* S^1-*action which is free on* ∂X, *the* $SU(n)$-*action is standard on* D^{2n} *and trivial on* X, *and*

$$\pi_1(X) = \pi_1(M) .$$

Furthermore, if

$$H^*(M; \boldsymbol{Z}) = H^*(P_{n+k}(\boldsymbol{C}); \boldsymbol{Z}) ,$$

then X is acyclic over integers, the S^1-*action on X is semi-free, and*

$$H^*(F; \boldsymbol{Z}) = H^*(P_k(\boldsymbol{C}); \boldsymbol{Z}) .$$

Corollary 2.2, *Let* $n \geqslant 7$ *and* $0 \leqslant k < n-4$. *Let M be a compact connected smooth* $2(n+k)$-*manifold which is homotopy equivalent to* $P_{n+k}(\boldsymbol{C})$. *If M admits a non-trivial smooth* $SU(n)$-*action, then M is diffeomorphic to* $P_{n+k}(\boldsymbol{C})$.

Proof of Theorem 2.1. By Lemma 1.5, Lemma 1.6 and the assumption $n \geqslant 7$ and $0 \leqslant k < n-4$, the identity component of each isotropy group of the given $SU(n)$-action on M is conjugate to one of the following

$$SU(n),\ SU(n\text{-}l) \quad \text{and} \quad NSU(n-1).$$

(i) First we show that the stationary point set $F=F(SU(n),M)$ is non-empty. Assume $F=\emptyset$, then by (1.1) there is a smooth fibre bundle

$$F(SU(n-1),M) \to M \to P_{n-1}(\boldsymbol{C}).$$

Thus

$$\chi(M) = \chi(P_{n-1}(\boldsymbol{C}))\cdot\chi(F(SU(n-1)M))$$

and hence

$$k+1\equiv 0 \,(\mathrm{mod}\ n)\,.$$

This is impossible by the assumption $0\leqslant k<n-4$. Thus $F\neq\emptyset$. Then by (1.4) there is an equivariant diffeomorphism

$$M=\partial(D^{2n}\times_{S^1}X)/S^1=D^{2n}\times_{S^1}\partial X\cup S^{2n-1}\times_{S^1}X$$

as $SU(n)$-manifolds. Here X is a compact connected orientable $(2k+2)$-manifold with smooth S^1-action which is free on ∂X.

(ii) Next we show that X is acyclic over rationals. Since

$$D^{2n}\times_{S^1}\partial X\to\partial X/S^1=F$$

is a $2n$-disk bundle, there is an isomorphism

$$H^i(M,\ S^{2n-1}\times_{S^1}X;\boldsymbol{Q})=H^{i-2n}(F;\boldsymbol{Q})\,.$$

Thus

(2.3) $$H^i(M;Q)=H^i(S^{2n-1}\times_{S^1}X;Q) \qquad \text{for } i\leqslant 2n-2\,.$$

Now we show that the euler class $e(p)$ of the principal S^1-bundle

$$p:\ \partial(D^{2n}\times X)\to M$$

is non-zero in $H^2(M;Q)$. Assume $e(p)=0$, then the euler class of the principal S^1-bundle

$$S^{2n-1}\times X\to S^{2n-1}\times_{S^1}X$$

is zero in $H^2(S^{2n-1}\times_{S^1}X;\mathrm{Q})$, and hence there is an isomorphism

$$H^*(S^{2n-1};\boldsymbol{Q})\otimes H^*(X;Q)=H^*(S^1;\boldsymbol{Q})\otimes H^*(S^{2n-1}\times_{S^1}X;\boldsymbol{Q})\,.$$

Therefore

$$H^i(X;\boldsymbol{Q})=\boldsymbol{Q} \qquad \text{for } 0\leqslant i\leqslant 2n-2$$

by (2.3) and the assumption

$$H^*(M;\boldsymbol{Q}) = H^*(P_{n+k}(\boldsymbol{C});\boldsymbol{Q})\,.$$

But

$$\dim X = 2k+2 \leqslant 2n-2\,.$$

Thus $H^{2k+2}(X;\boldsymbol{Q})=\boldsymbol{Q}$ and this is a contradiction, since the connected manifold X has a non-empty boundary. Therefore $e(p)\neq 0$ and hence

$$(2.4) \qquad H^*(\partial(D^{2n}\times X)\boldsymbol{Q}) = H^*(S^{2n+2k+1};\boldsymbol{Q})\,.$$

There is an isomorphism

$$H^i(D^{2n}\times X;\boldsymbol{Q}) = H_{2n+2k+2-i}(D^{2n}\times X,\ \partial(D^{2n}\times X);Q)$$

by the Poincaré-Lefschetz duality, and the homomorphism

$$H_{2n+2k+2-i}(D^{2n}\times X;\ Q) \to H_{2n+2k+2-i}(D^{2n}\times X,\partial(D^{2n}\times X);Q)$$

is onto for $0<i<2n+2k+2$ by (2.4). Since X is a connected $(2k+2)$-manifold with a non-empty boundary,

$$H_{2n+2k+2-i}(D^{2n}\times X;\boldsymbol{Q}) = 0 \qquad \text{for } i\leqslant 2n,$$

and hence

$$H^i(X;\boldsymbol{Q}) = 0 \qquad \text{for } 0<i\leqslant 2n\,.$$

Therefore X is acyclic over rationals. Then

$$H^*(\partial X;\boldsymbol{Q}) = H^*(S^{2k+1};\boldsymbol{Q})\,,$$

by the Poincaré-Lefschetz duality, and hence

$$H^*(F;Q) = H^*(P_k(\boldsymbol{C});Q)\,.$$

Furthermore $F(S^1, X)$ consists just one point by the P.A. Smith theory (see [2], chapter IV) from the fact that X is acyclic over rationals and the S^1-action is free on ∂X.

(ii) Next we show $\pi_1(X)=\pi_1(M)$. Since $F(S^1, X) = \{x_0\}$, there is an S^1-map

$$s\colon S^{2n-1} \to \partial(D^{2n}\times X)$$

given by $s(y)=(y,x_0)$. Then we have an isomorphism

$$\pi_1(M) = \pi_1(\partial(D^{2n}\times X))$$

from the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(S^1) & \longrightarrow & \pi_1(S^{2n-1}) & & \\ \downarrow id & & \downarrow s_* & & \\ \pi_1(S^1) & \longrightarrow & \pi_1(\partial(D^{2n}\times X)) & \xrightarrow{p_*} & \pi_1(M)\,. \end{array}$$

Applying the van Kampen theorem (see [5], p. 63) to the decomposition

$$\partial(D^{2n}\times X) = D^{2n}\times\partial X \cup S^{2n-1}\times X\ ,$$

we have

$$\pi_1(X) = \pi_1(\partial(D^{2n}\times X))\ ,$$

and hence

$$\pi_1(X) = \pi_1(M)\,.$$

(iv) Finally we show that the assumption

$$H^*(M; Z) = H^*(P_{n+k}(C)\ \ Z)$$

implies $H^*(X, x_0; \boldsymbol{Z})=0$. There is a commutative diagram:

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{s} & \partial(D^{2n}\times X) \\ \downarrow p_0 & & \downarrow p \\ P_{n-1}(C) & \xrightarrow{t} & M\,. \end{array}$$

Since $t^*e(p)=e(p_0)$ is a generator of $H^*(P_{n-1}(C);Z)$, $e(p)$ is a generator of $H^*(M; Z)$. Therefore

$$H^*(\partial(D^{2n}\times X);Z) = H^*(S^{2n+2k+1}; Z)$$

by the Gysin seqence for the principal S^1-bundle

$$p\colon \partial(D^{2n}\times X)\to M\,,$$

and hence X is acyclic over integers and

$$H^*(F; \boldsymbol{Z}) = H^*(P_k(C); \boldsymbol{Z})$$

by the same argument as in (ii). Then the S^1-action on X is semi-free by the P.A. Smith theory from the fact that X is acyclic over integers and the S^1-action is free on ∂X. This completes the proof of Theorem 2.1.

Proof of Corollary 2.2. If M admits a non-trivial smooth $SU(n)$-action, then by Theorem 2.1, there is an equivariant diffeomorphism

$$M = \partial(D^{2n}\times X)/S^1$$

as $SU(n)$-manifolds. Here X is a compact contractible $(2k+2)$-manifold with smooth semi-free S^1-action with just one stationary point x_0. Therefore the

S^1-action on $D^{2n}\times X$ is semi-free and its stationary point is only $(0, x_0)$. Let U be an invariant closed disk around the point $(0, x_0)$. One may assume that the S^1-action on U is linear. Put

$$W=(D^{2n}\times X-\text{int } U)/S^1.$$

Then

$$\partial W = \partial U/S^1 \cup \partial(D^{2n}\times X)/S^1 = P_{n+k}(\boldsymbol{C})\cup M.$$

Since

$$\pi_1(M)=\pi_1(W)=0,$$
$$H_*(W, M \ \boldsymbol{Z})=0$$

and

$$\dim W=2n+2k+1\geqslant 6,$$

we have

$$M=P_{n+k}(\boldsymbol{C})$$

by applying the h-cobordism theorem (see [10], Theorem 9.1) to the triad $(W; \text{M}, P_{n+k}(\boldsymbol{C}))$. This completes the proof of Corollary 2.2.

3. Construction of $SU(n)$-actions

In this section we construct $SU(n)$-actions on cohomology complex projective spaces, and we have the following results.

Theorem 3.1. *Let $n\geqslant 2$, $k\geqslant 1$ and $p\geqslant 1$. Then there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M)=\boldsymbol{Z}/p\boldsymbol{Z} \quad \text{and} \quad H^*(M;\boldsymbol{Q})=H^*(P_{n+k}(\boldsymbol{C});\ \boldsymbol{Q})$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n), M) = P_k(\boldsymbol{C}).$$

Theorem 3.2. *Let $n\geqslant 2$ and $k\geqslant 3$. Let G be a finitely presentable group with $H_1(G;\ \boldsymbol{Z})=H_2(G;\boldsymbol{Z})=0$. Then*

(a) *there is a compact orientable $2(n+k)$-manifold M such that*

$$\pi_1(M) = G \quad \text{and} \quad H^*(M;\ Z) = H^*(P_{n+k}(\boldsymbol{C});\boldsymbol{Z})$$

and M admits a smooth $SU(n)$-action with

$$F(SU(n),\ M) = P_k(\boldsymbol{C}),$$

(b) *there is a smooth $SU(n)$-action on $P_{n+k}(\boldsymbol{C})$ such that*

$$\pi_1(F)= \text{G} \quad \text{and} \quad H^*(F;\ Z) = H^*(P_k(\boldsymbol{C});\boldsymbol{Z}),$$

where $F=F(SU(n), P_{n+k}(C))$.

First we prepare the following lemma. It is proved by a similar argument as in the proof of Theorem 2.1 and Corollary 2.2, so we omit the proof.

Lemma 3.3. *Let X be a compact orientable $(2k+2)$-manifold which is acyclic over Z (resp. Q). Assume that X admits a smooth S^1-action which is free on ∂X. If $n\geqslant 2$, then*
(a) $M=\partial(D^{2n}\times X)/S^1$ *is a cohomology* $P_{n+k}(C)$ *over Z (resp. Q),*
(b) $\pi_1(M)=\pi_1(X)$.
Moreover if $n+k\geqslant 3$ and X is contractible, then $M=P_{n+k}(C)$.

Now we construct an acyclic S^1-manifold. Let W be a closed orientable smooth homology $(2k+1)$-sphere over Z (resp. Q) and let

(3.4) $$Y=P_k(C)\times[0,1]\#W, \qquad (k\geqslant 1).$$

Then F is a compact connected orientable smooth $(2k+1)$-manifold with boundary

$$\partial Y=P_k(C)\times 0\cup P_k(C)\times 1.$$

It is easily seen that

(3.5) $$\pi_1(Y)=\pi_1(W),$$

(3.6) $$H^i(Y;Z)=H^i(P_k(C);Z)\oplus H^i(W;Z), \qquad (0<i\leqslant 2k).$$

Furthermore there is a smooth principal S^1-bundle

$$p: E\rightarrow Y$$

such that $\partial_i E\rightarrow P_k(C)\times i$, $(i=0, 1)$ is equivalent to the Hopf bundle $S^{2k+1}\rightarrow P_k(C)$, where $\partial_i E=p^{-1}(P_k(C)\times i)$. Then

(3.7) $$\pi_1(E)=\pi_1(Y),$$

(3.8) $$H^*(E, \partial_1 E; A)=0$$

where $A=Z$ (resp. Q), by (3.6) and the Gysin sequence for S^1-bundles. Furthermore

$$X=E\bigcup_{\partial_1 E}D^{2k+2}$$

is a compact orientable manifold with a semi-free smooth S^1-action which is linear and free on $\partial X=\partial_0 E=S^{2k+1}$. It is easily seen that

(3.9) $\pi_1(X)=\pi_1(W)$, by (3.5) and (3.7),

(3.10) X is acyclic over Z (resp. Q), by (3.8).

Proof of Theorem 3.1. Put $W=S^{2k+1}/\boldsymbol{Z}_p$, a lens space, in (3.4). Then there is a compact orientable $(2k+2)$-manifold X with a semi-free smooth S^1-action which is linear and free on $\partial X=S^{2k+1}$, such that $\pi_1(X)=\boldsymbol{Z}_p$ and X is acyclic over $\boldsymbol{Q}$. Then by Lemma 3.3, the $SU(n)$-manifold

$$M = \partial(D^{2n}\times X)/S^1$$

is a compact orientable $2(n+k)$-manifold such that

$$\pi_1(M) = \boldsymbol{Z}_p,\ H^*(M;Q) = H^*(P_{n+k}(\boldsymbol{C});Q)$$

and

$$F(SU(n),M) = \partial X/S^1 = P_k(\boldsymbol{C})\,. \qquad \text{q.e.d.}$$

REMARK 3.11. It is known that if G is a finitely presentable group with $H_1(G;\boldsymbol{Z})=H_2(G\boldsymbol{Z})=0$, then for each $m\geqslant 7$, there is a compact contractible smooth n-manifold P such that

$$\pi_1(\partial P) = G \qquad \text{(see [12]).}$$

It is known that there are infinitely many groups satisfying the above condition.

Proof of Theorem 3.2 (a). Let $k\geqslant 3$. Put $W=\partial P$, a smooth homology $(2k+1)$-sphere over Z with $\pi_1(\partial P)=G$, in (3.4). Then there is a compact orientable $(2k+2)$-manifold X with a semi-free smooth S^1-action which is linear and free on $\partial X=S^{2k+1}$, such that $\pi_1(X)=G$ and X is acyclic over Z. Then by Lemma 3.3, the $SU(n)$-manifold

$$M = \partial(D^{2n}\times X)/S^1$$

is a compact orientable $2(n+k)$-manifold such that

$$\pi_1(M) = G,\ \text{ff}^*(M;Z) = H^*(P_{n+k}(\boldsymbol{C});Z)$$

and

$$F(SU(n),M) = P_k(\boldsymbol{C})\,. \qquad \text{q.e.d.}$$

Proof of Theorem 3.2 (b). Let $k\geqslant 3$. For a given group G satisfying the hypothesis, there is a compact contractible smooth $(2k+1)$-manifold p such that

$$\pi_1(\partial P) = G$$

by Remark 3.11. Let

$$Y = P_k(\boldsymbol{C})\times[0,1]\# P\,,$$

a boundary connected sum with boundary

$$\partial Y = P_k(\boldsymbol{C})\#\partial P\cup P_k(\boldsymbol{C})\times 1\,.$$

Then $P_k(C)\times 1$ is a deformation retract of Y, and hence there is a smooth principal S^1-bundle

$$p : E \to Y,$$

such that $\partial_1 E \to P_k(C)\times 1$ is equivalent to the Hopf bundle $S^{2k+1}\to P_k(C)$, where $\partial_1 E = p^{-1}(P_k(C)\times 1)$. Then

$$X = E \bigcup_{\partial_1 E} D^{2k+2}$$

is a compact contractible $(2k+2)$-manifold with a semi-free smooth S^1-action. Then by Lemma 3.3, the $SU(n)$-manifold

$$M = \partial(D^{2n}\times X)/S^1$$

is diffeomorphic to $P_{n+k}(C)$ for $n \geqslant 2$, and

$$F(SU(n), M) = \partial X/S^1 = P_k(C)\#\partial P.$$

Therefore there is a smooth $SU(n)$-action on $P_{n+k}(C)$ such that

$$\pi_1(F) = G \quad \text{and} \quad H^*(F; Z) = H^*(P_k(C); Z),$$

where $F = F(SU(n), P_{n+k}(C))$. q.e.d.

4. Signature of certain smooth G-manifolds

The purpose of this section is to study a signature of closed orientable manifold which admits a smooth G-action with isotropy groups of uniform dimension. We have the following result.

Theorem 4.1. *Let G be a compact Lie group and H a closed connected subgroup. Let M be a compact orientable manifold without boundary. Assume that M admits a smooth G-action such that the identity component of an isotropy group G_x is conjugate to H in G for each point x of M. Then $F(H, M)$, the stationary point set with respect to the H-action, is orientable, and*

(a) *if* $\dim N(H) \neq \dim H$, *then* $\mathrm{Sign}(M)=0$,

(b) *if* $\dim N(H) = \dim H$, *then*

$$|N(H)/H|\,\mathrm{Sign}(M) = \mathrm{Sign}(G/H)\,\mathrm{Sign}(F(H, M)).$$

Here $N(H)$ is the normalizer of H in G, $|N(H)/H|$ is the order of the finite group $N(H)/H$.

The result is a generalization of the fact that $\mathrm{Sign}(M)=0$ if M admits a smooth circle action without stationary points.

Lemma 4.2. *Let G be a compact Lie group and H a closed connected subgroup. Let M be a smooth G-manifold such that the identity component of G_x is* *conjugate to* H *in* G *for each point* x *of* M. *Then*

(a) *the* $W(H)$*-action on* $F(H,M)$ *is almost free (i.e. all isotropy groups are discrete), where* $W(H)=N(H)/H$,

(b) *there is an equivariant diffeomorphism*

$$M = G \underset{N(H)}{\times} F(H, M) = G/H \underset{W(H)}{\times} F(H, M),$$

(c) *if* M *is orientable, then* $F(H, M)$ *is orientable.*

Proof. By the assumption, the identity component of G_x is equal to H for each point x of $F(H, M)$, and the mapping

$$f\colon\quad G\times F(H, M) \to M$$

given by $f(g, x)=gx$ is surjective. Moreover $f(g, x)$ is in $F(H, M)$ if and only if $g\in N(H)$, thus $W(H)$ acts on $F(H, M)$ naturally and (b) is proved. Next, if an isotropy group $W(H)_x$ is not discrete for a point x of $F(H, M)$, then

$$\dim G_x \neq \dim H .$$

This contradicts our assumption, and (a) is proved. By (b), the product manifold $G/H\times F(H,M)$ is a total space of a principal $W(H)$-bundle over M. Therefore $G/H\times F(H,M)$ is orientable, if M is orinetable, and hence $F(H, M)$ is orientable. q.e.d.

Lemma 4.3. *Let* G *be a compact Lie group which is not discrete. Let* M *be a compact orientable smooth manifold without boundary. Then,* $\operatorname{Sign}(M)=0$ *if* M *admits an almost free smooth* G*-action.*

Proof. G contains a circle subgroup and the circle action on M has no stationary points. Therefore $\operatorname{Sign}(M)=0$. q.e.d.

Proof of Theorem 4.1. Denote by $W(H)^0$, the identity component of $W(H)$. Then

$$G/H \underset{W(H)^0}{\times} F(H, M)$$

is a total space of a principal $W(H)/W(H)^0$-bundle over M by Lemma 4.2. (b). Therefore

$$|W(H)/W(H)^0|\cdot\operatorname{Sign}(M) = \operatorname{Sign}(G/H \underset{W(H)^0}{\times} F(H, M)) .$$

Next, $G/H \underset{W(H)^0}{\times} F(H,M)$ is a total space of a smooth fibre bundle over an orientable manifold $(G/H)/W(H)$ with a fibre $F(H, M)$ and a structure group $W(H)^0$ which is connected. Therefore

$$\mathrm{Sign}(G/H \underset{W(H)^0}{\times} F(H, M)) = \mathrm{Sign}((G/H)/W(H)^0)\cdot \mathrm{Sign}(F(H,M))$$

for a certain orientation of $F(H, M)$ by [4]. By the above equations,

$$|W(H)/W(H)^0|\mathrm{Sign}(M) = \mathrm{Sign}((G/H)/W(H)^0)\ \mathrm{Sign}(F(H,M)) .$$

Now, if $\dim W(H)\neq 0$ then $\mathrm{Sign}(F(H,M))=0$ by Lemma 4.2 (a) and Lemma 4.3. If $\dim W(H)=0$, then

$$|W(H)|\,\mathrm{Sign}(M) = \mathrm{Sign}(G/H)\ \mathrm{Sign}(F(H, M)) .$$

This completes the proof.

REMARK 4.4. Let G be an arbitrary compact connected Lie group and T be a maximal torus. Then $\mathrm{Sign}(G/T)=0$, since G/T is stably parallelizable (see [3], section 5.4).

REMARK 4.5. Let G be a compact connected Lie group and H a closed connected subgroup. Then $\mathrm{Sign}(G/H)=0$ if

$$\mathrm{rank}\, G\neq \mathrm{rank}\, H \qquad \text{(see [7])}.$$

Because the left translation on G/H of a maximal torus of G has no stationaly points.

5. $SU(3)$-actions on orientable 8-manifolds

The purpose of this section is to prove the following result.

Theorem 5.1. *Let M be a closed connected orientable 8-manifold. Assume that M admits a non-trivial smooth $SU(3)$-action with a principal isotropy type (H). Then*

(a) $H^4(M; \boldsymbol{Q})=0$, *if* $\dim H=0$,
(b) $\mathrm{Sign}(M)=0$, *if* $\dim H=1$ *and M has not isotropy types* $(NSU(2))$ *and* $(T_{(2)})$,
(c) $\mathrm{Sign}(M)=0$, *if* $\dim H=2$,
(d) $H^4(M; \boldsymbol{Q})=0$, *if* $\dim H=3$ *and M has not an isotropy type* $(NSU(2))$,
(e) $M=P_2(\boldsymbol{C})\times F(NSU(2),M)$, *if* $\dim H=4$.

Here $NSU(2)$ is the normalizer of $SU(2)$ in $SU(3)$, the identity component of $T_{(2)}$ is a maximal torus of $SU(3)$ and $T_{(2)}$ has 2-components.

First we recall an additivity property of the signature due to S.P. Novikov (see [1], p. 588). Suppose that Y is a compact oriented $4n$-manifold with boundary ∂Y. Let $\hat{H}^{2n}(Y; \boldsymbol{Q})$ denote the image of the natural homomorphism

$$j^*: H^{2n}(Y, \partial Y; \boldsymbol{Q}) \to H^{2n}(Y; \boldsymbol{Q}) .$$

Then the bilinear form B on $\hat{H}^{2n}(Y; \boldsymbol{Q})$ defined by

$$B(j^*(a), j^*(b)) = ab[Y]$$

is symmetric and non-degenerate by Poincare-Lefschetz duality. We can now define $\mathrm{Sign}(Y)$ as the signature of B. Suppose now that Y' is another compact oriented $4n$-manifold with boundary $\partial Y' = -\partial Y$. Then $X = Y \underset{\partial Y}{\cup} Y'$ is a closed orineted $4n$-manifold and

$$\mathrm{Sign}(X) = \mathrm{Sign}(Y) + \mathrm{Sign}(Y'). \tag{5.2}$$

REMARK 5.3. Let ξ be an orientable k-plane bundle over a closed orientable manifold X. Denote by $t(\xi)$, $e(\xi)$ and $D(\xi)$, the Thom class, the Euler class and the disk bundle of ξ, respectively. Then $D(\xi)$ is a compact orientable manifold and there is a commutative diagram:

$$\begin{array}{ccc} H^*(D(\xi), \partial D(\xi)) & \xrightarrow{j^*} & H^*(D(\xi)) \\ \cong \uparrow \psi & & \cong \uparrow \pi^* \\ H^*(X) & \xrightarrow{\cdot e(\xi)} & H^*(X). \end{array}$$

Here ψ is the Thom isomorphism defined by

$$\psi(a) = \pi^*(a) \cdot t(\xi).$$

There is an equation

$$\psi(a) \cdot \psi(b) = (-1)^{kp} \psi(ab \cdot e(\xi)) \qquad \text{for } b \in H^p(X).$$

Therefore we can calculate $\mathrm{Sign}(D(\xi))$ from the information about the cohomology ring $H^*(X)$ and the Euler class $e(\xi)$.

Now we prepare the following results.

Lemma 5.4.

(a) $H^*(SU(3); \boldsymbol{Z}) = \wedge_{\boldsymbol{Z}}(x_3, x_5)$, *deg* $x_i = i$, $(i = 3, 5)$.

(b) $H^*(SU(3)/SU(2)\,\boldsymbol{Z}) = H^*(S^5; \boldsymbol{Z})$ *and the right translation of* $NSU(2)/SU(2) = S^1$ *induces a trivial action on* $H^*(SU(3)/SU(2); Z)$.

(c) $H^*(SU(3)/SO(3)\,\boldsymbol{Q}) = H^*(S^5; \boldsymbol{Q})$, *and the right translation of* $NSO(3)/SO(3) = \boldsymbol{Z}_3$ *induces a trivial action on* $H^*(SU(3)/SO(3); \boldsymbol{Q})$.

(d) $H^*(SU(3)/T; \boldsymbol{Z}) = \boldsymbol{Z}[u_1, u_2, u_3]/(s_1, s_2, s_3)$, *where T is a maximal torus of $SU(3)$ consists of all diagonal matrices, s_k is the k-th elementary symmetric polynomials, and deg $u_i = 2$, $(i = 1, 2, 3)$. Furthermore the induced action of $N(T)/T = S_3$, the symmetric group on 3-elements, is given by*

$$a^*(u_i) = u_{a(i)}, \qquad a \in S_3.$$

(e) $H^*(SU(3)/D(m,n); \boldsymbol{Q}) = \wedge_{\boldsymbol{Q}}(x_2, x_5)$, *deg* $x_i = i$, $(i = 2, 5)$. *Here $D(m, n)$ is a closed one-dimensional subgroup defined by*

$$D(m, n) = \left\{ \begin{pmatrix} z^m & & \\ & z^n & \\ & & z^{-(m+n)} \end{pmatrix} ; z \in C, \ |z| = 1 \right\}$$

for any pair of integers $(m, n) \neq (0, 0)$.

Proof. Since $SU(3)/SU(2)=S^5$, (b) is true. (a) is proved by making use of the Gysin sequence for

$$SU(2) \rightarrow SU(3) \rightarrow S^5 .$$

(c) is proved from

$$\pi_1(SU(3)/SO(3)) = 0 \text{ and } \pi_2(SU(3)/SO(3)) = Z_2 .$$

(d) is a classical result (see [9]). In fact $u_i = p_i^*(u)$, where u is a generator of $H^2(P_2(C); Z)$ and p_i ; $SU(3)/T \rightarrow P_2(C)$ is defined by

$$p_i((x_{ab}) \cdot T) = (x_{1i} : x_{2i} : x_{3i}) .$$

Finally (e) is proved from the fact that the Euler class of principal S^1-bundle π: $SU(3)/D(m, n) \rightarrow SU(3)/T$ is

$$e(\pi) = nu_1 + mu_2 ,$$

and hence the homomorphism

$$H^2(SU(3)/T; Q) \xrightarrow{\cdot e(\pi)} H^4(SU(3)/T; Q)$$

is an isomorphism. q.e.d.

Lemma 5.5.

(a) *Let* φ *be an 8-dimensional non-trivial real representation of* $SU(3)$. *Let* (H_φ) *be the principal isotropy type of the linear action given by* φ. *Then there are only the following cases:*

(i) $\varphi = Ad_{SU(3)}$, $H_\varphi = T$: *a maximal torus of* $SU(3)$,

(ii) $\varphi = \rho_3 +$ *trivial summand,* $H_\varphi = SU(2)$,

where ρ_3: $SU(3) \rightarrow O(6)$ *is the standard representation.*

(b) *Let* ψ *be a 4-dimensional non-trivial real representation of* $NSU(2)$. *Let* (H_ψ) *be the principal isotropy type of the linear action given by* ψ. *Then there are only the following cases:*

(i) $\psi = Ad_{NSU(2)}$, $H_\psi = T$: *a maximal torus of* $NSU(2)$,

(ii) $\psi = \sigma_k$, $H_\psi = D(k-1, -k)$, $(k \in Z)$,

where the representation σ_k: $NSU(2) \rightarrow U(2) \subset O(4)$ *is given by*

$$\sigma_k\begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} y^k x_{11} & y^k x_{12} \\ y^k x_{21} & y^k x_{22} \end{pmatrix}.$$

(iii) *ψ is induced from a non-trivial real representation of S^1, via the natural projection $NSU(2) \to NSU(2)/SU(2) \cong S^1$, and $H_\psi^0 = SU(2)$, where H_ψ^0 is the identity component of H_ψ.*

We omit the proof (see [8], Theorem I).

From now on we assume that M is a closed connected orientable smooth 8-manifold and M admits a non-trivial smooth $SU(3)$-action with a principal isotropy type (H). Then $SU(3)/H$ is orientable by the differentiable slice theorem (see [11], Lemma 3.1).

We will prove Theorem 5.1 by the following many propositions.

Proposition 5.6. *Assume that $SU(3)_x^0$ is conjugate to H^0 in $SU(3)$ for each $x \in M$. Here G^0 is the identity component of G and $SU(3)_x$ is the isotropy group at x. Then,*

(a) $\mathrm{Sign}(M)=0$, *if* $\dim H=1$ *or* 2,

(b) $H^4(M; \boldsymbol{Q})=0$, *if* $\dim H=0$ *or* 3,

(c) $M=P_2(\boldsymbol{C}) \times F(NSU(2), M)$, *if* $\dim H=4$.

Proof. If $\dim H=1$ or 2, then $\mathrm{Sign}(M)=0$ by Theorem 4.1 and Remarks 4.4, 4.5. If $\dim H=0$, then $M=SU(3)/H$ and hence $H^4(M; \boldsymbol{Q})=0$ by Lemma 5.4 (a). By Lemma 4.2, there is an equivariant diffeomorphism

$$M = SU(3)/H^0 \times_K F, K = N(H^0)/H^0, F = F(H^0, M).$$

If $\dim H=4$, then H^0 is conjugate to $NSU(2)$ in $SU(3)$ and $N(NSU(2))= NSU(2)$. Therefore

$$M = P_2(\boldsymbol{C}) \times F(NSU(2), M).$$

Finally if $\dim H=3$, then H^0 is conjugate to $SO(3)$ or $SU(2)$ in $SU(3)$. If $H^0=SO(3)$, then $\dim F=3$ and

$$H^4(M; \boldsymbol{Q}) = H^4(SU(3)/SO(3) \times F \boldsymbol{Q}) = 0$$

by Lemma 5.4 (c). Next if $H^0=SU(2)$, then $\dim F=4$, F admits a smooth S^1-action without stationary points and there is an equivariant diffeomorphism

$$M = S^5 \times_{S^1} F.$$

There is a sufficiently large integer n such that the $S^1/\boldsymbol{Z}_n$-action on the orbit space $F/\boldsymbol{Z}_n$ is free. Then there is an isomorphism

$$H^*(M; \boldsymbol{Q}) = H^*(M'; \boldsymbol{Q}),$$

where

$$M' = (S^5/\boldsymbol{Z}_n \times F/\boldsymbol{Z}_n)/(S^1/\boldsymbol{Z}_n),$$

and there is a fibre bundle

$$S^5/\boldsymbol{Z}_n \to M' \to F/S^1$$

with a structure group $S^1/\boldsymbol{Z}_n$. Here $F/S^1=(F/\boldsymbol{Z}_n)/(S^1/\boldsymbol{Z}_n)$ is a 3-dimensional rational cohomology manifold. Therefore

$$H^4(M;\boldsymbol{Q}) = H^4(M'; \boldsymbol{Q}) = 0 . \qquad \text{q.e.d.}$$

REMARK 5.7. Now Theorem 5.1 is proved for dim $H=0$ or 4. Moreover, Theorem 5.1 is proved for the case $H^0=SO(3)$, since $SO(3)$ is not conjugate to any subgroup of $NSU(2)$ in $SU(3)$ and H with $H^0=SO(3)$ is not a principal isotropy group of any 8-dimensional real representation of $SU(3)$ by Lemma 5.5.

Proposition 5.8. *Suppose* dim $H=1$. *Then* Sign$(M)=0$, *if M has not isotropy types* $(NSU(2))$ *and* $(T_{(2)})$.

Proof. By Proposition 5.6 (a), one may assume that there is an isotropy type (K_1) with dim $K_1>1$. Then by making use of the differentiable slice theorem, there is an isotropy type (K_2) and there is an equivariant decomposition

$$M = D(\nu_1) \cup D(\nu_2),$$

where $D(\nu_i)$ is an equivariant normal disk bundle of an embedding $SU(3)/K_i \subset M$, and

$$\partial D(\nu_1) = -\partial D(\nu_2) = SU(3)/H .$$

Thus

$$\text{Sign}(M) = \text{Sign}(D(\nu_1)) + \text{Sign}(D(\nu_2)) .$$

Since

$$H^4(SU(3)/K; Q) = 0 \qquad \text{for } \dim K \neq 2,\ 4,$$

by Lemma 5.4, Sign$(D(\nu_i))=0$ for dim $K_i \neq 2$, 4. Let K be a 2-dimensional closed subgroup of $SU(3)$. Then K is conjugate to one of the following

$$T,\ T_{(2)},\ T_{(3)} \text{ and } N(T) = T_{(6)}.$$

Here $T^0_{(i)}=T$ and $T_{(i)}$ has i-components. By Lemma 5.4 (d),

$$H^4(SU(3)/T; \mathrm{Q}) = \boldsymbol{Q} \oplus \boldsymbol{Q},$$
$$H^4(SU(3)/T_{(2)}; Q) = Q,$$
$$H^4(SU(3)/T_{(3)}; Q) = H^4(SU(3)/N(T)\, Q) = 0 .$$

Thus $\mathrm{Sign}(D(\nu_i))=0$, if $K_i=T_{(3)}$ or $N(T)$. If $K_i=T$, then $\mathrm{Sign}(D(\nu_i))=0$ from Lemma 5.4 (d) and Remark 5.3. q.e.d.

REMARK 5.9. If $\dim H=2$ in Theorem 5.1, then $H=T$ or $T_{(3)}$, since $SU(3)/T_{(2)}$ and $SU(3)/N(T)$ are non-orientable by Lemma 5.4(d). Theorem 5.1 is proved for $H=T_{(3)}$ by Proposition 5.6, since $T_{(3)}$ is not conjugate to any subgroup of $NSU(2)$ in $SU(3)$ and $T_{(3)}$ is not a principal isotropy group of any 8-dimensional real representation of $SU(3)$ by Lemma 5.5. Therefore, it remains to prove Theorem 5.1 for the cases $H=T$ and $H^0=SU(2)$.

Proposition 5.10. *Suppose* $H=T$. *Then* $\mathrm{Sign}(M)=0$.

Proof. If $F(NSU(2), M)$ is empty, then $\mathrm{Sign}(M)=0$ by Proposition 5.6. Now we assume that $F(NSU(2),M)$ is not empty. Then

$$\dim F(NSU(2), M)=1$$

by Lemma 5.5, and any stationary point (if exists) of $SU(3)$ is isolated by Lemma 5.5 (a). Let

$$F(SU(3), M)=\{x_1, \cdots, x_k\}, \qquad (k\geqslant 0)$$

and let D_i be an invariant closed disk around x_i, such that

$$D_i\cap D_j=\mathbf{0} \qquad \text{for} \quad i\neq j.$$

Let $D=D_1\cup \quad \cup D_k$ and $E=M-\mathrm{int}\, D$. Then

$$D_i\cap F(NSU(2), E)\neq\emptyset, \qquad (i=1, \cdots, k)$$

by Lemma 5.5 (a). Let

$$E_0=\{x\in E\,|\,(SU(3)_x)=(NSU(2))\},$$

let U_0 be an invariant closed tubular neighborhood of E_0 in E, and let $U=U_0\cup D$. Then $M-\mathrm{int}\ U$ is connected and

$$(SU(3)_x^0)=(T), \qquad \text{for} \quad x\in M-\mathrm{int}\ U.$$

Therefore, there is an equivariant diffeomorphism

$$M-\mathrm{int}\ U=SU(3)/T\underset{N(T)/T}{\times}F, \quad F=F(T, M-\mathrm{int}\ U)$$

by Lemma 4.2, and there is a commutative diagram:

$$\begin{array}{ccc}
H^4(M-\mathrm{int}\ U;\ \boldsymbol{Q}) & \xrightarrow{i^*} & H^4(\partial(M-\mathrm{int}\ U);\ \boldsymbol{Q}) \\
\cong\Big\downarrow p^* & & \cong\Big\downarrow p^* \\
H^4(SU(3)/T\times F;\ \boldsymbol{Q})^{N(T)/T} & \xrightarrow[i_0^*]{} & H^4(SU(3)/T\times\partial F;\ \boldsymbol{Q})^{N(T)/T}
\end{array}$$

Here i_0^* is injective, since $H^{odd}(SU(3)/T;\boldsymbol{Q})=0$ by Lemma 5.4 (d), $\dim F=2$, and each connected component of F has non-empty boundary from the connectedness of $M-\text{int }U$. Thus

$$\hat{H}^4(M-\text{int}\,U;\boldsymbol{Q})=0\,,$$

and hence $\text{Sign}(M-\text{int }U)=0$. Next, let $U_1, \cdots, U_n$ be connected components of U. Then we can prove that

$$\hat{H}^4(U_i;\boldsymbol{Q})=0\,, \qquad \text{if } U\cap D=\emptyset\,,$$
$$H^4(U_i;\boldsymbol{Q})=0\,, \qquad \text{if } U_i\cap D\neq\emptyset\,,$$

and hence

$$\text{Sign}(U)=\text{Sign}(U_1)+\cdots+\text{Sign}(U_n)=0\,.$$

Therefore

$$\text{Sign}(M)=\text{Sign}(M-\text{int}\,U)+\text{Sign}(U)=0\,. \qquad \text{q.e.d.}$$

We recall the following result which is essentially proved in the proof of Proposition 5.6 (b).

Lemma 5.11. *Let X be a compact connected orientable smooth n-manifold (∂X is empty or not). Let $n=7$ or 8. Assume that X admits a smooth $SU(3)$-action with*

$$(SU(3)_x^0)=(SU(2)) \qquad \textit{for} \quad x\in X.$$

Then

$$H^{n-4}(X;Q)=0\,.$$

Proposition 5.12. *Assume that $H^0=SU(2)$ and M has not an isotropy type $(NSU(2))$. Then $H^4(M;\boldsymbol{Q})=0$.*

Proof. If $F(SU(3),M)=\emptyset$, then $H^4(M;\boldsymbol{Q})=0$ by Lemma 5.11. Next if $F(SU(3),M)\neq\emptyset$, then $\dim F(SU(3),M)=2$ by Lemma 5.5 (a). Let U be an invariant closed tubular neighborhood of $F(SU(3),M)$ in M. Then there is an exact sequence:

$$H^3(\partial U;Q)\to H^4(M;Q)\to H^4(U;\boldsymbol{Q})\oplus H^4(M-\text{int}U;\boldsymbol{Q})\,.$$

Here

$$H^3(\partial U;Q)=H^4(M-\text{int}\,U;\boldsymbol{Q})=0$$

by Lemma 5.11, and

$$H^4(U;\mathrm{Q})=H^4(F(SU(3),M);Q)=0\,.$$

Therefore

$$H^4(M;Q)=0\,. \qquad \text{q.e.d.}$$

This completes the proof of Theorem 5.1.

6. SZ7(3)-actions on cohomology $P_4(C)$

In the previous paper [13] we have considered smooth $SU(3)$-actions on homotopy $P_3(C)$. In this section, first we prove the following result as an application of Theorem 5.1.

Theorem 6.1. *Let M be a compact connected orientable 8-manifoldsuch that*

$$H^*(M; Q) = H^*(P_4(C); Q).$$

Then for any non-trivial smooth SU(3)-action on M, the stationary point set is a 2-sphere and the principal isotropy type is $(SU(2))$. Furthermore there is an equivariant diffeomorphism

$$M = \partial(D^6 \times X)/S^1.$$

Here X is a compact connected orientable 4-manifolawhich is acyclic over rationals, X admits a smooth S^1-action which isfree on ∂X, the SU(3)-action is standard on D^6 and trivial on X.

Proof. Denote by (H), the principal isotropy type of the given $SU(3)$-action on M. Since $\text{Sign}(M) \neq 0$, the following are the only possible cases from Theorem 5.1,

(a) dim $H=l$ and M has an isotropy type $(NSU(2))$or $(T_{(2)})$,

(b) $H^0=SU(2)$and M has an isotropy type $(NSU(2))$,

(c) $H=NSU(2)$and $M=P_2(C)$ X $F(NSU(2),M)$.

If $H=NSU(2)$, then $\chi(M)=5$ is divisible by $\chi(P_2(C))=3$, and this is a contradiction. Next if dim $H=1$, then there is a decomposition

$$M = D(\nu_1) \cup D(\nu_2)$$

as in the proof of Proposition 5.8, where $D(\nu_i)$ is a normal disk bundle over $SU(3)/K_i$. One may assume $K_1=$ *NSU(2)* or $T_{(2)}$, and hence

$$\chi(SU(3)/K_1) = 3$$

by Lemma 5.4. On the other hand,

$$5 = \chi(M) = \chi(SU(3)/K_1)+\chi(SU(3)/K_2)$$

Thus $\chi(SU(3)/K_2) = 2$, and hence $K_2 = T_{(3)}$ by Lemma 5.4. Since $H^2(SU(3)/T_{(3)};Q)=0$, there is a contradiction in the following exact sequence of rational cohomology groups:

$$H^1(\partial D(\nu_1)) \to H^2(M) \to H^2(SU(3)/K_1) \oplus H^2(SU(3)/K_2)$$
$$\to H^2(\partial D(\nu_1)) \to H^3(M).$$

Therefore we obtain $H^0=SU(2)$. If $F(SU(3),M)=\emptyset$, then there is a fibre bundle

$$F(SU(2),M)\to M\to P_2(C)\ .$$

Thus $\chi(M)=5$ is divisible by $\chi(P_2(C))=3$, and this is a contradiction. Hence $F(SU(3),M)\neq\emptyset$ and this implies $H=SU(2)$ by Lemma 5.5 (a). Let U be an invariant tubular neighborhood of $F(SU(3),\mathrm{M})$ in M. Then

$$X=F(SU(2),M-\mathrm{int}\ U)$$

is a compact connected orientable 4-manifold with the natural action of $NSU(2)/SU(2)=S^1$ which is free on ∂X. Furthermore there is an equivariant diffeomorphism

$$M=\partial(D^6\times X)/S^1\ ,$$

and X is acyclic over rationals by the same argument as in the proof of Theorem 2.1. Finally,

$$F(SU(3),\ \mathrm{M})\ =\ \partial X/S^1=\ S^2\ . \qquad \text{q.e.d.}$$

Next, as a complementary part of Theorem 5.1, we give examples of certain $SU(3)$-actions on 8-manifolds with non-zero signature.

Let $\psi: NSU(2)\to U(3)$ be a unitary representation of $NSU(2)$. Then ψ induces a smooth $NSU(2)$-action ψ_* on $P_2(C)$. Denote by $M(\psi)$, the orbit manifold of the free smooth action of $NSU(2)$ on $SU(3)\times P_2(C)$ given by

$$h\cdot(g,\ x)=(gh^{-1},\ \psi_*(h,\ x)),\ g\in SU(3),\ h\in NSU(2),\ x\in P_2(C)\ .$$

Then the compact connected orientable 8-manifold $M(\psi)$ admits a natural smooth $SU(3)$-action without stationary points and

$$\mathrm{Sign}(M(\psi))=1\ .$$

EXAMPLE 6.2. Let $\alpha_k: NSU(2)\to U(3)$ be a unitary representation given by

$$\alpha_k\begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & y\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & y^k\end{pmatrix}.$$

Then $M(\alpha_k)$ has just two isotropy types

$$(SU(2)_{(k)})\quad\text{and}\quad(NSU(2))\ ,$$

where $SU(2)_{(k)}$ has k-components and its identity component is $SU(2)$. (see Theorem 5.1 (d))

EXAMPLE 6.3. Let β_k: $NSU(2)\to U(3)$ be a unitary representation given by

$$\beta_k\begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & y^k \end{pmatrix}.$$

Then $M(\beta_k)$ has just three isotropy types

$$(D(k, -k-1)), (T) \text{ and } (NSU(2)),$$

where $D(k, -k-1)$ is a closed one-dimensional subgroup defined in Lemma 5.4. (see Theorem 5.1 (b))

EXAMPLE 6.4. Let γ: $NSU(2) \to U(3)$ be a unitary representation given by

$$\gamma\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ \sqrt{2}ac & ad+bc & \sqrt{2}bd \\ c^2 & \sqrt{2}cd & d^2 \end{pmatrix}.$$

Then $M(\gamma)$ has just three isotropy types

$$(D(1, 1)_{(2)}), (T) \text{ and } (T_{(2)}),$$

where $G_{(2)}$ is a subgroup of $SU(3)$ such that $G_{(2)}$ has 2-components and its identity component is G. (see Theorem 5.1 (b))

7. Classification of smooth $SU(n)$-actions on orientable $2n$-manifolds

Let M be a compact connected $2n$-manifold with non-trivial smooth $SU(n)$-action, then the identity component of each isotropy group is conjugate to one of the following

$$SU(n), SU(n\text{-}l) \text{ and } NSU(n-1),$$

for $n \geqslant 5$. This is proved similarly as Lemma 1.5. Therefore there is an equivariant diffeomorphism

$$M = \partial(D^{2n} \times X)/S^1$$

as $SU(n)$-manifolds by (1.1) and (1.4). Here X is a compact connected 2-dimensional S^1-manifold and the S^1-action on dX is free if dX is non-empty. Furthermore if M is orientable, then X is also orientable. Next we remark that for orientable 2-dimensional S^1-manifold X, if the isotropy group $S^1_x \neq S^1$ for $x \in X$, then S^1_x is a principal isotropy group by the differentiable slice theorem, and hence the S^1-space $X - F(S^1, X)$ has just one isotropy type.

(i) If X has just one isotropy type (S^1), then $\partial X = \emptyset$ and

$$M = P_{n-1}(C) \times X.$$

(ii) If X has just one isotropy type $(\boldsymbol{Z}_k)$, then

$$M = S^{2n} \qquad \text{if } \partial X \neq \emptyset ,$$
$$M = L^{2n-1}(k) \times S^1 \qquad \text{if } \partial X = \emptyset .$$

Here $L^{2n-1}(k) = S^{2n-1}/\boldsymbol{Z}_k$ is a standard lens space.

(iii) If X has just two isotropy types $(\boldsymbol{Z}_k)$ and (S^1), then

$$M = P_n(\boldsymbol{C}) \qquad \text{if } \partial X \neq \emptyset ,$$
$$M = S^{2n-1} \times_{S^1} S^2_{(k)} \qquad \text{if } \partial X = \emptyset .$$

Here $S^2_{(k)}$ is a 2-sphere with the S^1-action given by

$$e^{i\theta}(x_0, x_1, x_2) = (x_0, x_1 \cos k\theta + x_2 \sin k\theta, -x_1 \sin k\theta + x_2 \cos k\theta) .$$

This completes the classification.

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