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4-FOLD TRANSITIVE GROUPS IN WHICH ANY 2-ELEMENT AND ANY 3-ELEMENT OF A STABILIZER OF FOUR POINTS ARE COMMUTATIVE

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1. Introduction

The known 4-fold transitive groups in which the stabilizer of four points is nilpotent are S_4 , S_5 , S_6 , A_6 , A_7 , M_{11} and M_{12} .

In this paper we shall prove the following

Theorem. Let G be a 4-fold transitive group on Ω . If any 2-element and any 3-element of a stabilizer of four points in G are commutative, then G is S_4 , S_5 , S_6 , A_6 , A_7 , M_{11} or M_{12} .

As a corollary of this theorem, we have that a 4-fold transitive group in which the stabilizer of four points is nilpotent is one of the groups listed in our theorem.

Our notation is standard (cf. Wielandt [9]). For a subgroup X of G and a subset Δ of Ω , X_{Δ} is the point-wise stabilizer of Δ in X and if X fixes Δ as a set, the restriction of X on Δ will be denoted by X^{Δ} . F(X) is the set of points in $\Omega = \{1, 2, \dots, n\}$ fixed by every element of X.

2. Proof of the theorem

We proceed by way of contradiction and divide the proof in ten steps. From now on we assume that G^{Ω} is a counterexample to our theorem of the least possible degree. We set $D=G_{1234}$. Let P and R be a Sylow 2-subgroup and Sylow 3-subgroup of D, respectively.

By [2]–[8], we know the following.

(1) $F(D) = \{1, 2, 3, 4\}, |F(P)| = 4 \text{ or } 5, R \neq 1 \text{ and } P^{\Omega - F(P)} \text{ is not semi-regular.}$ (2) |F(P)| = 4.

Proof. From (1), we have only to show that $|F(P)| \neq 5$. Suppose |F(P)| = 5. We set $F(P) = \{1, 2, 3, 4, i\}$ and $L = \langle R^d | d \in D \rangle$. By assumption, P centralizes L and so $i \in F(L)$. Then it follows from $L \leq D$ that $i^p \subseteq F(L)$.

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Since $2 \not| |D: D_i|$, $3 \not| |D: D_i$ and $|D: D_i| \neq 1$ by (1), we have $|D: D_i| \geq 5$ and so $|F(L)| \geq 9$. By the Witt's theorem, $N_G(L)^{F(L)}$ is a 4-fold transitive group. Furthermore $|F(L)| < n = |\Omega|$ since $R \neq 1$. Hence by our minimal choice of Ω we have $N_G(L)^{F(L)} \simeq M_{11}$ or M_{12} because $|F(L)| \geq 9$. Since |F(P)| = 5 and [P, L] = 1, we have |F(L)| is odd and so $N_G(L)^{F(L)} \simeq M_{11}$. Clearly $P^{F(L)}$ is a Sylow 2-subgroup of a stabilizer of four points in $N_G(L)^{F(L)}$, so $P^{F(L)} = 1$ by the structure of M_{11} , hence F(L) is a subset of F(P), which is contrary to |F(P)| = 5. Thus we get (2).

Now let *m* be the maximal number of $|P_i|$ with $i \in \Omega - F(P)$. There exists some $j \in \Omega - F(P)$ such that $|P_i| = m$. We set $P_i = Q$. Then we have

(3) $1 \neq Q \neq P$.

Proof. By (1), $P^{\Omega-F(L)}$ is not semi-regular, and so $Q \neq 1$. It is clear that $Q \neq P$.

(4) If Q^* is a 2-subgroup of G containing Q properly, then we have $|F(Q^*)| \leq 4$.

Proof. Suppose that $|F(Q^*)| > 4$. Then there exists an element $g \in G$ with $(Q^*)^g \le D$. Since P is a Sylow 2-subgroup of D, $(Q^*)^{gd} \le P$ holds for some $d \in D$. By assumption we can choose an element k in $F((Q^*)^{gd})$ with $k \in F(P)$. Then we have $m = |Q| < |Q^*| = |(Q^*)^{gd}| \le |P_k|$, which is contrary to the choice of Q.

(5) $N_G(Q)^{F(Q)}$ is 4-fold transitive.

Proof. In the lemma 6 of [1], we put $G^{\alpha} = N_{G}(Q)^{F(Q)}$, p=2 and k=4, then (5) follows immediately from (2) and (4).

(6) $N_G(Q)^{F(Q)} \simeq S_6 \text{ or } M_{12}$.

Proof. By (5), $N_G(Q)^{F(Q)}$ satisfies the assumption of our theorem. Since n is even by (2) and $Q \neq 1$, $n > |F(Q)| \ge 6$. By the minimal choice of Ω , we have $N_G(Q)^{F(Q)} \simeq A_6$, S_6 or M_{12} . On the other hand, we have $N_P(Q) > Q$ by (3), and so it follows from (4) that $|F(N_P(Q))| = 4$. Thus $N_G(Q)^{F(Q)} \neq A_6$, which shows (6).

(7) Let S be a nontrivial 3-subgroup of D. Then F(S)=F(Q). Furthermore $R^{\Omega-F(R)}$ is semi-regular and the set F(Q) depends only on D but is independent of the choice of P and Q.

Proof. Since S centralizes Q, S is contained in $N_D(Q)$. By (6), $S^{F(Q)}=1$, that is, F(Q) is a subset of F(S), hence $|F(S)| \ge 6$.

In the lemma 6 of [1], we put $G^{\alpha} = N_{c}(S)^{F(S)}$, p=2 and k=4, then we get

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 $N_{G}(S)^{F(S)}$ is 4-fold transitive. On the other hand $S \neq 1$, hence $n > |F(S)| \ge 6$. By the minimality of Ω , $N_{G}(S)^{F(S)} \simeq A_{6}$, S_{6} , A_{7} , M_{11} or M_{12} .

Suppose $F(S) \neq F(Q)$. Then we have $|F(S)| \geq 7$. Since P centralizes S, P acts on F(S) - F(P) and so |F(S)| is even by (2). Hence $N_c(S)^{F(S)} \simeq M_{12}$. Therefore $Q^{F(S)} = 1$ by the structure of M_{12} . Hence F(S) is a subset of F(Q), a contradiction. Thus we conclude F(Q) = F(S).

From this, the latter half of (7) immediately follows.

(8) Let T be a Sylow 2-subgroup of $N_{c}(Q)$ and R^{*} be an arbitrary Sylow 3-subgroup of D. Then $[T, R^{*}]=1$.

Proof. By (6), there is a 2-element x in $N_G(Q)$ such that $|F(x) \cap F(Q)| = 4$ and $\langle x^g | g \in N_G(Q) \rangle^{F(Q)} = N_G(Q)^{F(Q)}$. We set $\langle x^g | g \in N_G(Q) \rangle = M$, $(N_G(Q))_{F(Q)} = K$. Then $N_G(Q) = MK$. Let T^* be an arbitrary Sylow 2subgroup of M, then since by (4) Q is a unique Sylow 2-subgroup of K, T^*Q is a Sylow 2-subgroup of $N_G(Q)$. Applying (7), for any $u \in N_G(Q)$ we get $F(Q) = F(R^*) = F((R^*)^u)$, hence $(R^*)^u \leq K$. Since $|F(x^g) \cap F((R^*)^u)| = |F(x^g) \cap$ F(Q)| = 4 for any $g \in N_G(Q)$, x^g centralizes $(R^*)^u$. Hence M centralizes $(R^*)^u$, so $[T^*Q, (R^*)^u] = 1$. Since T^*Q is a Sylow 2-subgroup of $N_G(Q)$, there exists an element $v \in N_G(Q)$ such that $T = (T^*Q)^v$. Thus we get $[T, (R^*)^{uv}] = 1$. Put $u = v^{-1}$. Then (8) holds.

(9) There exists an involution in Q. Let t be an involution in Q, then $|F(t)| \equiv 0 \pmod{3}$.

Proof. From (3), the first statement is clear. Since $F(R) = F(Q) \subseteq F(t)$, |F(Q)| = 6 or 12 and [R, t] = 1, we have $|F(t)| = |F(R)| + |F(t) - F(R)| \equiv 0 \pmod{3}$.

(10) We have now a contradiction in the following way.

Let t be an involution in Q. For $i \in \Omega - F(t)$, we set $i^t = j$. Then t normalizes G_{12ij} . There exists an element z in G such that $G_{12ij} = D^z$. By (7), G_{12ij} fixes $F(Q^z)$ as a set. We set $N = (G_{12ij})_{F(Q^z)}$, then by (4) Q^z is a Sylow 2-subgroup of N. Again by (7), t normalzes N, hence t normalizes at least one of the Sylow 2-subgroups of N, say Q^{zm} where m is an element of N. Now $t^{m^{-1}z^{-1}}$ normalizes Q, so $t^{m^{-1}z^{-1}}$ centralizes R by (8), hence t centralizes R^{zm} . Since R^z is a subgroup of G_{12ij} , $R^z \leq N$ by (7). Hence $R^{zm} \leq N$. By (6), $N_G(Q^{zm})^{F(Q^{zm})} \simeq S_6$ or M_{12} . Since the set $F(R^{zm}) \cap F(t)$ is not empty and $F(R^{zm}) \cap F(t) \neq F(R^{zm}) =$ $F(Q^{zm})$, we have $|F(R^{zm}) \cap F(t)| \equiv 2$ or 4. Since R^{zm} centralizes t, it follows from (7) that $|(\Omega - F(R^{zm})) \cap F(t)| \equiv 0 \pmod{3}$. Hence $|F(t)| = |F(t) \cap F(R^{zm})| +$ $|(\Omega - F(R^{zm})) \cap F(t)| \equiv 1$ or 2 (mod 3), contrary to (9).

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