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Osaka University
1-CONFORMALLY FLAT STATISTICAL SUBMANIFOLDS

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Introduction

We study 1-conformally flat statistical submanifolds of flat statistical manifolds. Let \( \phi \) be a function on a domain \( \Omega \) in an affine space \( \mathbb{A}^{n+1} \). Denoting by \( \bar{D} \) the canonical flat affine connection on \( \mathbb{A}^{n+1} \), we can consider a Hessian domain \( (\Omega, \bar{D}, \bar{g} = \bar{D}d\phi) \) a flat statistical manifold. In this paper, we show that, if \( \bar{g} \) is positive definite, \( n \)-dimensional level surfaces of \( \phi \) are 1-conformally flat statistical submanifolds of \( (\Omega, \bar{D}, \bar{g}) \), and that a 1-conformally flat statistical manifold with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold.

The concept of \( \alpha \)-conformally equivalence was first treated in [9] with respect to sequential estimation theory. On the realization of statistical manifolds in the affine space, see [5][6][7].

1. Theorems

Let \( \bar{D} \) be the canonical flat affine connection on an \( (n+1) \)-dimensional real affine space \( \mathbb{A}^{n+1} \), and \( \{x^1, \ldots, x^{n+1}\} \) the canonical affine coordinate on it, i.e., \( \bar{D}dx^i = 0 \). If the Hessian \( \bar{D}d\phi = \sum_{i,j}(\partial^2 \phi)/(\partial x^i \partial x^j)dx^i dx^j \) of a function \( \phi \) on a domain \( \Omega \) in \( \mathbb{A}^{n+1} \) is non-degenerate, we call \( (\Omega, \bar{D}, \bar{g} = \bar{D}d\phi) \) a Hessian domain.

For a torsion-free affine connection \( \nabla \) and a pseudo-Riemannian metric \( h \) on a manifold \( N \), the triple \( (N, \nabla, h) \) is called a statistical manifold if \( \nabla h \) is symmetric. If the curvature tensor \( R \) of \( \nabla \) vanishes, \( (N, \nabla, h) \) is said to be flat. A Hessian domain \( (\Omega, \bar{D}, \bar{g} = \bar{D}d\phi) \) is a flat statistical manifold. Conversely, a flat statistical manifold is locally a Hessian domain [1][10].

For \( \alpha \in \mathbb{R} \), statistical manifolds \( (N, \nabla, h) \) and \( (N, \bar{\nabla}, \bar{h}) \) are said to be \( \alpha \)-conformally equivalent if there exists a function \( \phi \) on \( N \) such that

\[
\bar{h}(X, Y) = e^{\phi}h(X, Y),
\]

\[
h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \frac{1+\alpha}{2}d\phi(Z)h(X, Y)
\]

\[
+ \frac{1-\alpha}{2}(d\phi(X)h(Y, Z) + d\phi(Y)h(X, Z))
\]

for \( X, Y, Z \in \mathcal{X}(N) \), where \( \mathcal{X}(N) \) is the set of all tangent vector fields on \( N \). A
statistical manifold \((N, \nabla, h)\) is called \(\alpha\)-conformally flat if \((N, \nabla, h)\) is locally \(\alpha\)-conformally equivalent to a flat statistical manifold [6].

For a pseudo-Riemannian manifold \((\tilde{N}, \tilde{h})\) and a submanifold \(N\) of \(\tilde{N}\), we call \((N, \nabla, h)\) a statistical submanifold of \((\tilde{N}, \tilde{h})\) if \((N, \nabla, h)\) is a statistical manifold, where \(\nabla\) is an affine connection on \(N\) and \(h\) the induced pseudo-Riemannian metric for \(\tilde{h}\). Let \(\tilde{\nabla}\) be an affine connection on \(\tilde{N}\). We denote by \(T_xN \oplus T_xN^\perp\) the orthogonal decomposition of \(T_xN\) with respect to \(\tilde{h}\), where \(T_xN\) and \(T_xN^\perp\) are the set of all tangent vectors at \(x\) on \(\tilde{N}\) and on \(N\), respectively. If \((\nabla_X Y)_x\) is the \(T_xN\)-component of \((\nabla_X Y)_x\) for \(X, Y \in \mathcal{X}(N)\) and an arbitrary \(x\) in \(N\), we call \((N, \nabla, h)\) the statistical submanifold realized in \((\tilde{N}, \tilde{\nabla}, \tilde{h})\).

Amari said that, if \((N, \nabla, h)\) is a statistical manifold for a Riemannian metric \(h\) and a submanifold \(N\) of \(\tilde{N}\), \((\tilde{N}, \tilde{\nabla}, \tilde{h})\) is a statistical manifold for the above induced connection \(\nabla\) and the induced metric \(h\) [1]. For a pseudo-Riemannian metric \(\tilde{h}\), \((N, \nabla, h)\) is a statistical manifold if \(h\) is non-degenerate. Then, through this paper, we call a statistical submanifold realized in a statistical manifold \((\tilde{N}, \tilde{\nabla}, \tilde{h})\), simply, a statistical submanifold of \((\tilde{N}, \tilde{\nabla}, \tilde{h})\).

In this paper we aim to prove the next theorems.

**Theorem 1.** Let \(M\) be a simply connected \(n\)-dimensional level surface of \(\varphi\) on an \((n + 1)\)-dimensional Hessian domain \((\Omega, D, \tilde{g} = \tilde{D}d\varphi)\) with a Riemannian metric \(\tilde{g}\), and suppose that \(n \geq 2\). If we consider \((\Omega, D, \tilde{g})\) a flat statistical manifold, \((M, D, g)\) is a \(1\)-conformally flat statistical submanifold of \((\Omega, D, \tilde{g})\), where we denote by \(D\) and \(g\) the connection and the Riemannian metric on \(M\) induced by \(\tilde{D}\) and \(\tilde{g}\).

**Theorem 2.** An arbitrary \(1\)-conformally flat statistical manifold of \(\text{dim } n \geq 2\) with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold of \(\text{dim } (n + 1)\).

We shall show a corollary of Theorem 1 with relation to projectively flat connections and dual-projectively flat connections in the last section.

### 2. Statistical Manifolds and Affine Differential Geometry

In this section, we study a level surface \(M\) of \(\varphi\) on an \((n + 1)\)-dimensional Hessian domain \((\Omega, \tilde{D}, \tilde{g})\), using affine differential geometry and the concept of statistical submanifold. A level surface \(M\) of \(\varphi\) is an \(n\)-dimensional submanifold of \(\Omega\) if and only if \(d\varphi_x \neq 0\) for all \(x \in M\). Henceforward, we suppose that \(n \geq 2\), that \(\tilde{g}\) is a Riemannian metric, and that \(d\varphi_x \neq 0\) for all \(x \in M\).

Let \(\tilde{E}\) be the gradient vector field on \(\Omega\) defined by

\[
\tilde{g}(\tilde{X}, \tilde{E}) = d\varphi(\tilde{X}) \quad \text{for } \tilde{X} \in \mathcal{X}(\Omega).
\]
Since \( g \) is positive definite and \( d\varphi_x \neq 0 \) for all \( x \in M \), \( d\varphi(\bar{E}) \) does not vanish on \( M \) and a vector \( \bar{E}_x \) is vertical to \( T_x M \) with respect to \( \bar{g} \), where \( T_x M \) is the set of all tangent vectors at \( x \) on \( M \). We set \( E = -d\varphi(\bar{E})^{-1}\bar{E} \) on \( M \). Then the vector field \( \bar{E} \) is transversal to \( M \), and so is \( E \).

Let \( \iota \) be a canonical immersion of \( M \) into \( \Omega \). For \( \bar{D} \) and an affine immersion \( (\iota, E) \), we can define the induced affine connection \( D^E \), the fundamental form \( g^E \), the shape operator \( S^E \) and the transversal connection form \( \tau^E \) on \( M \) by

\[
\begin{align*}
\bar{D}_X Y &= D^E_X Y + g^E(X, Y)E \\
\bar{D}_X E &= S^E(X) + \tau^E(X)E \quad \text{for } X, Y \in \mathcal{X}(M).
\end{align*}
\]

We denote by \((M, D, g)\) the statistical submanifold of \((\Omega, \bar{D}, \bar{g})\), considering \((\Omega, \bar{D}, \bar{g})\) a statistical manifold. Then the next holds.

**Lemma 2.1.** A statistical submanifold \((M, D, g)\) coincides with a manifold \((M, D^E, g^E)\) induced by an affine immersion \( (\iota, E) \), i.e.,

\[
D = D^E, \quad g = g^E \quad \text{on } M.
\]

Proof. Let \( D^E \) be the induced affine connection, \( g^E \) the fundamental form, \( S^E \) the shape operator, and \( \tau^E \) the transversal connection form, for \( \bar{D} \) and \( \bar{E} \). Since \( E_x \) and \( \bar{E}_x \) are vertical to \( T_x M \) for \( x \in M \) with respect to \( \bar{g} \), \( D = D^E = D^\bar{E} \) holds. From (1) and \( \bar{D}_X Y = D^E_X Y + g^E(X, Y)\bar{E} \), we have

\[
\begin{align*}
g^E &= -d\varphi(\bar{E})^{-1} g^E.
\end{align*}
\]

By [3] we know that

\[
\begin{align*}
g^E &= -d\varphi(\bar{E})^{-1} g.
\end{align*}
\]

From (3) and (4) \( g = g^E \) holds. \( \square \)

Since \( g \) is non-degenerate, so is \( g^E \). Then \( (\iota, E) \) is called a non-degenerate immersion. Moreover, the immersion \( (\iota, E) \) has the following property.

**Lemma 2.2.** An affine immersion \( (\iota, E) \) is equiaffine, i.e.,

\[
\tau^E = 0 \quad \text{on } M.
\]

Proof. We have

\[
\begin{align*}
\tau^E &= (d \log |d\varphi(\bar{E})|)(X)
\end{align*}
\]
by [3]. Calculating the right-hand side of (5), we have

$$\tau^E = d\varphi(\bar{E})^{-1}X(d\varphi(\bar{E})).$$

Thus, we obtain

$$\bar{D}_X E = -\bar{D}_x (d\varphi(\bar{E})^{-1})E$$
$$= -X(d\varphi(\bar{E})^{-1})\bar{E} - d\varphi(\bar{E})^{-1}D_x \bar{E}$$
$$= d\varphi(\bar{E})^{-2} X(d\varphi(\bar{E}))\bar{E} - d\varphi(\bar{E})^{-1}\{S^E(X) + \tau^E(X)\bar{E}\}$$
$$= -d\varphi(\bar{E})^{-1}S^E(X).$$

Hence $S^E = -d\varphi(\bar{E})^{-1}S^E$ and $\tau^E = 0$ hold.

It is known that the structure induced by a non-degenerate equiaffine immersion is the statistical manifold structure. Conversely, Kurose proved the next proposition.

**Proposition 2.3 ([6]).** A simply connected statistical manifold can be realized in $\mathbb{A}^{n+1}$ by a non-degenerate equiaffine immersion if and only if it is 1-conformally flat. Such an immersion is uniquely determined up to affine transformations of $\mathbb{A}^{n+1}$.

Proposition 2.3 can be proved by projectively flatness of the dual connection of a given connection [2]. Finally, let us show Theorem 1.

Proof of Theorem 1. By Lemma 2.2 and Proposition 2.3 a statistical manifold $(M, D^E, g^E)$ is 1-conformally flat. Thus Theorem 1 holds by Lemma 2.1.

3. Proof of Theorem 2

Let $(N, \nabla, h)$ be a 1-conformally flat flat statistical manifold of $\dim n \geq 2$ with a Riemannian metric $h$. By Proposition 2.3 $(N, \nabla, h)$ can be realized by a non-degenerate equiaffine immersion. We denote by $(i, E)$ a non-degenerate equiaffine immersion into $\mathbb{A}^{n+1}$ which realizes $(N, \nabla, h)$. Then we can immerse $(N, \nabla, h)$ into a flat statistical manifold as the next lemma.

**Lemma 3.1.** For a simply connected open subset $U$ of $N$ and a small $\varepsilon > 0$, we define a function $\phi$ on $\bar{U} = \bigcup_{q \in U} \{i(q) \oplus (-\varepsilon, \varepsilon) \cdot E_q\}$ by

$$\phi(p) = e^{-t} \text{ for } p = i(p_0) + tE_{p_0}, \quad p_0 \in U, \quad t \in (-\varepsilon, \varepsilon).$$

Then $(U, \nabla, h)$ is a statistical submanifold of a flat statistical manifold $(\bar{U}, \bar{D}, \bar{D}d\phi)$. 
Proof. For $X, Y \in \mathcal{X}(U)$, we have
\[ d\phi(X) = 0, \quad d\phi(E) = -1, \]
and
\[
(\tilde{D}_X d\phi)(Y) = X(d\phi(Y)) - d\phi(\tilde{D}_X Y) \\
= -d\phi(\nabla_X Y + h(X, Y)E) \\
= -h(X, Y)d\phi(E) \\
= h(X, Y).
\]
Thus, $(U, \nabla, h)$ is a submanifold of $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$.

We also denote by $E$ a vector field on $\tilde{U}$ whose value is $E_{p_0}$ at $p = t(p_0) + tE_{p_0}$. On $\iota(U)$ we have
\[ E(d\phi(E)) = 1, \quad \tilde{D}_E E = 0, \]
and
\[
(\tilde{D}_E d\phi)(E) = E(d\phi(E)) - d\phi(\tilde{D}_E E) = 1.
\]
Thus $(\tilde{D}d\phi)_{(p_0)}$ is positive definite for $p_0 \in U$. From continuity of a function $\phi$, $\tilde{D}d\phi$ is a Riemannian metric on $\tilde{U}$ for a small $\varepsilon$. Hence $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$ is a flat statistical manifold.

4. Dual-Projectively Flat Connections

In this section, we describe dual-projectively flatness of an affine connection $D$ on a level surface $M$ and projectively flatness of the dual-connection $D'$ of $D$.

Let $(N, h)$ be a pseudo-Riemannian manifold. Torsion free affine connections $\nabla$ and $\tilde{\nabla}$ on $N$ are projectively equivalent if there exists a 1-form $\kappa$ such that
\[ \tilde{\nabla}_X Y = \nabla_X Y + \kappa(X)Y + \kappa(Y)X \]
for $X, Y \in \mathcal{X}(N)$. An affine connection $\nabla$ is called projectively flat if $\nabla$ is locally projectively equivalent to a flat affine connection. Torsion free affine connections $\nabla$ and $\tilde{\nabla}$ on $N$ are dual-projectively equivalent if there exists a 1-form $\kappa$ such that
\[ h(\tilde{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \kappa(Z)h(X, Y) \]
for $X, Y, Z \in \mathcal{X}(N)$. An affine connection $\nabla$ is called dual-projectively flat if $\nabla$ is locally dual-projectively equivalent to a flat affine connection [4].
For a statistical manifold \((N, \nabla, h)\) there exists the torsion free affine connection \(\nabla'\) on \(N\) such that

\[
Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z)
\]

The connection \(\nabla'\) is said to be the dual connection of \(\nabla\), and \((N, \nabla', h)\) the dual statistical manifold of \((N, \nabla, h)\). If \((N', \nabla', h')\) and \((N'', \nabla'', h'')\) are statistical submanifolds of \((N, \nabla, h)\) and \((N, \nabla', h)\), respectively, \((N^s, \nabla^s, h^s)\) is the dual statistical manifold of \((N', \nabla', h')\) [1].

Statistical manifolds \((N, \nabla, h)\) and \((N, \tilde{\nabla}, \tilde{h})\) are \(\alpha\)-conformally equivalent if and only if the dual statistical manifolds \((N, \nabla', h)\) and \((N, \tilde{\nabla}', \tilde{h})\) are \((-\alpha)\)-conformally equivalent. Especially, a statistical manifold \((N, \nabla, h)\) is \(1\)-conformally flat if and only if the dual statistical manifold \((N, \nabla', h)\) is \((-1)\)-conformally flat [6].

Moreover, Kurose showed that, by Proposition 9.1 in [8], a statistical manifold \((N, \nabla', h)\) is \((-1)\)-conformally flat if and only if \(\nabla'\) is a projectively flat connection with symmetric Ricci tensor, and that

**Proposition 4.1** ([6]). A statistical manifold \((N, \nabla, h)\) is \(1\)-conformally flat if and only if the dual connection \(\nabla'\) is a projectively flat connection with symmetric Ricci tensor.

On projectively flatness, Ivanov described the next proposition on section 2 in [4].

**Proposition 4.2** ([4]). A statistical manifold \((N, \nabla, h)\) is \(1\)-conformally flat if and only if \(\nabla\) is a dual-projectively flat connection with symmetric Ricci tensor.

For a level surface of a Hessian domain, we obtain the next corollary of Theorem 1 by Proposition 4.1 and 4.2.

**Corollary 4.3.** Let \(M\) be a simply connected \(n\)-dimensional level surface of \(\phi\) on an \((n+1)\)-dimensional Hessian domain \((\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\phi)\) with a Riemannian metric \(\tilde{g}\), and suppose that \(n \geq 2\). Let \((M, D, g)\) be a statistical submanifolds of \((\Omega, \tilde{D}, \tilde{g})\) and \(D'\) the dual connection of \(D\). Then, \(D\) is a dual-projectively flat connection with symmetric Ricci tensor and \(D'\) is a projectively flat connection with symmetric Ricci tensor.

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**References**

[2] F. Dillen, K. Nomizu and L. Vrancken: *Conjugate connections and Radon's theorem in affine*
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