Yagi, A.
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ON A CLASS OF LINEAR EVOLUTION EQUATIONS OF "HYPERBOLIC" TYPE IN REFLEXIVE BANACH SPACES

ATSUSHI YAGI

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1. Introduction

We are concerned with the Cauchy problem for linear evolution equations

$$du/dt+A(t)u=f(t),\ 0\leqslant t\leqslant T, \tag{1.1}$$

of "hyperbolic" type in a Banach space E. "Hyperbolic" type means that the linear opeators $-A(t)$ are the infinitesimal generators of C_0-semigroups on E. In this paper, as T. Kato [1], [2], [3] and [4], we deal with the class that there exists a certain dense linear manifold F in E contained in all the domains $D(A(t))$.

Roughly speaking, our assumptions consist of the reflexivity of E, strong continuity in t of $A(t)$ and its dual $A(t)'$, the stability of $\{A(t)\}$ on E and F (see §2) and the existence of a mollifying operator for $\{A(t)\}$ (see §3). Those are closely related among others to [3]. The main difference lies in weakening the smoothness condition of $A(t)$ in t instead of adding the reflexivity of E. In [3] the norm-continuity of $A(t)$: $F\rightarrow E$ is assumed.

In the proof of our theorem essential use is made of the energy estimates as S. Mizohata [7]. Hence the proof is quite different from [3] in which the integral equations take effect. The author wonders if, even under such a weak smoothness condition of $A(t)$, one can prove *a priori* the strong convergence of $U_n(t, s)$ in §4.

We note here some notations and terminology used in the sequel. The norm of a Banach space E is denoted by $||\cdot||_E$. The inner product by $(\cdot, \cdot)_E$, if E is Hibert. E' is the dual space of E, and $\langle\cdot, \cdot\rangle$ is the scalar product of E' and E. E_w is the locally convex space endowed with the weak topology. Let F be another Banach space. $\mathcal{L}(E; F)$ is the Banach space of all bounded linear operators of E to F with the uniform norm $||\cdot||_{F,E}$, and $\mathcal{L}_s(E; F)$ is the locally convex space with the strong topology. We will abbreviate $\mathcal{L}(E; E)$ as $\mathcal{L}(E)$, $||\cdot||_{E,E}$ as $||\cdot||_E$ and so forth. For a linear operator A of E and a linear manifold $G\subset D(A)$ in E, $A_{|G}$ is the restriction of A to G. A' is the dual of $A\in\mathcal{L}(E; F)$. A^* is the ajoint of A, if A is a densely defined linear operator in a Hilbert space. For an interval $[a, b]$ and a Banach space E, $\mathcal{L}_p([a, b]; E)$, $1\leqslant p<+\infty$, is the Banach space of all measurable functions f such that $\|f(\cdot)\|_E^p$ is integrable on $[a, b]$. $\mathcal{W}_p([a, b]; E)$ is the set of all absolutely continuous functions f such that

$$f(t)=f(a)+\int_a^t g(\tau)d\tau,\ a\leqslant t\leqslant b\,,$$

with some $g\in\mathcal{L}_p([a, b]; E)$. For a metric space D and a locally convex space E, $C(D; E)$ is the set of all continuous mappings of D to E. $C^1([a, b]; E)$ is the set of all continuously differentiable functions. $\mathcal{B}^{1,0}(R^n\times[a, b])$ is the set of all complex valued functions defined on $(x, t)\in R^n\times[a, b]$ which are uniformly bounded and continuous in (x, t), and are differentiable in x with uniformly bounded and continuous derivatives.

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2. Stable families

Let E be a Banach space. In this section $\{A(t)\}_{0\leqslant t\leqslant T}$ denotes a family of densely defined, closed linear operators in E. Following [3], we say $\{A(t)\}$ is stable if there are an open interval (β, ∞) contained in all $\rho(-A(t))$ and a constant M such that

$$\|\prod_{j=1}^{k}(\lambda+A(t_j))^{-1}\|_E\leqslant M(\lambda-\beta)^{-k},\ \lambda>\beta\,,$$

for any finite family $\{t_j\}_{1\leqslant j\leqslant k}$ with $0\leqslant t_1\leqslant\cdots\leqslant t_k\leqslant T$. The product $\prod$ is time-ordered. If $\{A(t)\}$ is such a family, then we write simply

$$\{A(t)\}\subset\mathcal{G}(E; M, \beta)\,.$$

A few criteria for stability have been given in §3 [3]. Specializing one of them in the case of E being Hilbert, we state the following:

Proposition 2.1. *Let E be a Hilbert space and $\{N(t)\}_{0\leqslant t\leqslant T}$ be a family of isomorphisms of E to itself such that*

$$\|N(t)N(s)^{-1}\|_E\leqslant e^{C|t-s|},\ 0\leqslant s,\ t\leqslant T\,, \tag{2.1}$$

with some constant C. Assume that

$$\bar{A}(t)=N(t)A(t)N(t)^{-1},\ 0\leqslant t\leqslant T\,, \tag{2.2}$$

are almost anti-symmetric (namely $D(\bar{A}(t))=D(\bar{A}(t)^*)$, $\bar{A}(t)+\bar{A}(t)^*\in\mathcal{L}(E)$) and that there is a constant β such that

$$\|\bar{A}(t)+\bar{A}(t)^*\|_E\leqslant 2\beta,\ 0\leqslant t\leqslant T\,.$$

Then

$$\{A(t)\} \subset \mathcal{G}(E;\, M,\, \beta)$$

with $M = \|N(T)^{-1}\|_E e^{CT} \|N(0)\|_E$.

Proof. We have for each $\lambda > \beta$ and $t \in [0, T]$

$$\|(\lambda + \bar{A}(t))y\|_E \geqslant (\lambda - \beta)\|y\|_E, \quad y \in D(\bar{A}(t)); \tag{2.3}$$

$$\|(\lambda + \bar{A}(t)^*)y\|_E \geqslant (\lambda - \beta)\|y\|_E, \quad y \in D(\bar{A}(t)^*). \tag{2.4}$$

(2.3) and (2.4) imply

$$(\beta, \infty) \subset \rho(-\bar{A}(t)), \; 0 \leqslant t \leqslant T, \tag{2.5}$$

and

$$\|(\lambda + \bar{A}(t))^{-1}\|_E \leqslant (\lambda - \beta)^{-1}, \; \lambda > \beta. \tag{2.6}$$

Whereas by (2.2) and (2.5) we obtain

$$(\beta, \infty) \subset \rho(-A(t)), \; 0 \leqslant t \leqslant T,$$

and

$$(\lambda + A(t))^{-1} = N(t)^{-1}(\lambda + \bar{A}(t))^{-1} N(t). \tag{2.7}$$

(2.1), (2.6) and (2.7), then, show the desired result.

3. Mollifying operators

Let E and F be Banach spaces such that F is densely and continuously embedded in E. We consider families of bounded operators $\rho \in \mathcal{L}(E;\, F)$.

A faimly $\{\rho_n(t)\}_{0 \leqslant t \leqslant T, n=1,2,\dots}$ is said to be a mollifying operator if ρ_n belongs to

$$\mathcal{C}([0,\, T];\, \mathcal{L}_s(E;\, F)) \cap \mathcal{C}^1([0,\, T];\, \mathcal{L}_s(E)) \tag{3.1}$$

for each n, and

$$\lim_{n \to \infty} \rho_n(t) = I \quad \text{in } \mathcal{L}_s(F), \tag{3.2}$$

for each $t \in [0, T]$.

Let $\{A(t)\}_{0 \leqslant t \leqslant T}$ be a family of $A(t) \in \mathcal{L}(F;\, E)$ and $\{\rho_n(t)\}$ be a mollifying operator. We define the family $\{C_n(t)\}_{0 \leqslant t \leqslant T, n=1,2,\dots}$ of elements of $\mathcal{L}(F;\, E)$ by

$$C_n(t) = A(t)\rho_n(t) - \rho_n(t)A(t) + d\rho_n(t)/dt.$$

$\{\rho_n(t)\}$ is said to be a mollifying operator for the family $\{A(t)\}$ if C_n belongs to

$$\mathcal{C}([0,\, T];\, \mathcal{L}_s(E)) \cap \mathcal{C}([0,\, T];\, \mathcal{L}_s(F)) \tag{3.3}$$

for each n, which implies $C_n(t) \in \mathcal{L}(E) \cap \mathcal{L}(F)$, and

$$\lim_{n \to \infty} C_n(t) = C(t) \quad \text{in } \mathcal{L}_s(F) \tag{3.4}$$

for each $t \in [0, T]$ with

$$\sup_{0 \leqslant t \leqslant T, n=1,2,\cdots} \|C_n(t)\|_F < \infty . \tag{3.5}$$

The following example shows that the mollifying operator can be interpreted as an abstract version of Friedrichs' mollifier for the operator theory.

EXAMPLE 3.1. Let $E = \mathcal{L}_2(R^m)$, $F = \mathcal{H}^1(R^m)$ and $\{\rho_\varepsilon *\}_{\varepsilon > 0}$ be Friedrichs' mollifier. Then $\{\rho_n(t)\}$ defined by

$$\rho_n(t)u = \rho_{1/n} * u, \quad u \in E ,$$

becomes a mollifying operator for any family $\{A(t)\}_{0 \leqslant t \leqslant T}$ of first order differential operators

$$A(t)u = \sum_{j=1}^{m} a_j(x, t)\partial u/\partial x_j + b(x, t)u, \quad u \in F ,$$

with the coefficients

$$a_j, b \in \mathcal{B}^{1,0}(R^m \times [0, T]) .$$

There is a somewhat interesting way to construct mollifying operators (cf. Theorem 6.1 [3]).

Proposition 3.2. *Let $\{S(t)\}_{0 \leqslant t \leqslant T}$ be a family of isomorphisms of F to E such that*

$$S \in C^1([0, T]; \mathcal{L}_s(F; E))$$

and that, when we regard them as closed operators in E, we can choose a sequence $\{\lambda_n\}_{n=1,2,\cdots}$, $0 < \lambda_1 < \lambda_2 < \cdots \to \infty$, of $\bigcap_{0 \leqslant t \leqslant T} \rho(S(t))$ satisfying the estimate

$$\|(\lambda_n - S(t))^{-1}\|_E \leqslant C/\lambda_n, \; n = 1, 2, \cdots ,$$

with some constant C independent of t. Assume

$$S(t)A(t)S(t)^{-1} \supset A(t) + B(t), \; 0 \leqslant t \leqslant T , \tag{3.6}$$

with some $B \in C([0, T]; \mathcal{L}_s(E))$, then $\{\rho_n(t)\}_{0 \leqslant t \leqslant T, n=1,2,\cdots}$ defined by

$$\rho_n(t) = (1 - \lambda_n^{-1} S(t))^{-1}$$

is a mollifying operator for $\{A(t)\}$.

Proof. Actually we can observe that

$$\rho_n \in C^1([0, T]; \mathcal{L}_s(E; F)) ,$$

$$d\rho_n(t)/dt = \lambda_n^{-1}(1 - \lambda_n^{-1}S(t))^{-1}(dS(t)/dt)(1 - \lambda_n^{-1}S(t))^{-1} . \tag{3.7}$$

Since (3.6) implies

$$S(t)A(t)z = A(t)S(t)z + B(t)S(t)z, \quad z \in S(t)^{-1}(F),$$

and we have

$$S(t)^{-1}(F) = (1-\lambda_n^{-1}S(t))^{-1}(F) = \rho_n(t)\,(F),$$

we can write for $y \in F$

$$\begin{aligned}&(A(t)\rho_n(t) - \rho_n(t)A(t))y \\ &= \rho_n(t)\{(1-\lambda_n^{-1}S(t))A(t) - A(t)\,(1-\lambda_n^{-1}S(t))\}\,\rho_n(t)y \\ &= -\lambda_n^{-1}\rho_n(t)B(t)S(t)\rho_n(t)y\,. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we can conclude

$$\begin{aligned}&C_n \in \mathcal{C}([0, T]; \mathcal{L}_s(E; F)), \\ &\lim_{n\to\infty} C_n(t) = 0 \ \text{ in } \ \mathcal{L}_s(F).\end{aligned}$$

4. Construction of the evolution operator

Theorem 4.1. *Let E and F be reflexive Banach spaces such that F is densely and continuously embedded in E, S be an isomorphism of F to E, and $\{A(t)\}_{0\leqslant t\leqslant T}$ be a family of closed linear operators in E with all $D(A(t))$ containing F. Assume*

(I) $A(\cdot)_{|F} \in \mathcal{C}([0, T]; \mathcal{L}_s(F; E))$,
$(A(\cdot)_{|F})' \in \mathcal{C}([0, T]; \mathcal{L}_s(E'; F'))$.

(II) $\{A(t)\} \subset \mathcal{G}(E; M, \beta)$.

(III) *$SA(t)S^{-1}$ are all densely defined,*
$\{SA(t)S^{-1}\} \subset \mathcal{G}(E; \tilde{M}, \tilde{\beta})$.

(IV) *There is a mollifying operator $\{\rho_n(t)\}$ for $\{A(t)_{|F}\}$.*

Then we can construct the evolution operator $\{U(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ with the following properties:

a) $U \in \mathcal{C}(\{(t, s); 0\leqslant s\leqslant t\leqslant T\}; \mathcal{L}_s(E))$,
$\|U(t, s)\|_E \leqslant Me^{\beta(t-s)}$.

b) $U \in \mathcal{C}(\{(t, s); 0\leqslant s\leqslant t\leqslant T\}; \mathcal{L}_s(F; F_w))$;
$U(\cdot, s) \in \mathcal{C}([s, T]; \mathcal{L}_s(F)),\ 0\leqslant s< T$;
$\|U(t, s)\|_F \leqslant \tilde{M}\|S^{-1}\|_{F,E}\|S\|_{E,F}e^{\tilde{\beta}(t-s)}$.

c) $U(t, s)U(s, r) = U(t, r),\ 0\leqslant r\leqslant s\leqslant t\leqslant T$;
$U(s, s) = I,\ 0\leqslant s\leqslant T$.

d) $U(\cdot, s) \in \mathcal{C}^1([s, T]; \mathcal{L}_s(F; E)),\ 0\leqslant s< T$;
$(\partial/\partial t)U(t, s) = -A(t)U(t, s)$.

e) $U(t, \cdot) \in \mathcal{C}^1([0, t]; \mathcal{L}_s(F; E)),\ 0< t\leqslant T$;

$$(\partial/\partial s)U(t, s) = U(t, s)A(s) .$$

Proof. For the partition $\{t_k\}_{0\leqslant k\leqslant n}$, $t_k=kT/n$, of the interval $[0, T]$ we define the approximation $\{A_n(t)\}_{0\leqslant t\leqslant T}$ to $\{A(t)\}$ by

$$A_n(t) = A(t_k),\ t_k\leqslant t<t_{k+1} ,$$

and the evolution operator $\{U_n(t, s)\}_{0\leqslant s\leqslant t\leqslant T}$ for $\{A_n(t)\}$ by

$$U_n(t, s) = \exp(-(t-t_j)A(t_j)) \cdots \exp(-(t_{i+1}-s)A(t_i)) ,$$
$$t_i\leqslant s<t_{i+1} \cdots t_j\leqslant t<t_{j+1} .$$

Putting

$$\tilde{A}(t) = SA(t)S^{-1},\ 0\leqslant t\leqslant T ,$$

we approximate $\{\tilde{A}_n(t)\}$ to $\{\tilde{A}(t)\}$ in the similar way and define the evolution operator $\{V_n(t, s)\}$

$$V_n(t, s) = \exp(-(t-t_j)\tilde{A}(t_j)) \cdots \exp(-(t_{i+1}-s)\tilde{A}(t_i)) ,$$
$$t_i\leqslant s<t_{i+1} \cdots t_j\leqslant t<t_{j+1} .$$

By (II) and (III) we have

$$||U_n(t, s)||_E\leqslant Me^{\beta(t-s)},\ ||V_n(t, s)||_E\leqslant \tilde{M}e^{\tilde{\beta}(t-s)} . \tag{4.1}$$

Lemma 4.2. *For each* $t\in[0, T]$

$$\exp(-rA(t))\supset S^{-1}\exp(-r\tilde{A}(t))S,\ r\geqslant 0 .$$

For $y\in D(\tilde{A}(t))$

$$(\partial/\partial s)\exp(-(r-s)A(t))S^{-1}\exp(-s\tilde{A}(t))y = 0,\ 0\leqslant s\leqslant r ,$$

therefore

$$\exp(-rA(t))S^{-1}y = S^{-1}\exp(-r\tilde{A}(t))y . \tag{4.2}$$

Since $D(\tilde{A}(t))$ is dense in E, (4.2) holds for any $x\in E$, hence the result is obtained.

The preceding lemma asserts that

$$\exp(-rA(t))\,(F)\subset F,\ r\geqslant 0 ,$$

and $\{\exp(-rA(t))\}_{r\geqslant 0}$ is a C_0-semigroup on F for each $t\in[0, T]$. Thus we have obtained

$$U_n(t, s)\in\mathcal{L}(F),\ ||U_n(t, s)||_F\leqslant M'e^{\tilde{\beta}(t-s)} , \tag{4.3}$$

where $M'=\tilde{M}||S^{-1}||_{F,E}||S||_{E,F}$.

Lemma 4.3. *Let* $s \in [0, T)$ *and* $f \in \mathcal{L}_1([s, T]; F) \cap C([s, T]; E)$. *If* $u \in C([s, T]; F) \cap C^1([s, T]; E)$ *is the solution of*

$$du/dt + A(t)u = f(t), \; s \leqslant t \leqslant T, \tag{4.4}$$

then for each $t \in [s, T]$

$$u(t) = \lim_{n\to\infty} \{U_n(t, s)u(s) + \int_s^t U_n(t, \tau) f(\tau) d\tau\} \tag{4.5}$$

both in E *and* F_w, *and hence*

$$||u(t)||_F \leqslant M' \{e^{\tilde{\beta}(t-s)} ||u(s)||_F + \int_s^t e^{\tilde{\beta}(t-\tau)} ||f(\tau)||_F d\tau\} .$$

For $\tau \in [s, t] - \{t_k\}$ we have

$$(\partial/\partial\tau) U_n(t, \tau) u(\tau) = U_n(t, \tau) \{A_n(\tau) - A(\tau)\} u(\tau) + U_n(t, \tau) f(\tau) . \tag{4.6}$$

Since $U_n(t, \cdot)u(\cdot)$ is continuous in $\tau \in [s, t]$, integrating (4.6) on $[s, t]$, we get

$$u(t) - U_n(t, s)u(s) - \int_s^t U_n(t, \tau) f(\tau) d\tau = \int_s^t U_n(t, \tau) \{A_n(\tau) - A(\tau)\} u(\tau) d\tau .$$

According to Lebesgue's dominated convergence theorem, (I) and (4.1) imply

$$\lim_{n\to\infty} ||\int_s^t U_n(t, \tau) \{A_n(\tau) - A(\tau)\} u(\tau) d\tau||_E = 0 ,$$

which shows (4.5) in E. Next we have by (4.3)

$$||U_n(t, s)u(s) + \int_s^t U_n(t, \tau) f(\tau) d\tau||_F \leqslant M' \{e^{\tilde{\beta}(t-s)} ||u(s)||_F + \int_s^t e^{\tilde{\beta}(t-\tau)} ||f(\tau)||_F d\tau\} , \tag{4.7}$$

therefore there is a subsequence $\{n'\}$ such that

$$y = \lim_{n'\to\infty} \{U_{n'}(t, s)u(s) + \int_s^t U_{n'}(t, \tau) f(\tau) d\tau\} \quad \text{in } F_w$$

with some $y \in F$. But, since the convergence in F_w implies that in E_w, y must be equal to $u(t)$. Thus we conclude

$$u(t) = \lim_{n\to\infty} \{U_n(t, s)u(s) + \int_s^t U_n(t, \tau) f(\tau) d\tau\} \quad \text{in } F_w .$$

The last inequality is the immediate consequence of (4.7) and (4.5) in F_w.

Now, let $s \in [0, T)$ and $p \in (1, \infty)$ be fixed and define

$$X = \mathcal{L}_p([s, T]; E),\ Y = \mathcal{L}_p([s, T]; F).$$

It is known that

$$X' = \mathcal{L}_q([s, T]; E'),\ Y' = \mathcal{L}_q([s, T]; F')$$

with $p^{-1}+q^{-1}=1$, so that X and Y are also reflexive. For a fixed $y \in F$, we put

$$u_n(t) = U_n(t, s)y,\ s \leqslant t \leqslant T.$$

We have evidently

$$du_n/dt \in X,\quad \sup_n \|du_n/dt\|_X < \infty;$$
$$u_n \in Y,\quad \sup_n \|u_n\|_Y < \infty. \tag{4.8}$$

Therefore there is a subsequence $\{n'\}$ such that

$$v = \lim_{n' \to \infty} u_{n'} \quad \text{in } Y_w, \tag{4.9}$$
$$w = \lim_{n' \to \infty} du_{n'}/dt \quad \text{in } X_w. \tag{4.10}$$

For any $g \in E'$ let $G \in X'$ be the step function

$$G(\tau) = \begin{cases} g, & s \leqslant \tau \leqslant t \\ 0, & t < \tau \leqslant T, \end{cases}$$

then, since

$$\langle du_n/dt, G\rangle = \langle \int_s^t (du_n(\tau)/d\tau)d\tau, g\rangle$$
$$= \langle u_n(t) - y, g\rangle,$$

we have

$$\lim_{n' \to \infty} \langle u_{n'}(t), g\rangle = \langle y, g\rangle + \langle w, G\rangle$$
$$= \langle y + \int_s^t w(\tau)d\tau, g\rangle,$$

namely

$$y + \int_s^t w(\tau)d\tau = \lim_{n' \to \infty} u_{n'}(t) \quad \text{in } E_w,\ s \leqslant t \leqslant T. \tag{4.11}$$

Putting

$$u(t) = y + \int_s^t w(\tau)d\tau,\ s \leqslant t \leqslant T, \tag{4.12}$$

we will prove that u is a solution of (4.4) with $f=0$.

Clearly we see

$$u \in \mathcal{W}_p([s, T]; E) \tag{4.13}$$

on account of

$$du/dt = w \in X\,. \tag{4.14}$$

Since

$$u = \lim_{n' \to \infty} u_{n'} \quad \text{in } X_w$$

follows from (4.1) and (4.11), and (4.9) implies

$$v = \lim_{n' \to \infty} u_{n'} \quad \text{in } X_w\,,$$

we obtain $u=v \in Y$. Hence we can define $Au \in X$ by

$$(Au)(t) = (A(t)_{|F})u(t), \text{ a.e. } t \in [s, T]\,.$$

For any $G \in X'$

$$\langle A_{n'}u_{n'} - Au, G\rangle$$
$$= \langle u_{n'}, \{(A_{n'} - A)_{|F}\}'G\rangle + \langle u_{n'} - u, (A_{|F})'G\rangle\,. \tag{4.15}$$

According to the dominated convergence theorem, (I) and (4.3) imply that the first term of (4.15) tends to 0 as $n' \to \infty$. The second also goes to 0 by (4.9). Therefore we get

$$\lim_{n' \to \infty} \langle A_{n'}u_{n'}, G\rangle = \langle Au, G\rangle, G \in X'\,,$$

or equivalently

$$Au = \lim_{n' \to \infty} A_{n'}u_{n'} \quad \text{in } X_w\,. \tag{4.16}$$

Letting $n' \to \infty$ in the equality

$$du_{n'}/dt = -A_{n'}u_{n'} \quad \text{in } X$$

in view of (4.10), (4.14) and (4.16), we obtain

$$du/dt = -Au \quad \text{in } X\,. \tag{4.17}$$

Next, we operate the mollifying operator $\rho_n(t)$ to $u(t)$. Obviously

$$\rho_n u = \rho_n(\cdot)u(\cdot) \in C([s, T]; F)\,.$$

By (3.1) and (4.13) $\rho_n u$ is, as an E valued function, absolutely continuous and

$$d\rho_n(t)u(t)/dt = -A(t)\rho_n(t)u(t) + C_n(t)u(t)\,, \tag{4.18}$$
$$\text{a.e. } t \in [s, T]\,.$$

Since the right hand side of (4.18) is continuous by (3.3), we conclude $\rho_n u \in C^1([s, T]; E)$. Noting

$$C_n u \in Y \subset \mathcal{L}_1([s, T]; F),$$

we can apply Lemma 4.3 to $(\rho_n - \rho_m)u$, and obtain

$$\begin{aligned} \|\{\rho_n(t)-\rho_m(t)\}u(t)\|_F \leqslant M'\{e^{\tilde{\beta}(t-s)}\|\{\rho_n(s)-\rho_m(s)\}y\|_F \\ +\int_s^t e^{\tilde{\beta}(t-\tau)}\|\{C_n(\tau)-C_m(\tau)\}u(\tau)\|_F d\tau\}. \end{aligned} \tag{4.19}$$

(4.19) shows by (3.2), (3.4) and (3.5) that $\{\rho_n u\}_{n=1,2,\dots}$ is a Cauchy sequence in $C([s, T]; F)$ with the uniform topology, from which we can deduce

$$u(t) = \lim_{n\to\infty} \rho_n(t)u(t) \quad \text{in } F, s\leqslant t\leqslant T.$$

Hence u belongs to $C([s, T]; F)$, moreover to $C^1([s, T]; E)$ on account of (4.12), (4.14) and (4.17). Thus we have proved

$$\begin{cases} du(t)/dt = -A(t)u(t), s\leqslant t\leqslant T, \\ u(s) = y. \end{cases} \tag{4.20}$$

Lemma 4.4. *For any* $0\leqslant s\leqslant t\leqslant T$ *and* $x\in E$, $\{U_n(t, s)x\}_{n=1,2,\dots}$ *converges in* E.

Let u be the solution of (4.20) whose existence was already established. Lemma 4.3 claims

$$u(t) = \lim_{n\to\infty} U_n(t, s)y \quad \text{in } E, y\in F, \tag{4.21}$$

in particular the existence of the limit. The convergence for general $x\in E$ follows from (4.1) and the density of F in E.

We define the linear operator $U(t, s)$, $0\leqslant s\leqslant t\leqslant T$, of E

$$U(t, s)x = \lim_{n\to\infty} U_n(t, s)x \quad \text{in } E, x\in E.$$

It suffices to prove that $\{U(t, s)\}$ has the properties a)$\sim$e). First of all, d) is obtained by (4.21). The inequality in Lemma 4.3 asserts

$$U(t, s)\in \mathcal{L}(F), \|U(t, s)\|_F\leqslant M' e^{\tilde{\beta}(t-s)}. \tag{4.22}$$

c) is clear from (4.1). (4.1) implies also

$$U(t, s)\in \mathcal{L}(E), \|U(t, s)\|_E\leqslant M e^{\beta(t-s)}. \tag{4.23}$$

Letting $n\to\infty$ in

$$U_n(t, s)y = y-\int_s^t U_n(t, \sigma)A_n(\sigma)y d\sigma, \quad y\in F,$$

in view of

$$U(t, \cdot)A(\cdot)y\in \mathcal{L}_1([0, t]; E),$$

we get

$$U(t, s)y = y - \int_s^t U(t, \sigma)A(\sigma)y d\sigma , \tag{4.24}$$

which shows e). By d) and e) we see

$$U(\cdot, \cdot)y \in C(\{(t, s); 0 \leqslant s < t \leqslant T\}; E), \quad y \in F ,$$

on the other hand (4.24) gives the continuity at $t=s$, therefore a) follows from (4.23) and the density of F. It is not difficult to verify the first assertion of b) by a), (4.22) and the reflexivity of F. The rest of b) have already been proved above.

We give here a formulation on the Cauchy problem for (1.1) without its standard proof.

Theorem 4.5. *Let $\{A(t)\}_{0 \leqslant t \leqslant T}$ satisfy the assumptions made in Theorem* 4.1. *Then for any $y \in F$ and any $f \in C([0, T]; F)$, the problem*

$$\begin{cases} du/dt + A(t)u = f(t), \; 0 \leqslant t \leqslant T , \\ u(0) = y \end{cases}$$

has the unique solution in $C([0, T]; F) \cap C^1([0, T]; E)$, which is given by

$$u(t) = U(t, 0)y + \int_0^t U(t, \tau)f(\tau)d\tau, \; 0 \leqslant t \leqslant T .$$

5. Application to hyperbolic systems

As an application of our preceding theorems we consider the Cauchy problem for a system of linear partial differential equations

$$\begin{cases} \partial u/\partial t + \sum_{j=1}^{n} a_j(x, t)\partial u/\partial x_j + b(x, t)u = f(x, t), \; (x, t) \in R^n \times [0, T] , \\ u(x, 0) = \phi(x) . \end{cases} \tag{5.1}$$

Here $u=(u_1, \cdots, u_m)^t$ is an m-row vector of unknown functions of (x, t), $a_j(x, t)$ and $b(x, t)$ are $m \times m$ matrix functions. We assume

$$a_j, b \in \mathcal{B}^{1,0}(R^n \times [0, T]) .$$

Regarding (5.1) as an evolution equation in the Hilbert space $E = \mathcal{L}_2(R^n)$, we define $\{A(t)\}_{0 \leqslant t \leqslant T}$ as follows:

$$\begin{cases} D(A(t)) = \{u \in E; \sum_{j=1}^{n} a_j \partial u/\partial x_j \in E\} , \\ A(t)u = \sum_{j=1}^{n} a_j \partial u/\partial x_j + bu . \end{cases}$$

$\mathcal{H}^1(R^n)$ is adopted as F. Obviously $A(t)$ are closed linear operators in E with all $D(A(t))$ containing F.

Theorem 5.1. *Let* $\{s(t)\}_{0\leqslant t\leqslant T}$ *be a family of* $m\times m$ *Hermitian symbols of singular integral operators of type* C_1^∞ *such that*

$$p(s(t)-s(t'))\leqslant C_p|t-t'|,\ 0\leqslant t,\ t'\leqslant T, \tag{5.2}$$

for any seminorm p *of* C_1^∞, *and that*

$$(s(t, x, \xi)\eta, \eta)\geqslant\delta|\eta|^2,\ \eta\in R^m, \tag{5.3}$$

with some positive constant δ *independent of* (t, x, ξ). *Assume that for the symbol*

$$a(t, x, \xi) = 2\pi\sum_{j=1}^{n} a_j(x, t)\xi_j/|\xi|,$$

$s(t, x, \xi)a(t, x, \xi)s(t, x, \xi)^{-1}$ *are Hermitian for all* (t, x, ξ). *Then* $\{A(t)\}_{0\leqslant t\leqslant T}$ *defined above satisfies the assumptions* (I)~(IV) *in Theorem* 4.1.

Proof. Let $A_*(t)\in\mathcal{L}(\mathcal{L}_2(R^n); \mathcal{H}^{-1}(R^n))$ be defined by

$$A_*(t)v = -\sum_{j=1}^{n}\partial/\partial x_j\{a_j(x, t)^*v\}+b(x, t)^*v,$$

then we can observe

$$(A(t)_{|F})' = kA_*(t)h^{-1},$$

where h (resp. k) is the canonical isometry of $\mathcal{L}_2(R^n)$ (resp. $\mathcal{H}^{-1}(R^n)$) to E' (resp. F'). From this (I) follows.

To see (II) we make use of Proposition 2.1. Let $\mathcal{S}_1(t)$, $\mathcal{S}_2(t)$ be the singular integral operators with symbols

$$\sigma(\mathcal{S}_1(t)) = s(t, x, \xi),$$
$$\sigma(\mathcal{S}_2(t)) = s(t, x, \xi)^{-1}.$$

It is known that (5.3) implies

$$\mathrm{Re}(\mathcal{S}_1(t)u, u)_E\geqslant\delta'\|u\|_E^2-\gamma((\Lambda+1)^{-1}u, u)_E,\ u\in E;$$
$$\mathrm{Re}(\mathcal{S}_1(t)^*u, u)_E\geqslant\delta'\|u\|_E^2-\gamma((\Lambda+1)^{-1}u, u)_E,\ u\in E,$$

with some constants $\delta'>0$, $\gamma\geqslant0$ and $\Lambda\in\mathcal{L}(F; E)$

$$\Lambda u = \bar{\mathcal{F}}[|\xi|\mathcal{F}[u]],\ u\in F.$$

Therefore

$$N(t) = \mathcal{S}_1(t)+\gamma(\Lambda+1)^{-1},\ 0\leqslant t\leqslant T,$$

are all isomorphisms of E. (5.2) implies

$$\|N(t)N(t')^{-1}\|_E\leqslant\exp(C_1|t-t'|),\ 0\leqslant t,\ t'\leqslant T,$$

with some constant C_1 determined by δ' and some C_p alone. Put

$$\tilde{A}(t) = N(t)A(t)N(t)^{-1}, 0 \leqslant t \leqslant T,$$

then the theory of singular integral operators (*A.P.* Calderón and *A.* Zygmund [5], [6]) teaches us that $\tilde{A}(t)$ and $\tilde{A}(t)^*$ can be described on F as follows:

$$\tilde{A}(t)u = i(\mathcal{S}_1(t)\circ\mathcal{A}(t)\circ\mathcal{S}_2(t))\Lambda u + \mathcal{B}_1(t)u, \quad u\in F;$$
$$\tilde{A}(t)^*v = -i(\mathcal{S}_1(t)\circ\mathcal{A}(t)\circ\mathcal{S}_2(t))^{\#}\Lambda v + \mathcal{B}_2(t)v, \quad v\in F,$$

where $\mathcal{A}(t)$ are the singular integral operators with the symbols $a(t, x, \xi)$, $\mathcal{B}_1(t)$ and $\mathcal{B}_2(t)$ are bounded operators on E. Therefore we have

$$\|(\tilde{A}(t)+\tilde{A}(t)^*)u\|_E \leqslant C_2\|u\|_E, \; u\in F,$$

with some constant C_2 independent of t. From this we can conclude that $\tilde{A}(t)$ are almost anti-symmetric, whose tedious proof is omitted. Thus Proposition 2.1 establishes the stability of $\{A(t)\}$, the condition (II).

To verify (III) and (IV) we prepare the following lemma.

Lemma 5.2. *Let $S=(1+\Lambda^2)^{1/2}$ be the isomorphism of F to E. Then*

$$SA(t)S^{-1} = A(t)+B(t), 0\leqslant t\leqslant T,$$

with some $B\in C([0, T]; \mathcal{L}_s(E))$.

Actually we have for $u\in F$

$$\{SA(t)S^{-1}-A(t)\}u = \sum_{j=1}^{n} [S, a_j]\,(\partial/\partial x_j)S^{-1}u - [S, b]S^{-1}u, \tag{5.4}$$

where $[S, T]=ST-TS$ denotes the commutator. The right hand side of (5.4) can be extended to the bounded operators on E, from which our lemma follows.

It is not difficult to see that the perturbed family $\{A(t)+B(t)\}_{0\leqslant t\leqslant T}$ is still stable and belongs to $\mathcal{G}(E; M, \beta+BM)$ with $B=\sup_{0\leqslant t\leqslant T}\|B(t)\|_E$. Hence Lemma 5.2 shows (III).

It is observed immediately by Proposition 3.2 and Lemma 5.2 that

$$\rho_n(t) = \rho_n = (1+n^{-1}S)^{-1}, n = 1, 2, \cdots,$$

is the mollifying operator for $\{A(t)\}$, hence (IV).

REMARK 5.3. If (5.1) is a symmetric hyperbolic system, $a(t, x, \xi)$ are Hermitian, therefore the preceding theorem is applicable with $s(t, x, \xi)=I$.

In the case of (5.1) being regularly hyperbolic, there are invertible symbols $n(t)\in C_1^\infty$ such that

$$n(t, x, \xi)a(t, x, \xi)n(t, x, \xi)^{-1} = d(t, x, \xi)$$

are real diagonal. Let

$$s(t, x, \xi) = \{n(t, x, \xi)^* n(t, x, \xi)\}^{1/2}$$

be positive definite Hermitian symbols, then

$$sas^{-1} = s^{-1} n^* dns^{-1}$$

are Hermitian. Hence Theorem 5.1 can be applied to this case provided that n is Lipschitz continuous in t.

Finally, we give an elementary example which is neither symmetric nor regularly hyperbolic.

EXAMPLE 5.4. In (5.1) let $m=2$, $n=1$ and

$$a_1(x, t) = \begin{pmatrix} \phi_1(x, t) & \alpha(t)\psi(x, t) \\ \beta(t)\psi(x, t) & \phi_2(x, t) \end{pmatrix}, \ (x, t) \in R \times [0, T],$$

where ϕ_1, ϕ_2 and ψ are real valued functions in $\mathscr{B}^{1,0}(R \times [0, T])$, and α, β are Lipschitz continuous functions on $[0, T]$ such that

$$\alpha(t)\beta(t) > 0, \ 0 \leqslant t \leqslant T. \tag{5.5}$$

Then the symbols

$$s(t, x, \xi) = s(t) = \begin{pmatrix} |\beta(t)|^{1/2} & 0 \\ 0 & |\alpha(t)|^{1/2} \end{pmatrix}$$

satisfy the assumptions of Theorem 5.1. In fact

$$sas^{-1} = 2\pi \begin{pmatrix} \phi_1 & \alpha|\beta|(\alpha\beta)^{-1/2}\psi \\ |\alpha|\beta(\alpha\beta)^{-1/2}\psi & \phi_2 \end{pmatrix} \frac{\xi}{|\xi|}$$

are Hermitian, because

$$|\alpha(t)|\overline{\beta(t)} = \alpha(t)|\beta(t)|, \ 0 \leqslant t \leqslant T,$$

follows from (5.5).

OSAKA UNIVERSITY

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