Miyanishi, M.
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# REGULAR SUBRING OF A POLYNOMIAL RING

MASAYOSHI MIYANISHI*)

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**Introduction.** The purpose of this article is to prove the following two theorems:

**Theorem 1.** *Let $k$ be an algebraically closed field of characteristic zero, and let $A$ be a $k$-subalgebra of a polynomial ring $B:=k[x, y]$ such that $B$ is a flat $A$-module of finite type. Then $A$ is a polynomial ring in two variables over $k$.*

**Theorem 2.** *Let $k$ be an algebraically closed field of characteristic zero, and let $B:=k[x, y, z]$ be a polynomial ring in three variables over $k$. Assume that there is given a nontrivial action of the additive group $G_a$ on the affine 3-space $A_k^3:=\mathrm{Spec}(B)$ over $k$. Let $A$ be the subring of $G_a$-invariant elements in $B$. Assume that $A$ is regular. Then $A$ is a polynomial ring in two variables over $k$.*

Theorem 1 was formerly proved in part under one of the following additional conditions (cf. [7; pp. 139–142]):

(1) $B$ is etale over $A$,

(2) $A$ is the invariant subring in $B$ with respect to an action of a finite group.

In proofs of both theorems, substantial roles will be played by the following theorem, which is a consequence of the results obtained in Fujita [1], Miyanishi-Sugie [8] and Miyanishi [6]:

**Theorem 0.** *Let $k$ be an algebraically closed field of characteristic zero, and let $X=\mathrm{Spec}(A)$ be a nonsingular affine surface defined over $k$. Then the following assertions hold true:*

(1) *$X$ contains a nonempty cylinderlike open set, i.e., there exists a dominant morphism $\rho: X \to C$ from $X$ to a nonsingular curve $C$ whose general fibers are isomorphic to the affine line $A_k^1$, if and only if $X$ has the logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$.*

(2) *$X$ is isomorphic to the affiine plane $A_k^2$ if and only if $X$ has the logarithmic Kodaira dimension $\bar{\kappa}(X)=-\infty$, $A$ is a unique factorization domain, and $A^*=k^*$, where $A^*$ is the set of invertible elements in $A$ and $k^*=k-(0)$.*

In this article, the ground field $k$ is always assumed to be an algebraically

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closed field of characteristic zero. For the definition and relevant results on logarithmic pluri-genera and the logarithmic Kodaira dimension of an algebraic variety, we refer to Iitaka [3]. An algebraic action of the additive group $G_a$ on an affine scheme Spec$(B)$ over $k$ can be interpreted in terms of a locally nilpotent $k$-derivation $\Delta$ on $B$. In particular, the subring $A$ of $G_a$-invariant elements in $B$ is identified with the set of elements $b$ of $B$ such that $\Delta(b)=0$. For results on $G_a$-actions necessary in the subsequent arguments, we refer to [7; pp. 14–24]. The Picard group, i.e., the divisor class group, of a nonsingular variety $V$ over $k$ is denoted by Pic$(V)$; for a $k$-algebra $A$, $A^*$ denotes the multiplicative group of all invertible elements of $A$; the affine $n$-space over $k$ is denoted by $\boldsymbol{A}_k^n$; $\boldsymbol{Q}$ (resp. $\boldsymbol{Z}$) denotes the field of rational numbers (resp. the ring of rational integers).

## 1. Proof of Theorem 1

**1.1.** We shall begin with

**Lemma.** *Let $Y$ be a nonsingular, rational, affine surface, and let $f\colon Y\to C$ be a surjective morphism from $Y$ onto a nonsingular rational curve whose general fibers are isomorphic to the affine line $\boldsymbol{A}_k^1$. Then we have:*

(1) *Let $F$ be a fiber of $f$. If $F$ is irreducible and reduced, then $F$ is isomorphic to the affine line $\boldsymbol{A}_k^1$. If $F$ is singular, i.e., $F\not\cong\boldsymbol{A}_k^1$, then $F_{red}$ is a disjoint union of the affine lines.*

(2) *For a point $P\in C$, we denote by $\mu_P$ the number of irreducible components of the fiber $f^{-1}(P)$. If $C$ is isomorphic to the projective line $\boldsymbol{P}_k^1$, we have*

$$\operatorname{rank}_{\boldsymbol{Q}} \operatorname{Pic}(Y)\otimes_{\boldsymbol{Z}}\boldsymbol{Q} = 1+\sum_{P\in C}(\mu_P-1)\,.$$

*If $C$ is isomorphic to the affine line $\boldsymbol{A}_k^1$, we have*

$$\operatorname{rank}_{\boldsymbol{Q}} \operatorname{Pic}(Y)\otimes_{\boldsymbol{Z}}\boldsymbol{Q} = \sum_{P\in C}(\mu_P-1)\,.$$

Proof. There exists a nonsingular projective surface $W$ such that $W$ contains $Y$ as a dense open set and that the boundary curve $E:=W-Y$ has only normal crossings as singularities. The surjective morphism $f\colon Y\to C$ defines an irreducible linear pencil $\Lambda$ on $W$. By eliminating base points of $\Lambda$ by a succession of quadratic transformations, we may assume that $\Lambda$ is free from base points. Then the general members of $\Lambda$ are nonsingular rational curves, and the pencil $\Lambda$ defines a surjective morphism $\varphi\colon W\to\boldsymbol{P}_k^1$. The boundary curve $E$ contains a unique irreducible component $E_0$ which is a cross-section of $\varphi$, and the other components of $E$ are contained in the fibers of $\varphi$. We may assume that a fiber of $\varphi$ lying outside of $Y$ is irreducible. Let $S$ be a fiber of $\varphi$ such that $S\cap Y\neq\phi$. If $S$ is irreducible then $S\cong\boldsymbol{P}_k^1$ and $S\cap Y=S-S\cap E_0\cong\boldsymbol{A}_k^1$.

Suppose $S$ is reducible. If $S \cap Y$ is irreducible and reduced, we may contract all irreducible components of $S$ lying outside of $Y$ without losing generalities (cf. [7; Lemma 2.2, p. 115]). Hence, $S \cap Y \cong \boldsymbol{A}_k^1$. We may apply Lemma 1.3 of Kambayashi-Miyanishi [5] to obtain the same conclusion. Assume that $S \cap Y$ is singular. Then $S_{\text{red}}$ has the following decomposition into irreducible components,

$$S_{\text{red}} = \sum_i T_i + \sum_j Z_j ,$$

where $T_i \cap Y \neq \phi$ and $Z_j \cap Y = \phi$. By virtue of [7; ibid.], every component of $S_{\text{red}}$ is a nonsingular rational curve, $S_{\text{red}}$ is a connected curve, and the dual graph of $S_{\text{red}}$ contains no circular chains. On the other hand, since $Y$ is affine, $Y$ does not contain any complete curve and $E$ is connected, whence we know that if some $T_i$ meets the cross-section $E_0$ then $S_{\text{red}}$ has no components lying outside of $Y$. Suppose $S \cap Y$ is irreducible. Then $S_{\text{red}} \cap Y \cong \boldsymbol{A}_k^1$, for, if otherwise, the dual graph of $S_{\text{red}}$ would contain a circular chain. Suppose $S \cap Y$ is reducible. If either $T_i \cap Y \not\cong \boldsymbol{A}_k^1$ for some component $T_i$ or $T_i \cap T_l \cap Y \neq \phi$ for distinct components $T_i$ and $T_l$, the dual graph of $S_{\text{red}}$ would contain a circular chain. Therefore, every irreducible component of $S \cap Y$ is a connected component, and isomorphic to $\boldsymbol{A}_k^1$. This proves the first assertion.

Next, we shall prove the second assertion. Let $L$ be an irreducible fiber of $\varphi$. Then the $\boldsymbol{Q}$-vector space $\text{Pic}(W) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ has a basis of the divisor classes of the following curves:

(i) $E_0$, (ii) $L$, (iii) all irreducible components of a singular fiber $S$ of $\varphi$ except one component meeting $Y$, where $S$ ranges over all singular fibers of $\varphi$. The $\boldsymbol{Q}$-vector space $\text{Pic}(Y) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is generated by a part of the above basis consisting of all classes of curves which meet $Y$; when $C \cong \boldsymbol{A}_k^1$, we take $L$ to be the unique irreducible fiber lying outside of $Y$. Then we obtain immediately the equalities in the second assertion. Q.E.D.

**1.2.** With the notations in Theorem 1, set $V := \text{Spec}(A)$ and $W := \text{Spec}(B)$. The inclusion $A \hookrightarrow B$ induces a finite flat morphism $\pi: W \to V$. Since $\pi$ is finite, $\pi$ is faithfully flat. Therefore, $V$ is a nonsingular, rational affine surface. We have the following

**Lemma.** (1) *$V$ has the logarithmic Kodaira dimension $\bar{\kappa}(V) = -\infty$.* (2) $A^* = k^*$.

Proof. The second assertion is clear. As for the first assertion, note that $\pi: W \to V$ is a dominant morphism. Denote by $\bar{P}_m(V)$ the logarithmic $m$-th genus of $V$ for an integer $m \geqq 0$. Then we have

$$0=\bar{P}_m(W)\geqq\bar{P}_m(V)\quad\text{for every}\quad m\gg 0\,,$$

(cf. Iitaka [3]). Hence, $\bar{\kappa}(V)=-\infty$. Q.E.D.

**1.3.** We shall prove the following

**Lemma.** *A is a polynomial ring in two variables over k.*

Proof. Our proof consists of the paragraphs 1.3.1~1.3.5.

**1.3.1.** By virtue of Theorem 0 and Lemma 1.2, $V$ contains a nonempty cylinder-like open set. Namely, there exists a dominant morphism $\rho: V\rightarrow \boldsymbol{P}_k^1$ such that general fibers of $\rho$ are isomorphic to $\boldsymbol{A}_k^1$.

**1.3.2.** Let $n:=\deg\pi$. Then we claim:

*For every divisor D on V, nD is linearly equivalent to* 0, i.e., $nD\sim 0$.

In effect, $\pi^*(D)\sim 0$ because $\mathrm{Pic}(W)=0$. Since $\pi$ is proper, we have $\pi_*\pi^*(D)=nD\sim 0$ by the projection formula.

**1.3.3.** We claim that:

(1) $\rho(V)\cong \boldsymbol{A}_k^1$;

(2) *If* $\rho^{-1}(P)$ *is a singular fiber of* $\rho$, *it is of the form* $\rho^{-1}(P)=n_PC_P$, *where* $n_P\geqq 2$, $n_P|n$ *and* $C_P\cong\boldsymbol{A}_k^1$.

Proof. (1) By 1.3.2, we have $\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=0$. Suppose $\rho(V)=\boldsymbol{P}_k^1$. Then, Lemma 1.1 implies that

$$\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=1+\sum_{P\in\boldsymbol{P}_k^1}(\mu_P-1)\,.$$

Since $\mu_P\geqq 1$, we have $\mathrm{rank}_{\boldsymbol{Q}}\mathrm{Pic}(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}>0$, which is a contradiction. Hence $\rho(V)$ is an affine open set of $\boldsymbol{P}_k^1$. Since $A^*=k^*$, $\rho(V)$ must be isomorphic to $\boldsymbol{A}_k^1$.

(2) By Lemma 1.1, we have

$$\mathrm{rank}_{\boldsymbol{Q}}\,\mathrm{Pic}\,(V)\underset{\boldsymbol{Z}}{\otimes}\boldsymbol{Q}=\sum_{P\in\rho(V)}(\mu_P-1)=0\,.$$

Hence $\mu_P=1$ for all points $P\in\rho(V)$. This implies that a singular fiber of $\rho$ (if it exists at all) is of the form

$$\rho^{-1}(P)=n_PC_P\,,\quad\text{where}\quad n_P\geqq 2\quad\text{and}\quad C_P\cong\boldsymbol{A}_k^1\,.$$

Let $m$ be the order of $C_P$, i.e., $m$ is the least positive integer such that $mC_P\sim 0$. Since $n_PC_P\sim 0$, we have $m|n_P$. Write $n_P=sm$. Let $t$ be an inhomogeneous coordinate of $\rho(V)$. Then $t$ is everywhere defined on $V$; we may assume that $P\in\rho(V)$ is defined by $t=0$. Since $mC_P\sim 0$ and $n_PC_P=(t)$ (=the divisor defined by $t=0$ on $V$), there exists an element $t'\in A$ such that $t=(t')^s$. If $s>1$ then $t-\alpha=(t')^s-\alpha$ has distinct $s$ components for every $\alpha\in k^*$. This contradicts the irreducibility of general fibers of $\rho$. Hence $s=1$, i.e., $m=n_P$. Since $nC_P\sim 0$, we have $n_P|n$. Q.E.D.

**1.3.4.** Let $t$ be the same as defined in 1.3.3. Suppose $\rho$ has a singular fiber $\rho^{-1}(P)$. Since $\pi^*C_P\sim 0$, there exists an element $\tau\in B$ such that $t=\tau^{n_P}$. Let $A'=A\otimes_{k[t]}k[\tau]$. Since $A$ is flat over $k[t]$ (cf. [EGA (IV, 15.4.2)]), $A'$ is identified with a $k$-subalgebra $A[\tau]$ of $B$. Let $\tilde{A}$ be the normalization of $A'$ is $B$. Let $\tilde{V}=\mathrm{Spec}\,(\tilde{A})$. Then the morphism $\pi\colon W\to V$ factors as

$$\pi\colon W\xrightarrow{\pi_1}\tilde{V}\xrightarrow{\pi_2}V\,.$$

Let $n_1=\deg\pi_1$ and $n_2=\deg\pi_2$. Then $n=n_1\cdot n_2$, and $n_1\tilde{D}\sim 0$ for every divisor $\tilde{D}$ on $\tilde{V}$. We claim that:

(1) *$\tilde{V}$ is a nonsingular, rational, affine surface endowed with a dominant morphism $\tilde{\rho}\colon\tilde{V}\to A_k^1:=\mathrm{Spec}\,(k[\tau])$, which is induced by $\rho$. Hence, general fibers of $\tilde{\rho}$ are isomorphic to $A_k^1$.*

(2) *The fibration $\tilde{\rho}$ has a singular fiber with two or more irreducible components.*

Proof. Let $Q$ be a point on $C_P$. There exist local parameters $\xi$, $\eta$ of $V$ at $Q$ such that $C_P$ is defined locally by $\xi=0$. Then $t=u\xi^{n_P}$ for an invertible element $u$ of $\mathcal{O}_{Q,V}$. Let $\theta:=\tau/\xi$. Then $\tilde{V}$ is analytically isomorphic to a hypersurface $\theta^{n_P}=u$ in the $(\theta,\xi,\eta)$-space in a neighborhood of $\pi_2^{-1}(Q)$, and $\tilde{V}$ is smooth at every point of $\pi_2^{-1}(Q)$ by the Jacobian criterion. Since $Q$ is arbitrary on $C_P$, $\tilde{V}$ is smooth along $(\rho\cdot\pi_2)^{-1}(P)$. Let $\rho^{-1}(P')$ be another fiber of $\rho$. Then $\rho^{-1}(P')=n_{P'}C_{P'}$, where $n_{P'}\geqq 1$ and $C_{P'}\cong A_k^1$. Let $Q'$ be a point on $C_{P'}$. Then there exist local parameters $\xi'$, $\eta'$ such that $C_{P'}$ is defined locally by $\xi'=0$ and $t-\alpha=u'(\xi')^{n_{P'}}$ where $\alpha\in k^*$ and $u'$ is an invertible element of $\mathcal{O}_{Q',V}$. Then $\tilde{V}$ is analytically isomorphic to a hypersurface $\tau^{n_P}-\alpha=u'(\xi')^{n_{P'}}$ in the $(\tau,\xi',\eta')$-space. By the Jacobian criterion, $\tilde{V}$ is smooth at every point of $\pi_2^{-1}(Q')$. Since $Q'$ is arbitrary on $C_{P'}$, $\tilde{V}$ is smooth along $(\rho\cdot\pi_2)^{-1}(P')$. Thus we know that $\tilde{V}$ is smooth.

Let $\tilde{\rho}\colon\tilde{V}\to A_k^1:=\mathrm{Spec}(k[\tau])$ be the canonical morphism induced by $\rho$. The generic fibers of $\rho$ and $\tilde{\rho}$ are isomorphic to $\mathrm{Spec}(A\otimes_{k[t]}k(t))$ and $\mathrm{Spec}(A\otimes_{k[t]}k(\tau))$, respectively. Since $\mathrm{Spec}(A\otimes_{k[t]}k(t))\cong A_{k(t)}^1$, we know that $\mathrm{Spec}(A\otimes_{k[t]}k(\tau))\cong A_{k(\tau)}^1$. Hence, general fibers of $\tilde{\rho}$ are isomorphic to $A_k^1$.

Let $P$ be the point on $\mathrm{Spec}(k[\tau])$ lying above $P$ on $\mathrm{Spec}(k[t])$. Then the fiber $\tilde{\rho}^{-1}(\bar{P})$ has $n_P$ analytic branches over any point $Q$ of $C_P$, for $\tilde{\rho}^{-1}(\tilde{P})$ is analytically defined by $\theta^{n_P}=u$ as shown in the proof of the first assertion.

Hence, by Lemma 1.1, $\tilde{\rho}^{-1}(\tilde{P})$ has $n_P$ connected components, each of which is isomorphic to $\boldsymbol{A}_k^1$. Q.E.D.

**1.3.5.** As remarked in 1.3.4, $\mathrm{Pic}(\tilde{V})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=0$. However, if $\rho: V\rightarrow\boldsymbol{A}_k^1$ has a singular fiber, Lemma 1.1 implies that $\mathrm{rank}_{\boldsymbol{Q}}\mathrm{Pic}(\tilde{V})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}>0$. This is a contradiction. Therefore, $\rho$ has no singular fibers. Then, by virtue of [5; Th. 1], $V$ is an $\boldsymbol{A}^1$-bundle over $\rho(V)=\boldsymbol{A}_k^1$. Hence, we know that $V\cong\boldsymbol{A}_k^2$. Namely, $A$ is a polynomial ring in two variables over $k$. This completes a proof of Lemma 1.3 as well as a proof of Theorem 1.

**1.4.** REMARK. Theorem 1 is a generalization of the following result in the case of dimension 1:

*Let k be a field, and let A be a normal, 1-dimensional k-subalgebra of a polynomial ring over k. Then A is a polynomial ring over k.*

## 2. Proof of Theorem 2

**2.1.** We retain the notations and the assumptions of Theorem 2. Let $L:=k(x, y, z)$, and let $K$ be the invariant subfield of $L$ with respect to the induced $G_a$-action on $L$. Then, $A=B\cap K$. Since $B$ is normal and $\mathrm{trans.deg}_k K=2$, we know by Zariski's Theorem (cf. Nagata [9; Th. 4, p. 52]) that $A$ is finitely generated over $k$. By assumption, $A$ is regular. By virtue of [7; Lemma 1.3.1, p. 16], we know that $A$ is a unique factorization domain and $A^*=k^*$.

**2.2.** Let $W=\mathrm{Spec}(B)$, let $V=\mathrm{Spec}(A)$, and let $\pi: W\rightarrow V$ be the dominant morphism induced by the injection $A\hookrightarrow B$. We shall prove

**Lemma.** *$\pi: W\rightarrow V$ is a faithfully flat, equi-dimensional morphism of dimension* 1.

Proof. (1) We shall show that $B$ is flat over $A$. Let $\mathfrak{q}$ be a prime ideal of $B$ and let $\mathfrak{p}=\mathfrak{q}\cap A$. Then $B_{\mathfrak{q}}$ dominates $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is regular and $B_{\mathfrak{q}}$ is Cohen-Macaulay, $B_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ by virtue of [EGA (IV, 15.4.2)]. Hence $B$ is flat over $A$.

(2) We shall show that $\pi$ is surjective. Suppose $\pi$ is not surjective. Then there exists a maximal ideal $\mathfrak{m}$ of $A$ such that $\mathfrak{m}B=B$. Let $(\mathfrak{O}, t\mathfrak{O})$ be a discrete valuation ring of $K$ such that $\mathfrak{O}$ dominates $A_{\mathfrak{m}}$. Let $R:=B\otimes_A\mathfrak{O}$. Since $B$ is $A$-flat, $R$ is identified with a subring of $L$. Let $\Delta$ be a locally nilpotent derivation on $B$ associated to the given $G_a$-action. Then $\Delta$ extends to a locally nilpotent $\mathfrak{O}$-derivation, and $\mathfrak{O}$ is the ring of $\Delta$-invariants in $R$, i.e., $\mathfrak{O}=\{r\in R; \Delta(r)=0\}$. By assumption, we have $tR=R$, where $t$ is a uniformisant of $\mathfrak{O}$. Hence $tr=1$ for some element $r\in R$. Then $t\Delta(r)=0$, whence $r\in\mathfrak{O}$. This is a contradiction.

(3) Note that general fibers of $\pi$ are isomorphic to $A_k^1$. Hence, each irreducible component has dimension $\geqq 1$. Suppose that some component $T$ of a fiber $\pi^{-1}(P)$ (with $P \in V$) has dimension 2. Since $B$ is factorial, there exists an irreducible element $b \in B$ such that $T$ is defined by $b=0$. Since $T$ is invariant with respect to the $G_a$-action, we know that $b \in A$. Let $C=\operatorname{Spec}(A/bA)$. Then $C$ is an irreducible curve and $\pi^{-1}(C)=T \subset \pi^{-1}(P)$. This is a contradiction because $\pi$ is surjective. Thus, $\pi$ is a faithfully flat, equi-dimensional morphism of dimension 1. Q.E.D.

**2.3.** Let $U$ be the subset of all points $P$ of $V$ such that $\pi^{-1}(P)$ is irreducible and reduced. Then $U$ is a dense open set of $V$. By virtue of [5; Th. 1], $\pi^{-1}(U):=W \times_V U$ is an $A^1$-bundle over $U$.

**2.4.** We shall prove the following:

**Lemma.** *$V$ has the logarithmic Kodaira dimension $\bar{\kappa}(V)=-\infty$.*

Proof. We follow the arguments of Iitaka-Fujita [4]. Let $X$ be a hyperplane in $W \cong A_k^3$ such that $X \cap \pi^{-1}(U) \neq \phi$. Suppose that $\bar{\kappa}(V) \geqq 0$. Let $C$ be a prime divisor in $X$. Consider a morphism:

$$\varphi\colon\ C \times A_k^1 \hookrightarrow X \times A_k^1 = W \overset{\pi}{\to} V\,,$$

and assume that $\varphi$ is a dominant morphism. Since $\dim(C \times A_k^1)=\dim V=2$, we know by [3; Prop. 1] that

$$0=\bar{P}_m(C \times A_k^1) \geqq \bar{P}_m(V) \geqq 1 \quad \text{for for every} \quad m \gg 0\,.$$

This is a contradiction. Hence $\varphi$ is not a dominant morphism. Let $D$ be the closure of $\varphi(C \times A_k^1)$ in $V$. Then $C \times A_k^1 \subset \pi^{-1}(D)$. Suppose $C \cap \pi^{-1}(U) \neq \phi$. Then the general fibers of $\pi\colon \pi^{-1}(D) \to D$ are isomorphic to $A_k^1$. Hence $\pi^{-1}(D)$ is irreducible and reduced. Since $\dim(C \times A_k^1)=\dim \pi^{-1}(D)=2$, we have $C \times A_k^1=\pi^{-1}(D)$.

Let $Q$ be a point on $X$, and let $C_1, \cdots, C_r$ be prime divisors of $X$ such that $C_1 \cap \cdots \cap C_r=\{Q\}$ and that $C_i \cap \pi^{-1}(U) \neq \phi$ for every $1 \leqq i \leqq r$. For any point $Q$ of $X$, we can find such a set of prime divisors. In effect, $X$ is the affine plane $A_k^2$ and $X \cap (W-\pi^{-1}(U))$ has dimension $\leqq 1$. Thus, we have only to take a set of suitably chosen lines on $X$ passing through $Q$. Let $D_i$ be the irreducible curve which is the closure of $\varphi(C_i \times A_k^1)$ in $V$, where $1 \leqq i \leqq r$. Then $C_i \times A_k^1=\pi^{-1}(D_i)$ for $1 \leqq i \leqq r$. Since we have

$$\begin{aligned}(Q) \times A_k^1 &= (C_1 \cap \cdots \cap C_r) \times A_k^1 = (C_1 \times A_k^1) \cap \cdots \cap (C_r \times A_k^1)\\ &= \pi^{-1}(D_1) \cap \cdots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \cdots \cap D_r)\,,\end{aligned}$$

we know that $D_1 \cap \cdots \cap D_r = \{P\}$, a point in $V$. The correspondence $Q \mapsto P$ defines a morphism $\psi: X \to V$. If $\psi$ is a dominant morphism, we have

$$0 = \bar{P}_m(X) \geqq \bar{P}_m(V) \geqq 1 \quad \text{for every} \quad m \gg 0,$$

which is a contradiction. Hence $\psi$ is not a dominant morphism. Let $F$ be the closure of $\psi(X)$ in $V$. Then, for every point $P$ of $F$, we have $\dim \psi^{-1}(P) \geqq 1$, and $\pi(\psi^{-1}(P) \times A_k^1) = \psi(\psi^{-1}(P)) = P$. This contradicts Lemma 2.2. Therefore, $\bar{\kappa}(V) = -\infty$. Q.E.D.

**2.5.** As observed in the above arguments, $V$ is a nonsingular, rational, affine surface such that $\bar{\kappa}(V) = -\infty$, $A$ is a unique factorization domain and $A^* = k^*$. Then, by virtue of Theorem 0, $V$ is isomorphic to the affine plane $A_k^2$. Namely, $A$ is a polynomial ring in two variables over $k$. This completes a proof of Theorem 2.

**2.6.** We do not know whether or not the regularity condition on $A$ can be eliminated. As a view-point of practical use, the following result might be interesting:

**Proposition.** *Let the notations and the assumptions be the same as in Theorem 2. Instead of the regularity condition on $A$, we assume that $A$ contains a coordinate, say $z$. Then $A$ is a polynomial ring in two variables over $k$.*

Proof. (1) Let $H_\alpha$ be the hyperplane $z = \alpha$ in $W := \mathrm{Spec}(B)$ for every $\alpha \in k$. Since $z$ is $G_a$-invariant, the hyperplane $H_\alpha$ is $G_a$-invariant. Set $B_\alpha := B/(z-\alpha)B \cong k[x, y]$. Let $R_\alpha$ be the invariant subring of $B_\alpha$ with respect to the induced $G_a$-action on $H_\alpha$. Then it is clear that we have the following inclusions:

$$A_\alpha := A/(z-\alpha)A \hookrightarrow R_\alpha \hookrightarrow B_\alpha .$$

For a general element $\alpha \in k$, the induced $G_a$-action on $H_\alpha$ is nontrivial. Hence $R_\alpha$ is a one-parameter polynomial ring over $k$ (cf. [7; Lemma, p. 39]).

(2) Let $\mathfrak{k} = k(z)$, let $B_{\mathfrak{k}} := \mathfrak{k}[x, y]$ and let $R_{\mathfrak{k}}$ be the $G_a$-invariant subring of $B_{\mathfrak{k}}$ with respect to the induced $G_a$-action on $B_{\mathfrak{k}}$. Then it is not hard to show that $R_{\mathfrak{k}} = A \underset{k[z]}{\otimes} \mathfrak{k}$ and that $R_{\mathfrak{k}}$ is a one-parameter polynomial ring over $\mathfrak{k}$ (cf. Remark 1.4 and [7; ibid.]). Now, set $Y := \mathrm{Spec}(A)$ and let $f: Y \to A_k^1 = \mathrm{Spec}(k[z])$ be the morphism associated to the inclusion $k[z] \hookrightarrow A$. Then the foregoing observations imply the following:

(i) The generic fiber of $f$ is $\mathrm{Spec}(R_{\mathfrak{k}})$, which is isomorphic to $A_{\mathfrak{k}}^1$;

(ii) For every $\alpha \in k$, the fiber $f^{-1}(\alpha)$ is geometrically integral (cf. the inclusions $A_\alpha \hookrightarrow R_\alpha \hookrightarrow B_\alpha$ in the step (1)).

Since $f$ is clearly faithfully flat morphism, we know by [5; Th.1] that $f: Y \to A_k^1$ is an $A^1$-bundle over $A_k^1$. Hence $Y$ is isomorphic to the affine plane $A_k^2$.

Namely, $A$ is a polynomial ring in two variables over $k$. Q.E.D.

**2.7. Corollary.** *Let the notations and the assumptions be the same as in Theorem 2. Instead of the regularity condition on $A$, assume that $G_a$ acts linearly on $\boldsymbol{A}_k^3$ via the canonical action of $GL(3, k)$ on the vector space $kx+ky+kz$. Then $A$ is a polynomial ring in two variables over $k$.*

Proof. Since $G_a$ is a unipotent group, we may assume, after a change of coordinates $x$, $y$, $z$, that $G_a$ acts on $\boldsymbol{A}_k^3$ via the subgroup of upper triangular matrices in $GL(3, k)$. Then, one of coordinates, say $z$, is $G_a$-invariant. Then Proposition 2.5 implies that our assertion holds true. Q.E.D.

**2.8.** In the case of an algebraic action of $G_a$ on a polynomial ring $k[x, y]$ in two variables over $k$, $k[x, y]$ is a one-parameter polynomial ring over the subring of $G_a$-invariant elements (cf. [7; p. 39]). However, this does not hold in the case of an algebraic $G_a$-action on a polynomial ring of dimension $\geqq 3$ over $k$, as is shown by the following:

EXAMPLE. *Let $\Delta$ be a $k$-derivation on a polynomial ring $B:=k[x, y, z]$ defined by $\Delta(x)=y$, $\Delta(y)=z$ and $\Delta(z)=0$. Then $\Delta$ is locally nilpotent, whence $\Delta$ defines an algebraic $G_a$-action on $B$. The subring $A$ of $G_a$-invariant elements is $A=k[z, zx-\frac{1}{2}y^2]$. However, $B$ is not a one-parameter polynomial ring over $A$.*

Proof. We shall prove only the last assertion. Suppose that $B=A[t]$ for some element $t$ of $B$. Set $W:=\mathrm{Spec}(B)$ and $V:=\mathrm{Spec}(A)$. Let $f: W\to V$ be the morphism defined by the inclusion $A\hookrightarrow B$. Let $P$ be a point of $V$ such that $z=\alpha$ and $zx-\frac{1}{2}y^2=\beta$, where $\alpha$, $\beta\in k$. If $\alpha\neq 0$, the fiber $f^{-1}(P)$ is isomorphic to $\boldsymbol{A}_k^1$. If $\alpha=0$ and $\beta\neq 0$, $f^{-1}(P)$ is a disjoint union of two irreducible components isomorphic to $\boldsymbol{A}_k^1$. If $\alpha=\beta=0$, $f^{-1}(P)$ is a non-reduced curve with only one irreducible component isomorphic to $\boldsymbol{A}_k^1$ counted twice. But, if $B=A[t]$, all fibers of $f$ should be isomorphic to $\boldsymbol{A}_k^1$. Therefore, we have a contradiction. Q.E.D.

## References

[1] T. Fujita: *On Zariski Problem,* Proc. Japan Acad. **55**, Ser. A (1979), 106–110.

[2] A. Grothendieck et J. Dieudonné: Eléments de géométrie algébrique, Publ. Math. Inst. Hautes Etudes Sci., Vol. **4**, **8**, ..., referred to as (EGA).

[3] S. Iitaka: *On logarithmic Kodaira dimension of algebraic varieties,* Complex analysis and algebraic geometry, 175–189, Iwanami Shoten and Cambridge Univ. Press, 1977.

[4] S. Iitaka and T. Fujita: *Cancellation theorem for algebraic varieties,* J. Fac. Sci.

Univ. of Tokyo, Sec. IA, **24** (1977), 123–127.

[5] T. Kambayashi and M. Miyanishi: *On flat fibrations by affine line,* Illinois J. Math. **22** (1978), 662–671.

[6] M. Miyanishi: *An algebraic characterization of the affine plane,* J. Math. Kyoto Univ. **15** (1975), 169–184.

[7] M. Miyanishi: Lectures on curves on rational and unirational surfaces, Tata Institute of Fundamental Research, Vol. 60, Berlin-Heidelberg-New York, Springer Verlag, 1978.

[8] M. Miyanishi and T. Sugie: *Affine surfaces containing cylinderlike open sets,* J. Math. Kyoto Univ. **20** (1980), 11–42.

[9] M. Nagata: Lectures on the fourteenth problem of Hilbert, Tata Institute of Fundamental Research, Vol. 31, Bombay, 1965.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan