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REGULAR SUBRING OF A POLYNOMIAL RING

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Introduction. The purpose of this article is to prove the following two theorems:

Theorem 1. Let $k$ be an algebraically closed field of characteristic zero, and let $A$ be a $k$-subalgebra of a polynomial ring $B := k[x, y]$ such that $B$ is a flat $A$-module of finite type. Then $A$ is a polynomial ring in two variables over $k$.

Theorem 2. Let $k$ be an algebraically closed field of characteristic zero, and let $B := k[x, y, z]$ be a polynomial ring in three variables over $k$. Assume that there is given a nontrivial action of the additive group $G_a$ on the affine 3-space $A^3 := \text{Spec}(B)$ over $k$. Let $A$ be the subring of $G_a$-invariant elements in $B$. Assume that $A$ is regular. Then $A$ is a polynomial ring in two variables over $k$.

Theorem 1 was formerly proved in part under one of the following additional conditions (cf. [7; pp. 139-142]):

1. $B$ is etale over $A$,
2. $A$ is the invariant subring in $B$ with respect to an action of a finite group.

In proofs of both theorems, substantial roles will be played by the following theorem, which is a consequence of the results obtained in Fujita [1], Miyanishi-Sugie [8] and Miyanishi [6]:

Theorem 0. Let $k$ be an algebraically closed field of characteristic zero, and let $X = \text{Spec}(A)$ be a nonsingular affine surface defined over $k$. Then the following assertions hold true:

1. $X$ contains a nonempty cylinderlike open set, i.e., there exists a dominant morphism $\rho: X \to C$ from $X$ to a nonsingular curve $C$ whose general fibers are isomorphic to the affine line $A_1$, if and only if $X$ has the logarithmic Kodaira dimension $\kappa(X) = -\infty$.
2. $X$ is isomorphic to the affine plane $A_2$ if and only if $X$ has the logarithmic Kodaira dimension $\kappa(X) = -\infty$, $A$ is a unique factorization domain, and $A^* = k^*$, where $A^*$ is the set of invertible elements in $A$ and $k^* = k - (0)$.

In this article, the ground field $k$ is always assumed to be an algebraically

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closed field of characteristic zero. For the definition and relevant results on logarithmic pluri-genera and the logarithmic Kodaira dimension of an algebraic variety, we refer to Itaka [3]. An algebraic action of the additive group $G_a$ on an affine scheme $\text{Spec}(B)$ over $k$ can be interpreted in terms of a locally nilpotent $k$-derivation $\Delta$ on $B$. In particular, the subring $A$ of $G_a$-invariant elements in $B$ is identified with the set of elements $b$ of $B$ such that $\Delta(b) = 0$.

For results on $G_a$-actions necessary in the subsequent arguments, we refer to [7; pp. 14–24]. The Picard group, i.e., the divisor class group, of a nonsingular variety $V$ over $k$ is denoted by $\text{Pic}(V)$; for a $k$-algebra $A$, $A^*$ denotes the multiplicative group of all invertible elements of $A$; the affine $n$-space over $k$ is denoted by $A^n_k$; $\mathbb{Q}$ (resp. $\mathbb{Z}$) denotes the field of rational numbers (resp. the ring of rational integers).

1. Proof of Theorem 1

1.1. We shall begin with

**Lemma.** Let $Y$ be a nonsingular, rational, affine surface, and let $f: Y \to C$ be a surjective morphism from $Y$ onto a nonsingular rational curve whose general fibers are isomorphic to the affine line $A^1_\mathbb{A}$. Then we have:

1. Let $F$ be a fiber of $f$. If $F$ is irreducible and reduced, then $F$ is isomorphic to the affine line $A^1_\mathbb{A}$. If $F$ is singular, i.e., $F \cong A^1_\mathbb{A}$, then $F_{\text{red}}$ is a disjoint union of the affine lines.

2. For a point $P \in C$, we denote by $\mu_P$ the number of irreducible components of the fiber $f^{-1}(P)$. If $C$ is isomorphic to the projective line $\mathbb{P}^1_\mathbb{A}$, we have

$$\text{rank}_\mathbb{Q} \text{Pic}(Y) \otimes \mathbb{Q} = 1 + \sum_{P \in C} (\mu_P - 1).$$

If $C$ is isomorphic to the affine line $A^1_\mathbb{A}$, we have

$$\text{rank}_\mathbb{Q} \text{Pic}(Y) \otimes \mathbb{Q} = \sum_{P \in C} (\mu_P - 1).$$

Proof. There exists a nonsingular projective surface $W$ such that $W$ contains $Y$ as a dense open set and that the boundary curve $E := W - Y$ has only normal crossings as singularities. The surjective morphism $f: Y \to C$ defines an irreducible linear pencil $\Lambda$ on $W$. By eliminating base points of $\Lambda$ by a succession of quadratic transformations, we may assume that $\Lambda$ is free from base points. Then the general members of $\Lambda$ are nonsingular rational curves, and the pencil $\Lambda$ defines a surjective morphism $\varphi: W \to \mathbb{P}^1_\mathbb{A}$. The boundary curve $E$ contains a unique irreducible component $E_0$ which is a cross-section of $\varphi$, and the other components of $E$ are contained in the fibers of $\varphi$. We may assume that a fiber of $\varphi$ lying outside of $Y$ is irreducible. Let $S$ be a fiber of $\varphi$ such that $S \cap Y \neq \varnothing$. If $S$ is irreducible then $S \cong \mathbb{P}^1_\mathbb{A}$ and $S \cap Y = S \setminus E_0 = A^1_\mathbb{A}$. 


Suppose $S$ is reducible. If $S \cap Y$ is irreducible and reduced, we may contract all irreducible components of $S$ lying outside of $Y$ without losing generalities (cf. [7; Lemma 2.2, p. 115]). Hence, $S \cap Y \approx A^1$. We may apply Lemma 1.3 of Kambayashi-Miyanishi [5] to obtain the same conclusion. Assume that $S \cap Y$ is singular. Then $S_{\text{red}}$ has the following decomposition into irreducible components,

$$S_{\text{red}} = \sum T_i + \sum Z_j,$$

where $T_i \cap Y \neq \phi$ and $Z_j \cap Y = \phi$. By virtue of [7; ibid.], every component of $S_{\text{red}}$ is a nonsingular rational curve, $S_{\text{red}}$ is a connected curve, and the dual graph of $S_{\text{red}}$ contains no circular chains. On the other hand, since $Y$ is affine, $Y$ does not contain any complete curve and $E$ is connected, whence we know that if some $T_i$ meets the cross-section $E_0$ then $S_{\text{red}}$ has no components lying outside of $Y$. Suppose $S \cap Y$ is irreducible. Then $S_{\text{red}} \cap Y \approx A^1$, for, if otherwise, the dual graph of $S_{\text{red}}$ would contain a circular chain. Suppose $S \cap Y$ is reducible. If either $T_i \cap Y \approx A^1$ for some component $T_i$ or $T_i \cap T_j \cap Y \neq \phi$ for distinct components $T_i$ and $T_j$, the dual graph of $S_{\text{red}}$ would contain a circular chain. Therefore, every irreducible component of $S \cap Y$ is a connected component, and isomorphic to $A^1$. This proves the first assertion.

Next, we shall prove the second assertion. Let $L$ be an irreducible fiber of $\phi$. Then the $Q$-vector space $\text{Pic}(W) \otimes Q$ has a basis of the divisor classes of the following curves:

(i) $E_0$, (ii) $L$, (iii) all irreducible components of a singular fiber $S$ of $\phi$ except one component meeting $Y$, where $S$ ranges over all singular fibers of $\phi$.

The $Q$-vector space $\text{Pic}(Y) \otimes Q$ is generated by a part of the above basis consisting of all classes of curves which meet $Y$; when $C \approx A^1$, we take $L$ to be the unique irreducible fiber lying outside of $Y$. Then we obtain immediately the equalities in the second assertion. Q.E.D.

1.2. With the notations in Theorem 1, set $V := \text{Spec}(A)$ and $W := \text{Spec}(B)$. The inclusion $A \hookrightarrow B$ induces a finite flat morphism $\pi: W \rightarrow V$. Since $\pi$ is finite, $\pi$ is faithfully flat. Therefore, $V$ is a nonsingular, rational affine surface. We have the following

**Lemma.** (1) $V$ has the logarithmic Kodaira dimension $\kappa(V) = -\infty$. (2) $A^* = k^*$.

Proof. The second assertion is clear. As for the first assertion, note that $\pi: W \rightarrow V$ is a dominant morphism. Denote by $P_m(V)$ the logarithmic $m$-th genus of $V$ for an integer $m \geq 0$. Then we have
We shall prove the following lemma. A is a polynomial ring in two variables over k.

Proof. Our proof consists of the paragraphs 1.3.1~1.3.5.

1.3.1. By virtue of Theorem 0 and Lemma 1.2, V contains a nonempty cylinder-like open set. Namely, there exists a dominant morphism \( \rho: V \to \mathbb{P}^1_k \) such that general fibers of \( \rho \) are isomorphic to \( \mathbb{A}^1_k \).

1.3.2. Let \( n := \deg \pi \). Then we claim:

For every divisor \( D \) on \( V \), \( nD \) is linearly equivalent to 0, i.e., \( nD \sim 0 \).

In effect, \( \pi^*(D) \sim 0 \) because \( \text{Pic}(W) = 0 \). Since \( \pi \) is proper, we have \( \pi_\ast \pi^*(D) = nD \sim 0 \) by the projection formula.

1.3.3. We claim that:

1. \( \rho(V) \cong \mathbb{A}^1_k \);
2. If \( \rho^{-1}(P) \) is a singular fiber of \( \rho \), it is of the form \( \rho^{-1}(P) = n_p C_p \), where \( n_p \geq 2 \), \( n_p \mid n \) and \( C_p \cong \mathbb{A}^1_k \).

Proof. (1) By 1.3.2, we have \( \text{rank}_q \text{Pic}(V) \otimes \mathbb{Q} = 0 \). Suppose \( \rho(V) = \mathbb{P}^1_k \). Then, Lemma 1.1 implies that

\[
\text{rank}_q \text{Pic}(V) \otimes \mathbb{Q} = 1 + \sum_{P \in \mathbb{P}^1_k} (\mu_p - 1).
\]

Since \( \mu_p \geq 1 \), we have \( \text{rank}_q \text{Pic}(V) \otimes \mathbb{Q} > 0 \), which is a contradiction. Hence \( \rho(V) \) is an affine open set of \( \mathbb{P}^1_k \). Since \( \mathbb{A}^* = k^* \), \( \rho(V) \) must be isomorphic to \( \mathbb{A}^1_k \).

(2) By Lemma 1.1, we have

\[
\text{rank}_q \text{Pic}(V) \otimes \mathbb{Q} = \sum_{P \in \rho(V)} (\mu_p - 1) = 0.
\]

Hence \( \mu_p = 1 \) for all points \( P \in \rho(V) \). This implies that a singular fiber of \( \rho \) (if it exists at all) is of the form

\( \rho^{-1}(P) = n_p C_p \), where \( n_p \geq 2 \) and \( C_p \cong \mathbb{A}^1_k \).

Let \( m \) be the order of \( C_p \), i.e., \( m \) is the least positive integer such that \( mC_p \sim 0 \). Since \( n_p C_p \sim 0 \), we have \( m \mid n_p \). Write \( n_p = sm \). Let \( t \) be an inhomogeneous coordinate of \( \rho(V) \). Then \( t \) is everywhere defined on \( V \); we may assume that \( P \in \rho(V) \) is defined by \( t = 0 \). Since \( mC_p \sim 0 \) and \( n_p C_p = (t) \) (= the divisor...
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Defined by \( t = 0 \) on \( V \), there exists an element \( t' \in A \) such that \( t = (t')' \). If \( s > 1 \) then \( t - \alpha = (t')' - \alpha \) has distinct \( s \) components for every \( \alpha \in k^* \). This contradicts the irreducibility of general fibers of \( \rho \). Hence \( s = 1 \), i.e., \( m = n_p \).

Therefore, \( n C_p \cong 0 \), we have \( n_p | n \).

\section{1.3.4.}
Let \( t \) be the same as defined in 1.3.3. Suppose \( \rho \) has a singular fiber \( \rho^{-1}(P) \). Since \( \pi^* C_p \cong 0 \), there exists an element \( \tau \in B \) such that \( t = \tau^p \).

Let \( A' = A \otimes k[\tau] \). Since \( A \) is flat over \( k[t] \) (cf. [EGA (IV, 15.4.2)]) \( A' \) is identified with a \( k \)-subalgebra \( A[\tau] \) of \( B \). Let \( \bar{A} \) be the normalization of \( A' \) is \( B \). Let \( \bar{V} = \text{Spec}(\bar{A}) \). Then the morphism \( \pi: W \rightarrow V \) factors as

\[ \pi: W \xrightarrow{\pi_1} \bar{V} \xrightarrow{\pi_2} V. \]

Let \( n_1 = \deg \pi_1 \) and \( n_2 = \deg \pi_2 \). Then \( n = n_1 \cdot n_2 \), and \( n_D \cong 0 \) for every divisor \( D \) on \( V \).

We claim that:

1. \( \bar{V} \) is a nonsingular, rational, affine surface endowed with a dominant morphism \( \bar{\rho}: \bar{V} \rightarrow A_1 := \text{Spec}(k[\tau]) \), which is induced by \( \rho \). Hence, general fibers of \( \bar{\rho} \) are isomorphic to \( A_1 \).

2. The fibration \( \bar{\rho} \) has a singular fiber with two or more irreducible components.

Proof. Let \( Q \) be a point on \( C_p \). There exist local parameters \( \xi, \eta \) of \( V \) at \( Q \) such that \( C_p \) is defined locally by \( \xi = 0 \). Then \( t = u \xi \eta^p \) for an invertible element \( u \) of \( O_{Q, Y} \). Let \( \theta := \tau | \xi \). Then \( \bar{V} \) is analytically isomorphic to a hypersurface \( \theta^p = u \) in the \((\theta, \xi, \eta)\)-space in a neighborhood of \( \pi^{-1}(Q) \), and \( \bar{V} \) is smooth at every point of \( \pi^{-1}(Q) \) by the Jacobian criterion. Since \( Q \) is arbitrary on \( C_p \), \( \bar{V} \) is smooth along \( (\rho \pi_2)^{-1}(P) \). Let \( \rho^{-1}(P') \) be another fiber of \( \rho \). Then \( \rho^{-1}(P') = n_p C_p' \), where \( n_p \geq 1 \) and \( C_p' \cong A_1 \). Let \( Q' \) be a point on \( C_p' \). Then there exist local parameters \( \xi', \eta' \) such that \( C_p' \) is defined locally by \( \xi' = 0 \) and \( t - \alpha = u'(\xi')^{n_p'} - \alpha \in k^* \) and \( u' \) is an invertible element of \( O_{Q', Y} \). Then \( \bar{V} \) is analytically isomorphic to a hypersurface \( \theta^{n_p'} - \alpha = u'(\xi')^{n_p'} \) in the \((\tau, \xi', \eta')\)-space. By the Jacobian criterion, \( \bar{V} \) is smooth at every point of \( \pi^{-1}(Q') \). Since \( Q' \) is arbitrary on \( C_p' \), \( \bar{V} \) is smooth along \( (\rho \pi_2)^{-1}(P') \). Thus we know that \( \bar{V} \) is smooth.

Let \( \bar{\rho}: \bar{V} \rightarrow A_1 : = \text{Spec}(k[\tau]) \) be the canonical morphism induced by \( \rho \). The generic fibers of \( \rho \) and \( \bar{\rho} \) are isomorphic to \( \text{Spec}(A \otimes k(t)) \) and \( \text{Spec}(A \otimes k(\tau)) \), respectively. Since \( \text{Spec}(A \otimes k(\tau)) \cong A_{k(\tau)} \), we know that \( \text{Spec}(A \otimes k(\tau)) \cong A_{k(t)} \). Hence, general fibers of \( \bar{\rho} \) are isomorphic to \( A_1 \).

Let \( P \) be the point on \( \text{Spec}(k[\tau]) \) lying above \( P \) on \( \text{Spec}(k[t]) \). Then the fiber \( \bar{\rho}^{-1}(P) \) has \( n_p \) analytic branches over any point \( Q \) of \( C_p \), for \( \bar{\rho}^{-1}(P) \) is analytically defined by \( \theta^{n_p} = u \) as shown in the proof of the first assertion.
Hence, by Lemma 1.1, $\tilde{P}$ has $n_F$ connected components, each of which is isomorphic to $A^1_k$. Q.E.D.

1.3.5. As remarked in 1.3.4, $\text{Pic}(\tilde{V})\otimes \mathbb{Q} = 0$. However, if $\rho: V \to A^1_k$ has a singular fiber, Lemma 1.1 implies that $\text{rank}_\mathbb{Q} \text{Pic}(\tilde{V}) \otimes \mathbb{Q} > 0$. This is a contradiction. Therefore, $\rho$ has no singular fibers. Then, by virtue of [5; Th. 1], $V$ is an $A^1$-bundle over $\rho(V) = A^1_k$. Hence, we know that $V \cong A^1_k$. Namely, $A$ is a polynomial ring in two variables over $k$. This completes a proof of Lemma 1.3 as well as a proof of Theorem 1.

1.4. Remark. Theorem 1 is a generalization of the following result in the case of dimension 1:

Let $k$ be a field, and let $A$ be a normal, 1-dimensional $k$-subalgebra of a polynomial ring over $k$. Then $A$ is a polynomial ring over $k$.

2. Proof of Theorem 2

2.1. We retain the notations and the assumptions of Theorem 2. Let $L := k(x, y, z)$, and let $K$ be the invariant subfield of $L$ with respect to the induced $G_2$-action on $L$. Then, $A = B \cap K$. Since $B$ is normal and trans.deg$_k K = 2$, we know by Zariski's Theorem (cf. Nagata [9; Th. 4, p. 52]) that $A$ is finitely generated over $k$. By assumption, $A$ is regular. By virtue of [7; Lemma 1.3.1, p. 16], we know that $A$ is a unique factorization domain and $A^* = k^*$.

2.2. Let $W = \text{Spec}(B)$, let $V = \text{Spec}(A)$, and let $\pi: W \to V$ be the dominant morphism induced by the injection $A \hookrightarrow B$. We shall prove

**Lemma.** $\pi: W \to V$ is a faithfully flat, equi-dimensional morphism of dimension 1.

Proof. (1) We shall show that $B$ is flat over $A$. Let $q$ be a prime ideal of $B$ and let $\mathfrak{p} = q \cap A$. Then $B_q$ dominates $A_p$. Since $A_p$ is regular and $B_q$ is Cohen-Macaulay, $B_q$ is flat over $A_p$ by virtue of [EGA (IV, 15.4.2)]. Hence $B$ is flat over $A$.

(2) We shall show that $\pi$ is surjective. Suppose $\pi$ is not surjective. Then there exists a maximal ideal $m_A$ of $A$ such that $m_B = B$. Let $(\mathcal{O}, t\mathcal{O})$ be a discrete valuation ring of $K$ such that $\mathcal{O}$ dominates $A_m$. Let $R := B \otimes A^\mathcal{O}$. Since $B$ is $A$-flat, $R$ is identified with a subring of $L$. Let $\Delta$ be a locally nilpotent derivation on $B$ associated to the given $G_2$-action. Then $\Delta$ extends to a locally nilpotent $\mathcal{O}$-derivation, and $\mathcal{O}$ is the ring of $\Delta$-invariants in $R$, i.e., $\mathcal{O} = \{r \in R; \Delta(r) = 0\}$. By assumption, we have $tR = R$, where $t$ is a uniformisant of $\mathcal{O}$. Hence $tr = 1$ for some element $r \in R$. Then $t\Delta(r) = 0$, whence $r \in \mathcal{O}$. This is a contradiction.
(3) Note that general fibers of \( \pi \) are isomorphic to \( A_i \). Hence, each irreducible component has dimension \( \geq 1 \). Suppose that some component \( T \) of a fiber \( \pi^{-1}(P) \) (with \( P \in V \)) has dimension 2. Since \( B \) is factorial, there exists an irreducible element \( b \in B \) such that \( T \) is defined by \( b = 0 \). Since \( T \) is invariant with respect to the \( G_x \)-action, we know that \( b \in A \). Let \( C = \text{Spec} (A/bA) \). Then \( C \) is an irreducible curve and \( \pi^{-1}(C) = T \subseteq \pi^{-1}(P) \). This is a contradiction because \( \pi \) is surjective. Thus, \( \pi \) is a faithfully flat, equi-dimensional morphism of dimension 1. Q.E.D.

2.3. Let \( U \) be the subset of all points \( P \) of \( V \) such that \( \pi^{-1}(P) \) is irreducible and reduced. Then \( U \) is a dense open set of \( V \). By virtue of [5; Th. 1], \( \pi^{-1}(U) = W \times U \) is an \( A^1 \)-bundle over \( U \).

2.4. We shall prove the following:

**Lemma.** \( V \) has the logarithmic Kodaira dimension \( \kappa(V) = -\infty \).

Proof. We follow the arguments of Iitaka-Fujita [4]. Let \( X \) be a hyperplane in \( W \cong A_i^3 \) such that \( X \cap \pi^{-1}(U) \neq \emptyset \). Suppose that \( \kappa(V) \geq 0 \). Let \( C \) be a prime divisor in \( X \). Consider a morphism:

\[ \varphi: C \times A_i^1 \hookrightarrow X \times A_i^1 = W \xrightarrow{\pi} V, \]

and assume that \( \varphi \) is a dominant morphism. Since \( \dim(C \times A_i^1) = \dim V = 2 \), we know by [3; Prop. 1] that

\[ 0 = P_m(C \times A_i) \supseteq P_m(V) \supseteq 1 \quad \text{for every} \quad m \gg 0. \]

This is a contradiction. Hence \( \varphi \) is not a dominant morphism. Let \( D \) be the closure of \( \varphi(C \times A_i^1) \) in \( V \). Then \( C \times A_i^1 \subseteq \pi^{-1}(D) \). Suppose \( C \cap \pi^{-1}(U) = \emptyset \). Then the general fibers of \( \pi: \pi^{-1}(D) \to D \) are isomorphic to \( A_i^1 \). Hence \( \pi^{-1}(D) \) is irreducible and reduced. Since \( \dim(C \times A_i^1) = \dim \pi^{-1}(D) = 2 \), we have \( C \times A_i^1 = \pi^{-1}(D) \).

Let \( Q \) be a point on \( X \), and let \( C_1, \cdots, C_r \) be prime divisors of \( X \) such that \( C_1 \cap \cdots \cap C_r = \{Q\} \) and that \( C_i \cap \pi^{-1}(U) \neq \emptyset \) for every \( 1 \leq i \leq r \). For any point \( Q \) of \( X \), we can find such a set of prime divisors. In effect, \( X \) is the affine plane \( A_i^2 \) and \( X \cap (W - \pi^{-1}(U)) \) has dimension \( \leq 1 \). Thus, we have only to take a set of suitably chosen lines on \( X \) passing through \( Q \). Let \( D_i \) be the irreducible curve which is the closure of \( \varphi(C_i \times A_i^1) \) in \( V \), where \( 1 \leq i \leq r \). Then \( C_i \times A_i^1 = \pi^{-1}(D_i) \) for \( 1 \leq i \leq r \). Since we have

\[ (Q) \times A_i^1 = (C_1 \cap \cdots \cap C_r) \times A_i^1 = (C_1 \times A_i^1) \cap \cdots \cap (C_r \times A_i^1) \]

\[ = \pi^{-1}(D_1) \cap \cdots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \cdots \cap D_r), \]

\[ \boxdot \]
we know that \( D_1 \cap \cdots \cap D_r = \{ P \} \), a point in \( V \). The correspondence \( Q_i \mapsto P \) defines a morphism \( \psi : X \to V \). If \( \psi \) is a dominant morphism, we have

\[
0 = \tilde{P}_m(X) \supseteq \tilde{P}_m(V) \supseteq 1 \quad \text{for every } m \geq 0,
\]
which is a contradiction. Hence \( \psi \) is not a dominant morphism. Let \( F \) be the closure of \( \psi(X) \) in \( V \). Then, for every point \( P \) of \( F \), we have \( \dim \psi^{-1}(P) \geq 1 \), and \( \pi(\psi^{-1}(P) \times \mathbb{A}_k^1) = \psi(\psi^{-1}(P)) = P \). This contradicts Lemma 2.2. Therefore, \( \kappa(V) = -\infty \). Q.E.D.

2.5. As observed in the above arguments, \( V \) is a nonsingular, rational, affine surface such that \( \kappa(V) = -\infty \), \( A \) is a unique factorization domain and \( A^* = k^* \). Then, by virtue of Theorem 0, \( V \) is isomorphic to the affine plane \( \mathbb{A}_k^2 \). Namely, \( A \) is a polynomial ring in two variables over \( k \). This completes a proof of Theorem 2.

2.6. We do not know whether or not the regularity condition on \( A \) can be eliminated. As a view-point of practical use, the following result might be interesting:

**Proposition.** Let the notations and the assumptions be the same as in Theorem 2. Instead of the regularity condition on \( A \), we assume that \( A \) contains a coordinate, say \( z \). Then \( A \) is a polynomial ring in two variables over \( k \).

Proof. (1) Let \( H_\alpha \) be the hyperplane \( z=\alpha \) in \( W := \text{Spec}(B) \) for every \( \alpha \in k \). Since \( z \) is \( G_\alpha \)-invariant, the hyperplane \( H_\alpha \) is \( G_\alpha \)-invariant. Set \( B_\alpha := B/(z-\alpha)B \approx k[x,y] \). Let \( R_\alpha \) be the invariant subring of \( B_\alpha \) with respect to the induced \( G_\alpha \)-action on \( H_\alpha \). Then it is clear that we have the following inclusions:

\[
A_\alpha := A/(z-\alpha)A \hookrightarrow R_\alpha \hookrightarrow B_\alpha.
\]
For a general element \( \alpha \in k \), the induced \( G_\alpha \)-action on \( H_\alpha \) is nontrivial. Hence \( R_\alpha \) is a one-parameter polynomial ring over \( k \) (cf. [7; Lemma, p. 39]).

(2) Let \( \mathfrak{t} = k(z) \), let \( B_\mathfrak{t} := \mathfrak{t}[x,y] \) and let \( R_\mathfrak{t} \) be the \( G_\mathfrak{t} \)-invariant subring of \( B_\mathfrak{t} \) with respect to the induced \( G_\mathfrak{t} \)-action on \( B_\mathfrak{t} \). Then it is not hard to show that \( R_\mathfrak{t} := A \otimes_{k[z]} \mathfrak{t} \) and that \( R_\mathfrak{t} \) is a one-parameter polynomial ring over \( \mathfrak{t} \) (cf. Remark 1.4 and [7; ibid.]). Now, set \( Y := \text{Spec}(A) \) and let \( f : Y \to \mathbb{A}_k^1 = \text{Spec}(k[z]) \) be the morphism associated to the inclusion \( k[z] \hookrightarrow A \). Then the foregoing observations imply the following:

(i) The generic fiber of \( f \) is \( \text{Spec}(R_\mathfrak{t}) \), which is isomorphic to \( \mathbb{A}_k^1 \);

(ii) For every \( \alpha \in k \), the fiber \( f^{-1}(\alpha) \) is geometrically integral (cf. the inclusions \( A_\alpha \hookrightarrow R_\alpha \hookrightarrow B_\alpha \) in the step (1)).

Since \( f \) is clearly faithfully flat morphism, we know by [5; Th.1] that \( f : Y \to \mathbb{A}_k^1 \) is an \( \mathbb{A}_k^1 \)-bundle over \( \mathbb{A}_k^1 \). Hence \( Y \) is isomorphic to the affine plane \( \mathbb{A}_k^2 \).
Namely, $A$ is a polynomial ring in two variables over $k$. Q.E.D.

2.7. Corollary. Let the notations and the assumptions be the same as in Theorem 2. Instead of the regularity condition on $A$, assume that $G_\alpha$ acts linearly on $A^1_k$ via the canonical action of $GL(3, k)$ on the vector space $kx + ky + kz$. Then $A$ is a polynomial ring in two variables over $k$.

Proof. Since $G_\alpha$ is a unipotent group, we may assume, after a change of coordinates $x, y, z$, that $G_\alpha$ acts on $A^1_k$ via the subgroup of upper triangular matrices in $GL(3, k)$. Then, one of coordinates, say $z$, is $G_\alpha$-invariant. Then Proposition 2.5 implies that our assertion holds true. Q.E.D.

2.8. In the case of an algebraic action of $G_\alpha$ on a polynomial ring $k[x, y]$ in two variables over $k$, $k[x, y]$ is a one-parameter polynomial ring over the subring of $G_\alpha$-invariant elements (cf. [7; p. 39]). However, this does not hold in the case of an algebraic $G_\alpha$-action on a polynomial ring of dimension $\geq 3$ over $k$, as is shown by the following:

Example. Let $\Delta$ be a $k$-derivation on a polynomial ring $B := k[x, y, z]$ defined by $\Delta(x) = y, \Delta(y) = z$ and $\Delta(z) = 0$. Then $\Delta$ is locally nilpotent, whence $\Delta$ defines an algebraic $G_\alpha$-action on $B$. The subring $A$ of $G_\alpha$-invariant elements is $A = k[z, zx - \frac{1}{2} y^2]$. However, $B$ is not a one-parameter polynomial ring over $A$.

Proof. We shall prove only the last assertion. Suppose that $B = A[t]$ for some element $t$ of $B$. Set $W := \text{Spec}(B)$ and $V := \text{Spec}(A)$. Let $f: W \to V$ be the morphism defined by the inclusion $A \to B$. Let $P$ be a point of $V$ such that $z = \alpha$ and $zx - \frac{1}{2} y^2 = \beta$, where $\alpha, \beta \in k$. If $\alpha \neq 0$, the fiber $f^{-1}(P)$ is isomorphic to $A^1_k$. If $\alpha = 0$ and $\beta \neq 0$, $f^{-1}(P)$ is a disjoint union of two irreducible components isomorphic to $A^1_k$. If $\alpha = \beta = 0$, $f^{-1}(P)$ is a non-reduced curve with only one irreducible component isomorphic to $A^1_k$ counted twice. But, if $B = A[t]$, all fibers of $f$ should be isomorphic to $A^1_k$. Therefore, we have a contradiction. Q.E.D.

References
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