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NORMAL SUBGROUPS AND MULTIPLICITIES OF INDECOMPOSABLE MODULES

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Introduction

Let G be a finite group and $(K,\boldsymbol{o},F)$ be a p-modular system, where p is a prime number. We assume that K contains the $|G|$-th roots of unity and F is algebraically closed and we put $R=\boldsymbol{o}$ or F. For an R-free finitely generated indecomposable RG-module M and a normal subgroup N of G, let V be an indecomposable component of M_N, where M_N is the restriction of M to N. In this paper we give some results on the multiplicity of V as a component of M_N and from them we obtain properties of heights of indecomposable modules and irreducible characters. This study is inspired by Murai [8, 9].

Throughout this paper N is a fixed normal subgroup of G and v is the p-adic valuation such that $v(p)=1$. All RG-modules are assumed to be R-free of finite rank. For an indecomposable RG-module M, let $\mathrm{vx}(M)$ denote a vertex of M. As is well known $v(\mathrm{rank}_R M)\geq v(|G:\mathrm{vx}(M|)$. We refer to Feit[1, Chap.3] and Nagao-Tsushima [10, Chap.4] for the vertex-source theory in modular representations of finite groups.

1. p-parts of multiplicities

In this section we study the p-parts of multiplicities of indecomposable RN-modules in an indecomposable decomposition of M_N. The following is a key result of this paper.

Theorem 1. *Let V be a G-invariant indecomposable RN-module. Let M be an indecomposable RG-module with vertex Q and n be the multiplicity of V in an indecomposable decomposition of M_N. Then we have $v(n)\geq v(|G:QN|)$.*

Proof. Let L be a subgroup of G such that L/N is a Sylow p-subgroup of G/N and let

$$M_L=M_1\oplus M_2\oplus\cdots\oplus M_s,$$

where each M_i is an indecomposable RL-module. By Mackey decomposition

M_i is $(Q^{x_i}\cap L)$-projective for some $x_i\in G$. We have

$$\nu(|L:(Q^{x_i}\cap L)N|)=\nu(|G:(Q^{x_i}\cap L)N|)\geq\nu(|G:Q^{x_i}N|)=\nu(|G:QN|).$$

Let n_i be the multiplicity of V as an indecomposable component of M_{i_N} and Q_i be a vertex of M_i. We have $n=\sum_{i=1}^{s}n_i$. If Theorem holds for each M_i, then we have

$$\nu(n_i)\geq\nu(|L:Q_iN|)\geq\nu(|L:(Q^{x_i}\cap L)N|)\geq\nu(|G:QN|)$$

and hence $\nu(n)\geq\nu(|G:QN|)$. So we may assume that $G=L$. Then by a theorem of Green, there exists an indecomposable $R(QN)$-module M_0 such that M is isomorphic to M_0^G. Then we have $M_N=\sum_x M_0\otimes x$, where x ranges over a set of representatives for the QN-cosets $(QN)x$ of G. Since $M_0\otimes x$ is an RN-module which is G-conjugate to M_{0_N}, n is divisible by $|G:QN|$. This completes the proof.

Proposition 1. *Let M be an indecomposable RG-module such that* $\nu(\text{rank}_R M)=\nu(|G:\text{vx}(M)|)$, *then there exists an indecomposable component V of M_N which satisfies the following.*

(i) $\nu(\text{rank}_R V)=\nu(|N:\text{vx}(V)|)$,

(ii) *Let n be the multiplicity of V in an indecomposable decomposition of M_N. There exists a vertex P of M such that $P\cap N$ is a vertex of V, $T(V)\supset P$ and $\nu(n)=\nu(|T(V):PN|)$, where $T(V)$ is the inertial group of V in G.*

Proof. Let $\{V_1,V_2,\cdots,V_t\}$ be a set of representatives (up to isomorphism) for the G-conjugacy classes of indecomposable components of M_N and $\tilde{V}_i$ be the direct sum of all RN-modules which is G-conjugate to V_i. We can set

$$M_N\cong\sum_{i=1}^{s}\oplus n_i\tilde{V}_i,$$

where n_i is the multiplicity of V_i. Here we fix some i for a while and let T_i be the inertial group of V_i. We put

$$M_{T_i}=M_1\oplus M_2\oplus\cdots\oplus M_t\oplus L_1\oplus L_2\oplus\cdots\oplus L_u,$$

where M_j is an indecomposable RT_i-module such that V_i is a component of M_{j_N}, and L_j is an indecomposable RT_i-module such that V_i is not a component of L_{j_N}. Let Q be a vertex of M. By Mackey decomposition, M_j is $(Q^{y_j}\cap T_i)$-projective for some $y_j\in G$. By Theorem 1 we have

$$\nu(n_i)\geq\min_{1\leq j\leq t}\{\nu(|T_i:\text{vx}(M_j)N|)\}\geq\min_{1\leq j\leq t}\{\nu(|T_i:(Q^{y_j}\cap T_i)N|)\}.$$

Hence we have

$$(1) \qquad \nu(n_i \mathrm{rank}_R \tilde{V}_i) = \nu(n_i) + \nu(|G: T_i|) + \nu(\mathrm{rank}_R V_i)$$
$$\geq \min_{1 \leq j \leq t} \{\nu(|G: (Q^{y_j} \cap T_i)N|)\} + \nu(\mathrm{rank}_R V_i)$$
$$\geq \nu(|G: QN|) + \nu(|N: \mathrm{vx}(V_i)|).$$

On the other hand V_i is $(Q^{x_i} \cap N)$-projective for some $x_i \in G$. Therefore we have

$$\nu(n_i \mathrm{rank}_R \tilde{V}_i) \geq \nu(|G: QN|) + \nu(|N: (Q^{x_i} \cap N)|)$$
$$= \nu(|G: QN|) + \nu(|Q^{x_i}N: Q^{x_i}|) = \nu(|G: Q|).$$

By the assumption we may assume that $\nu(n_i \mathrm{rank}_R \tilde{V}_i) = \nu(|G: Q|)$. Then $\nu(\mathrm{rank}_R V_i) = \nu(|N: \mathrm{vx}(V_i)|)$ and $|\mathrm{vx}(V_i)| = |Q \cap N|$. On the other hand for some M_j we have $\nu(n_i \mathrm{rank}_R \tilde{V}_i) - \nu(\mathrm{rank}_R V_i) = \nu(|G: \mathrm{vx}(M_j)N|) = \nu(|G: (Q^{y_j} \cap T_i)N|) = \nu(|G: Q^{y_j}N|)$. This implies $Q^{y_j} \subset T_i$, $\nu(n_i) = \nu(|T_i: Q^{y_j}N|)$ and $PN = Q^{y_j}N$, where P is a vertex of M_j which is contained in Q^{y_j}. Since V_i is a component of M_{j_N}, we have $|Q \cap N| = |\mathrm{vx}(V_i)| \leq |P \cap N|$. Hence we have $Q^{y_j} \cap N = P \cap N$, so $Q^{y_j} = P$ and $P \cap N$ is a vertex of V_i. This completes the proof.

As a corollary of the proposition we have the following for N-projective indecomposable modules (see Karpilovsky [3, Chapter 12]).

Corollary 1. *Let M be an N-projective indecomposable RG-module and V be an indecomposable component of M_N with multiplicity n. Then* $\mathrm{vx}(V)$ *is a vertex of M and* $\nu(\mathrm{rank}_R M) \geq \nu(|G: N|) + \nu(\mathrm{rank}_R V)$. *Moreover if* $\nu(\mathrm{rank}_R M) = \nu(|G: \mathrm{vx}(M)|)$ *then* $\nu(n) = \nu(|T(V): N|)$ *and* $\nu(\mathrm{rank}_R V) = \nu(|N: \mathrm{vx}(V)|)$.

Proof. Let $\tilde{V}$ be the direct sum of the G-conjugates of V. By the assumption $\mathrm{vx}(V)$ is a vertex of M and $M_N \cong \oplus n\tilde{V}$. Hence from the arguments in the proof of the above proposition we have $\nu(\mathrm{rank}_R M) \geq \nu(|G: \mathrm{vx}(M)N|) + \nu(\mathrm{rank}_R V) = \nu(|G: N|) + \nu(\mathrm{rank}_R V)$. The latter also follows from it.

From the above corollary we have the following, which is shown implicitly in Knörr [6].

Corollary 2. *Let M be an indecomposable RG-module with source S. If* $\nu(\mathrm{rank}_R M) = \nu(|G: \mathrm{vx}(M)|)$, *then* $p \nmid \mathrm{rank}_R S$.

Proof. Let Q be a vertex of M. By Green correspondence we may assume that Q is normal and S is an RQ-module. Here we can put $N = Q$ and $V = S$ in

Corollary 1. By the assumption and since $Q=\mathrm{vx}(V)$ we have $p\nmid \mathrm{rank}_R S$.

Y. Tsushima and M. Murai pointed out independently that if G is p-solvable the converse of Corollary 2 is true. This follows from Green correspondence and the fact that if G is p-solvable then the equality $\nu(n)=\nu(|T(V):N|)$ holds in Corollary 1.

Moreover if G is p-solvable then for an irreducible FG-module M, $\nu(\dim_F M)=\nu(|G:\mathrm{vx}(M)|)$ by Hemernik-Michler [2, Theorem 2.1]. Hence Proposition 1 and Corollary 2 combined with Clifford's theorem imply the following.

Corollary 3. *Suppose that G is p-solvable. Let M be an irreducible FG-module and V be an irreducible constituent of M_N with multiplicity n. Then V has $P\cap N$ as a vertex and $\nu(n)=\nu(|T(V):PN|)$, where P is a vertex of M. Moreover if S is a source of M, then $p\nmid \dim_F S$.*

Let B be a p-block of G with defect group D. In [8] Murai extends the heights of characters to RG-modules. For an RG-module U in B the height $\mathrm{ht}(U)$ is defined by $\mathrm{ht}(U)=\nu(\mathrm{rank}_R U)-\nu(|G:D|)$. In particular when U is indecomposable, U is of height 0 if and only if $\nu(\mathrm{rank}_R U)=\nu(|G:\mathrm{vx}(U)|)$ and $\mathrm{vx}(U)$ is G-conjugate to D. Let b be a p-block of N covered by B. Since by Knörr [5, Prop.4.2], a defect group of b is G-conjugate to $D\cap N$, we see by Proposition 1 that if an indecomposable RG-module U lying in B is of height 0 then an indecomposable component V of U_N lying in b is of height 0 (see [8, Theorem 4.11]). We can also get this fact from the following, which Murai proved by using the arguments of the proof of Proposition 1.

Proposition 2. *Let B and b be as in the above, and M be an indecomposable RG-module lying in B. Let*

$$M_N \cong \sum_{i=1}^{t} \oplus n_i V_i$$

be a decomposition of M_N to the sum of indecomposable RN-submodules. Then we have $\mathrm{ht}(M)\geq \min\{\mathrm{ht}(V_i)\mid 1\leq i\leq t\}$

Proof. We may assume that $\{V_1, V_2, \cdots, V_s\}$ $(s\leq t)$ is a set of representatives for the G-conjugacy classes of indecomposable components of M_N and that V_i $(1\leq i\leq s)$ belongs to b. Let D be a defect group of B such that $D\cap N$ is a defect group of b. Using the notations in the proof of Proposition 1, from (1) we have

$$\begin{aligned}\nu(n_i \mathrm{rank}_R \tilde{V}_i) &\geq \nu(|G:QN|)+\nu(\mathrm{rank}_R V_i)\\ &\geq \nu(|G:DN|)+\nu(|N:D\cap N|)+\mathrm{ht}(V_i)\end{aligned}$$

$$=\nu(|G:D|)+\mathrm{ht}(V_i)$$

where $1 \leq i \leq s$. This implies the inequality in the proposition.

2. Heights of irreducible characters

Let χ be an irreducible character in B and ζ be an irreducible constituent of χ_N in b, where B and b are as in §1. Let X be an indecomposable oG-lattice affording χ and let Z be an indecomposable component of X_N which lies in b and $\mathrm{ht}(X) \geq \mathrm{ht}(Z)$ (Z exists by Proposition 2). Then $\mathrm{rank}_o Z$ is a multiple of $\zeta(1)$, and hence $\mathrm{ht}(Z) \geq \mathrm{ht}(\zeta)$. Since $\mathrm{ht}(\chi)=\mathrm{ht}(X)$, we have $\mathrm{ht}(\chi) \geq \mathrm{ht}(\zeta)$ as in [9, Lemma 2.2]. On the other hand, by Proposition 2, for an irreducible FG-module M in B and an irreducible constituent V of M_N in b, we have $\mathrm{ht}(M) \geq \mathrm{ht}(V)$ ([9, Lemma 3.2]). We shall show that $\mathrm{ht}(\chi)=\mathrm{ht}(\zeta)$ and $\mathrm{ht}(M)=\mathrm{ht}(V)$ when a defect group D of B is contained in N.

The following is shown from the results of Külshammer-Robinson [7], and the converse is proved in Robinson [11, Lemma 4.4].

Lemma 1 (Külshammer-Robinson). *Let χ be an irreducible character of G and ζ be an irreducible constituent of χ_N with multiplicity n. If χ is afforded by an N-projective oG-lattice M then we have $\nu(n)=\nu(|T(\zeta):N|)$.*

Proof. Suppose that χ is afforded by an N-projective indecomposable oG-lattice M and let V be an indecomposable component of M_N. Then from the argument in the proof Corollary 1 we have $\mathrm{rank}_o M = m\,\mathrm{rank}_o V$, where m is a natural number with $\nu(m) \geq \nu(|G:N|)$. Since $\nu(\chi(1)/\zeta(1)) \geq \nu(m) \geq \nu(|G:N|)$ because m divides $\chi(1)/\zeta(1)$, we see $\nu(n) \geq \nu(|T(\zeta):N|)$. On the other hand, as is well known n divides $|T(\zeta):N|$. Therefore we have $\nu(n)=\nu(|T(\zeta):N|)$.

Lemma 2. *Let M be an N-projective irreducible FG-module and V be an irreducible component of M_N with multiplicity n. Then we have $\nu(n)=\nu(|T(V):N|)$. Moreover n is equal to the multiplicity m of M as an indecomposable component of V^G.*

Proof. We may assume that V is G-invariant. We put $E=\mathrm{End}_{FG}(V^G)$ and let e be a primitive idempotent of E corresponding to M, i.e., $M=eV^G=(eE)V$. Then as is well known E is isomorphic to a twisted group algebra $F(\bar{G},\varphi)$ over F with factor set φ, where $\bar{G}=G/N$. Moreover $\dim_F M=(\dim_F(eE))(\dim_F V)$ and hence $n=\dim_F(eE)$. By Humphreys [3], there exists a central p'-extension $\hat{G}$ of $\bar{G}$ such that $F(\bar{G},\varphi)$ is isomorphic to a direct sum of some block ideals of $F\hat{G}$. Now as M is irreducible, eE is irreducible. Hence n is equal to the dimension of an irreducible and projective $F\hat{G}$-module, so we have $\nu(n)=\nu(|\hat{G}|)=\nu(|G:N|)$.

By the way m is equal to the dimension of the irreducible E-module corresponding to eE. But eE is irreducible, hence m is equal to $\dim_F(eE)$. This completes the proof.

Proposition 3. *Let B be a p-block of G with defect group D and b be a p-block of N covered by B. Assume that D is contained in N. Then for an irreducible character χ in B and for an irreducible constituent ζ of χ_N in b, we have* $\mathrm{ht}(\chi)=\mathrm{ht}(\zeta)$. *We also have* $\mathrm{ht}(M)=\mathrm{ht}(V)$ *for an irreducible FG-module M in B and an irreducible constituent V of M_N in b.*

Proof. We may assume D is a defect group of b. By the assumption and Lemma 1 we have $v(\chi(1))=v(|G:N|)+v(\zeta(1))=v(|G:D|)+\mathrm{ht}(\zeta)$. This implies $\mathrm{ht}(\chi)=\mathrm{ht}(\xi)$. By the former of Lemma 2 we also have $v(\dim_F M)=v(|G:N|)+v(\dim_F V)=v(|G:D|)+\mathrm{ht}(V)$. Hence $\mathrm{ht}(M)=\mathrm{ht}(V)$. This completes the proof.

For a p-block B of G let $\mathrm{Ker}(B)$ be the kernel of B and let mod-$\mathrm{Ker}(B)=\cap \mathrm{Ker}\, M$, where M runs over the irreducible FG-modules in B. After [8], let $\mathrm{Irr}^0(B)$ be the set of irreducible characters of height 0 in B and let $\mathrm{Ker}^0(B)=\cap \mathrm{Ker}\,\chi$, where χ runs through $\mathrm{Irr}^0(B)$. As is well known, $\mathrm{Ker}(B)$ is a p'-group and mod-ker(B) is p-nilpotent. By [8, Lemma 5.1], we have $\mathrm{Ker}(B) \subset \mathrm{Ker}^0(B) \subset$ mod-$\mathrm{Ker}(B)$.

Theorem 2. *Let B be a p-block of G with defect group D. Then we have* $\mathrm{Ker}^0(B)=\bigcap_{x\in G}(\mathrm{Ker}(B)D')^x$, *where D' is the commutator subgroup of D. In particular if D is abelian then* $\mathrm{Ker}^0(B)=\mathrm{Ker}(B)$.

Proof. Let Q be a Sylow p-subgroup of $\mathrm{Ker}^0(B)$. Then $\mathrm{Ker}^0(B)=\mathrm{Ker}(B)Q$ by [8, Lemma 5.1]. Since Q is contained in a vertex of an $\boldsymbol{o}G$-lattice affording $\chi\in\mathrm{Irr}^0(B)$, we may assume that $Q\subset D$. Let B_0 be the Brauer correspondent of B in $N_G(D)$. For any $\zeta\in\mathrm{Irr}^0(B_0)$ there is $\chi\in\mathrm{Irr}^0(B)$ such that ζ is a constituent of $\chi_{N_G(D)}$ (cf. [8, Prop.1.8]). Hence $\mathrm{Ker}^0(B)\cap N_G(D)\subset \mathrm{Ker}^0(B_0)$. In particular $Q\subset \mathrm{Ker}^0(B_0)\cap D$. By Proposition 3 for any $\zeta\in\mathrm{Irr}(B_0)$, ζ belongs to $\mathrm{Irr}^0(B_0)$ if and only if an irreducible constituent of ζ_D is linear. Therefore we see $\mathrm{Ker}^0(B_0)\cap D=D'$. So we have $Q\subset D'$.

Put $H=\bigcap_{x\in G}(\mathrm{Ker}(B)D')^x$ and let χ be any element of $\mathrm{Irr}^0(B)$. Then there exists $\zeta\in\mathrm{Irr}^0(B_0)$ such that ζ is a constituent of $\chi_{N_G(D)}$. By the above argument $\mathrm{Ker}\,\zeta\supset D'$. Hence $\chi_{\mathrm{Ker}(B)D'}$ has the trivial character of $\mathrm{Ker}(B)D'$ as an irreducible constituent, so χ_H has the trivial character of H as an irreducible constituent. Therefore $\mathrm{Ker}\,\chi\supset H$ and hence we have $\mathrm{Ker}^0(B)\supset H$. Since $\mathrm{Ker}^0(B)\subset H$, we have $\mathrm{Ker}^0(B)=H$.

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