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NONISOMORPHIC ALGEBRAIC MODELS OF NASH MANIFOLDS AND COMPACTIFIABLE C^∞ MANIFOLDS

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1. Introduction

A well-known theorem of A. Tognoli [9] asserts that every compact C^∞ submanifold M of \mathbf{R}^n with $2 \dim M + 1 \leq n$, one can find a C^∞ imbedding $e: M \rightarrow \mathbf{R}^n$, arbitrarily close in the C^∞ topology to the inclusion map $M \rightarrow \mathbf{R}$, such that $e(M)$ is a nonsingular algebraic subset of \mathbf{R}^n . In particular, M admits an algebraic model. Here an algebraic model of M means a nonsingular algebraic subset of some Euclidean space diffeomorphic to M . J. Bochnak and W. Kucharz showed in [4] that M has a continuous family of birationally nonequivalent algebraic models when M is a connected closed manifold with $\dim M \geq 1$. In this paper, we consider algebraic models of a given affine Nash manifold and a given compactifiable C^∞ manifold. Here we say that a C^∞ manifold M is *compactifiable* if there exists a compact C^∞ manifold Y with boundary such that M is C^∞ diffeomorphic to the interior of Y . M. Shiota proved in [8, Remark 6.2.11] that any affine Nash manifold admits an algebraic model. We prove that either any Nash manifold or any compactifiable C^∞ manifold have an infinite family of birationally nonequivalent algebraic models. More precisely, we prove the following.

Theorem 1. *Each affine Nash manifold M with $\dim M \geq 1$ has an infinite family of nonsingular algebraic subsets $\{X_n\}_{n \in \mathbf{N}}$ of some Euclidean space such that each X_n is Nash diffeomorphic to M and that X_n is not birationally equivalent to X_m for $n \neq m$.*

Theorem 2. *Every compactifiable C^∞ manifold M with $\dim M \geq 1$ has an infinite family of nonsingular algebraic subsets $\{X_n\}_{n \in \mathbf{N}}$ of some Euclidean space such that each X_n is C^∞ diffeomorphic to M and that X_n is not birationally equivalent to X_m for $n \neq m$.*

Theorem 2 is a refinement of [3, Corollary 3.3]. We have next

corollary because a nonsingular algebraic subset is an affine Nash submanifold.

Corollary 3. *Any nonsingular algebraic subset with $\dim M \geq 1$ has infinitely many birationally nonequivalent algebraic models in some Euclidean space.*

2. Proof of Results

Recall the notation of [3]. For any two positive integers n and d , let $P(n, d)$ denote the projective space associated with the vector space of all homogeneous polynomials in $\mathbf{R}[T_0, \dots, T_n]$ of degree d . For H in $P(n, d)$ and its representative K , we set

$$V(H) = \{[x_0 : \dots : x_n] \in \mathbf{RP}^n \mid K(x_0, \dots, x_n) = 0\},$$

$$V(H, \mathbf{C}) = \{[x_0 : \dots : x_n] \in \mathbf{CP}^n \mid K(x_0, \dots, x_n) = 0\}.$$

We will identify H with K .

A subset Σ of $P(n, d)$ is said to be *thin* if it is contained in the union of countably many algebraic subsets of $P(n, d)$ different from $P(n, d)$. We prepare next lemma to prove Theorem 1.

Lemma. *Let X be a compact irreducible nonsingular algebraic set. Let Y and Z_i ($1 \leq i \leq s$) be nonsingular algebraic hypersurfaces of X satisfying the following seven conditions:*

- (1) *The dimension of Y is greater than or equal to 2.*
- (2) *Each $W_i := Y \cap Z_i$ is nonsingular.*
- (3) *Every Z_i intersects Y transversally at any point of W_i .*
- (4) *Any W_i intersects transversally in Y one another.*
- (5) *There exist two C^∞ submanifolds Y_1 and Y_2 of Y with same boundary such that Y is the attaching space of Y_1 and Y_2 by their boundaries.*
- (6) *The union $\bigcup_{i=1}^s W_i$ is contained in Y_2 .*
- (7) *A C^∞ smoothing of the C^∞ manifold with cornered boundary $Y_3 = \{x \in Y_2 \mid \text{dist}(x, \bigcup_{i=1}^s W_i) \geq \varepsilon\}$ for small $\varepsilon > 0$ is C^∞ diffeomorphic to the cartesian product of the boundary of Y_2 and $[0, 1]$.*

Then, we can find an infinite family of nonsingular algebraic subsets $\{S_m\}_{m \in \mathbf{N}}$ of some Euclidean space such that each S_m is Nash diffeomorphic to $Y - \bigcup_{i=1}^s W_i$ and that S_m is not birationally equivalent to $S_{m'}$ for $m \neq m'$.

Proof. By [3, Proposition 2.5], there exists an algebraic imbedding

$f: X \rightarrow \mathbf{R}P^n$ (for some n), such that the Zariski closure U of $f(X)$ in $\mathbf{C}P^n$ is nonsingular. Then we identify X , Y and Z_i ($1 \leq i \leq s$) with $f(X)$, $f(Y)$ and $f(Z_i)$ ($1 \leq i \leq s$), respectively. Let h be a homogeneous polynomial of $\mathbf{R}[T_0, \dots, T_n]$ so that $Y = X \cap V(h)$. For each positive integer l , set

$$h_l = (T_0^2 + \dots + T_n^2)^l h.$$

Observe that $X \cap V(h_l) = X \cap V(h) = Y$.

By [6, Theorem 6.5 and Theorem 7.5], there exists a positive integer l_0 such that for every integer $l > l_0$, one can find a thin subset Σ_l of $P(n, \deg h_l)$ with the property that for every $H_l \in P(n, \deg h_l) - \Sigma_l$, the complex hypersurface $V(H_l, \mathbf{C})$ is nonsingular and transverse to U , and that the variety $U(l) = U \cap V(H_l, \mathbf{C})$ is irreducible. If H_l is sufficiently close to h_l then $X(l)$ defined by

$$X(l) = X \cap V(H_l) = U(l) \cap \mathbf{R}P^n$$

is a nonsingular irreducible real algebraic hypersurface of X , and the Zariski closure of $X(l)$ in $\mathbf{C}P^n$ is equal to $U(l)$. Since the sequence of the Hodge numbers $h^{d-1,0}(U(l))$, where $d = \dim X$, diverges infinity as l increases [cf 6, Proof of Lemma 1. p240], one can find an increasing sequence $\{l_m\}_{m \in \mathbf{N}}$ of positive integers so that the sequence $h^{d-1,0}(U(l_m))$ is also increasing. In particular, the varieties $U(l_m)$ and $U(l_{m'})$ are not birationally equivalent for $m \neq m'$, because the Hodge numbers $h^{p,0}$ are birational invariants [5, p190 Exercise 8.8]. It follows that $X(l_m)$ and $X(l_{m'})$ are not birationally equivalent for $m \neq m'$. If the approximation is sufficiently close, one can easily check that the triple $(X, X(l_m), \{X(l_m) \cap Z_i\}_i)$ satisfies the conditions in the lemma. Put

$$S_m = X(l_m) - \bigcup_{i=1}^s (X(l_m) \cap Z_i).$$

By the proof of [8, Remark 6.2.11], $Y - \bigcup_{i=1}^s W_i$ is Nash diffeomorphic to S_m . On the other hand, $X(l_m)$ and $X(l_{m'})$ are birationally equivalent if and only if S_m and $S_{m'}$ are birationally equivalent. Therefore $\{S_m\}_{m \in \mathbf{N}}$ is the required one. \square

We are in a position to prove Theorem 1.

Proof of Theorem 1.

If M is compact then the theorem follows from [3](or [4]) and the fact that two compact Nash manifolds are Nash diffeomorphic if and only if they are C^∞ diffeomorphic. We now suppose that M is not compact. Using [7], we can assume that M is the interior of a compact

C^∞ manifold with boundary L_1 . By [2], there exist a compact C^∞ manifold with boundary L_2 in some \mathbf{R}^n and compact C^∞ submanifolds X_i ($1 \leq i \leq s$) of $\text{Int } L_2$ such that the following three conditions are satisfied: (1) the boundary of L_2 is C^∞ diffeomorphic to the boundary of L_1 , (2) each X_i intersects transversally one another, and (3) a C^∞ smoothing of the C^∞ manifold with cornered boundary $\{x \in L_2 \mid \text{dist}(x, X) \geq \varepsilon\}$ for small $\varepsilon > 0$ is C^∞ diffeomorphic to the cartesian product of the boundary of L_2 and $[0, 1]$. Here $X = \bigcup_{i=1}^s X_i$.

Let L denote the attaching space of L_1 and L_2 by the above diffeomorphism between their boundaries. We regard M, L_1, L_2 and X_i as submanifolds of L .

On the other hand, if $\dim M = 1$ then, we can apply the method of the proof of [4] to (L, X) because X consists of finite points. Therefore we have a continuous family of birationally nonequivalent algebraic models of M . But in general we do not know whether we are able to apply or not.

We return to the proof of Theorem 1. We may assume $\dim M \geq 2$. According to a relative Nash theorem [1], one can imbed L in some \mathbf{R}^l so that L is a nonsingular algebraic subset of \mathbf{R}^l and each X_i is nonsingular algebraic subset of L . Moreover, by blowing up all X_i , we may suppose that every X_i is of codimension 1 in L . By the proof of [8, Remark 6.2.11], M is Nash diffeomorphic to $L - X$.

We now construct an irreducible algebraic model Y of L so that each $f(X_i)$ is a nonsingular algebraic subset of Y , where f is a diffeomorphism from L to Y . Fix an algebraic imbedding $L \rightarrow \mathbf{R}P^l$ and we identify L , X and X_i with its images, respectively. It follows from the proof of [3, Corollary 3.3] that there exists an irreducible nonsingular algebraic curve C in L satisfying the following three conditions: (1) C has a trivial normal bundle in L , (2) C is a disjoint s C^∞ curves C_1, \dots, C_s , each connected component of L containing precisely one C_i and (3) C does not intersect X .

Consider the triple $C \subset L \subset \mathbf{R}P^l$. Since $H_*^{alg}(\mathbf{R}P^l, \mathbf{Z}_2) = H_*(\mathbf{R}P^l, \mathbf{Z}_2)$ ($\mathbf{R}P^l$ has totally algebraic homology), it follows from the proof of [2, Theorem 3.1] that there exists a positive integer k and a Nash imbedding $e: L \times \{0\} \rightarrow \mathbf{R}P^l \times \mathbf{R}^k$ such that $Y := e(L)$ is a nonsingular algebraic subset of $\mathbf{R}P^l \times \mathbf{R}^k$, each $Z_i := e(X_i \times \{0\})$ is a nonsingular algebraic subset of $\mathbf{R}P^l \times \mathbf{R}^k$, and $C \times \{0\} \subset Y$. Since each connected component of Y contains a connected component of $C \times \{0\}$, Y is the required irreducible algebraic model of L .

Applying Lemma to the triple $(Y \times \mathbf{R}P^1, Y \times \{a\}, Z_i \times \mathbf{R}P^1)$ (for some $a \in \mathbf{R}P^1$), we have the desired family. \square

Proof of Theorem 2.

It is known that any compactifiable C^∞ manifold admits a nonaffine Nash manifold structure [7]. To apply Theorem 1, we show that each nonaffine Nash manifold M is C^∞ imbeddable into some Euclidean space as an affine Nash manifold. We may assume that M is connected. Using [7], we have a compact C^∞ manifold Y' with boundary K so that M is C^∞ diffeomorphic to the interior of Y' . Let Y be the double of Y' . By a relative Nash theorem [1], there exist nonsingular algebraic sets Z and Z' such that (Y, K) is pairwise C^∞ diffeomorphic to (Z, Z') . Since $Z - Z'$ is an affine Nash manifold, M is C^∞ diffeomorphic to a connected component of $Z' - Z$. An application of Theorem 1 completes the proof. \square

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