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ON THE ABEL-JACOBI MAP
FOR NON-COMPACT VARIETIES

DONU ARAPURA and KYUNGHO OH

(Received July 10, 1996)

1. Introduction

Let \( X \) be a smooth projective variety over \( \mathbb{C} \) of dimension \( n \) and \( S \) be a reduced normal crossing divisor on \( X \). Then the generalized Jacobian \( J(X-S) \) is a group \( H^{n-1}(X,\omega_X(S))/H_{2n-1}(X-S,\mathbb{Z}) \). When \( X \) is a curve, this fits into an exact sequence of algebraic groups:

\[
1 \rightarrow (\mathbb{C}^\times)^{\sigma-1} \rightarrow J(X-S) \rightarrow J(X) \rightarrow 0
\]

where \( \sigma \) is the number of points in \( S \) and \( J(X) \) is the usual Jacobian of \( X \). Let \( \text{Div}^0(X-S) \) be the set of divisors of degree 0 on \( X \) which does not intersect with \( S \). Then integration determines the Abel-Jacobi homomorphism \( \alpha : \text{Div}^0(X-S) \rightarrow J(X-S) \). We will prove an analogue of Abel's theorem (due to Rosenlicht [8] for curves) that the kernel of \( \alpha \) is the following subgroup \( \text{Prin}_S(X) \) of \( S \)-principal divisors:

\[
\text{Prin}_S(X) = \{(f) \in \text{Div}(X-S) \mid f \in K(X) \text{ and } f = 1 \text{ on } S\}.
\]

A proof is a variation of our previous work [1], which involves reinterpretation of the Abel-Jacobi map in the language of mixed Hodge structures and their extensions. As a further application of this technique, we prove a Torelli theorem for a non-compact curve, which states that if \( X \) is the complement of at least 2 points in a nonhyperelliptic curve, then it is determined by the graded polarized mixed Hodge structure on \( H^1(X,\mathbb{Z}) \).

We would like to thank the referee for thoughtful comments.

2. Hodge Structures

**Definition 2.1.** A (pure) Hodge structure \( H \) of weight \( m \) consists of a finitely generated abelian group \( H_{\mathbb{Z}} \) and a decreasing filtration \( F^* \) of \( H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C} \) such that \( H_{\mathbb{C}} = F^p \oplus F^{m-p+1} \).

1991 Mathematics Subject Classification: 14H40, 14C30.
Example 1. The Hodge structure of Tate $\mathbb{Z}(-1)$ is defined to be the Hodge structure of weight 2 with $H_{2} = \frac{1}{2\pi \sqrt{-1}} \in \mathbb{C} = F^1 H_{\mathbb{C}}$.

The most natural example of Hodge structure of weight $k$ is the $k$-th integral cohomology of a compact Kähler manifold. A differential form lies in $F^p$ if in local coordinate it has at least $p$ "$dz$'s". To extend Hodge theory to any (singular or non-projective) complex algebraic varieties $X$, Deligne [3] introduced the notion of a mixed Hodge structure. He showed that the cohomology of any variety carries such a structure.

Definition 2.2. A mixed Hodge structure (MHS) $H$ consists of a triple $(H_{\mathbb{Z}}, W_{\star}, F^{\star})$, where

1. $H_{\mathbb{Z}}$ is a finitely generated abelian group. (In practice $H_{\mathbb{Z}}$ will be free and we will identify it with a lattice in $H_{Q} := H_{\mathbb{Z}} \otimes \mathbb{Q}$.)
2. $W_{\star}$ is an increasing filtration of $H_{Q}$, called the weight filtration.
3. $F^{\star}$ is a decreasing filtration of $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$, called the Hodge filtration.

The Hodge filtration $F^{\star}$ is required to induce a (pure) Hodge structure of weight $m$ on each of the graded pieces

$$G_{r_{m}}^{W_{\star}} = W_{m}/W_{m-1}$$

Example 2. Let $D$ be a divisor on a smooth projective variety $X$ over $\mathbb{C}$. Set $U = X - D$. By Hironaka, there exists a birational map $\pi: \tilde{X} \to X$, with $\tilde{X}$ non-singular such that $\tilde{D} = \pi^{-1}(D)$ is a normal crossing divisor. Then $H^1(U, \mathbb{Z})$ carries a mixed Hodge structure and the Hodge filtration is given by

$$F^0 = H^1(U, \mathbb{C}), \quad F^1 = H^0(\tilde{X}, \Omega^1(\log \tilde{D})), \quad F^2 = 0.$$

We will denote $H^0(\tilde{X}, \Omega^1(\log \tilde{D}))$ by $H^0(X, \Omega^1(\log D))$. This group does not depend on the choice of $\tilde{X}$.

Given two mixed Hodge structures $A$ and $B$, we write $B > A$ if there exists $m_0$ such that $W_{m}A_{Q} = A_{Q}$ for all $m \geq m_0$ and $W_{m}B_{Q} = 0$ for all $m < m_0$.

Finally, we define the $p$-th Jacobian of a mixed Hodge structure of $H$ to be the generalized torus

$$J^p H = H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^p H_{\mathbb{C}}.$$

The set of mixed Hodge structures forms an abelian category with an internal Hom. Thus one can form the abelian group of extension classes of two objects. Carlson [2] described the structure of this extension group in terms of the Jacobian.
Theorem 2.1 (Carlson). Let $A$ and $B$ be mixed Hodge structures with $B > A$ and $B$ torsion free. Then there is a natural isomorphism.

$$\text{Ext}^1_{MHS}(B, A) \cong J^0\text{Hom}(B, A).$$

3. Homologically trivial divisors

Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$ and $S$ be a reduced normal crossing divisor on $X$. Let $\text{Div}(X - S)$ be the group of divisors on $X$ which do not intersect $S$. Moreover, we set

1. $\text{Prin}_S(X) = \{(f) \in \text{Div}(X - S) | f \in K(X) \text{ and } f = 1 \text{ on } S\}$
2. $\text{Cl}_S(X) = \text{Div}(X - S)/\text{Prin}_S(X)$.

The kernel of the cycle map [5, §19.1]

$$cl : \text{Div}(X - S) \to H_{2n-2}(X - S, \mathbb{Z})$$

will be called the group of homologically trivial divisors and it will be denoted by $\text{Div}^0(X - S)$. Note that $\text{Prin}_S(X) \subset \text{Div}^0(X - S)$.

Let $\mathcal{K}^*$ be the sheaf of invertible rational functions on $X$ and $\mathcal{K}^*(-S)$ be the subsheaf of $\mathcal{K}^*$ consisting of functions which are 1 on $S$. Similarly, we define $\mathcal{O}^*(-S)$ to be the subsheaf of $\mathcal{O}^*$ consisting of functions which are 1 on $S$. Consider the following exact sequence

$$1 \longrightarrow \mathcal{O}^*(-S) \longrightarrow \mathcal{K}^*(-S) \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{Q}$ is the quotient sheaf. Then one can prove that $H^0(X, \mathcal{K}^*(-S)) = \text{Prin}_S(X)$ and $H^0(X, \mathcal{Q}) = \text{Div}(X - S)$ as in [7, II, 6.11]. Let

$$\text{Cl}^0_S(X) = \text{Div}^0(X - S)/\text{Prin}_S(X).$$

Consider the following diagram:

$$
\begin{array}{ccc}
H^0(X, \mathcal{Q}) & \longrightarrow & H^1(X, \mathcal{O}^*(-S)) \\
\downarrow & & \downarrow \frac{1}{2\pi i}d\log \\
\text{Div}(X - S) & \longrightarrow & H_{2n-2}(X - S, \mathbb{Z})
\end{array}
$$

The map $1/2\pi id\log$ is the connecting homomorphism associated to the exponential sequence:

$$
\begin{array}{c}
0 \longrightarrow j_!\mathbb{Z} \longrightarrow \mathcal{O}(-S) \xrightarrow{\exp(2\pi i)} \mathcal{O}^*(-S) \longrightarrow 1
\end{array}
$$
where \( j \) is the natural inclusion from \( X - S \) to \( X \). By Lefschetz duality [9, Theorem 6.2.19], the right vertical arrow is an isomorphism. Moreover, the diagram is commutative since the cycle map is compatible with Chern class map. Therefore, \( Cl^0_S(X) \) is isomorphic to a subgroup of the kernel of the connecting homomorphism

\[
H^1(X, \mathcal{O}^*(-S)) \xrightarrow{\frac{1}{2\pi i} \log} H^2(X, j_!\mathbb{Z}) = H^2(X, S).
\]

So, \( Cl^0_S(X) \) is isomorphic to a subgroup of \( H^1(X, \mathcal{O}(-S))/H^1(X, S; \mathbb{Z}) \). By duality, we can identify \( H^1(X, \mathcal{O}(-S))/H^1(X, S; \mathbb{Z}) \) with \( H^{n-1}(X, \omega_X(S))/H_{2n-1}(X - S, \mathbb{Z}) \) where \( \omega_X(S) = \wedge^n \Omega_X \otimes \mathcal{O}_X(S) \). Thus we obtain an injection

\[
(5) \quad \beta : Cl^0_S(X) \to H^{n-1}(X, \omega_X(S))/H_{2n-1}(X - S, \mathbb{Z}),
\]

which will be identified with the Abel-Jacobi map later.

**Lemma 3.1.** In the diagram,

\[
\begin{array}{ccc}
\text{Div}(X - S) & \xrightarrow{\text{cl}_{X-S}} & H_{2n-2}(X - S) \\
\downarrow \text{cl}_X & & \downarrow \text{H}_{2n-2}(X) \\
& \text{H}_{2n-2}(X) &
\end{array}
\]

the kernel of the map \( \text{cl}_{X-S} \) is equal to the kernel of the map \( \text{cl}_X \).

**Proof.** Clearly, we have \( \ker \text{cl}_{X-S} \subset \ker \text{cl}_X \). We will prove the converse in dual form. Let \( D \in \ker \text{cl}_X \). Consider a long exact sequence of Mixed Hodge structures, so called a "Thom-Gysin" sequence, associated to a triple \( S \subset X - |D| \subset X \);

\[
(6) \quad 0 \to H^1(X, S) \to H^1(X - |D|, S) \to H^2(X, X - |D|) \xrightarrow{\text{Gysin}} H^2(X, S)
\]

Note that \( H^2(X, X - |D|) \cong H^2_D(X) \). By Fujiki [4], we have

\[
H^2(X, X - |D|) = \text{Hom}(H^{2n-2}(D), \mathbb{Z}(-n))
\]

as a mixed Hodge structure. Also observe that \( H^{2n-2}(D) = \bigoplus \mathbb{Z}(-n + 1) \) where the sum is over all irreducible components of \( D \). Thus \( H^2(X, X - |D|) \) has a pure Hodge structure of weight 2. On the other hand, it follows from the long exact sequence of cohomologies associated to the pair \( (X, S) \) that \( Gr^W_2 H^2(X, S) \) injects into \( H^2(X) \). Hence if the class of \( D \) in \( H^2(X) \) vanishes, then so does the class of \( D \) in \( H^2(X, S) \).
4. Extensions of MHS

Let $D \in \text{Div}^0(X - S)$ be a homologically trivial divisor. From the sequence (6), we get an extension of mixed Hodge structures

$$0 \rightarrow H^1(X, S) \rightarrow H^1(X - |D|, S) \rightarrow K \rightarrow 0$$

where $K = \ker[H^2(X, X - |D|) \xrightarrow{\text{Gysin}} H^2(X, S)]$. Let

$$\phi_D : \mathbb{Z}(-1) \rightarrow \bigoplus \mathbb{Z}(-1) = H^2(X, X - |D|)$$

be a morphism of Hodge structures defined by $\phi_D(1/2\pi\sqrt{-1}) = \sum D_i$ where $D_i$ are irreducible components of $D$. Since $D$ is homologically trivial, $\phi_D$ factors through $K$. By pulling back the extension (7) along $\phi_D$, we get a new extension of mixed Hodge structures:

$$0 \rightarrow H^1(X, S) \rightarrow E_D \rightarrow \mathbb{Z}(-1) \rightarrow 0.$$ 

Thus this corresponds to an element in $\text{Ext}^1(\mathbb{Z}(-1), H^1(X, S))$. By a theorem of Carlson, $\text{Ext}^1(\mathbb{Z}(-1), H^1(X, S))$ is isomorphic to

$$J^0\text{Hom}(\mathbb{Z}(-1), H^1(X, S)) = H^1(X, S; \mathbb{C})/H^1(X, S; \mathbb{Z}) + F^1H^1(X, S; \mathbb{C}),$$

which will be denoted by $J(X - S)$. Note that $J(X - S)$ is independent of the choice of a compactification of $X - S$.

**Lemma 4.1.** The Jacobian $J(X - S)$ is naturally isomorphic to

$$H^1(X, \mathcal{O}(-S))/H^1(X, \mathbb{Z}).$$

**Proof.** Consider an exact sequence of cohomologies on $X$.

$$\ldots \rightarrow H^0(X, \mathcal{C}_S) \rightarrow H^1(X, j_! \mathcal{C}) \rightarrow H^1(X, \mathcal{C}) \rightarrow \ldots$$

where $j$ is the natural inclusion from $X - S$ to $X$. Since this is an exact sequence of mixed Hodge structures and the Hodge filtrations are strictly preserved by the maps, this induces an exact sequence:

$$\ldots \rightarrow \text{Gr}^0_F H^0(X, \mathcal{C}_S) \rightarrow \text{Gr}^0_F H^1(X, j_! \mathcal{C}) \rightarrow \text{Gr}^0_F H^1(X, \mathcal{C}) \rightarrow \ldots$$

Now consider the following diagram of cohomologies on $X$:

$$\begin{array}{cccccccc}
\text{Gr}^0_F H^0(X, \mathcal{C}_X) & \rightarrow & \text{Gr}^0_F H^0(\mathcal{C}_S) & \rightarrow & \text{Gr}^0_F H^1(j_! \mathcal{C}) & \rightarrow & \text{Gr}^0_F H^1(\mathcal{C}_X) & \rightarrow & \text{Gr}^0_F H^1(\mathcal{C}_S) \\
\downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_1 & & \downarrow \alpha_1 & & \downarrow \beta_1 \\
H^0(\mathcal{O}_X) & \rightarrow & H^0(\mathcal{O}_S) & \rightarrow & H^1(\mathcal{O}(-S)) & \rightarrow & H^1(\mathcal{O}_X) & \rightarrow & H^1(\mathcal{O}_S)
\end{array}$$
The vertical arrows $\beta_0$ and $\beta_1$ are isomorphisms because the spectral sequence associated to the Hodge filtration on $H^*(S, \mathbb{Z}_S)$ degenerates at $E_1$ \cite[(1.5)]{3}, (8.2.1), (8.1.12), (8.1.9). The vertical arrows $\alpha_0$ and $\alpha_1$ are isomorphisms by the $E_1$-degeneration of the usual Hodge to DeRham spectral sequence. Hence by 5-lemma, the map $\gamma_1$ is an isomorphism. It follows that the sequence

$$0 \to H^0(d\mathcal{O}(-S)) \to H^1(j_!\mathcal{C}) \to H^1(\mathcal{O}(-S)) \to 0$$

is exact. Note that $F^1H^1(j_!\mathcal{C}) = H^0(d\mathcal{O}(-S))$. This completes the proof. \hfill \square

Thus for a homologically trivial divisor $D \in \text{Div}^0(X - S)$, we can associate an element in the Jacobian $H^1(X, \mathcal{O}(-S))/H^1(X, S; \mathbb{Z})$. By duality, the Jacobian can be identified with

$$H^{n-1}(X, \omega_X(S))/H_{2n-1}(X - S; \mathbb{Z}).$$

The map

$$\alpha : \text{Div}^0(X - S) \to H^{n-1}(X, \omega_X(S))/H_{2n-1}(X - S; \mathbb{Z})$$

obtained in this way will be called the \textit{Abel-Jacobi} map. We will show that the Abel-Jacobi map $\alpha$ can be realized in the following way;

\textbf{Theorem 4.2.} \textit{Given a cohomology class in} $H^{n-1}(X, \omega_X(S))$, \textit{choose a} $(n, n-1)$-\textit{form} $\omega$ \textit{representing this cohomology class. Then} $\alpha(D)$ \textit{is given by}

$$\omega \mapsto \int_{\Gamma_D} \omega$$

\textit{where} $\Gamma_D$ \textit{is a} $(2n-1)$-\textit{chain in} $X - S$ \textit{whose boundary is} $D$.

\textbf{Proof.} After a birational change of $X$, we may assume that the support of $D$ is a reduced normal crossing divisor. To each holomorphically trivial divisor $D \in \text{Div}^0(X - S)$, one can associate a form $\eta_D \in H^0(X, \Omega_X^1(\log |D|)) = F^1H^1(X - |D|, \mathbb{C})$ with $\text{Res}\eta_D = D$ since the map in the sequence (8) strictly preserves the Hodge filtration $F^*$ and $F^1H^1(X - |D|, S; \mathbb{C}) \subset F^1H^1(X - |D|, \mathbb{C})$. To construct a retraction $r : E_D \to H^1(X, S; \mathbb{Z})$, choose a set $\{\xi_1, \cdots, \xi_m\}$ of differential $(2n-1)$-forms on $X - S$ representing a basis of $H^{2n-1}(X - S, \mathbb{Z})$ such that $\xi_i$ vanishes in a neighborhood $N(D)$ of $|D|$. This is possible since we have a surjection $H^{2n-1}(X - S, D) \to H^{2n-1}(X - S)$. Let $\{\xi_1, \cdots, \xi^m\}$ be the dual basis of $H^1(X, S; \mathbb{Z})$. We now set

$$r(\eta) = \sum_i \int_X (\eta \wedge \xi_i)\xi^i.$$
Let $B(D)$ be a small tubular neighborhood of $|D|$ in $X - S$ such that the closure of $B(D)$ is contained in $N(D)$. We can write

$$\omega = \sum_{i=1}^{m} c_i \xi_i + d\phi$$

where $\phi$ is a $C^\infty(2n-2)$-form on $X - S$. Set $\eta = \eta_D$. Via the isomorphism given in Theorem 2.1 (cf. [1, Theorem 6.2]), $\alpha(D)$ is given by sending $\omega$ to

$$\int_X r(\eta) \wedge \omega$$

$$= \int_X r(\eta) \wedge \sum_i c_i \xi_i = \int_X \left( \sum_i \left( \int_X \eta \wedge \xi_i \right) \xi_i \right) \wedge \left( \sum_j c_j \xi_j \right)$$

$$= \int_X \eta \wedge \sum_i c_i \xi_i = \int_{X-B(D)} \eta \wedge (\omega - d\phi)$$

$$= \int_{X-B(D)} -\eta \wedge d\phi$$

(since $\eta \wedge \omega = 0$ on $X - B(D)$)

$$= \int_{\partial B(D)} \eta \wedge \phi$$

(by Stokes' Theorem.)

$$= \int_{\partial B(D)} \eta \wedge \int_\omega$$

($\int_\omega$ is a primitive of $\omega$ on $B(D)$)

$$= \int_\omega$$

Since $D$ is also algebraically equivalent to zero [6, p. 462] it is enough to consider the following case: Let $T$ be a non-singular curve and $\mathcal{D}$ be an irreducible divisor on $X \times T$, flat over $T$. $D$ is given by $p_{2*}(p_1^*(0) - p_1^*(1))$ for some points $0, 1 \in T$.

$$\mathcal{D} \subset X \times T \xrightarrow{p_2} X$$

$$\xrightarrow{p_1}$$

$$T$$

Let $\mathcal{D}$ be a desingularization of $\mathcal{D}$ and $p'_1$ be the composition of $\mathcal{D} \to \mathcal{D}$ and $p_1$. Let
be the Stein factorization of \( \tilde{D} \rightarrow X \). So \( f \) has connected fibers and \( g \) is a finite surjective map. Now choose a path \( \gamma \) from 0 to 1 in \( T \) such that \( p_1' \) is smooth over \( \gamma - \partial \gamma \). Let \( \Gamma_D = p_1'^{-1}(\gamma), \Gamma_D' = f_* \Gamma_D \) and \( \Gamma_D = g_* \Gamma_D' \). Take a division \( \gamma = \Sigma_i \gamma_i \) of \( \gamma \) so that \( p_1' \) is trivial over \( \gamma_i \). Set \( (\tilde{\Gamma}_D)_i = p_1'^{-1}(\gamma_i), (\Gamma_D')_i = f_*(\tilde{\Gamma}_D)_i \). Then each \( (\tilde{\Gamma}_D)_i \) shrinks to a fiber, hence \( H^{2n-1}((\tilde{\Gamma}_D)_i, \mathbb{C}) = 0 \) and so \( H^{2n-1}((\Gamma_D')_i, \mathbb{C}) = 0 \). Therefore we have

\[
\int_D \int \omega = g_* \left( \int_{\Gamma_D} g^* \left( \int \omega \right) \right)
\]

where \( D' = f_*(p_1'^*(0) - p_1'^*(1)) \)

\[
= g_* \left( \Sigma_i \int_{\partial(\Gamma_D')_i} g^* \left( \int \omega \right) \right)
\]

\[
= g_* \left( \Sigma_i \int_{\Gamma_D'} g^* \omega \right) \quad \text{by Stokes' theorem}
\]

\[
= g_* \left( \int_{\Gamma_D} g^* \omega \right) = \int_{\Gamma_D} \omega
\]

Note that \( g^*(\int \omega) \) is extendable to \( \Gamma_D' \) since \( H^{2n-1}(\Gamma_D', \mathbb{C}) = 0 \).

Note that when \( S = \emptyset \), our Abel-Jacobi map agrees with the classical Abel-Jacobi map. The initial step of the proof contains a useful method for calculating \( \alpha \). Under the original definition

\[
J(X - S) = H^1(X, S; \mathbb{C})/H^1(X, S; \mathbb{Z}) + F^1 H^1(X, S; \mathbb{C})
\]

\( \alpha(D) \) is represented by \( r(\eta_D) \), where \( \eta_D \in F^1 H^1(X - D, S; \mathbb{C}) \) is given by a logarithmic 1-form with \( \text{Res} \eta_D = D \) and \( r \) an integral retraction onto \( H^1(X, S; \mathbb{C}) \). Note that a form in \( H^0(\Omega^1_X(\log |D|)) \) defines a class in \( H^1(X - D, S) \) if and only if it vanishes on \( S \).

5. Abel's Theorem

We will establish Abel's Theorem by showing that the two definitions of the Abel-Jacobi map \( \alpha \) and \( \beta(S) \) agree up to sign.
Theorem 5.1. \( \alpha \) and \( \beta \) coincide up to sign.

Proof. First, we will give an explicit description of the map \( \beta \) in (5). By construction \( \beta \) is a composite of the injection

\[
\beta': Cl^0_S(X) \longrightarrow H^1(O(-S))/H^1(X,S;\mathbb{Z})
\]

and the duality map

\[
H^1(O(-S))/H^1(X,S;\mathbb{Z}) \cong H^{n-1}(X,\omega_X(S))/H_{2n-1}(X-S,\mathbb{Z}).
\]

Let \( D \) be a homologically trivial divisor on \( X-S \). Choose a finite open covering \( \{U_i\}_{i=0,\ldots,m} \) of \( X \) such that \( D \) is defined by \( f_i = 0 \) on \( U_i \) and \( U_0 = X - |D| \). As \( S \) and \( D \) are disjoint, there is no loss in assuming that \( f_i = 1 \) on \( S \). Then the cohomology class \([D]\) of \( D \) in \( H^1(O^*(-S)) \) can be represented by \( \{f_i/f_j\} \). Since \( D \) is homologically trivial, there is a cocycle \( \phi_{ij} \in Z^1(O(-S)) \) such that \( \exp(2\pi i \phi_{ij}) = f_i/f_j \). Then \( \beta'(D) \) is represented by \( \phi_{ij} \).

Next we calculate \( \alpha(D) \). We make use of the identification

\[
J(X-S) \cong H^1(O(-S))/H^1(X,S;\mathbb{Z})
\]

to view \( \alpha(D) \) as an element of the latter group. By degeneration of the Hodge to De Rham spectral sequence, there exists \( \psi_i \in H^0(O_X(U_i)) \) such that \( d\psi_i = \psi_j - \psi_i \). Therefore \( \eta = 1/2\pi id\log f_i + \psi_i \) is a globally defined logarithmic 1-form satisfying \( \text{Res}\eta = D \) which also vanishes on \( S \). As explained in the remarks at the end of the last section, \( \alpha(D) \) is represented by \( r(\eta) \), and in fact we are free to modify \( \eta \) by adding an element of \( F^1H^1(X,S) \). Note that \( H^1(X-D,S;\mathbb{C}) \) is isomorphic to the first hypercohomology group of \( \Omega^1_X(\log D+S)(-S) \), and this can be described using Cech methods. In particular \((\phi_{ij}, d\log f_i)\) is a cocycle defining a class \( \Phi \in H^1(X-D,S;\mathbb{C}) \). We claim that \( \Phi \) can be decomposed as a sum \( \Phi_1 + \Phi_2 \) where the first term lies in \( L = H^1(X-D,S;\mathbb{C}) \) and the second in \( F^1H^1(X-D,S) \). To see this, first observe that the quotient \( H^1(X, O(-S)) \) by the image of \( L \) is isomorphic to the quotient of \( J(X-S) \) by the subgroup of homologically trivial divisors with support in \( |D| \) (under \( \beta' \)). Therefore as the image of \( \Phi \) in \( J(X-S) \) is \( D \), it follows that modulo \( L, \Phi \) can be represented by a form in \( F^1H^0(X-D,S) \). As \( \Phi \) has integral residues, it follows that after subtracting off an addition element of \( L \), the difference lies in \( F^1H^1(X,S) \). In other words, we have obtained the desired decomposition of \( \Phi \). Now set \( \eta' = \eta - \Phi_2 \). Let \( E_D \subset H^1(X-D,S) \) be the extension defined in (8). Consider the unique retraction \( \eta:E_D \to H^1(X,S;\mathbb{Z}) \) with kernel \( Z\Phi_1 \). Then \( \eta' = a\Phi_1 + r(\eta') \), and by matching residues, we see that \( a = 1 \). Therefore \( r(\eta') = \eta - \Phi = -(\phi_{ij}, \psi_i) \) represents \( \alpha(D) \). But of course \( \alpha(D) \) is the image of this class in \( J(X-S) \) and this is represented by \( -\phi_{ij} \), or \(-\beta'(D)\). \( \square \)
6. Hodge Theoretic Proof of Abel’s Theorem

We give an alternative proof of Abel’s theorem based on Carlson’s theorem.

**Theorem 6.1.** A homologically trivial divisor $D \in \text{Div}^0(X - S)$ is $S$-principal if and only if there exists $\eta \in H^0(X, \Omega^1_X(\log |D|))$ such that

1. $\text{Res}_D \eta = D$
2. $\eta$ has integral periods for any closed loop in $X - |D|$
3. $\int_\gamma \eta \in \mathbb{Z}$ where $\gamma$ is a path in $X - S$ connecting points of $S$.

By the way, the statement (2) is included in (3).

**Proof.** Given $\eta$ as above, set

$$f(z) = \exp\left(2\pi \sqrt{-1} \int_{z_0}^{z} \eta \right).$$

Conversely, if $D = (f) \in \text{Prin}_S(X)$, let

$$\eta = \frac{1}{2\pi \sqrt{-1}} \frac{df}{f}.$$

**Corollary 6.2.** $\alpha(D) = 0$ if and only if $D$ is $S$-principal.

**Proof.** $\alpha(D) = 0$ if and only if the extension (8) splits in the category of Mixed Hodge Structures. Hence $\alpha(D) = 0$ if and only if $\eta_D$ represents an integral class in $H^1(X - |D|, S)$. Thus $\alpha(D) = 0$ if and only if $\eta_D$ satisfies the conditions in Theorem 6.1.

7. Non-compact Curves

When $X$ is a curve, $J(X - S)$ is an extension of the classical Jacobian $J(X)$ by the complex multiplicative group.

**Lemma 7.1.** Let $X$ be a smooth projective curve and $S$ be a set of distinct points. Then we have an exact sequence of algebraic groups:

$$1 \rightarrow (\mathbb{C}^*)^{\sigma - 1} \rightarrow J(X - S) \rightarrow J(X) \rightarrow 0$$

where $\sigma$ is the number of points in $S$ and $J(X)$ is the usual Jacobian of $X$.

**Proof.** Consider an exact sequence of cohomologies on $X$:

$$0 \rightarrow H^0(\Omega^1_X) \rightarrow H^0(\Omega^1_X(\log S)) \rightarrow H^0(O_S) \rightarrow H^\infty(\otimes^\infty_X) \rightarrow H^\infty(\otimes^\infty_X(\log S)) \rightarrow 1$$
By Serre duality, \( H^1(X, \Omega^1_X(\log S)) = H^0(X, \mathcal{O}(-S)) = 1 \) and \( h^1(X, \Omega^1_X) = 1 \). This sequence fits into the following diagram:

\[
0 \rightarrow H_2(X, X-S)/H_2(X, \mathbb{Z}) \rightarrow H_1(X-S, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z}) \rightarrow 0
\]

The cokernels of the vertical arrows will give the desired sequence. The cokernel of the leftmost arrow is identified with the multiplicative group \((\mathbb{C}^*)^\sigma-1\) via the exponential map \( \exp(2\pi i(\ )) \).

**Example 3.** Let \( X = \mathbb{P}^1 \) and \( S = \{0, \infty\} \). Then \( H^0(X, \omega_X(S)) \) is generated by \( dz/z \). By the above Lemma, we have \( J(X-S) = \mathbb{C}^* \). By Theorem 4.2, the Abel-Jacobi map \( \alpha : \text{Div}^0(X-S) \rightarrow \mathbb{C}^* \) is the natural linear extension of

\[
\alpha(x-1) = \exp \int_1^x \frac{dz}{z} = x, \quad \text{for } x \in X-S
\]

if we choose \( 1 \in X-S \) as a base point. Thus \( \ker \alpha = \{\sum n_p - (\sum n_p) \cdot 1 \in \text{Div}^0(X-S) | \Pi \Pi^p = 1\} \). On the other hand, a rational function \( f \) on \( X \) is in \( \text{Prin}_S(X) \) iff

\[
f(z) = \frac{\Pi_{i=1}^n(z - a_i)}{\Pi_{i=1}^n(z - b_i)}
\]

with \( \Pi a_i = \Pi b_i \neq 0, \infty \). As expected by our Abel-Jacobi theorem, \( \ker \alpha = \text{Prin}_S(X) \).

As an application, we give a version of Torelli theorem for noncompact curves. A similar result for complete singular curves was obtained by Carlson [2]. Let \( X \) be a smooth non-compact curve and \( \bar{X} \) its unique smooth compactification. Then the mixed Hodge structure on \( H^1(X, \mathbb{Z}) \) carries a natural graded polarization given as follows: The polarization on \( \text{Gr}^W_1H^1(X, \mathbb{Z}) \) is induced by the polarized Hodge structure on \( H^1(\bar{X}, \mathbb{Z}) \), which is determined by the intersection product of one-cycles on \( \bar{X} \). For \( \text{Gr}^W_2H^1(X, \mathbb{Z}) \), choose the unique symmetric bilinear form on \( \bigoplus_{i=1}^n \mathbb{Z}(-1) \) so that \( \{e_i\} \) forms an orthonormal basis. Then restrict this polarization to \( \text{Gr}^W_2H^1(X, \mathbb{Z}) \).

**Theorem 7.2.** Let \( X \) be a smooth non-compact curve and \( \bar{X} \) its unique smooth compactification. Suppose \( \bar{X} \) is non-hyperelliptic of genus \( > 1 \) and the number of points in \( \bar{X} - X \) at least 2. Then \( X \) is determined by the graded polarized MHS on \( H^1(X, \mathbb{Z}) \).
Proof. Let $\tilde{X} - X = \{p_1, \ldots, p_n\}$. Consider the ‘Thom-Gysin’ sequence:

$$
0 \longrightarrow H^1(\tilde{X}, \mathbb{Z}) \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}(-1) \longrightarrow H^2(\tilde{X}, \mathbb{Z}) = \mathbb{Z}(-1)
$$

where each point $p_j$ contributes to the $j$-th component vector $\{e_j\}$ of $\bigoplus_{i=1}^n \mathbb{Z}(-1)$. Note that $K = \ker(\bigoplus_{i=1}^n \mathbb{Z}(-1) \longrightarrow H^2(\tilde{X}, \mathbb{Z}) = \mathbb{Z}(-1))$ is just $Gr^W_2 H^1(X, \mathbb{Z})$ and $H^1(\tilde{X}, \mathbb{Z}) = Gr^W_1 H^1(X, \mathbb{Z})$. Now we provide a polarization on $H^1(X, \mathbb{Z})$.

First, by the classical Torelli theorem, the polarization on $Gr^W_1 H^1(X, \mathbb{Z})$ determines $\tilde{X}$. Second, define a map $\phi_{ij} : \mathbb{Z}(-1) \longrightarrow \bigoplus_{i=1}^n \mathbb{Z}(-1)$ sending $1/2\pi\sqrt{-1}$ to $e_i - e_j$. Then these maps are all possible maps from $\mathbb{Z}(-1)$ to $\bigoplus_{i=1}^n \mathbb{Z}(-1)$ which factors through $K$ and minimizes the length of the image of the generator $1/2\pi\sqrt{-1}$. By pulling back along $\phi_{ij}$, we get an element in $\text{Ext}^1(\mathbb{Z}(-1), H^1(\tilde{X}, \mathbb{Z})) \cong J(\tilde{X})$, which depends only on the polarized Hodge structure on $Gr^W_2 H^1(X, \mathbb{Z})$. This corresponds to $\alpha(p_i - p_j) \in J(\tilde{X})$ under the Abel-Jacobi map $\alpha$ [1, Theorem 6.2]. As $\tilde{X}$ is not hyperelliptic, $\alpha(p_i - p_j)$ uniquely determines $p_i$ and $p_j$. Otherwise, there exists a meromorphic function $f$ on $\tilde{X}$ such that $(f) = p_i + p - p_j - q$ by the classical Abel’s theorem. Therefore the graded polarized MHS on $H^1(X, \mathbb{Z})$ determines $X$. 

References

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