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ON THE STRUCTURE OF A BOUNDED DOMAIN WITH A SPECIAL BOUNDARY POINT, II

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Introduction. This is a continuation of our previous paper [10]. We shall establish some extensions of Wong's characterization [19] of the open unit ball $\mathcal{B}^n$ in $\boldsymbol{C}^n$. Also we generalize a theorem of Behrens [2] derived from our result [9], and finally improve our main theorem in [10].

As a generalization of the notion of strictly pseudoconvex domains with C^2-smooth boundaries, we introduced in [10] the notion of domains with piecewise C^2-smooth boundaries of special type (see section 1). Now Wong [19] has given characterizations of the open unit ball $\mathcal{B}^n$ in $\boldsymbol{C}^n$ among bounded strictly pseudoconvex domains with C^∞-smooth boundaries. Our first purpose of this paper is to show that analogous characterizations are still valid for our domains with piecewise C^2-smooth boundaries of special type. In fact, by a direct application of our result [10], we shall establish the following extension of the Wong's result [19]:

Theorem I. *Let D be a bounded domain in $\boldsymbol{C}^n(n>1)$ with piecewise C^2-smooth boundary of special type and let $Aut(D)$ be the Lie group of all biholomorphic automorphisms of D. Then the following statements are mutually equivalent*:

(i) *D is biholomorphically equivalent to $\mathcal{B}^n$.*
(ii) *D is homogeneous.*
(iii) *$Aut(D)$ is non-compact.*
(iv) *There exists a compact subset K of D such that $Aut(D)\cdot K=D$.*

Corollary 1. *Let D be a bounded domain in $\boldsymbol{C}^n(n>1)$ with piecewise C^2-smooth boundary of special type. We assume that the boundary ∂D of D is not C^2-smooth globally, that is, ∂D has a corner. Then $Aut(D)$ is compact.*

Corollary 2. *Let D be a bounded circular domain in $\boldsymbol{C}^n(n>1)$ with piecewise C^2-smooth, but not smooth, boundary of special type and assume $o\in D$, where o denotes the origin of $\boldsymbol{C}^n$. Then every element of $Aut(D)$ keeps o fixed and hence is linear.*

Next we assume that a complex manifold M can be exhausted by biholomorphic images of a complex manifold D, that is, for any compact subset K of M there exists a biholomorphic mapping f_K from D into M such that $K \subset f_K(D)$. Then, how can we describe M using the data of D? In connection with this question, we have obtained in [10; Theorem II] the following result:

Let M be a connected hyperbolic manifold of complex dimension n in the sense of Kobayashi [7] and let D be a bounded domain in $\boldsymbol{C}^n$ with piecewise C^2-smooth boundary of special type. Assume that M can be exhausted by biholomorphic images of D. Then M is biholomorphically equivalent either to D or to some Siegel domain $\mathcal{D}(\boldsymbol{R}_+^k, H)$ in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k} (1 \leqq k \leqq n)$.

The second purpose of this paper is to study the case where M is not hyperbolic in the statement above. In such a case we show that the zero set of the infinitesimal Kobayashi metric F_M on M is an $(n-l)$-dimensional holomorphic vector bundle over M (see Proposition in section 3). Consequently, by the proof of the Main Theorem of Fornaess and Sibony [3] we obtain the following

Theorem II. *Let M be a connected σ-compact complex manifold of complex dimension n and let D be a bounded domain in $\boldsymbol{C}^n$ with piecewise C^2-smooth boundary of special type. We assume that*

1) *M can be exhausted by biholomorphic images of D;*

2) *the zero set of the infinitesimal Kobayashi metric F_M on M is a holomorphic line bundle over M.*

Then there exists a closed connected complex submanifold A of codimension one of D or of some Siegel domain $\mathcal{D}(\boldsymbol{R}_+^k, H)$ in $\boldsymbol{C}^n \times \boldsymbol{C}^{n-k}$ such that M is biholomorphically equivalent to the total space of a holomorphic line bundle over A.

This is a generalization of Behrens [2]. Indeed, combining the methods of Fornaess and Sibony [3] with our previous result published in a preprint form [9], Behrens has derived the above theorem in the case where D is a bounded strictly pseudoconvex domain with C^2-smooth boundary.

Before proceeding, one terminology is to be introduced. Let D be a domain in $\boldsymbol{C}^n$ with the Kobayashi pseudodistance d_D. For a point p of $\bar{D}$, the topological closure of D in $\boldsymbol{C}^n$, we say that D is *hyperbolically imbedded at p* if, for any neighborhood W of p in $\boldsymbol{C}^n$, there exists a neighborhood V of p in $\boldsymbol{C}^n$ such that

$$\bar{V} \subset W \quad \text{and} \quad d_D(D \cap (\boldsymbol{C}^n \setminus W), D \cap V) > 0 .$$

Note that, if D is relatively compact in $\boldsymbol{C}^n$ and hyperbolically imbedded at every point of $\bar{D}$, then D is said to be hyperbolically imbedded in $\boldsymbol{C}^n$ in the sense of Kiernan [5], [6]. Obviously D is hyperbolic if and only if it is hyperbolically imbedded at every point of D. Now, our final purpose is to prove the following theorem, which is an unbounded version of [10; Theorem I] (see Theorem III′ in section 1):

Theorem III. *Let D be a hyperbolic, not necessarily bounded, domain in $\boldsymbol{C}^n (n>1)$ with a boundary point $p \in \partial D$ satisfying the conditions (C.1) through (C.5) described in section 1 for some open neighborhood U of p and C^2-functions $\rho_i\colon U \to \boldsymbol{R}$, $i=1, \cdots, k$. Assume that the following two conditions are satisfied:*

(*) *There exist a compact set K in D, a sequence $\{k_\nu\}$ in K and a sequence $\{f_\nu\}$ in $Aut(D)$ such that $\lim_{\nu\to\infty} f_\nu(k_\nu)=p$.*

(**) *D is hyperbolically imbedded at p.*

Then D is biholomorphically equivalent to some Siegel domain $\mathfrak{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$. Conversely, every Siegel domain $\mathfrak{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$ is a hyperbolic domain and the conditions (C.1)~(C.5), (*) and (**) are all satisfied at the point $p=o \in \partial\mathfrak{D}(\boldsymbol{R}^k_+, H)$, where o stands for the origin of $\boldsymbol{C}^n = \boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$.

As a special case, let us consider a bounded domain D in $\boldsymbol{C}^n$. Then D is hyperbolically imbedded in $\boldsymbol{C}^n$ in the sense of Kiernan [5], [6; Theorem 1]. Hence the condition (**) of Theorem III is automatically satisfied for any boundary point p of D. Therefore Theorem III is a generalization of [10; Theorem I]. Moreover, considering the case where D is a bounded domain and $k=1$ in Theorem III, we obtain a well-known result of Rosay [15].

After some preliminaries in section 1, Theorem I and its corollaries will be proven in section 2. Sections 3 and 4 are devoted to proving Theorems II and III. In the final section 5, as concluding remarks we mention the analogues of Theorems I, II and III in the case where D is a domain in a complex manifold X, and discuss also the condition (**) of Theorem III.

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1. Preliminaries

In this section we recall first some definitions and a fundamental result in [10].

A bounded domain D in $\boldsymbol{C}^n$ is said to have a *piecewise C^2-smooth boundary* if there exist a finite open covering $\{U_j\}_{j=1}^N$ of an open neighborhood V of ∂D and C^2-functions $\rho_j\colon U_j \to \boldsymbol{R}$, $j=1, \cdots, N$, such that

(1) $D \cap V = \{z \in V\colon$ for $j=1, \cdots, N$, either $z \notin U_j$ or $z \in U_j$, $\rho_j(z)<0\}$;

(2) for every set $\{j_1, \cdots, j_i\}$ with $1 \leqq j_1 < \cdots < j_i \leqq N$, the differential form

$$d\rho_{j_1} \wedge \cdots \wedge d\rho_{j_i}(z) \neq 0 \qquad \text{for all} \quad z \in \bigcap_{l=1}^{i} U_{j_l}\,.$$

We call $\{U_j;\ \rho_j\}_{j=1}^N$ a *defining system for D*.

Let D be a bounded domain in $\boldsymbol{C}^n$ with piecewise C^2-smooth boundary and let $\{U_j;\ \rho_j\}_{j=1}^{N}$ be its defining system. Then ∂D is said to be *of special type* if the following conditions are satisfied: For an arbitrary given point $p\in\partial D$, one can find a subset J of $\{1, \cdots, N\}$ consisting of k elements with $1\leqq k\leqq n$, say $J=\{1, \cdots, k\}$ for the sake of simplicity, and an open neighborhood U of p with $U\subset\bigcap_{i=1}^{k} U_i$ such that

(C.1) $$\rho_i(p)=0 \quad \text{for} \quad i=1, \cdots, k;$$

(C.2) $$D\cap U=\{z\in U:\ \rho_i(z)<0 \quad \text{for} \quad i=1, \cdots, k\};$$

(C.3) $$\bar{\partial}\rho_1\wedge\cdots\wedge\bar{\partial}\rho_k(z)\neq 0 \quad \text{for all} \quad z\in U;$$

(C.4) $$\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\rho_i}{\partial z_\alpha\partial\bar{z}_\beta}(p)\,\xi_\alpha\bar{\xi}_\beta\geqq 0, \quad \xi=(\xi_\alpha)\in T \quad \text{for} \quad i=1, \cdots, k,$$

where

$$T=\{\xi=(\xi_\alpha)\in \boldsymbol{C}^n:\ \sum_{\alpha=1}^{n}\frac{\partial\rho_i}{\partial z_\alpha}(p)\,\xi_\alpha=0 \quad \text{for} \quad i=1, \cdots, k\};$$

(C.5) for some constant $A\geqq 0$, the function $\rho=\sum_{i=1}^{k}\rho_i+A\sum_{i=1}^{k}(\rho_i)^2$ is strictly plurisubharmonic on U.

Such a system $(U;\ \rho_1, \cdots, \rho_k)$ is called a *defining system for D in the neighborhood U of p*.

It is obvious that any bounded strictly pseudoconvex domain with C^2-smooth boundary is a typical example of domains with piecewise C^2-smooth boundary of special type. Here it should be remarked that not all functions ρ_j: $U_j\to\boldsymbol{R}$ in the definition above need to be strictly plurisubharmonic. In fact, consider the following domain

$$D=\{(z, w)\in\boldsymbol{C}^2:\ \rho_1(z, w)<0,\ \rho_2(z, w)<0\}$$

in $\boldsymbol{C}^2$, where

$$\rho_1(z, w)=|z|^2+|w|^4-1, \quad \rho_2(z, w)=4|z|^2+75|w-\frac{2}{5}|^2-16$$

for $(z, w)\in\boldsymbol{C}^2$. Then, setting

$$S=\{(z, 0)\in\boldsymbol{C}^2:\ |z|=1\}\subset\{(z, w)\in\boldsymbol{C}^2:\ \rho_1(z, w)=\rho_2(z, w)=0\},$$

we can see that the Levi-form $L(\rho_1)$ of ρ_1 is degenerate at each point of S, and is strictly positive definite at every point of $\partial D\backslash S$. On the other hand, it is clear that the Levi-form $L(\rho_1+\rho_2)$ of $\rho_1+\rho_2$ is strictly positive definite at every point of $\boldsymbol{C}^2$. Keeping these facts in mind, we can check that D is, in fact, a bounded domain with piecewise C^2-smooth boundary of special type.

Now, for the open convex cone

$$\boldsymbol{R}_+^k = \{(y_1, \cdots, y_k) \in \boldsymbol{R}^k : y_i > 0 \quad \text{for} \quad i = 1, \cdots, k\}$$

in $\boldsymbol{R}^k$, $1 \leqq k \leqq n$, and an $\boldsymbol{R}_+^k$-hermitian form H: $\boldsymbol{C}^{n-k} \times \boldsymbol{C}^{n-k} \to \boldsymbol{C}^k$, we shall denote by $\mathcal{D}(\boldsymbol{R}_+^k, H)$ the Siegel domain in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$ associated to $\boldsymbol{R}_+^k$ and H in the sense of Pjateckii-Sapiro [14].

The following lemma guarantees us that bounded domains with piecewise C^2-smooth boundaries of special type as well as Siegel domains are taut in the sense of Wu [20].

Lemma 1. *Let X be a connected complex manifold, D a bounded domain in $\boldsymbol{C}^n$ with piecewise C^2-smooth boundary of special type and $\mathcal{D}(\boldsymbol{R}_+^k, H)$ a Siegel domain in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k} = \boldsymbol{C}^n$. Let $f: X \to \boldsymbol{C}^n$ be a holomorphic mapping. Then we have:*

1) *If $f(X) \subset \bar{D}$, then either $f(X) \subset D$ or there exists a point $p \in \partial D$ such that $f(X) = \{p\}$.*

2) *If $f(X) \subset \overline{\mathcal{D}(\boldsymbol{R}_+^k, H)}$, then either $f(X) \subset \mathcal{D}(\boldsymbol{R}_+^k, H)$ or $f(X) \subset \partial\mathcal{D}(\boldsymbol{R}_+^k, H)$.*

Proof. First, assuming that $f(x_0) \in \partial D$ for some point $x_0 \in X$, we show that $f(x) = f(x_0)$ for all $x \in X$. To this end, choose a defining system $(U; \rho_1, \cdots, \rho_k)$ for D in an open neighborhood U of $f(x_0)$ and let us consider a strictly plurisubharmonic function $\rho = \sum_{i=1}^k \rho_i + A \sum_{i=1}^k (\rho_i)^2$ on U as in (C.5). After shrinking U if necessary, we can assume that $D \cap U \subset \{z \in U : \rho(z) < 0\}$. Now take a connected open neighborhood W of x_0 so small that $f(W) \subset U$ and consider the plurisubharmonic function $\rho \circ f$: $W \to \boldsymbol{R}$. Then

$$\rho \circ f(x_0) = 0 \quad \text{and} \quad \rho \circ f(x) \leqq 0 \qquad \text{for all} \quad x \in W$$

and hence by the maximum principle

$$\rho \circ f(x) = 0 \qquad \text{for all} \quad x \in W.$$

This combined with the strict plurisubharmonicity of ρ yields that $f(x) = f(x_0)$ on W, and accordingly on X by analytic continuation, as desired.

Next we consider the second case. With respect to the given coordinate system

$$z = (z', z'') = (z_1, \cdots, z_k, z_{k+1}, \cdots, z_n)$$

in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k} = \boldsymbol{C}^n$, the $\boldsymbol{R}_+^k$-hermitian form H is written as $H = (H_1, \cdots, H_k)$ and accordingly

$$\mathcal{D}(\boldsymbol{R}_+^k, H) = \{z \in \boldsymbol{C}^n : \tilde{\rho}_i(z) < 0 \qquad \text{for} \quad i = 1, \cdots, k\},$$

where

$$\tilde{\rho}_i(z) = H_i(z'', z'') - \operatorname{Im} z_i \qquad \text{for} \quad i = 1, \cdots, k.$$

Since every H_i is a positive semi-definite hermitian form on $\boldsymbol{C}^{n-k}$, every $\tilde{\rho}_i$ is a plurisubharmonic function on $\boldsymbol{C}^n$. Now, we assume that $f(x_0)\in\partial\mathcal{D}(\boldsymbol{R}_+^k, H)$ for some point $x_0\in X$. Then there exists an index i_0, $1\leqq i_0\leqq k$, such that $\tilde{\rho}_{i_0}\circ f(x_0)=0$. So, considering the plurisubharmonic function

$$\tilde{\rho}_{i_0}\circ f\colon X\to \boldsymbol{R}\,,$$

we can see in the same way as in the proof of 1) that

$$\tilde{\rho}_{i_0}\circ f(x)=0 \qquad \text{for all} \quad x\in X\,.$$

This combined with the assumption $f(X)\subset\overline{\mathcal{D}(\boldsymbol{R}_+^k, H)}$ assures that $f(X)\subset\partial\mathcal{D}(\boldsymbol{R}_+^k, H)$, as desired. Q.E.D.

We finish this section by recalling the following theorem, which is essential to the proof of Theorem I.

Theorem III′ ([10; Theorem I]). *Let D be a bounded domain in $\boldsymbol{C}^n (n>1)$ with a boundary point $p\in\partial D$ satisfying the conditions* (C.1) *through* (C.5) *for some open neighborhood U of p and C^2-functions $\rho_i\colon U\to\boldsymbol{R}$, $i=1, \cdots, k$. Assume that*:

(*) *There exist a compact set K in D, a sequence $\{k_\nu\}$ in K and a sequence $\{f_\nu\}$ in $Aut(D)$ such that* $\lim_{\nu\to\infty} f_\nu(k_\nu)=p$.

Then D is biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}_+^k, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$. Conversely, every Siegel domain $\mathcal{D}(\boldsymbol{R}_+^k, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ is biholomorphically equivalent to a bounded domain D in $\boldsymbol{C}^n$ satisfying all the conditions (C.1)$\sim$(C.5) *and* (*).

2. Proofs of Theorem I and its corollaries

Proof of Theorem I. From the definition of domains with piecewise C^2-smooth boundaries of special type, we see that the set of all C^2-smooth strictly pseudoconvex boundary points of D is open and dense in ∂D. Hence the equivalence of three statements (i), (ii) and (iv) follows immediately from [15] or [10; Corollary 2]. Since $Aut(D)$ is non-compact if D is biholomorphically equivalent to the open unit ball $\mathcal{B}^n$, in order to complete the proof we have only to show the converse. In the following, let us set, for $r>0$,

$$\Delta(r)=\{\eta\in\boldsymbol{C}\colon |\eta|<r\} \quad \text{and} \quad \mathfrak{H}=\{\xi\in\boldsymbol{C}\colon \operatorname{Im}\xi>0\}\,.$$

Now suppose that $Aut(D)$ is non-compact. Then, for an arbitrarily fixed point q of D, one can choose a sequence $\{f_\nu\}$ in $Aut(D)$ in such a way that the sequence $\{f_\nu(q)\}$ converges to some boundary point p of D [11; Proposition 6,

p. 82]. Consequently, taking a defining system $(U; \rho_1, \cdots, \rho_k)$ for D in an open neighborhood U of p, we obtain from Theorem III′ that D is biholomorphically equivalent to some Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$. We choose a biholomorphic mapping φ: $\mathcal{D}(\boldsymbol{R}^k_+, H) \to D$. Once it is shown that $k=1$, our proof will be finished, because any Siegel domain $\mathcal{D}(\boldsymbol{R}^1_+, H)$ in $\boldsymbol{C} \times \boldsymbol{C}^{n-1}$ is biholomorphically equivalent to the open unit ball $\mathcal{B}^n$. Assuming that $k \geqq 2$, we shall obtain a contradiction by using a similar method as in [11; Chap. 5].

With respect to the given coordinate system

$$z = (z', z'') = (z_1, \cdots, z_k, z_{k+1}, \cdots, z_n)$$

in $\boldsymbol{C}^n = \boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$, the $\boldsymbol{R}^k_+$-hermitian form H can be written in the form $H = (H_1, \cdots, H_k)$. Since $k \geqq 2$, there exists a boundary point

$$z_0 = (z_0', z_0'') = (z_1^0, \cdots, z_n^0) \in \partial \mathcal{D}(\boldsymbol{R}^k_+, H)$$

such that

$$\begin{aligned} &\operatorname{Im} z_1^0 - H_1(z_0'', z_0'') = 0 ; \\ &\operatorname{Im} z_i^0 - H_i(z_0'', z_0'') > 0 \qquad \text{for} \quad i = 2, \cdots, k . \end{aligned}$$

Let us take an $r>0$ so small that

$$\{(z_1^0+\xi, z_2^0+\eta, z_3^0, \cdots, z_n^0) \in \boldsymbol{C}^n : \xi \in \mathfrak{H}, \eta \in \Delta(r)\} \subset \mathcal{D}(\boldsymbol{R}^k_+, H)$$

and

$$\{(z_1^0+\xi, z_2^0+\eta, z_3^0, \cdots, z_n^0) \in \boldsymbol{C}^n : \xi \in \partial\mathfrak{H}, \eta \in \Delta(r)\} \subset \partial\mathcal{D}(\boldsymbol{R}^k_+, H) .$$

Then, for an arbitrary given point $a \in \boldsymbol{R} = \partial \mathfrak{H}$ and an arbitrary given sequence $\{b_\nu\}_{\nu=1}^\infty$ of positive numbers b_ν tending to 0, we can define a family of holomorphic mappings

$$F^\nu = (F_1^\nu, \cdots, F_n^\nu) \colon \Delta(r) \to \boldsymbol{C}^n \qquad \text{for} \quad \nu = 1, 2, \cdots$$

by setting

$$F^\nu(\eta) = \varphi(z_1^0 + a + \sqrt{-1}\, b_\nu, z_2^0+\eta, z_3^0, \cdots, z_n^0) \qquad \text{for} \quad \eta \in \Delta(r) ,$$

where φ: $\mathcal{D}(\boldsymbol{R}^k_+, H) \to D$ is the given biholomorphic mapping. Owing to the boundedness of D, we can select subsequence $\{F^{\nu_j}\}$ of $\{F^\nu\}$ which converges uniformly on every compact subset of $\Delta(r)$ to a holomorphic mapping F: $\Delta(r) \to \boldsymbol{C}^n$. Clearly we have $F(\Delta(r)) \subset \bar{D}$. Moreover, since

$$\lim_{\nu \to \infty} (z_1^0 + a + \sqrt{-1}\, b_\nu, z_2^0, \cdots, z_n^0) = (z_1^0 + a, z_2^0, \cdots, z_n^0) \in \partial \mathcal{D}(\boldsymbol{R}^k_+, H)$$

and since φ is a biholomorphic mapping from $\mathcal{D}(\boldsymbol{R}^k_+, H)$ onto D, we see that

$$\lim_{j\to\infty} F^{\nu_j}(0) = F(0) \in \partial D .$$

Hence we conclude by Lemma 1 that $F(\eta)=F(0)$ for all $\eta\in\Delta(r)$. Thus, after taking a subsequence and relabelling if necessary, we have that the sequence $\{F^\nu\}$ converges uniformly on compact subsets to a constant mapping. So it follows from a well-known Weierstrass' theorem that

$$\lim_{\nu\to\infty} \frac{\partial \varphi_j}{\partial z_2}(z_1^0+a+\sqrt{-1}\, b_\nu, z_2^0+\eta, z_3^0, \cdots, z_n^0) = \lim_{\nu\to\infty} \frac{dF_j^\nu}{d\eta}(\eta) = 0$$

uniformly on every compact set in $\Delta(r)$ for $j=1, \cdots, n$, where φ_j denotes the j-th component function of φ: $\mathcal{D}(\boldsymbol{R}_+^k, H)\to D$. In particular, if we consider the holomorphic functions h_j: $\mathfrak{H}\to \boldsymbol{C}, j=1, \cdots, n$, defined by

$$h_j(\xi) = \frac{\partial \varphi_j}{\partial z_2}(z_1^0+\xi, z_2^0, \cdots, z_n^0) \qquad \text{for} \quad \xi\in\mathfrak{H} ,$$

then

$$(\sharp) \qquad \lim_{b\to +0} h_j(a+\sqrt{-1}\, b) = 0 \qquad \text{for} \quad j = 1, \cdots, n;\, a\in\boldsymbol{R} .$$

On the other hand, since D is a bounded domain in $\boldsymbol{C}^n$, the Cauchy estimates tell us that every function h_j is bounded on $\mathfrak{H}$. Therefore, by composing h_j and the Cayley transformation C: $\Delta=\{w\in\boldsymbol{C}: |w|<1\}\to\mathfrak{H}$ defined by

$$C: w\mapsto \xi = \sqrt{-1}\,(1+w)\cdot(1-w)^{-1} \qquad \text{for} \quad w\in\Delta ,$$

we obtain the bounded holomorphic functions $f_j=h_j\circ C$ on Δ for $j=1, \cdots, n$. Here we can check easily by using ($\sharp$) and [17; Theorem VIII. 10., p. 306] that, for every $j=1, \cdots, n$ and an arbitrary point $\zeta\in\partial\Delta$ with $\zeta\neq 1$, we have $\lim_{w\to\zeta} f_j(w)=0$ when $w\to\zeta$ from the inside of any fixed Stolz domain with vertex at ζ. Hence, *F.* and M. Riesz' theorem [17; Theorem IV. 9., p. 137] guarantees us that

$$f_j(w) = 0 \qquad \text{for} \quad w\in\Delta;\, j = 1, \cdots, n$$

or equivalently

$$\frac{\partial \varphi_j}{\partial z_2}(z_1^0+\xi, z_2^0, \cdots, z_n^0) = 0 \qquad \text{for} \quad \xi\in\mathfrak{H};\, j=1, \cdots, n .$$

Thus the complex Jacobian determinant of the biholomorphic mapping φ: $\mathcal{D}(\boldsymbol{R}_+^k, H)\to D$ vanishes identically on the non-empty subset $\{(z_1^0+\xi, z_2^0, \cdots, z_n^0)\in \boldsymbol{C}^n: \xi\in\mathfrak{H}\}$ of $\mathcal{D}(\boldsymbol{R}_+^k, H)$, which is a contradiction. Q.E.D.

Proof of Corollary 1. Assume that $Aut(D)$ is non-compact. Then, by Theorem I D is biholomorphically equivalent to the open unit ball $\mathcal{B}^n$. In particular, D is a homogeneous domain. On the other hand, by our assumption there exists a non-smooth boundary point p of D, so that in a certain open neighborhood U of p, D has a defining system $(U; \rho_1, \cdots, \rho_k)$ with $k \geqq 2$. Under such conditions, we have already known from [13] or [10; Corollary 1] that D is biholomorphically equivalent to the direct product of the open unit balls $\mathcal{B}^{n_i}$ in C^{n_i} $(1 \leqq i \leqq k)$: $D \cong \mathcal{B}^{n_1} \times \cdots \times \mathcal{B}^{n_k}$, where each $n_i \geqq 1$ and $n_1 + \cdots + n_k = n$. However, this is a contradiction, because $\mathcal{B}^n$ is not biholomorphically equivalent to any direct product domain. Thus $Aut(D)$ must be compact.

Proof of Corollary 2. Assume that D is a bounded circular domain in C^n with piecewise C^2-smooth, but not smooth, boundary of special type and D contains the origin o of C^n. Let G denote the identity connected component of $Aut(D)$ and let D_0 be the G-orbit passing through the origin o. Then $D_0 = \{o\}$. In fact, by the proof of [8; Lemma 1.2] we know that D_0 is a complex submanifold of D. On the other hand, D_0 is compact by Corollary 1. Thus D_0 is a compact connected homogeneous hyperbolic manifold, so that it must reduce to $\{o\}$ [7; Theorem 2.1, p. 70], as desired. Next, by the compactness of $Aut(D)$ we can select finitely many elements $g_1, \cdots, g_k$ of $Aut(D)$ such that

$$Aut(D) = \bigcup_{i=1}^{k} g_i \cdot G \quad \text{(disjoint union)}$$

and accordingly

$$Aut(D) \cdot o = \{g_1 \cdot o, \cdots, g_k \cdot o\} .$$

Since $Aut(D)$ contains the rotational group

$$T_\theta : (z_1, \cdots, z_n) \mapsto (e^{\sqrt{-1}\theta} z_1, \cdots, e^{\sqrt{-1}\theta} z_n), \quad \theta \in \boldsymbol{R},$$

we now conclude that

$$g_i \cdot o = o \qquad \text{for} \quad i = 1, \cdots, k \quad \text{and hence} \quad Aut(D) \cdot o = \{o\} .$$

Therefore any element of $Aut(D)$ is linear by a well-known theorem of H. Cartan [11; Proposition 2, p. 67]. Q.E.D.

EXAMPLE. Let us consider the domain

$$D = \{(z, w) \in C^2 : a|z|^2 + b|w|^2 < 1, \quad b|z|^2 + a|w|^2 < 1\}$$

in C^2, where $a, b > 0$ and $a \neq b$. Then D is a bounded circular domain with piecewise C^2-smooth, but not smooth, boundary of special type. Let T be the group of the linear transformations

$$T_{(s,t)} : (z, w) \mapsto (e^{\sqrt{-1}s} z, e^{\sqrt{-1}t} w), \quad (s, t) \in \boldsymbol{R}^2$$

and let $\sigma_0 : (z, w) \mapsto (w, z)$. Then we have

$$Aut(D) = T \cup \sigma_0 \cdot T \quad \text{(disjoint union)}.$$

In fact, we know by our Corollaries 1 and 2 that $Aut(D)$ is a compact Lie subgroup of $GL(2; \boldsymbol{C})$. Hence $g_0 \cdot Aut(D) \cdot g_0^{-1} \subset U(2)$ for some element $g_0 \in GL(2; \boldsymbol{C})$. Replacing D by the circular domain $g_0(D)$ if necessary, we may assume that $Aut(D) \subset U(2)$. Now assume that dim $Aut(D) \geqq 3$. Then $Aut(D) \supset SU(2)$ and accordingly ∂D must be smooth, a contradiction. Therefore dim $Aut(D) \leqq 2$ and, in fact, we can see that the identity connected component of $Aut(D)$ coincides with T. Then, for an arbitrary given $\sigma \in Aut(D)$ there exists $(\alpha, \beta) \in \boldsymbol{R}^2 \setminus \{(0, 0)\}$ such that

$$\sigma \cdot T_{(s,0)} = T_{(\alpha s, \beta s)} \cdot \sigma \qquad \text{for all} \quad s \in \boldsymbol{R}.$$

It is now easy to deduce from this equality that $\sigma \in T$ or $\sigma \in \sigma_0 \cdot T$.

3. Proof of Theorem II

According to Fornaess and Sibony [3] and Behrens [2], the only thing which is to be proved now is the following

Proposition. *Let M be a connected σ-compact complex manifold of complex dimension n and let D be a bounded domain in $\boldsymbol{C}^n$ with piecewise C^2-smooth boundary of special type. We assume that M can be exhausted by biholomorphic images of D. Then the zero set of the infinitesimal Kobayashi metric F_M on M is an $(n-l)$-dimensional holomorphic vector bundle over M.*

Proof. Using our constructions of [10], we will proceed along the same line as in [2]. Throughout the proof we use the same notation as in [10], unless otherwise stated.

First we fix a family $\{M_j\}_{j=1}^{\infty}$ of relatively compact subdomains of M such that

$$M = \bigcup_{j=1}^{\infty} M_j \supset \cdots \supset M_{j+1} \supset M_j \supset \cdots \supset M_1.$$

By our assumption there exists a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ of biholomorphic mappings from D into M such that

$$M_\nu \subset \varphi_\nu(D) \qquad \text{for} \quad \nu = 1, 2, \cdots.$$

We set

$$\psi_\nu = \varphi_\nu^{-1}: \varphi_\nu(D) \to D \qquad \text{for} \quad \nu = 1, 2, \cdots.$$

Then we can assume that $\{\psi_\nu\}$ converges uniformly on every compact set in M to a holomorphic mapping $\psi: M \to \boldsymbol{C}^n$ with $\psi(M) \subset \bar{D}$. By virtue of Lemma 1 we have now two cases:

Case 1. $\psi(M)\subset D$ and *Case* 2. $\psi(M)=\{p\}\subset\partial D$.

Let us study for a while the second case. We fix a point $x_0\in M_1$ and an $M'=M_j$ arbitrarily, and consider the biholomorphic mappings

$$F^\nu = L^\nu\circ h^\nu\circ\psi_\nu \qquad \text{for} \quad \nu\geqq\nu(M')$$

as in the proof of [10; Theorem II]. Then

$$F^\nu(x_0) = (\underbrace{-1, \cdots, -1}_{k \text{ times}}, 0, \cdots, 0) \qquad \text{for all} \quad \nu\geqq\nu(M') .$$

Moreover we know [10] that there exist an unbounded domain $\mathcal{W}$ in $\boldsymbol{C}^n$ and a subsequence $\{F^{\nu_j}\}$ of $\{F^\nu\}$ satisfying the following conditions:

1) $\mathcal{W}$ is biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$, via the non-singular linear mapping L: $\boldsymbol{C}^n\to\boldsymbol{C}^n$ defined by

$$L(w', w'') = (-\sqrt{-1}\, w', w'') \qquad \text{for} \quad (w', w'')\in\boldsymbol{C}^k\times\boldsymbol{C}^{n-k} = \boldsymbol{C}^n \ ;$$

2) $\{F^{\nu_j}\}$ converges uniformly on compact subsets to a holomorphic mapping F: $M\to\overline{\mathcal{W}}\subset\boldsymbol{C}^n$ with

$$F(x_0) = (\underbrace{-1, \cdots, -1}_{k \text{ times}}, 0, \cdots, 0)\in\mathcal{W} . \tag{3.1}$$

Note that $F(M)\subset\mathcal{W}$ by Lemma 1. In the following, we shall make the identification:

$$\mathcal{W} = \mathcal{D}(\boldsymbol{R}^k_+, H)$$

via the bilinear mapping L: $\boldsymbol{C}^n\to\boldsymbol{C}^n$ and, changing the notation, we assume that $\{F^\nu\}$ itself converges uniformly on compact subsets to the holomorphic mapping F: $M\to\mathcal{W}$. Now, let us take the family $\{W_\nu\}_{\nu=1}^\infty$ of domains in $\boldsymbol{C}^n$ defined in (2.10) of [10] and set

$$G^\nu(w) = \varphi_\nu\circ(h^\nu)^{-1}\circ(L^\nu)^{-1}(w) , \quad w\in W_\nu$$

for $\nu=1, 2, \cdots$. Then we have by [10] that:

(3.2) For any compact set K in $\mathcal{W}$, there is an integer $\nu(K)$ such that $K\subset W_\nu$ for all $\nu\geqq\nu(K)$; and

(3.3) G^ν are biholomorphic mappings from W_ν into M such that $G^\nu\circ F^\nu=id$ and $F^\nu\circ G^\nu=id$ for all ν.

According to Fornaess and Sibony [3], we shall introduce the holomorphic mappings

$$\alpha_\nu = \begin{cases} \psi\circ\varphi_\nu\colon D \to D & \text{in Case 1} \\ F\circ G^\nu\colon W_\nu \to \mathcal{W} & \text{in Case 2.} \end{cases}$$

In the first case, by the boundedness of D we can assume that $\{\alpha_\nu\}$ converges uniformly on compact subsets of D to a holomorphic mapping $\alpha: D\to\bar{D}$ and, for any fixed point $x\in M$,

$$\alpha\circ\psi(x) = \lim_{\nu\to\infty} \alpha_\nu\circ\psi_\nu(x) = \psi(x)\in D .$$

Hence $\alpha(D)\subset D$ by Lemma 1. In the second case, we know that $\mathcal{W}$ is a taut domain and by (3.3)

$$\lim_{\nu\to\infty} \alpha_\nu(-1, \cdots, -1, 0, \cdots, 0) = (\underbrace{-1, \cdots, -1}_{k \text{ times}}, 0, \cdots, 0)\in\mathcal{W} .$$

Therefore, combining the fact (3.2) with the usual normal family argument, we can also assume that $\{\alpha_\nu\}$ converges uniformly on compact subsets of $\mathcal{W}$ to a holomorphic mapping

$$\alpha: \mathcal{W} \to \mathcal{W} \quad \text{in Case 2.}$$

Moreover, we can check easily that

$$\alpha\circ\psi(x) = \psi(x)\,, \quad x\in M \quad \text{in Case 1;}$$
$$\alpha\circ F(x) = F(x)\,, \quad x\in M \quad \text{in Case 2.}$$

From now on we want to consider simultaneously the both Cases 1 and 2. For this purpose, we define the objects

$$\Omega,\ \Omega_\nu,\ \Phi^\nu,\ \Psi^\nu,\ \Psi \qquad \text{for} \quad \nu = 1, 2, \cdots$$

by

$$\Omega = D,\ \Omega_\nu = D,\ \Phi^\nu = \varphi_\nu,\ \Psi^\nu = \psi_\nu,\ \Psi = \psi \qquad \text{in Case 1;}$$
$$\Omega = \mathcal{W},\ \Omega_\nu = W_\nu,\ \Phi^\nu = G^\nu,\ \Psi^\nu = F^\nu,\ \Psi = F \quad \text{in Case 2}$$

respectively. So, summing up the above, we obtain the into-biholomorphic mappings

$$\Phi^\nu: \Omega_\nu \to M \qquad \text{for} \quad \nu = 1, 2, \cdots$$

such that the sequence

$$\Psi^\nu = (\Phi^\nu)^{-1}\,, \quad \nu = 1, 2, \cdots$$

converges uniformly on every compact subset to the holomorphic mapping

$$\Psi: M \to \Omega\,.$$

Moreover, the sequence

$$\alpha_\nu = \Psi\circ\Phi^\nu\colon \Omega_\nu \to \Omega \quad \nu = 1, 2, \cdots$$

converges uniformly on every compact set in Ω to the holomorphic mapping

$$\alpha\colon \Omega \to \Omega \quad \text{with} \quad \alpha\circ\Psi = \Psi \quad \text{on } M.$$

We set as in [3]

$$Z = \{q\in\Omega\colon \alpha(q) = q\}$$

and let l be the maximal rank of Ψ on M. Then we have by [3; Lemmas 4.2~4.4] that

(3.4) Z is a connected closed l-dimensional complex submanifolds of Ω;

(3.5) α is a holomorphic retraction of Ω to Z;

(3.6) $\Psi(M)=Z$ and Ψ has constant rank l on M.

Therefore, by virtue of the hyperbolicity of Ω, in order to complete the proof of the proposition we have only to verify the equality

$$F_M(z_0; \zeta_0) = F_\Omega(\Psi(z_0); d\Psi_{z_0}(\zeta_0)) \tag{3.7}$$

for an arbitrary given element $(z_0; \zeta_0)$ of the holomorphic tangent bundle $\mathcal{T}M$ of M, where $d\Psi_{z_0}$ denotes the complex differential of Ψ at the point $z_0\in M$. To obtain the equality (3.7), let us recall here the following three facts:

(3.8) Every geometrically convex hyperbolic domain in $\boldsymbol{C}^n$ is taut [1], [4];

(3.9) for any taut complex manifold X, F_X is continuous on $\mathcal{T}X$ [16]; and

(3.10) $\mathcal{D}(\boldsymbol{R}^k_+,H)$ is a geometrically convex domain [18].

Now we shall consider the first case: $\Omega=D$. Since $M_\nu\subset\Phi^\nu(D)\subset M$ for all ν and $\{M_\nu\}$ increases to M monotonously, it follows that

$$F_M(z_0; \zeta_0) = \lim_{\nu\to\infty} F_{\Phi^\nu(D)}(z_0; \zeta_0)\,.$$

This combined with (3.9) yields the desired equality (3.7):

$$F_M(z_0; \zeta_0) = \lim_{\nu\to\infty} F_D(\Psi^\nu(z_0); d\Psi^\nu_{z_0}(\zeta_0)) = F_D(\Psi(z_0); d\Psi_{z_0}(\zeta_0))\,,$$

since

$$\lim_{\nu\to\infty} (\Psi^\nu(z_0); d\Psi^\nu_{z_0}(\zeta_0)) = (\Psi(z_0); d\Psi_{z_0}(\zeta_0)) \quad \text{in } \mathcal{T}D$$

by a well-known theorem of Weierstrass.

Next, let us consider the second case: $\Omega=\mathcal{W}$. We first fix a family $\{S_j\}_{j=1}^\infty$ of relatively compact subdomains of the taut domain $\mathcal{W}=\mathcal{D}(\boldsymbol{R}^k_+, H)$ such that

$$\mathcal{W} = \bigcup_{j=1}^\infty S_j\supset\cdots\supset S_{j+1}\supset S_j\supset\cdots\supset S_1\ni\Psi(z_0)\,.$$

Here we can assume by (3.10) and (3.8) that every S_j is geometrically convex and taut. Let us fix an arbitrary integer j. By (3.2) there is a large integer $\nu(j)$ such that

$$\Psi^{\nu}(z_0)\in S_j\subset W_{\nu} \qquad \text{for all} \quad \nu\geqq\nu(j)\,.$$

Thus the length decreasing property of infinitesimal Kobayashi metrics implies that

$$\begin{aligned} F_{S_j}(\Psi^{\nu}(z_0);\, d\Psi^{\nu}_{z_0}(\zeta_0)) &= F_{\Phi^{\nu}(S_j)}(\Phi^{\nu}\circ\Psi^{\nu}(z_0);\, d(\Phi^{\nu}\circ\Psi^{\nu})_{z_0}(\zeta_0)) \\ &= F_{\Phi^{\nu}(S_j)}(z_0;\, \zeta_0)\geqq F_M(z_0;\, \zeta_0) \end{aligned}$$

for all $\nu\geqq\nu(j)$. (Note that $\Phi^{\nu}(S_j)$ are subdomains of M.) Hence, letting ν tend to infinity, we have

$$F_{S_j}(\Psi(z_0);\, d\Psi_{z_0}(\zeta_0))\geqq F_M(z_0;\, \zeta_0)$$

because S_j is taut and so F_{S_j} is continuous on $\mathcal{T}S_j$ by (3.9). On the other hand, since $\{S_j\}$ increases monotonously to $\mathcal{W}$, we see that

$$\lim_{j\to\infty} F_{S_j}(q;\, \xi) = F_{\mathcal{W}}(q;\, \xi) \qquad \text{for every} \quad (q;\, \xi)\in\mathcal{T}\mathcal{W}\,.$$

Consequently

$$F_{\mathcal{W}}(\Psi(z_0);\, d\Psi_{z_0}(\zeta_0))\geqq F_M(z_0;\, \zeta_0)\,.$$

Thus, by the length decreasing property we also obtain the equality (3.7) in Case 2. Our proof is completed. Q.E.D.

4. Proof of Theorem III

Throughout this section we denote by D, $p\in\partial D$, $\{k_{\nu}\}\subset K$, $\{f_{\nu}\}\subset Aut(D)$ and U the same object as in the statement of Theorem III. Without loss of generality, we may assume that U is a small open Euclidean ball, so that it is taut in the sense of Wu [20]. By the compactness of K, we may further assume that

$$\lim_{\nu\to\infty} k_{\nu} = k_0 \quad \text{for some point} \quad k_0\in K\,.$$

Given a point $a\in D$ and a positive numbre r, we define the open subset $B(a;\, r)$ of D by

$$B(a;r) = \{z\in D\colon d_D(a, z)<r\}\,.$$

Under these assumptions, we show the following lemma, which is the first step of the proof of Theorem III:

Lemma 2. *The sequence $\{f_{\nu}\}$ contains a subsequence which converges uniform-*

ly on every compact subset of D to the constant mapping C_p: $D \to \boldsymbol{C}^n$ defined by $C_p(z)=p$ for all $z \in D$.

Proof. We will proceed in several steps.

1) *There exist an integer ν_0 and a positive number r_0 such that $f_\nu(B(k_0; r_0)) \subset U$ for all $\nu \geqq \nu_0$*: By our assumption (**) we can choose an open neighborhood V of p in such a way that $\bar{V} \subset U$ and $d_D(D \cap (\boldsymbol{C}^n \setminus U), D \cap V) > 0$. We set

$$r_0 = \frac{1}{3} d_D(D \cap (\boldsymbol{C}^n \setminus U), D \cap V)$$

and choose an integer ν_0 so large that

$$k_\nu \in B(k_0; r_0), f_\nu(k_\nu) \in V \qquad \text{for} \quad \nu \geqq \nu_0 .$$

Then

$$B(k_0; r_0) \subset B(k_\nu; 2r_0), \quad B(f_\nu(k_\nu); 2r_0) \subset U \qquad \text{for} \quad \nu \geqq \nu_0 ,$$

because every point outside U is at least $3r_0$ away from $D \cap V$. Since every automorphism f_ν is an isometry of D with respect to d_D [7], this implies that

$$f_\nu(B(k_0; r_0)) \subset f_\nu(B(k_\nu; 2r_0)) = B(f_\nu(k_\nu); 2r_0) \subset U$$

for all $\nu \geqq \nu_0$, as desired.

2) *Putting $F_\nu = f_{\nu | B(k_0 ; r_0)}$ for $\nu \geqq \nu_0$, the sequence $\{F_\nu\}_{\nu \geqq \nu_0}$ contains a subsequence which converges uniformly on compact subsets of $B(k_0; r_0)$ to the constant mapping $C_{p | B(k_0 ; r_0)}$*: By 1) we may regard $\{F_\nu\}$ as a sequence in $\mathrm{Hol}(B(k_0; r_0), U)$, the set of all holomorphic mappings from $B(k_0; r_0)$ into U. Hence it forms a normal family, because U is taut. Moreover, since $\lim_{\nu \to \infty} k_\nu = k_0 \in B(k_0; r_0)$ and $\lim_{\nu \to \infty} F_\nu(k_\nu) = p \in U$, $\{F_\nu\}$ is not compactly divergent. Thus some subsequence $\{F_{\nu_j}\}$ of $\{F_\nu\}$ converges uniformly on compact subsets of $B(k_0; r_0)$ to a holomorphic mapping F: $B(k_0; r_0) \to U$. Clearly $F(B(k_0; r_0)) \subset \bar{D} \cap U$. Let $\rho = \sum_{i=1}^{k} \rho_i + A \sum_{i=1}^{k} (\rho_i)^2$ be a strictly plurisubharmonic function defined on U as in (C.5). Replacing U by a smaller ball if necessary, we may assume without loss of generality that $D \cap U \subset \{z \in U: \rho(z) < 0\}$. Then, considering the plurisubharmonic function $\rho \circ F$: $B(k_0; r_0) \to \boldsymbol{R}$, we can show with exactly the same arguments as in Lemma 1 that $F = C_{p | B(k_0 ; r_0)}$.

3) *There exists a subsequence $\{f_{\nu_j}\}$ of $\{f_\nu\}$ such that $\lim_{j \to \infty} f_{\nu_j}(z) = p$ for each fixed yoint $z \in D$*: By passing to a subsequence if necessary, we may assume by 2) that

$$\lim_{\nu \to \infty} f_\nu(z) = p \quad \text{for any point} \quad z \in B(k_0; r_0) .$$

Therefore

$$S=\{z\in D: \lim_{\nu\to\infty} f_\nu(z)=p\}$$

is a non-empty subset of D. To show our assertion 3), it is enough to prove that S is open and closed in D. First we verify the openness of S. For each point $z_0\in S$, we claim that there exists an open ball $B(z_0;\delta)$ contained in S. To this end, we fix an open neighborhood W of p arbitrarily, and choose an open neighborhood V of p so small that $\bar{V}\subset W$ and $\delta=d_D(D\cap(\boldsymbol{C}^n\backslash W), D\cap V)>0$. Take a point $z\in B(z_0;\delta)$ arbitrarily. Then, for all sufficiently large ν we have

$$d_D(f_\nu(z), D\cap V)\leqq d_D(f_\nu(z), f_\nu(z_0))=d_D(z, z_0)<\delta,$$

which means that $f_\nu(z)\in W$. Since W is arbitrary, this implies that $\lim_{\nu\to\infty} f_\nu(z)=p$ and accordingly $B(z_0;\delta)\subset S$, as desired. Next, taking an arbitrary point $z_0\in\bar{S}$, we claim that $z_0\in S$. Otherwise, that is, if $\lim_{\nu\to\infty} f_\nu(z_0)\neq p$, then we can choose an open neighborhood W of p and a sequence $\{\nu_j\}\subset\boldsymbol{N}$ in such a way that $f_{\nu_j}(z_0)\notin W$ for all j. For such a W, let us fix a small neighborhood V of p so that $\delta=d_D(D\cap(\boldsymbol{C}^n\backslash W), D\cap V)>0$, and take a point $w_0\in B(z_0: \delta/2)\cap S$ arbitrarily. Then

$$\delta\leqq d_D(f_{\nu_j}(z_0), D\cap V)\leqq d_D(f_{\nu_j}(z_0), f_{\nu_j}(w_0))=d_D(z_0, w_0)<\frac{\delta}{2}$$

for a large integer j, since $\lim_{j\to\infty} f_{\nu_j}(w_0)=p$, which is a contradiction. Thus $\bar{S}\subset S$ and S is a closed subset of D.

4) *Some subsequence of* $\{f_\nu\}$ *converges uniformly on compact subsets of* D *to the constant mapying* C_p: By 3) we may assume that $\lim_{\nu\to\infty} f_\nu(z)=p$ for each fixed point $z\in D$. We claim that this convergence is uniform on every compact subset of D. To prove our claim, assume the contrary. Then, there exist a compact subset L of D and an open neighborhood W of p such that $f_\nu(L)\not\subset W$ for infinitely many ν. So we can extract two sequences $\{\nu_j\}\subset\boldsymbol{N}$ and $\{a_j\}\subset L$ in such a way that $f_{\nu_j}(a_j)\notin W$ for all j. We can assume that $\lim_{j\to\infty} a_j=a$ for some point $a\in L$. Then, choosing a neighborhood V of p so small that $\bar{V}\subset W$ and $d_D(D\cap(\boldsymbol{C}^n\backslash W), D\cap V)>0$, we have a contradiction:

$$0<d_D(D\cap(\boldsymbol{C}^n\backslash W), D\cap V)\leqq d_D(f_{\nu_j}(a_j), f_{\nu_j}(a))=d_D(a_j, a)\to 0$$

as $j\to\infty$, since $\lim_{j\to\infty} f_{\nu_j}(a)=p$.

We have thus completed the proof of Lemma 2. Q.E.D.

Proof of Theorem III. The following proof will be presented in outline, since the details of the steps can be filled in by consulting the corresponding passages in the proof of [10; Theorem I]. We shall use the same notation as in [10], unless otherwise stated.

First we assume that D is a hyperbolic domain and the conditions (C.1)$\sim$(C.5), (*) and (**) are fulfilled. Then, after taking a subsequence and relabelling if necessary, we have by Lemma 2 that the sequence $\{f_\nu\}$ converges uniformly on every compact subset of D to the constant mapping $C_p(z)=p$, $z\in D$. Thus, repeating the same arguments developed in the steps 2)$\sim$7) of the proof of [10; Theorem I], we can construct an unbounded domain $\mathcal{W}$ in $\boldsymbol{C}^n$ biholomorphically equivalent to some Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ and a holomorphic mapping $F\colon D\to\mathcal{W}$. Since any Siegel domain in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ is biholomorphically equivalent to a bounded domain in $\boldsymbol{C}^n$, once it is shown that $F\colon D\to\mathcal{W}$ is injective, we can regard D as a bounded domain in $\boldsymbol{C}^n$. Thus the final step 8) of the proof of [10; Theorem I] goes through without any change. Now, assume that $F(z')=F(z'')=w$ for $z', z''\in D$. Let D', W' be relatively compact subdomains of D, $\mathcal{W}$ respectively such that $z', z''\in D'$ and $F(\bar{D}')\subset W'$. Then the same reasoning as in the step 7) of the proof of [10; Theorem I] yields that F is injective on D' and so $z'=z''$, as desired. Thus we have shown that D is biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$.

In order to prove the converse assertion, let us take an arbitrary Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ and consider the functions ρ_i, $i=1, \cdots, k$, defined by

$$\rho_i(z) = \frac{-1}{|z_i+\sqrt{-1}|^2}\cdot(\operatorname{Im} z_i - H_i(z'', z''))$$

for

$$z = (z', z'') = (z_1, \cdots, z_k, z'')\in\boldsymbol{C}^n\setminus\bigcup_{i=1}^{k}\{z\in\boldsymbol{C}^n\colon z_i+\sqrt{-1} = 0\}\ ,$$

where H_i is the i-th component function of the $\boldsymbol{R}^k_+$-hermitian form H. Now we set $U=B_{1/2}(o)$, the open Euclidean 1/2-ball with center at the origin o. Then we can check by routine calculations that every function ρ_i is real analytic on U and the conditions (C.1)$\sim$(C.5) are satisfied for the system $(o; U; \rho_1, \cdots, \rho_k)$ [10]. Furthermore, considering the one-parameter subgroup

$$\varphi_t\colon (z', z'')\mapsto (e^t z', e^{(1/2)t} z'')\ ,\quad t\in\boldsymbol{R}$$

of $Aut(\mathcal{D}(\boldsymbol{R}^k_+, H))$, we obtain that

$$\lim_{t\to-\infty}\varphi_t(z_0) = o \quad \text{for any fixed point} \quad z_0\in\mathcal{D}(\boldsymbol{R}^k_+, H)\,.$$

Clearly this guarantees us that the condition (*) is satisfied.

It remains to show tthat $\mathcal{D}(\boldsymbol{R}^k_+, H)$ is hyperbolically imbedded at o. To this end, putting $\mathcal{D}=\mathcal{D}(\boldsymbol{R}^k_+, H)$ for simplicity, we recall that there exists a biholomorphic mapping C: $\mathcal{D}\to\mathcal{B}$ from $\mathcal{D}$ onto a certain bounded domain $\mathcal{B}$ in $\boldsymbol{C}^n$, which can be extended to a biholomorphic mapping $U=B_{1/2}(o)$ onto an open neighborhood V of the point $C(o)\in\partial\mathcal{B}$ [10; Lemma in section 1]. We denote this extended biholomorphic mapping by the same letter C. Now, for the verification of the condition (**) at o it is sufficient to show the following:

(4.1) For any neighborhood W of o with $W\subset U$, there exists a smaller neighborhood V' of o such that $\bar{V}'\subset W$ and $d_{\mathcal{D}}(\mathcal{D}\cap(\boldsymbol{C}^n\backslash W), \mathcal{D}\cap V')>0$.

Let us now fix such a neighborhood W arbitrarily and set $W'=C(W)$. Then W' is a neighborhood of $C(o)\in\partial\mathcal{B}$ contained in V, so that one can find a neighborhood V'' of $C(o)$ such that $\bar{V}''\subset W'$ and $d_{\mathcal{B}}(\mathcal{B}\cap(\boldsymbol{C}^n\backslash W'), \mathcal{B}\cap V'')>0$, because the bounded domain $\mathcal{B}$ is hyperbolically imbedded at $C(o)\in\partial\mathcal{B}$ [6]. Thus, recalling the fact C: $\mathcal{D}\to\mathcal{B}$ is an isometry [7], we can see that the set $V'=C^{-1}(V'')$ has the property of (4.1), as desired. Q.E.D.

5. Concluding remarks

5.1. Let D be a domain in a complex manifold X of complex dimension n and let p be a point of $\bar{D}$. Then we can define the hyperbolically imbeddedness of D at p in the same way as in the Euclidean case. Furthermore, the notion of domains in X with piecewise C^2-smooth boundaries of special type can be natually introduced.

REMARK 1. *The analogue of Theorem III is true in the case where D is a hyperbolic domain in a complex manifold X.*

In fact, since the open neighborhood U of p can be chosen as small as we wish, we may assume that U is a local coordinate neighborhood of p in X and there exists a biholomorphic mapping γ: $U\to\boldsymbol{C}^n$ such that $\gamma(p)=o$ and $\gamma(U)=\mathcal{B}^n$. Thus, by transfering back and forth between U and $\mathcal{B}^n$ via this coordinate mapping γ in the proof of Theorem III, we can prove the general case as above.

REMARK 2. By virtue of Remark 1, one can see that *the analogue of Theorem II is true in the case where D is a hyperbolically imbedded subdomain of a complex manifold X in the sense of Kiernan* [6] *and ∂D is a piecewise C^2-smooth boundary of special type.*

REMARK 3. *The analogue of Theorem I is also true in the case where D is a hyperbolically imbedded subdomain of a Stein manifold X in the sense of Kiernan* [6] *and ∂D is a piecewise C^2-smooth boundary of special type.*

In fact, it is obvious by Remark 1 that the assertions (i), (ii) and (iv) are mutually equivalent and (i) implies (iii) in Theorem I for the general case above. Therefore the only thing which has to be proven is the implication (iii)$\Rightarrow$(i).

We first notice by [6; Theorem 1] and Lemma 1 that our domain D is a taut subdomain of X. Therefore, assuming that $Aut(D)$ is non-compact, we can select a sequence $\{f_\nu\}$ in $Aut(D)$ such that $\{f_\nu(q)\}$ converges to some boundary point p of D, where q is an arbitrary given point of D. Then we can conclude by Remark 1 that D is biholomorphically equivalent to some Siegel domain $\mathfrak{D}(\boldsymbol{R}_+^k, H)$ in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$. We have to show that $k=1$. Retaining the notation in the proof of Theorem I, we now define a family of holomorphic mappings F^ν: $\Delta(r) \to X$, $\nu=1, 2, \cdots$, by

$$F^\nu(\eta) = \varphi(z_1^0 + a + \sqrt{-1}\, b_\nu, z_2^0 + \eta, z_3^0, \cdots, z_n^0) \qquad \text{for} \quad \eta \in \Delta(r)\,.$$

Then, since $\mathrm{Hol}(\Delta(r), D)$ is relatively compact in $\mathrm{Hol}(\Delta(r), X)$ by [6; Theorem 1], we can assume by the proof of Theorem I that $\{F^\nu\}$ converges uniformly on compact subsets of $\Delta(r)$ to the constant mapping $F(\eta) \equiv F(0) \in \partial D$. Since X is a Stein manifold, there are global functions $c_1, \cdots, c_n$ on X such that $c=(c_1, \cdots, c_n)$: $X \to \boldsymbol{C}^n$ gives a holomorphic imbedding of an open neighborhood V of $F(0) \in \partial D$ into $\boldsymbol{C}^n$. We now consider the bounded holomorphic functions $\tilde{\varphi}_j = c_j \circ \varphi$, $j=1, \cdots, n$, on $\mathfrak{D}(\boldsymbol{R}_+^k, H)$. Then, replacing φ_j by $\tilde{\varphi}_j$ in the proof of Theorem I, we can prove that

$$\frac{\partial \tilde{\varphi}_j}{\partial z_2}(z_1^0 + \xi, z_2^0, \cdots, z_n^0) = 0 \qquad \text{for} \quad \xi \in \mathfrak{H}, j = 1, \cdots, n\,.$$

Therefore, by setting $\tilde{D} = \mathfrak{D}(\boldsymbol{R}_+^k, H) \cap \varphi^{-1}(D \cap V)$ and $\tilde{\varphi} = (c \circ \varphi)_{|\tilde{D}}$, we see that the complex Jacobian determinant of the biholomorphic mapping $\tilde{\varphi}$: $\tilde{D} \to c(D \cap V) \subset \boldsymbol{C}^n$ vanishes identically on a non-empty set $\tilde{D} \cap \{(z_1^0 + \xi, z_2^0, \cdots, z_n^0) \in \boldsymbol{C}^n : \xi \in \mathfrak{H}\}$, a contradiction. Consequently $k=1$ and D must be biholomorphically equivalent to $\mathfrak{B}^n$.

5.2. Finally we give a remark on the second condition (**) in Theorem III. As previously mentioned in the introduction, this is automatically satisfied for any boundary point of D, provided that D is a bounded domain in $\boldsymbol{C}^n$. However, in contrast with this, for an unbounded domain D it does not seem easy to see whether the condition (**) is fulfilled or not at a given point $p \in \partial D$. In fact, this may be illustrated by the following example: Consider the domains D, B in $\boldsymbol{C}^2$ and a holomorphic mapping f: $\boldsymbol{C}^2 \to \boldsymbol{C}^2$ defined by

$$\begin{aligned} D &= \{(z_1, z_2) \in \boldsymbol{C}^2 : \ 2\,\mathrm{Re}\, z_2 + |z_1 z_2|^2 + |z_2|^2 < 0\}\ ; \\ B &= \{(w_1, w_2) \in \boldsymbol{C}^2 : \ 2\,\mathrm{Re}\, w_2 + |w_1|^2 + |w_2|^2 < 0\}\ : \\ (w_1, w_2) &= f(z_1, z_2) = (z_1 z_2, z_2) \qquad \text{for} \quad (z_1, z_2) \in \boldsymbol{C}^2 \end{aligned}$$

(see [12; p. 85]). Then B is a unit ball with center at $(0, -1)$ and f gives rise to a biholomorphic mapping from D onto B. In particular, D is a homogeneous hyperbolic domain. Now let us consider the set

$$S = \{(z_1, z_2) \in \boldsymbol{C}^2 : z_2 = 0\}\ .$$

Then it is obvious that ∂D is real analytic and $S\subset\partial D$. Here we assert that D is not hyperbolically imbedded at $(0, 0)\in S\subset\partial D$. To verify this, consider the holomorphic mappings f_ν: $\Delta=\{t\in \boldsymbol{C}: |t|<1\}\to \boldsymbol{C}^2$, $\nu=2, 3, \cdots$, defined by

$$f_\nu(t)=(t\sqrt{2\nu-1}, -1/\nu) \qquad \text{for} \quad t\in\Delta$$

and set

$$a_\nu=f_\nu(0), \quad b_\nu=f_\nu(1/\sqrt{2\nu-1}) \qquad \text{for} \quad \nu=2, 3, \cdots.$$

Then it is easy to see that

$$\{f_\nu\}\subset Hol(\Delta, D) \quad \text{and} \quad \lim_{\nu\to\infty}(a_\nu, b_\nu)=((0, 0), (1, 0))\in\partial D\times\partial D.$$

On the other hand, the distance decreasing property tells us that

$$d_D(a_\nu, b_\nu)=d_D(f_\nu(0), f_\nu(1/\sqrt{2\nu-1}))\leqq d_\Delta(0, 1/\sqrt{2\nu-1})\to 0$$

as $\nu\to\infty$. Obviously this implies that D is not hyperbolically imbedded at (0, 0), as asserted. Similarly we can, in fact, prove that this last conclusion is also true for any other point $p\in S$. On the other hand, by the fact that f is biholomorphic on $\boldsymbol{C}^2\backslash S$ we can deduce that the condition (**) is satisfied for any boundary point $p\in\partial D\backslash S$.

The example above shows also that (**) does not follow from (*). Of course, (**) does not imply (*). Therefore, two conditions (*) and (**) have no relevancy to each other in general.

References

[1] T.J. Barth: *Convex domains and Kobayashi hyperbolicity*, Proc. Amer. Math. Soc. **79** (1980), 556–558.

[2] M. Behrens: *A generalization of a theorem of Fornaess-Sibony*, Math. Ann. **273** (1985), 123–130.

[3] J.E. Fornaess and Sibony: *Increasing sequences of complex manifolds*, Math. Ann. **255** (1981), 351–360.

[4] P. Kiernan: *On the relations between taut, tight and hyperbolic manifolds*, Bull. Amer. Math. Soc. **76** (1970), 49–51.

[5] P. Kiernan: *Extensions of holomorphic maps*, Trans. Amer. Math. Soc. **172** (1972), 347–355.

[6] P. Kiernan: *Hyperbolically imbedded spaces and the big Picard theorem*, Math. Ann. **204** (1973), 203–209.

[7] S. Kobayashi: Hyperbolic manifolds and holomorphic mappings, Marcel Dekker, New York, 1970.

[8] A. Kodama: *On the equivalence problem for a certain class of generalized Siegel domains*, III, Osaka J. Math. **18** (1981), 481–499.

[9] A. Kodama: *On complex manifolds exhausted by biholomorphic images of a strictly pseudoconvex domain*, Preprint.

[10] A. Kodama: *On the structure of a bounded domain with a special boundary point*, Osaka J. Math. **23** (1986), 271–298.

[11] R. Narasimhan: Several complex variables, Univ. Chicago Press, Chicago and London, 1971.

[12] S.I. Pinčuk: *Holomorphic inequivalence of some classes of domains in* C^n, Mat. Sb. **111** (1980), 67–94; English transl. in Math. USSR Sb. **39** (1981), 61–86.

[13] S.I. Pinčuk: *Homogeneous domains with piecewise-smooth boundaries*, Mat. Zametki **32** (1982), 729–735; English transl. in Math. Notes **32** (1982), 849–852.

[14] I.I. Pjateckii-Sapiro: Géométrie des domaines classiques et théorie des fonctions automorphies, Dunod, Paris, 1966.

[15] J.P. Rosay: *Sur une caractérisation de la buole parmi les domaines de* C^n *par son groupe d'automorphismes*, Ann. Inst. Fourier (Grenoble) **29**, 4 (1979), 91–97.

[16] H.L. Royden: *Remarks on the Kobayashi metric*, Lecture Notes in Math. 185, Springer-Verlag, Berlin, Heidelberg and New York, 1971, 125–135.

[17] M. Tsuji: Potential theory in modern function theory, Chelsea Publishing Company, New York, 1975.

[18] E. Vesentini: *Holomorphic almost periodic functions and positive-definite functions on Siegel domains*, Ann. Mat. Pura. Appl. (4) **102** (1975), 177–202.

[19] B. Wong: *Characterization of the unit ball in* C^n *by its automorphism group*, Invent. Math. **41** (1977), 253–257.

[20] H. Wu: *Normal families of holomorphic mappings*, Acta Math. **119** (1967), 193–233.

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