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HOMOTOPY REPRESENTATIONS AND SPHERES OF REPRESENTATIONS

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0. Introduction

T. tom Dieck and T. Petrie have introduced and studied homotopy representations in [7] and [8]. Let G be a finite group. A G -CW-complex X is called a homotopy representation of G if the H -fixed point set X^H is homotopy equivalent to a $(\dim X^H)$ -dimensional sphere for each subgroup H of G . (If X^H is empty, then we set $\dim X^H = -1$ and S^{-1} is empty.) A homotopy representation of G is called linear if it is G -homotopy equivalent to a unit sphere of a real representation of G . (See [7].) We denote the set of G -homotopy classes of homotopy representations [resp. linear homotopy representations] of G by $V^+(G, h^\infty)$ [resp. $V^+(G, \mathcal{I})$]. These sets are commutative semi-groups with addition induced by join. Let $V(G, \lambda)$ be the Grothendieck group associated to $V^+(G, \lambda)$ for $\lambda = \mathcal{I}$ or h^∞ . We call $V(G, h^\infty)$ the homotopy representation group of G and $V(G, \mathcal{I})$ the linear homotopy representation group of G . The group $V(G, h^\infty)$ has been studied by tom Dieck and Petrie ([7], [8]) and the group $V(G, \mathcal{I})$ has been studied by many authors. (See [1], [4], [10], [11], [13], [15], [18] and [19].)

Let $\phi(G)$ be the set of conjugacy classes of subgroups of G and $C(G)$ be the ring of all integer valued functions on $\phi(G)$. For any homotopy representation X , $\text{Dim } X \in C(G)$ is defined by $(\text{Dim } X)(H) = \dim X^H + 1$, which is called the dimension function of X . Since $\text{Dim } X * Y = \text{Dim } X + \text{Dim } Y$ ($*$ means the join), the homomorphism $\text{Dim}: V(G, \lambda) \rightarrow C(G)$ is induced by the assignment $X \mapsto \text{Dim } X$. This homomorphism is called the dimension homomorphism of $V(G, \lambda)$. The kernel of Dim is denoted by $v(G, \lambda)$. tom Dieck and Petrie have shown that $v(G, h^\infty)$ is isomorphic to the Picard group $\text{Pic}(A(G))$ of the Burnside ring of G in [7].

We are interested in the difference between $V(G, \mathcal{I})$ and $V(G, h^\infty)$. We observe the homomorphisms which are induced from the inclusion $V^+(G, \mathcal{I}) \rightarrow V^+(G, h^\infty)$:

$$\begin{aligned} I_G: V(G, \mathcal{I}) &\rightarrow V(G, h^\infty) \\ i_G: v(G, \mathcal{I}) &\rightarrow v(G, h^\infty). \end{aligned}$$

These homomorphisms are injective ([7]). We obtain the following results.

Theorem A. *The homomorphism I_G is isomorphic if and only if G is a cyclic group or a dihedral 2-group D_n ($n \geq 2$).*

Here $D_n = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

Theorem B. *Suppose that G is nilpotent. Then the homomorphism i_G is isomorphic if and only if G is a cyclic group or a dihedral 2-group D_n ($n \geq 2$).*

T. Petrie has already announced the analogous theorem of Theorem A for oriented homotopy representations in [16]. Our Theorems are the un-oriented versions.

This paper is organized as follows. In section 1 we shall have the necessary conditions. In section 2 and section 3 we shall compute the order of $v(G, h^\infty)$. In section 4 the proofs of the main theorems will be completed.

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1. Necessary conditions

Using the results in [5] and [11], we shall show the following proposition.

Proposition 1.1. *Let G be an abelian group. Then the homomorphism i_G is an isomorphism if and only if G is a cyclic group or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. It is known that the group $v(G, h^\infty)$ is isomorphic to

$$\prod_{H \subseteq G} \mathbb{Z}_{|G/H|}^* / \{\pm 1\} \quad ([5])$$

and the group $v(G, \iota)$ is isomorphic to

$$\prod_{\substack{H \subseteq G \\ G/H : \text{cyclic}}} \mathbb{Z}_{|G/H|}^* / \{\pm 1\} \quad ([11]).$$

Here $\mathbb{Z}_{|G/H|}^*$ is the group of invertible elements in $\mathbb{Z}_{|G/H|}$. Hence it is easy to see this proposition since i_G is injective.

Let H be a normal subgroup of G and X a homotopy representation of G . Since X^H is a homotopy representation of G/H , the correspondence $X \mapsto X^H$ induces a homomorphism $f: v(G, \lambda) \rightarrow v(G/H, \lambda)$ for $\lambda = \iota$ or h^∞ . The following lemma is elementary.

Lemma 1.2. *Under the above situation,*

- i) *The homomorphism f is surjective.*
- ii) *The following diagram is commutative.*

$$\begin{array}{ccc} v(G, \mathcal{L}) & \xrightarrow{i_G} & v(G, h^\infty) \\ f \downarrow & & \downarrow f \\ v(G/H, \mathcal{L}) & \xrightarrow{i_{G/H}} & v(G/H, h^\infty) \end{array}$$

iii) If i_G is isomorphic, then $i_{G/H}$ is also isomorphic.

Proof. i): Let $\mathcal{L}: v(G/H, \lambda) \rightarrow v(G, \lambda)$ be the homomorphism which is induced by considering G/H -spaces as G -spaces via the canonical projection $G \rightarrow G/H$. Then, the composition $f \circ \mathcal{L}$ is the identity on $v(G, \lambda)$ and so f is surjective.

ii): This follows from the definitions of f and i_G .

iii): This follows directly from the injectivity of $i_{G/H}$, i) and ii).

Let us apply lemma 1.2 to the commutator subgroup $[G, G]$ of G . Then we have:

Proposition 1.3. *If i_G is an isomorphism, then $G/[G, G]$ is a cyclic group or $\mathbf{Z}_2 \times \mathbf{Z}_2$. Furthermore suppose that G is nilpotent, then G is one of the following groups;*

(1) *cyclic group*

(2) *dihedral 2-group D_n ($n \geq 2$)*

$$D_n = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$$

(3) *quaternion 2-group Q_n ($n \geq 3$)*

$$Q_n = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle$$

(4) *semi-dihedral 2-group SD_n ($n \geq 4$)*

$$SD_n = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle$$

Proof. The first half is clear from Proposition 1.1. Suppose that G is nilpotent. We may put $G = P_1 \times \cdots \times P_r$, where P_1 is a 2-group and P_i is a p_i -group (p_i : odd prime) for $i > 1$. If $G/[G, G]$ is cyclic, then G is also cyclic. In the case $G/[G, G] = \mathbf{Z}_2 \times \mathbf{Z}_2$, $P_i/[P_i, P_i]$ is trivial for $i > 1$. Hence P_i must be trivial for $i > 1$, so that G is 2-group. Therefore, if G is abelian, then G is $D_2 (= \mathbf{Z}_2 \times \mathbf{Z}_2)$. If G is not abelian, then G is D_n ($n \geq 3$), Q_n ($n \geq 3$) or SD_n ($n \geq 4$) by [9, Chap. 5, Theorem 4.5].

2. The order of the Picard group of the Burnside ring

The group $v(G, h^\infty)$ is isomorphic to the Picard group $\text{Pic}(A(G))$ of $A(G)$, where $A(G)$ is the Burnside ring of G . In this section we compute the order of $\text{Pic}(A(G))$. We recall the Burnside ring $A(G)$. The set of G -isomorphism

classes of finite G -sets becomes a commutative semi-ring with addition induced by disjoint union and multiplication induced by cartesian product. The Grothendieck ring of the semi-ring is called the Burnside ring. Let S be a finite G -set. The correspondence $S \mapsto |S^H|$ induces the ring homomorphism $\chi: A(G) \rightarrow \prod_{(H) \in \phi(G)} \mathbf{Z}_H$, where $\mathbf{Z}_H = \mathbf{Z}$. As is well-known, the ring homomorphism χ is injective and if we consider $A(G)$ as a subring of $\prod_{(H) \in \phi(G)} \mathbf{Z}_H$ via χ , then

$$A(G) = \{(d_H) \in \prod_{(H)} \mathbf{Z}_H \mid \text{congruences } (*)_H \text{ for } (H) \in \phi(G)\},$$

where

$$(*)_H: \sum_{(K)} |NH/NH \cap NK| \varphi(|K/H|) d_K \equiv 0 \pmod{|NH/H|}.$$

The sum is over the NH -conjugacy classes (K) such that H is a normal subgroup in K and K/H is cyclic. φ denotes the Euler function and NH denotes the normalizer of H in G . We can rewrite $(*)_H$ as the following:

$$(**)_H: \sum_{(K)} n(H, K) d_K \equiv 0 \pmod{|NH/H|},$$

where each $n(H, K)$ is a certain integer and $n(H, H) = 1$ for any $(H), (K) \in \phi(G)$ such that K/H is cyclic. The sum is over the G -conjugacy classes (K) such that H is normal in K and K/H is cyclic (see [6]).

We put $\phi(G) = \{(H_1), \dots, (H_n)\}$ and assume that $(H_i) \leq (H_j)$ implies $i \geq j$, where $(H_i) \leq (H_j)$ means that H_i conjugates to a certain subgroup of H_j . We set

$$R_k = \{(d_{H_i})_{1 \leq i \leq k} \in \prod_{i=1}^k \mathbf{Z}_{H_i} \mid (**)_{H_i}, 1 \leq i \leq k\}.$$

Note that $R_1 = \mathbf{Z}$ and $R_n = A(G)$.

Lemma 2.1. R_k is the subring of $\prod_{i=1}^k \mathbf{Z}_{H_i}$.

Proof. It is trivial that R_k is an additive subgroup of $\prod_{i=1}^k \mathbf{Z}_{H_i}$. We note that there exists $(d'_{H_i})_{1 \leq i \leq n} \in A(G)$ such that $d_{H_i} = d'_{H_i}$, $1 \leq i \leq k$, for any $(d_{H_i})_{1 \leq i \leq k} \in R_k$. Let $(d_{H_i})_{1 \leq i \leq k}$ and $(e_{H_i})_{1 \leq i \leq k}$ be any two elements in R_k . Since $A(G)$ is the ring, $(d'_{H_i} e'_{H_i})_{1 \leq i \leq n}$ is in $A(G)$. Hence $(d_{H_i} e_{H_i})_{1 \leq i \leq k}$ satisfies $(**)_{H_j}$, $1 \leq j \leq k$, since $(d'_{H_i} e'_{H_i})_{1 \leq i \leq n}$ satisfies $(**)_{H_j}$, $1 \leq j \leq n$. Therefore $(d_{H_i} e_{H_i})_{1 \leq i \leq k}$ is in R_k .

We define a map $p: R_{k-1} \rightarrow \mathbf{Z}_{|WH_k|}$, $WH_k = NH_k/H_k$, by

$$(d_{H_i})_{1 \leq i \leq k-1} \mapsto \sum_{\substack{j \\ 1 \neq H_j/H_k: \text{cyclic}}} n(H_k, H_j) d_{H_j} \pmod{|WH_k|}.$$

Lemma 2.2. p is the ring homomorphism.

Proof. It is trivial to be an additive homomorphism. For $(d_{H_i})_{1 \leq i \leq k}$ [resp. $(e_{H_i})_{1 \leq i \leq k}] \in R_{k-1}$, we choose $(d'_{H_i})_{1 \leq i \leq n}$ [resp. $(e'_{H_i})_{1 \leq i \leq n}] \in A(G)$ like the proof of

(2.1). Then

$$\begin{aligned} d'_{H_k} e'_{H_k} &\equiv - \sum_{1 \neq H_j/H_k : \text{cyclic}} n(H_k, H_j) d_{H_j} e_{H_j} \pmod{|WH_k|} \\ d'_{H_k} &\equiv - \sum_{1 \neq H_j/H_k : \text{cyclic}} n(H_k, H_j) d_{H_j} \pmod{|WH_k|} \\ e'_{H_k} &\equiv - \sum_{1 \neq H_j/H_k : \text{cyclic}} n(H_k, H_j) e_{H_j} \pmod{|WH_k|}. \end{aligned}$$

Hence $p((d_{H_i} e_{H_i})_{1 \leq i \leq k}) = p((d_{H_i})_{1 \leq i \leq k}) p((e_{H_i})_{1 \leq i \leq k})$

Let $s: R_k \rightarrow R_{k-1}$ and $r: R_k \rightarrow \mathbf{Z}$ be the ring homomorphisms defined by

$$\begin{aligned} s: (d_{H_i})_{1 \leq i \leq k} &\mapsto (d_{H_i})_{1 \leq i \leq k-1} \\ r: (d_{H_i})_{1 \leq i \leq k} &\mapsto d_{H_k}. \end{aligned}$$

We have the following lemma. (See [5].)

Lemma 2.3. *The following diagram is the pull-back of ring.*

$$\begin{array}{ccc} R_k & \xrightarrow{s} & R_{k-1} \\ r \downarrow & & \downarrow p \\ \mathbf{Z} & \xrightarrow{q} & \mathbf{Z}_{|WH_k|} \end{array} \quad (2 \leq k \leq n)$$

Here q is the canonical projection.

Proof. It is easy to show this lemma from the definitions of R_k , s , p and r . The next proposition is the main result in this section.

Proposition 2.4. *Let G be any finite group. Then*

$$|\text{Pic}(A(G))| = 2^{-n} |A(G)^*| \prod_{(H) \in \phi(G)} \varphi(|NH/H|),$$

where $n = |\phi(G)|$ and $A(G)^*$ is the unit group of $A(G)$.

Proof. The pull-back diagram in Lemma 2.3 yields the Mayer-Vietoris exact sequence of the Picard group [6]. That is, the sequence:

$$\begin{aligned} 0 \rightarrow R_k^* &\rightarrow R_{k-1}^* \oplus \mathbf{Z}^* \rightarrow \mathbf{Z}_{|WH_k|}^* \\ &\rightarrow \text{Pic } R_k \rightarrow \text{Pic } R_{k-1} \oplus \text{Pic } \mathbf{Z} \rightarrow \text{Pic } \mathbf{Z}_{|WH_k|} \end{aligned}$$

is exact, $2 \leq k \leq n$. Since $\text{Pic } \mathbf{Z} = 0$ and $\text{Pic } \mathbf{Z}_{|WH_k|} = 0$ by ([2], chap. 2, 5), we obtain the exact sequence:

$$\begin{aligned} 0 \rightarrow R_k^* &\rightarrow R_{k-1}^* \oplus \mathbf{Z}^* \rightarrow \mathbf{Z}_{|WH_k|}^* \\ &\rightarrow \text{Pic } R_k \rightarrow \text{Pic } R_{k-1} \rightarrow 0 \quad (2 \leq k \leq n). \end{aligned}$$

Inductively, $\text{Pic } R_k$ has a finite order and then we have

$$\frac{|\text{Pic } R_k|}{|\text{Pic } R_{k-1}|} = \frac{\varphi(|WH_k|)}{2} \frac{|R_k^*|}{|R_{k-1}^*|} \quad (2 \leq k \leq n).$$

Therefore,

$$\frac{|\text{Pic } R_n|}{|\text{Pic } R_1|} = \frac{1}{2^{n-1}} \frac{|R_n^*|}{|R_1^*|} \prod_{k=2}^n \varphi(|WH_k|).$$

Since $R_n = A(G)$, $R_1 = \mathbf{Z}$ and $\varphi(|WH_1|) = 1$, the desired result holds.

Corollary 2.5. $|v(G, h^\infty)| = 2^{-n} |A(G)^*| \prod_{(G)} \varphi(|NH/H|)$, $n = |\phi(G)|$.

3. Examples

We indeed compute the order of $\text{Pic}(A(G))$ (i.e. $v(G, h^\infty)$) for some groups.

EXAMPLE 3.1. Let G be D_n ($n \geq 2$) in Proposition 1.3. Then $|\text{Pic}(A(G))| = 2^N$, where $N = (n-2)(n-3)/2$.

Proof. Conjugacy classes of subgroups of D_n are the following:

$$\begin{aligned} (D_n) \\ \langle a^{2^i} \rangle, i = 0, 1, \dots, n-1 \\ \langle a^{2^i}, b \rangle, i = 1, 2, \dots, n-1 \\ \langle a^{2^i}, ab \rangle, i = 1, 2, \dots, n-1, \end{aligned}$$

and WH of these subgroups in G are the following:

$$\begin{aligned} WD_n &= 1 \\ W\langle a^{2^i} \rangle &= D_{i+1}, i = 0, 1, \dots, n-1 \\ W\langle a^{2^i}, b \rangle &= \mathbf{Z}_2, i = 1, 2, \dots, n-1 \\ W\langle a^{2^i}, ab \rangle &= \mathbf{Z}_2, i = 1, 2, \dots, n-1. \end{aligned}$$

Hence $|\phi(G)| = 3n-1$ and $\prod_{(G)} \varphi(|WH|) = 2^{n(n-1)/2}$. We see that $|A(G)^*|$ is 2^{n+2} by Theorem 4.1 or Example 4.8 in [14]. By Proposition 2.4,

$$\begin{aligned} |\text{Pic}(A(G))| &= 2^{-(3n-1)} \times 2^{n(n-1)/2} \times 2^{n+2} \\ &= 2^N. \end{aligned}$$

One can show the following Examples 3.2 and 3.3 by the same argument.

EXAMPLE 3.2. Let G be Q_n ($n \geq 3$) in Proposition 1.3. Then $|\text{Pic}(A(G))| = 2^{N+1}$, where $N = (n-2)(n-3)/2$.

EXAMPLE 3.3. Let G be SD_n ($n \geq 4$) in Proposition 1.3. Then $|\text{Pic}(A(G))|$

$=2^N$, where $N=(n-2)(n-3)/2$.

EXAMPLE 3.4. If $v(G, h^\infty)=0$, then $|G|=2^n$ or $2^n p_1 p_2 \cdots p_r$, where p_i , $1 \leq i \leq r$, are distinct odd primes of forms $2^e + 1$.

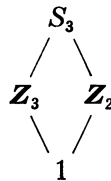
Proof. Put $|G|=2^n p_1^{e_1} \cdots p_r^{e_r}$, where p_i , $1 \leq i \leq r$, are distinct odd primes. Then $\varphi(|G|)=2^{n-1}(p_1-1)p_1^{e_1-1} \cdots (p_r-1)p_r^{e_r-1}$. If $v(G, h^\infty)=0$, then $\varphi(|G|)$ must be 2-power by Corollary 2.5. Hence $e_i=1$ and p_i-1 is 2-power for any i .

EXAMPLE 3.5. Let S_r be the symmetric group on r letters, where $r=3$, 4 or 5. We have the following table.

G	$V(G, \iota)$	$V(G, h^\infty)$
S_3	\mathbf{Z}^3	\mathbf{Z}^4
S_4	\mathbf{Z}^5	\mathbf{Z}^8
S_5	\mathbf{Z}^7	$\mathbf{Z}^{15} \oplus \mathbf{Z}_2$

$$\mathbf{Z}^n = \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \text{ (n-times)}$$

Proof. We note that $V(G, \lambda) = \text{Dim } V(G, \lambda) \oplus v(G, \lambda)$ and $\text{Dim } V(G, \lambda)$ is a free abelian group, $\lambda = \iota$ or h^∞ . The rank of $\text{Dim } V(G, \iota)$ is equal to the number of conjugacy classes of cyclic subgroups of G by [6]. The rank of $\text{Dim } V(G, h^\infty)$ is equal to the number of conjugacy classes (H) of subgroups H of G such that $H/[H, H]$ is cyclic by [7]. By these facts $\text{Dim } V(G, \lambda)$ is computable. For the symmetric group on n letters S_n , $v(S_n, \iota)=0$ by [12] or [6]. Now we compute $v(S_r, h^\infty)$ for $r=3$, 4 or 5. In case $r=3$, the diagram of conjugate subgroups of S_3 is the following:



and $WS_3=1$, $W\mathbf{Z}_3=\mathbf{Z}_3$, $W\mathbf{Z}_2=1$ and $W1=S_3$. Hence $|\phi(S_3)|=4$ and $\prod_{(H)} \varphi(|WH|)=2$. The order of $A(S_3)^*$ is 8 by using the congruences $(*)_H$ in section 2. Therefore $V(S_3, h^\infty)=0$. In cases $r=4$ and $r=5$, using the diagrams of conjugate subgroups of S_4 and S_5 in [14], one can show $|v(S_4, h^\infty)|=1$ and $|v(S_5, h^\infty)|=2$ respectively. Hence $v(S_4, h^\infty)=0$ and $v(S_5, h^\infty)=\mathbf{Z}_2$.

EXAMPLE 3.6. Let G be the symmetric group on n letters. Then i_G is

an isomorphism if $n \leq 4$ and is not an isomorphism if $n > 4$.

Proof. By [12], $v(S_n, \iota) = 0$ for any positive integer n . If $n \geq 6$, then $v(S_n, h^\infty) \neq 0$ by Example 3.4. Since $v(S_r, h^\infty) = 0$ ($r \leq 4$) and $v(S_5, h^\infty) = \mathbf{Z}_2$ by Example 3.5, the desired result follows.

REMARK. $v(G, h^\infty) = 0$ implies that the stable G -homotopy class of the homotopy representation X is decided by its dimension function $\text{Dim } X$, where homotopy representations X and Y are stable G -homotopy equivalent if there exists a homotopy representation Z such that $X * Z$ and $Y * Z$ are G -homotopy equivalent.

4. Proofs

We shall prove Theorem B by comparing orders of $v(G, \iota)$ and $v(G, h^\infty)$. Let m be the exponent of G and u_m a primitive m -th root of unity. Let Γ be the Galois group $\text{Gal}(\mathbf{Q}(u_m)/\mathbf{Q})$. Γ acts on the set $\text{Irr}(G, \mathbf{R})$ of real irreducible characters of G by $(\gamma \cdot \chi)(g) = \gamma(\chi(g))$ for $\chi \in \text{Irr}(G, \mathbf{R})$, $\gamma \in \Gamma$ and $g \in G$. We need the following theorem in [4] and [6] in order to compute the order of $v(G, \iota)$.

Theorem 4.1 (T. tom Dieck). *Let G be a p -group. Then $v(G, \iota)$ is isomorphic to $\bigoplus_{A \in X} \Gamma/\Gamma_A$, where $X = \text{Irr}(G, \mathbf{R})/\Gamma$ and $\Gamma_A = \{\gamma \in \Gamma \mid \gamma \cdot \chi = \chi\}$, ($\chi \in A$).*

REMARK. For any group G , $v(G, \iota)$ is isomorphic to a quotient group of $\bigoplus_{A \in X} \Gamma/\Gamma_A$. (See [4] or [6].)

Lemma 4.2. *For D_n , Q_n and SD_n in Proposition 1.3, we have $|v(D_n, \iota)| = 2^N$, $|v(Q_n, \iota)| = 2^N$ and $|v(SD_n, \iota)| = 2^{N-1}$. Here $N = (n-2)(n-3)/2$.*

Proof. We need the real irreducible character tables of D_n , Q_n and SD_n . By [17, Chap. 13, 2.], we have the next tables.

In the D_n case:

	a^k	$a^k b$
θ_1	1	1
θ_2	1	-1
θ_3	$(-1)^k$	$(-1)^k$
θ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$u_m^{hk} + u_m^{-hk}$	0

Here $1 \leq h < m/2$, $m = 2^{n-1}$.

In the Q_n case:

	a^k	$a^k b$
θ_1	1	1
θ_2	1	-1
θ_3	$(-1)^k$	$(-1)^k$
θ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$u_m^{2hk} + u_m^{-2hk}$	0
ψ_s	$\psi_s(a^k)$	0

Here $\psi_s(a^k) = 2(u_m^{(2s-1)k} + u_m^{-(2s-1)k})$
 $1 \leq h < m/4, 1 \leq s \leq m/4, m = 2^{n-1}.$

In the SD_n case:

	a^k	$a^k b$
θ_1	1	1
θ_2	1	-1
θ_3	$(-1)^k$	$(-1)^k$
θ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h	$u_m^{2hk} + u_m^{-2hk}$	0
ψ_s	$\psi_s(a^k)$	0

Here $\psi_s(a^k) = u_m^{(2s-1)k} + u_m^{-(2s-1)k}$
 $+ u_m^{(m/2-(2s-1))k} + u_m^{-(m/2-(2s-1))k}$
 $1 \leq h < m/4, 1 \leq s < m/8, m = 2^{n-1}.$

By the irreducible character tables of D_n , Q_n and SD_n , we have

$$\begin{aligned}
 X &= \text{Irr}(G, \mathbf{R})/\Gamma \\
 &= \begin{cases} \{\{\theta_i\}, A_i \mid 1 \leq i \leq 4, 1 \leq t \leq n-2\} & \text{if } G = D_n \\ \{\{\theta_i\}, A_i, B \mid 1 \leq i \leq 4, 1 \leq t \leq n-3\} & \text{if } G = Q_n \\ \{\{\theta_i\}, A_i, C \mid 1 \leq i \leq 4, 1 \leq t \leq n-3\} & \text{if } G = SD_n. \end{cases}
 \end{aligned}$$

Here $A_i = \{\chi_h \mid h \equiv 2^{i-1} \pmod{2^i}\}$

$$B = \{\psi_s \mid 1 \leq s \leq m/4\}$$

$$C = \{\psi_s \mid 1 \leq s \leq m/8\}.$$

Hence,

$$|\Gamma/\Gamma_{A_t}| = |A_t| = \begin{cases} 2^{n-2-t} & \text{if } G = D_n \\ 2^{n-3-t} & \text{if } G = SD_n \text{ or } Q_n, \end{cases}$$

$$|\Gamma/\Gamma_B| = 2^{n-3}$$

and

$$|\Gamma/\Gamma_C| = 2^{n-4}.$$

Therefore, by Theorem 4.1, we have

$$|v(G, \mathcal{L})| = \begin{cases} \prod_{t=1}^{n-1} |A_t| = 2^N & \text{if } G = D_n \\ |B| \prod_{t=1}^{n-3} |A_t| = 2^N & \text{if } G = Q_n \\ |C| \prod_{t=1}^{n-3} |A_t| = 2^{N-1} & \text{if } G = SD_n. \end{cases}$$

Proposition 4.3. *The homomorphism i_G is an isomorphism if $G = D_n$ ($n \geq 2$) and is not an isomorphism if $G = Q_n$ ($n \geq 3$) or SD_n ($n \geq 4$).*

Proof. The desired result follows from Examples 3.1, 3.2 and 3.3 and Lemma 4.2.

Corollary 4.4. *In the case $G = Q_n$ or SD_n , $v(G, \mathcal{L})$ is a subgroup of index 2 of $v(G, h^\infty)$.*

Proof of Theorem B. This follows directly from Propositions 1.1, 1.3 and 4.5.

Proof of Theorem A. Theorem A follows from Theorem B and the theorem of tom Dieck and Petrie ([3], [7]), that is, $\dim V(G, \mathcal{L}) = \dim V(G, h^\infty)$ if and only if G is nilpotent. Indeed, if I_G is an isomorphism, then G is nilpotent since $\dim V(G, \mathcal{L}) = \dim V(G, h^\infty)$. Since i_G is also isomorphic, G is a cyclic group or D_n by Theorem B.

Conversely, suppose that G is a cyclic group or D_n , then i_G is an isomorphism and $\dim V(G, \mathcal{L}) = \dim V(G, h^\infty)$. It is sufficient to show that I_G is surjective. Let x be any element of $V(G, h^\infty)$. Then there exists an element $u \in v(G, \mathcal{L})$ such that $\dim x = \dim u$. Hence $x - I_G(u)$ is in $v(G, h^\infty)$. Since i_G is an isomorphism, there exists an element $v \in v(G, \mathcal{L})$ such that $x - I_G(u) = i_G(v)$. Hence $x = I_G(u + v)$. Therefore I_G is surjective.

References

- [1] M.F. Atiyah and D.O. Tall: *Group representations, λ -rings and the J -homomorphism*, *Topology* **8** (1969), 253–297.
- [2] N. Bourbaki: *Algèbre commutative*, Hermann, Paris, 1961.
- [3] T. tom Dieck: *Homotopiedarstellung endlicher Gruppen: Dimensionsfunktionen*, *Invent. Math.* **67** (1982), 231–252.
- [4] T. tom Dieck: *Homotopy equivalent group representations*, *Crelles J. Reine Angew. Math.* **298** (1978), 182–195.
- [5] T. tom Dieck: *Homotopy equivalent group representations and Picard groups of the Burnside ring and the character ring*, *Manuscripta Math.* **26** (1978), 179–200.
- [6] T. tom Dieck: *Transformation groups and representation theory*, *Lecture Notes in Math.* **766**, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
- [7] T. tom Dieck and T. Petrie: *Homotopy representations of finite groups*, *Inst. Hautes Etudes Sci. Publ. Math.* **56** (1982), 129–169.
- [8] T. tom Dieck and T. Petrie: *The homotopy structure of finite group actions on spheres*, *Proc. of Waterloo topology Conference* (1978), 222–243.
- [9] D. Gorenstein: *Finite groups*, *Harper's Series in Modern Math.*, Haper & Row, New York, Evanston and London, 1968.
- [10] S. Kakutani: *On the groups $J_{\mathbb{Z}_{mq}}(*)$* , *Osaka J. Math.* **17** (1980), 512–534.
- [11] K. Kawakubo: *Equivariant homotopy equivalence of group representations*, *J. Math. Soc. Japan* **32** (1980), 105–118.
- [12] K. Kawakubo: *Weyl group actions and equivariant homotopy equivalence*, *Proc. Amer. Math. Soc.* **80** (1980), 172–176.
- [13] C.N. Lee and A.G. Wasserman: *On the group $JO(G)$* , *Mem. Amer. Math. Soc.* **159** (1975).
- [14] T. Matsuda: *On the unit groups of Burnside rings*, *Japan. J. Math.* **8** (1982), 71–93.
- [15] M. Morimoto: *On the groups $J_G(*)$ for $G=SL(2, p)$* , *Osaka J. Math.* **19** (1982), 57–78.
- [16] T. Petrie: *Transformation groups and representation theory*, *Proc. of Symposia in Pure Math.* **37** (1980), 621–631.
- [17] J.P. Serre: *Linear representations of finite groups*, *Graduate Texts in Math.*, Springer, New York-Heidelberg-Berlin, 1977.
- [18] V.P. Snaith: *J -equivalence of groups representations*, *Proc. Cambridge Philos. Soc.* **70** (1971), 9–14.
- [19] P. Traczyk: *On the G -homotopy equivalence of spheres of representations*, *Math. Z.* **161** (1978), 257–261.

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