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GENERALIZATIONS OF NAKAYAMA RING III

Dedicated to Professor Hisao Tominaga on his 60th birthday

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In the previous papers [6] and [7], we gave several rings generalized from Nakayama rings. We shall study the same problem, in this note, following those methods.

In the first two sections, we shall consider some right artinian rings with properties (∗, 1) and (∗, 2), respectively (see §1), and we shall give the complete types of US-4 algebras with $J^3=0$ over an algebraically closed field in the third section. In the final section, we shall give a structure of US-4 algebras with (∗, 1).

1. (∗, 1). In this paper, we shall study a right artinian ring R with identity, and every R-module is assumed a unitary right R-module. We denote the *Jacobson radical* and the *socle* of an R-module M by $\mathrm{J}(M)$ and $\mathrm{Soc}(M)$, respectively. Occasionally, we write $J=\mathrm{J}(R)$. $|M|$ means the length of a composition series of M.

We have studied the following condition in [5]:

(∗, n) *Every maximal submodule of a direct sum of any n hollow modules is also a direct sum of hollow modules.*

In [7] we have given characterizations of a right artinian ring with (∗, 3). We shall study, in this section, a right artinian ring R with (∗, 1). If R satisfies (∗, 1), $eJ=A_1\oplus A_2\oplus\cdots\oplus A_n$ for each primitive idempotent e, where the A_i are hollow. Since A_i is hollow, $\mathrm{J}(A_i)$ is a unique maximal submodule of A_i and $\mathrm{J}(A_i)=\sum_{k=1}^{ni}\oplus A_{ik}$ where the A_{ik} are hollow. Hence we obtain the diagram (see [4] for the meaning of the diagram).

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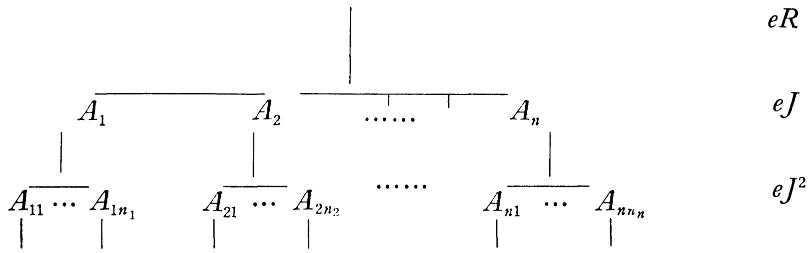


(1)

and eJ^i is a direct sum of hollow modules. Let M be a hollow module. Then $M \approx eR/A$ for some primitive idempotent e and some right ideal A in eR. If R satisfies $(*, 1)$, $\mathrm{J}(M) \approx eJ/A$ is a direct sum of hollow modules. Hence we have the following proposition:

Proposition 1. *R satisfies $(*, 1)$ if and only if eJ/A is a direct sum of hollow modules for any primitive idempotent e and any right ideal A in eJ.*

Following Proposition 1, we may study conditions under which eJ/A is a direct sum of hollow modules. Assume that an R-module D is a direct sum of two submodules E_1 and E_2, i.e., $D=E_1 \oplus E_2$. Let π_i: $D \to E_i$ be the projection. For any submodule B of D we put $B^{(i)}=\pi_i(B)$ and $B_{(i)}=B \cap E_i$ $(i=1, 2)$.

The following lemma is well known:

Lemma 2. $B^{(1)}/B_{(1)} \overset{f}{\approx} B^{(2)}/B_{(2)}$ *and* $B=\{(e_1+B_{(1)})+(f(e_1)+B_{(2)}) \mid e_1 \in B^{(1)}\}$.

By $B^{(1)}(f)B^{(2)}$ we denote the B in Lemma 2. If $B_{(2)}=0$, $B^{(1)}(f)B^{(2)}$ is nothing but the graph of $B^{(1)}$ with respect to f': $B^{(1)}/B_{(1)} \to B^{(2)}$ and we denote it by $B^{(1)}(f')$. We call B a *standard submodule* of D provided $B=B_{(1)} \oplus B_{(2)}$.

Lemma 3. *Assume that $D=A \oplus C$, where C is semi-simple. Let B be a submodule of D. Then $D/B \approx A/B' \oplus C'$, where B' is a submodule of A and C' is semi-simple.*

Proof. $(\tilde{D}=)\, D/(B_{(1)} \oplus B_{(2)}) \approx A/B_{(1)} \oplus C/B_{(2)}$. Since C is semi-simple, $C=C_1 \oplus B_{(2)}$. Hence $\tilde{D}=A/B_{(1)} \oplus C_1 \supset B/(B_{(1)} \oplus B_{(2)})$ $(=\tilde{B})$. Put $C_1=C_2 \oplus B^{(2)}/B_{(2)}$. Since f: $B^{(2)}/B_{(2)} \approx B^{(1)}/B_{(1)}$, $\tilde{B}=B^{(2)}/B_{(2)}(f)$. Therefore $\tilde{D}=A/B_{(1)} \oplus \tilde{B} \oplus C_2$, and so $D/B \approx \tilde{D}/\tilde{B}=A/B_{(1)} \oplus C_2$.

REMARK. In order to study $(*, 1)$, it is sufficient from Lemma 3 that we find a direct decomposition of A/B' whose direct summands are hollow, when $D=eJ$.

Theorem 4. *Let R be a right artinian ring. Then R satisfies $(*, 1)$ for any hollow module if and only if the following two conditions are fulfiled:*

1) $eJ=\sum_{i=1}^{n(e)}\oplus A_i$, *where e is any primitive idempotent in R and the* A_i *are hollow.*

2) *Let* $C_i\supset D_i$ *be two submodules of* A_i *such that* C_i/D_i *is simple. If* $f\colon C_i/D_i\approx C_j/D_j$ *for* $i\neq j$, f *or* f^{-1} *is extendible to an element in* $Hom_R(A_i/D_i, A_j/D_j)$ *or* $Hom_R(A_j/D_j, A_i/D_i)$.

Proof. Assume that (*, 1) is satisfied. Then $eJ=\sum_{i=1}^{n}\oplus A_i$ as 1) by assumption. Assume $f\colon C_1/D_1\approx C_2/D_2$. We shall consider $eJ/C_1(f)\,C_2$, and hence we may assume that $D_1=D_2=0$ and C_1, C_2 are simple. Put $C=C_1(-f)\oplus\sum_{j\geqslant 3}\oplus A_j$ and consider $eJ/C\approx(A_1\oplus A_2)/C_1(f)\,C_2$. Then $eJ/C=\sum_{i=1}^{t}\oplus D_i$ by assumption, where the D_i are hollow. We obtain the following exact sequence:

$$0\to C_1(-f)\to A_1\oplus A_2\to D_1\oplus\cdots\oplus D_t\to 0$$

We may assume that $C_i\neq A_i$ for $i=1, 2$. Then $C\subset J(A_1\oplus A_2)$, and so $t=2$. Let $\rho_i\colon eJ/C\to D_i$ be the projection. We may assume that A_1 and A_2 are submodules of eJ/C. Since D_i is hollow and $A_i\not\subset J(eJ/C)$, $D_i=\rho_i(A_1)$ or $\rho_i(A_2)$ and $\rho_1(A_i)=D_1$ or $\rho_2(A_i)=D_2$ for $i=1, 2$. Assume that $|A_1|\geqslant|A_2|$. We note that $|eJ/C|=|A_1|+|A_2|-1$. We shall show that $\rho_i|A_j$ is an isomorphism for some i and j. Contrarily assume that no-one of $\rho_i|A_j$ is an isomorphism. Now we have the following two cases from the remark above:

1) $\rho_1(A_2)=D_1$ and

2) $\rho_1(A_2)\neq D_1$, $\rho_1(A_1)=D_1$ and $\rho_2(A_2)=D_2$ (cf. [8], Lemma 2.1).

1) Since $\rho_1|A_2$ is not an isomorphism, $|D_1|\leqslant|A_2|-1$. If $\rho_2(A_2)=D_2$, $|D_2|\leqslant|A_2|-1$, and hence $|D_1|+|D_2|\leqslant|A_1|+|A_2|-2$, a contradiction. Hence $\rho_2(A_2)\neq D_2$, and so $\rho_2(A_1)=D_2$. Since $\rho_2|A_1$ is not an isomorphism, $|D_2|\leqslant|A_1|-1$, which is again a contradiction.

2) Since $\rho_1(A_1)=D_1$, $\rho_2(A_2)=D_2$, we have a contradiction as above. Hence some $\rho_i|A_j$ is an isomorphism. If so is $\rho_1|A_1$, $eJ/C=A_1\oplus D_2$. Then f^{-1} is extendible to $g|A_2$, where $g\colon eJ/C\to A_1$ is the projection (cf, [2], p. 771). We obtain a similar result for the remainder. Conversely, let S be any simple submodule of $eJ=\sum_{i=1}^{n}\oplus A_i$. We shall show that $eJ/S=\sum_{i=1}^{m}\oplus B_i$ such that $\{B_i\}_{i=1}^{m}$ fulfils the same condition as $\{A_i\}_{i=1}^{n}$. We may assume that S is of the form $\{s_1+f_2(s_1)+\cdots+f_i(s_1)\,|\,s_1\in \mathrm{Soc}(A_1)$ and $f_i\in \mathrm{Hom}_R(\mathrm{Soc}(A_1), \mathrm{Soc}(A_i))\}$. From the assumption, there exists p $(\leqslant i)$ such that $f_k f_p^{-1}$ is extendible to an element g_k in $\mathrm{Hom}_R(A_p, A_k)$ for all k $(\leqslant i)$. Put $A_p'=\{g_1(a)+\cdots+g_{p-1}(a)+a+g_{p+1}(a)+\cdots+g_i(a)\,|\,a\in A_p\}$ (cf. [1], p. 787). Then $eJ=A_1\oplus\cdots\oplus A_{p-1}\oplus A_p'\oplus A_{p+1}\oplus\cdots\oplus A_n$ and $A_p\approx A_p'\supset S$. Hence eJ/S has the desired direct decomposition. Let B be any submodule of eJ. Take a simple submodule S_1 in B. Then B/S_1 is a submodule of eJ/S_1 which has the same direct decomposition as eJ. Iterating this procedure, we know that eJ/B is a direct sum of hollow modules.

REMARK. We assume in 1) of Theorem 4 that all A_i are uniserial. Then, since A_i/D_i is uniform, we can show easily and similarly to the proof of Theorem 4 that either A_1 or A_2 is a direct summand of eJ/C without $|C_i/D_i|=1$. Hence if 2) of Theorem 4 is satisfied, we have the same without $|C_i/D_i|=1$, provided all A_i are uniserial.

Corollary 5. *Assume that* $eJ=\sum_{i=1}^{n(e)}\oplus A_i$ *for each* e, *where the* A_i *are hollow. Further assume that any sub-factor module of* A_i *is not isomorphic to any one of* A_j *for any pair* i, j $(i\neq j,\ i, j=1, 2, \cdots, n(e))$ *and for any* e, *then* R *satisfies* $(*, 1)$.

Proposition 6. *If* $J^2=0$, R *satisfies* $(*, 2)$ *and hence* $(*, 1)$.

Proof. This is clear from [4], Proposition 3.

EXAMPLE.

$$R=\left(\begin{array}{cccc|ccc} K & K & K & K & K & K & K \\ & K & K & K & & & \\ 0 & & K & 0 & & & 0 \\ & & & K & & & \\ \hline & & & & K & K & K \\ & 0 & & & 0 & K & 0 \\ & & & & 0 & 0 & K \end{array}\right)$$

Put $A=(0, K, K, K, 0, 0, 0)$ and $B=(0, 0, 0, 0, K, K, K)$. Then $eJ=A\oplus B$. Any subfactor module of A is not isomorphic to any one of B. Hence R satisfies $(*, 1)$ from Corollary 5, but R does not satisfy $(*, 3)$ and R is not US-3 by [3], Theorem 1 and [5], Lemma 1, since $J(A)$ is a direct sum of two simple modules. Further we know from Theorem 18 in §4 that R does not satisfy $(*, 2)$.

2. $(*, 2)$. We have shown in Proposition 6 that if $J^2=0$, R satisfies $(*, 2)$. In this section, we shall give a relationship between rings with $(*, 2)$ and ones with $(*, 3)$.

If R is an algebra over an algebraically closed field K_0, then $eRe/eJe=K_0\bar{e}$. Hence R satisfies

CONDITION II″ [4]. $eRe/eJe=\bar{e}K'$ *for each primitive idempotent* e, *where* K' *is a field contained in the center of* R.

In this case every unit element x in eRe is of the form

$$ek+j\,, \tag{2}$$

where $k\in K'$ and $j\in eJe$. Further $K'=\Delta(A)$ for any right ideal A in eJ (see [4]).

We always assume in this section that R satisfies Condition II″. If R satisfies further (*, 3), then

$$eJ = A_1 \oplus B_1 \quad \text{such that} \quad A_1/J(A_1) \not\approx B_1/J(B_1)\,, \tag{3}$$

where A_1, B_1 are hollow from [3], Proposition 26 and [4], Theorem 1. Next assume that R satisfies (*, 2) and the above condition (3) for each e. Then every proof given in [4], Lemmas 3∼18 is valid under (3). Hence eR has the structure given in [4], Theorem 1. Therefore R satisfies (*, n) for any n from [7], Theorems 2 and 3. Thus we obtain

Theorem 7. *Let R be an algebra over a field K with condition II″. Then the following conditions are equivalent:*

1) *R satisfies* (*, 2) *and* (3).
2) *R satisfies* (*, *n*) *for all n.*

Put

$$R = \begin{pmatrix} K & K\oplus K \\ 0 & K \end{pmatrix} \quad \text{and} \quad A_1 = (0,\ K\oplus 0)\,,\ B_1 = (0,\ 0\oplus K)\,.$$

Then $eJ = A_1 \oplus B_2$, $A_1 \approx B_1$. Hence (3) is not fulfiled and R satisfies (*, 2) but not (*, 3) from Proposition 6 and Theorem 7.

3. US-4 algebras with $J^3=0$. We have studied US-3 rings with (*, 1) or (*, 2) in [7]. We shall observe US-4 algebras over an algebraically closed field. We have defined in [6]

(**, n) *Every maximal submodule of a direct sum D of any n hollow modules contains a non-trivial direct summand of D.*

We call a ring R (*right*) *US-n* provided that (**, n) holds for any D [6]. Let K be a field, and put

$$R_n = \begin{pmatrix} K & K & K & \cdots & K \\ 0 & K & 0 & 0 \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & & K \end{pmatrix}$$

Then R_n is a US-m algebra, but not US-($m-1$) algebra from some m from [5], Corollaries 1 and 2 of Theorem 2.

Proposition 8. *Let R be an algebra of finite dimension over a field. Then the number of isomorphism classes of hollow modules is finite if and only if R is a* (*right*) *US-n algebra for some integer n. Hence an algebra of finite representation type is a US-n algebra.*

Proof. Assume that R is a US-n algebra. Let $\{A_1, A_2, \cdots, A_m\}$ be a set of R-modules in eR such that $|eR/A_i|=t$ for all i, where t is a fixed integer. If $m\geqslant n+1$, from [5], Corollary 2 of Theorem 2 there exists a unit x in eRe such that $xA_i\subset A_j$ for some pair (i, j). Since $|eR/A_i|=|eR/A_j|$, $eR/A_i\approx eR/A_j$. Therefore there exist at most n pairwise non-isomorphic, hollow modules eR/B with $|eR/B|=t$. Accordingly, R being artinian, the number of isomorphism classes of hollow modules is finite. Conversely, we assume that R is as above. Let m be the number of isomorphism classes of hollow modules eR/B for a fixed primitive idempotent e. Since $[R: K]=p<\infty$, $[\Delta: \Delta(B)]\leqslant p$, where $\Delta=eRe/eJe$ (see [4]). Let D be a direct sum of $(p+1)m$ hollow modules eR/A_i. Then there exists some direct summand eR/A_j, which appears at least $(p+1)$-times in D. Hence a direct sum of p-copies of eR/A_j satisfies $(**, p+1)$ by [5], Corollary 1 of Theorem 2. Hence D satisfies $(**, (p+1)m)$ by [5], Theorem 2. Repeating this argument for each primitive idempotent e, we know that R is a US-n algebra for some n.

From now on, we always assume, unless otherwise stated, that R is an algebra over a field with Condition II″.

Employing the similar argument given in the proof of [6], Proposition 1, we have

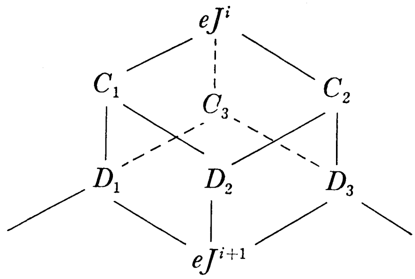
Lemma 9. *If R is a US-n algebra, $|eJ^i/eJ^{i+1}|\leqslant n-1$ and every intermediate submodules between eJ^i and eJ^{i+1} are characteristic for each primitive idempotent e.*

The following proposition is an immediate consequence of [5], Corollaries 1 and 2 of Theorem 2 for any right artinian ring.

Proposition 10. *Let R be a right artinian ring. Then R is a US-4 if and only if for any set of four submodules $\{A_i\}_{i=1}^4$ of eJ, there exists a pair (i, j) such that $A_i\sim A_j$. Further R is the algebra mentioned in the beginning, then there exist at most three maximal submodules in eJ^i and they are characteristic.*

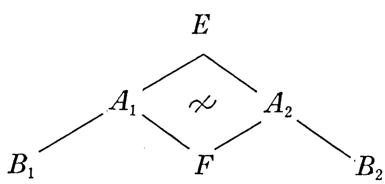
First we shall give the lattices of submodules in eJ, provided that R is a US-4 algebra which satisfies Condition II″.

Assume that $|eJ^i/eJ^{i+1}|=3$ and $eJ^i/eJ^{i+1}=\bar{D}_1\oplus\bar{D}_2\oplus\bar{D}_3$, where the D_i are intermediate submodules between eJ^i and eJ^{i+1} such that the $\bar{D}_i$ are simple. Since every maximal submodule in eJ^i is characteristic by Proposition 10, so is D_i and $D_i\not\sim D_j$ for $i\neq j$. Put $C_1=D_1+D_2$, $C_2=D_2+D_3$ and $C_3=D_1+D_3$. Then the C_i are characteristic, and $C_i\not\sim C_j$ for $i\neq j$. The set $\{C_i, D_j\}_{i,j=1}^3$ is the set of all intermediate submodules between eJ^i and eJ^{i+1}. Assume that no one of the D_i is hollow. Let E_i be a maximal submodule of D_i not equal to eJ^{i+1}. If $xE_3=E_1$ for some x in eRe, $E_1=xE_3\subset D_1\cap D_3=eJ^{i+1}$. Hence $E_k\not\sim E_{k'}$ for $k\neq k'$, a contradiction by Proposition 10. Therefore one of the D_i is hollow. Further, since the C_i and the D_j are characteristic and $D_1\not\sim D_2\not\sim D_3$, the C_i contains exactly two maximal submodules D_j. Thus we obtain the following diagram of submodules:



(4)

where each factor module given from connected modules is simple, (cf. [6]). Next assume that $|eJ^i/eJ^{i+1}|=2$. Consider a diagram:



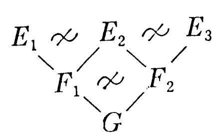
(5)

Lemma 11. $B_1 \cap F \neq B_1 \cap B_2$ *in* (5).

Proof. This is clear from the fact: $|B_1/(B_1 \cap F)|=1$ and $|B_1/(B_1 \cap B_2)|=2$.

Lemma 12. *If* $|F|=1$ *in the above, either* B_1 *or* B_2 *does not exist.*

Consider a diagram:

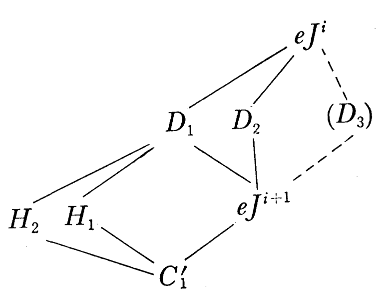


(6)

Similarly to Lemma 11, we have

Lemma 13. *Assume that* G *is a waist. Then* E_1 *and* E_3 *are hollow.*

We observe the following:

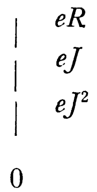


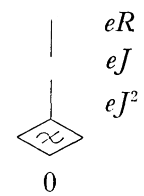
(7)

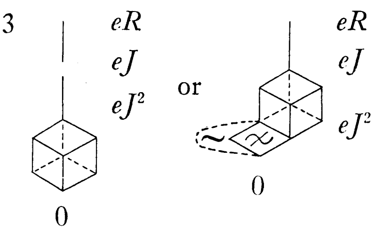
D_1 and C_1' are characteristic from Proposition 10. Put $\tilde{D}_1 = D_1/C_1' = \tilde{H}_1 \oplus \tilde{e}\tilde{J}^{i+1}$. We have two cases a) $\tilde{H}_1 \not\approx \tilde{e}\tilde{J}^{i+1}$ and b) $\tilde{H}_1 \approx \tilde{e}\tilde{J}^{i+1}$ (see R_3 in Example in [6]). There exists no H_2 in the case a). On the other hand, there exist many H_2 in the case b). If all H_2 are characteristic, we have a contradiction from Proposition 10. Hence we may assume that H_1 is not characteristic. If $eJeH_1 \subset C_1'$, H_1 is characteristic by (2), and so there exists j in eJe such that $jH_1 \not\subset C_1'$. Since $\tilde{H}_1$ is simple, $H_1 = hK + C_1'$ for $H_1 J = J(H_1) \subset C_1'$ (we may assume that R is basic). Let $j\tilde{h} = \tilde{h}k + \tilde{x}$, where $k \in K$ and $x \in eJ^{i+1}$. Since j is nil, $k=0$. Hence $j\tilde{h}$ is a generator of $\tilde{e}\tilde{J}^{i+1}$. Let $\tilde{h}_2$ be a generator of $\tilde{H}_2$. Then $\tilde{h}_2 = \tilde{h}k_1 + j\tilde{h}k_2$, and so $\tilde{h}_2 k_2^{-1} = (k_2^{-1}k_1 + j)\tilde{h}$. Hence $H_2 = (k_2^{-1}k_1 + j)H_1$, since C_1' is characteristic. Thus we obtain

Lemma 14. *$H_1 \sim H_2$ if $eJ^{i+1}/C_1' \approx H_1/C_1'$, and H_2 does not exist if $eJ^{i+1}/C_1' \not\approx H_1/C_1'$.*

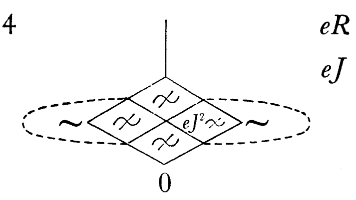
Now we shall give all lattices of the submodules in eR for each primitive idempotent e, provided R is a right US-4 algebra and $J^3=0$ ($J^2 \neq 0$).

1 

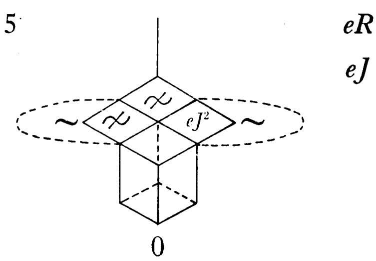
2 

3 

(Use Lemmas 12 and 14)

4 

(Use Lemmas 11, 12, 13 and 14)

5 

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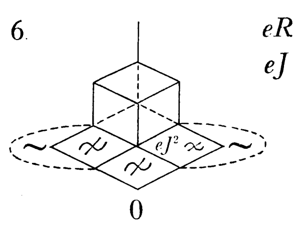
6 

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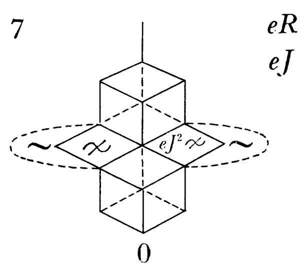
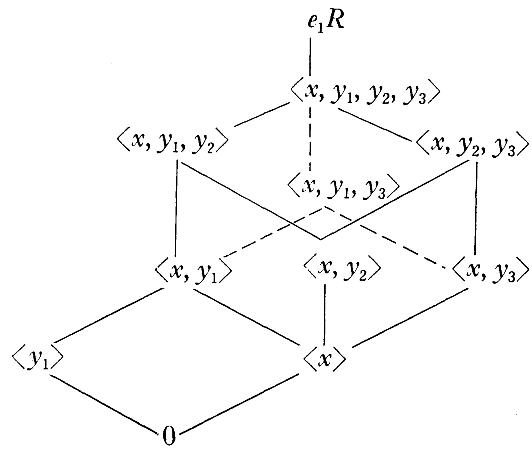
7 

(Use Lemmas 11, 12 and 14).

We shall give an example for each case. The types 1) and 2) are the cases of US-1 and US-2. 4): $eJ=A_1\oplus A_2$, where the A_i are uniserial such that any sub-factor module of A_1 is not isomorphic to one of A_2.
3): Let R be a vector space with basis $\{e_1, {}^1x^1, {}^1y_1{}^4, {}^1y_2{}^2, {}^1y_3{}^3, e_2, {}^2w^1, e_3, {}^3z^1, e_4\}$. Define $e_i\, e_j=\delta_{ij}\, e_i$ and ${}^1y^2$ means $e_1\,{}^1y^2\, e_2={}^1y^2$, and so on. Put $y_2w= x$, $y_3z= x$ and other products are zero. Then



7): $R=\langle e_1, {}^1y_1{}^2, {}^1y_2{}^4, {}^1y_3{}^3, {}^1x_1{}^1, {}^1x_2{}^2, {}^1x_3{}^3, e_2, {}^2z_2{}^2, {}^2z_3{}^3, e_3, {}^3w_1{}^1, {}^3w_2{}^2, e_4, {}^4v_1{}^1, {}^4v_2{}^2, {}^4v_3{}^3\rangle$ is a vector space. Define $y_1z_2= x_2$, $y_1z_3=x_3$, $y_2v_1=x_1$ $(y_3v_2=x_3)$, $y_3v_3=x_3$, $y_2w_1= x_1$, $y_2w_2=x_2$, $y_2w_3=x_3$ and other products are zero. Then we have the following lattice given in the next page.

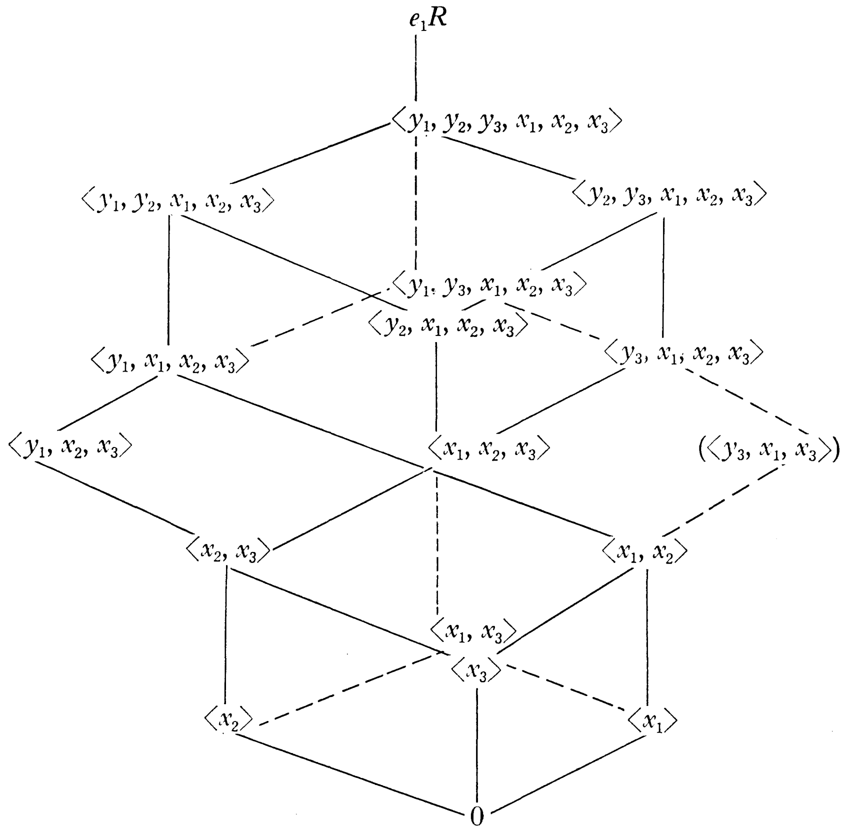
4. US-4 algebras with $(*, n)$, $n=1, 2$. We shall study, in this section, a US-4 algebra R over an infinite field K, which satisfies Condition II$''$ and $(*, 1)$.

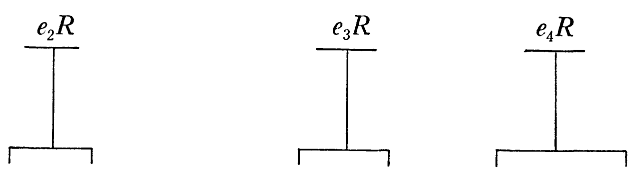
Lemma 15. *Let R be a US-n algebra over K as above. Assume that $eJ^i=\sum_{i=1}^{n}\oplus A_i$ with A_i hollow. Then, for any submodule B_j of A_j, $A_i/B_i\not\approx A_j/B_j$, provided $i\neq j$ and $A_i\neq B_i$.*

Proof. Assume that $f\colon A_1/B_1\approx A_2/B_2$ and $f\neq 0$. We may assume $|A_1/B_1|=1$. We shall show that there exist no units x in eRe such that $A_1(f)A_2= x(A_1(g)A_2)$ for any $(f\neq)\, g\colon A_1/B_1\to A_2/B_2$. Assume that there exists x as above. Let a be a generator of A_1 and $x=k+j$ as (2). Since $A_1(f)A_2=x(A_1(g)A_2)$,

$$a+f(a) = (k+j)\,(a'+g(a')+b) \tag{8}$$

where $a'\in A_2$, $b\in B_1\oplus B_2$ and $f(a)$, $g(a')$ are fixed representative elements of $f(a+A_1)$ and $g(a'+A_1)$, respectively. Let $\pi_i\colon \sum_{j=1}^{n}\oplus A_j\to A_i$ be the projection. Then





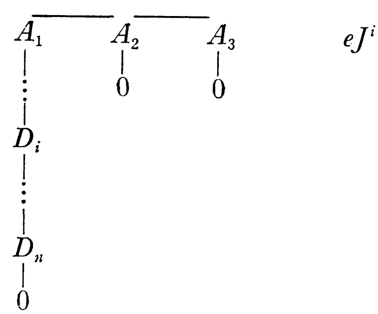
$$a = \pi_1(a+f(a)) = ka'+\pi_1(j(a'+g(a')+b)+kb)$$
$$f(a) = \pi_2(a+f(a)) = kg(a')+\pi_2(kb+j(g(a')+b))\,. \qquad (9)$$

Hence, since A_1 is hollow and $jeJ^i \subset eJ^{i+1}$,

$$\bar{a} = k\bar{a}' \quad \text{and} \quad f(\bar{a}) = kg(\bar{a}') = g(\bar{a}) \qquad (10)$$

where $\bar{a}$ is the class of a, which is a contradiction. Since K is infinite, there exist infinitely many isomorphisms $g=kf$. However, R is US-n, and hence we obtain a contradiction from [5], Corollaries 1 and 2 of Theorem 2 (cf. Proposition 10). Therefore $A_1=B_1$.

Now we assume that R is a US-4 algebra with $(*, 1)$. Then $|eJ^i/eJ^{i+1}| \leqslant 3$ by Lemma 9, and so $eJ^i = A_1 \oplus A_2 \oplus A_3$ from the diagram (1), where the A_i are hollow. Assume that all A_i are non-zero. First we assume that $\mathrm{J}(A_1) \neq 0$, $\mathrm{J}(A_2) \neq 0$. Then $\{A_1, A_2, A_3, \mathrm{J}(A_1 \oplus A_2)\}$ gives a contradiction to Proposition 10, since every unit in eRe is of the form (2). Hence we may assume that A_2 and A_3 are simple. If A_1 is not uniserial, there exist two submodules B_1, B_2 with $B_1 \not\sim B_2$ from the fact: $eJ^{i+j} = A_1 J^j$ and Lemma 9. Then $\{A_2, A_3, B_1, B_2\}$ gives a contradiction to Proposition 10. Therefore A_1 is uniserial and A_2, A_3 are simple, i.e.,



(11)

We shall show

$$D_i/D_{i+1} \not\approx A_2 \text{ for } i < n \tag{12}$$

Assume contrarily that f: $D_i/D_{i+1} \approx A_2$. Since $eJeA_2 \subset eJ^{i+1} = A_1 J$ and A_2 is simple, $eJeA_2 \subset \mathrm{Soc}(eJ^{i+1}) = D_n$. Put $g = kf$ $(k \neq 0 \in K)$, then $f \neq g$. Assume that there exists a unit x in eRe such that $D_i(f) = xD_i(g)$. Using (2), we know that $j(g(a') + a') \in D_n + D_{i+1} = D_{i+1}$ $(i < n)$, and so $(a + D_{i+1}) = k(a' + D_{i+1})$ and $f(a + D_i) = g(ka' + D_{i+1}) = g(a + D_{i+1})$, which is a contradiction. Hence $D_i(f) \not\sim D_i(g)$, provided $k \neq 0$. Since K is infinite, we obtain a contradiction from Proposition 10.

Next assume

$$f\colon D_n \approx A_2 \,. \tag{13}$$

Let g be a non-zero element in $\mathrm{Hom}_R(A_2, D_n)$ and assume $A_2(f) = xA_2(g)$ for some unit x in eRe. Since $eJeD_n = 0$ and $eJA_2 \subset D_n$, from (8) we have

$$a = ka' \quad \text{and} \quad (f-g)(a) = ja \tag{14}$$

where a, a' are in A_2. Now $A_1 \not\sim A_2$. Assume $A_2(f) \sim A_2$. Then $f = j_l$ (left multiplication of j) by putting $g = 0$ in (14). If $A_2(f) \not\sim A_2$, put $g = kf$ $(k \neq 0, 1)$. Since $A_2(f) \not\sim A_1$ from (2), $A_2(g)$ is related to one of $\{A_1, A_2, A_2(f)\}$ with respect to $\sim$ by Proposition 10. If $A_2(g) \sim A_2$, $f = kg = (-kj)_l$ by replacing f with 0 in (14). Finally assume $A_2(g) \sim A_2(f)$ (note $A_2(g) \not\sim A_1$). Then $f - g = (1-k)f = j_l$ from (14). Hence $f = ((1-k)^{-1}j)_l$. In any cases f is given by a left multiplication of an element in eJe. Thus we obtain

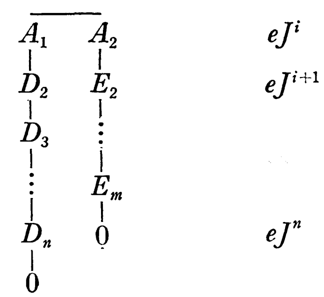
Lemma 16. *If R is a US-4 algebra with* $(*, 1)$ *and* $|eJ^i/eJ^{i+1}|=3$, *then* $eJ^i=A_1\oplus A_2\oplus A_3$ *such that*

1) A_1 *is uniserial, and A_2, A_3 are simple and they are not isomorphic to one another.*

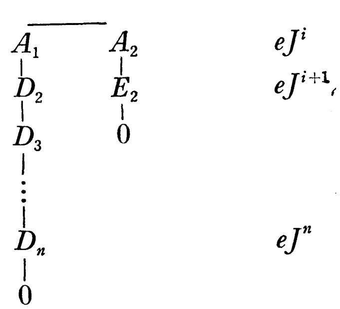
2) $D_i/D_{i+1}\not\approx A_2$ *(or A_3) provided* $D_i\neq \mathrm{Soc}(A_1)$.

3) *If* $D_n=\mathrm{Soc}(A_1)\approx A_2$ (or A_3), *this isomorphism is given by a left multiplication of an element in eJe.*

Next assume $|eJ^i/eJ^{i+1}|=2$ and $eJ^i=A_1\oplus A_2$. If A_1 is uniserial and A_2 is simple, we have the same property as in Lemma 16 for A_1 and A_2. Assume that neither A_1 nor A_2 is uniserial. Then there exist $C_1'\oplus C_2'$ in A_1 and $D_1'\oplus D_2'$ in A_2 such that $\{C_i', D_j'\}$ are not related one another with respect to $\sim$ from the diagram (1) and (2). Hence either A_1 or A_2 is uniserial. First we assume that both A_1 and A_2 are uniserial, i.e.,



We may assume $n\geqslant m$. Then any one of $\{A_1, D_2\oplus E_m, D_3\oplus E_{m-1}, \cdots, A_2\}$ is not related to another one with respect to $\sim$ and hence $m\leqslant 2$, i.e.,



(15)

In this case $\{A_1, A_2, D_2\oplus E_2\}$ are not related to one another with respect to $\sim$. If f: $E_2\approx D_i/D_{i+1}$, f is extendible to f': $A_2\to A_1/D_{i+1}$ by Theorem 4. Hence $A_2\approx D_{i-1}/D_{i+1}$. Therefore, if a sub-factor module of A_2 is isomorphic to one of A_2, we may assume f: $A_2/E_2\approx D_i/D_{i+1}$. Put $B=A_2(f)D_i$. Then $A_2\sim B$ by

Lemma 15 and Proposition 10, i.e., $xA_2 \subset B$ for a unit $x=e+j$ in eRe: $j \in eJe$. Let $A_2=a_2R$ and π_j: $eJ^i \to A_j$ the projection. Then $xa_2=(e+j)a_2=a_2r+f(a_2r)+d_{i+1}+e_2$: $r \in R$, $d_{i+1} \in D_{i+1}$, $e_2 \in E_2$ and $f(a_2r+E_2)=f(a_2r)+D_{i+1}$; $f(a_2r) \in D_i$. Hence

$$\begin{aligned} a_2+\pi_2(ja_2) &= a_2r+e_2 \\ \pi_1(ja_2) &= f(a_2r)+d_{i+1}\,. \end{aligned}$$

Since $jA_2 \subset D_{n-1} \oplus E_2$ (note $|A_2|=2$), $a_2 \equiv a_2r \pmod{E_2}$ and $f(a_2r) \in D_{i+1}$, provided $i+1 \leqslant n-1$. Hence $i \geqslant n-1$. Now assume $i=n$, and consider $\{A_1, D_n \oplus A_2, D_2 \oplus E_2, B\}$. Then $D_n \oplus A_2 \sim B$ by Proposition 10. Since $|D_n \oplus A_2|=|B|$, $x(A_2(f)D_n)=D_n \oplus A_2$. Similarly, we can show that every submodule in eJ^i is isomorphic to a standard submodule in eJ^i via x_l. We can express the above situation as the form (14). For instance, assume f: $A_2/E_2 \approx D_n$. Then $A_2(f)=xA_2$ as above, $(x=k+j)$. Hence for some a_2' in A_2

$$\begin{aligned} a_2 &= ka_2'+\pi_2(ja_2') \\ f(a_2) &= \pi_1(ja_2') = ja_2'-\pi_2(ja_2') \\ &= j(k^{-1}a_2-k^{-1}\,\pi_2(ja_2'))-\pi_2(ja_2') \end{aligned} \tag{16}$$

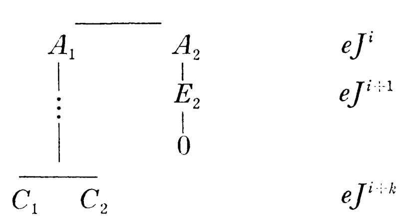
On the other hand, for an element e_2 in E_2

$$e_2 = e_2+f(e_2) = (k+j)\,a_2''\,;\ a_2'' \in A_2 \tag{17}$$

Hence $a_2'' \in E_2$. Since $jE_2 \subset D_n$, $ja_2''=0$ from (17). $E_2=a_2''R$ implies $jE_2=0$. In (16), $\pi_2(ja_2') \in E_2$, and so $f(a_2)=jk^{-1}a_2-\pi_2(ja_2')$. Hence

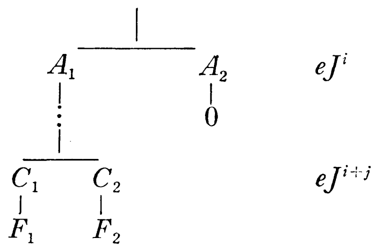
$$f(a_2) \equiv j'a_2 \pmod{E_2} \quad \text{and} \quad j'E_2 = 0 \tag{18}$$

Next consider a diagram

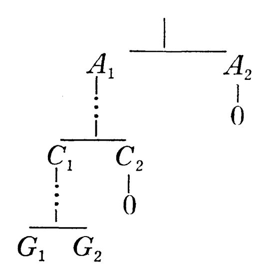


Consider $\{A_1, A_2, C_1 \oplus E_2, C_2 \oplus E_2\}$. Since R is US-4, $C_1 \oplus E_2 \sim C_2 \oplus E_2$, and hence $C_1 \oplus E_2 \approx C_2 \oplus E_2$. Therefore $C_1 \approx C_2$. However eJ^{i+k}/eJ^{i+k+1} contains at most three maximal submodules by Lemma 9 and Proposition 10. If $C_1 \approx C_2$, we can construct many submodules, which is a contradiction.

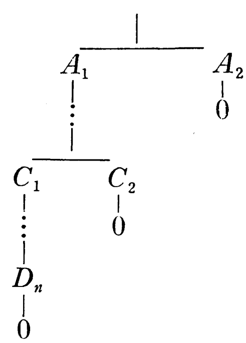
Further consider a diagram:



If $F_1 \neq 0$ and $F_2 \neq 0$, $\{A_2, C_1, C_2, F_1 \oplus F_2\}$ gives a contradiction. Hence either F_1 or F_2 is zero. Hence we obtain



Considering $\{G_1, G_2, C_2, A_2\}$, we know $G_2=0$. Therefore



(19)

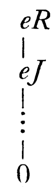
It is clear from the previous argument that A_2 (resp. C_2) is not isomorphic to a sub-factor module of A_1 except $A_2 \approx C_2$, $A_2 \approx D_n$ (resp. $C_2 \approx D_n$). Further those isomorphisms are given by left multiplications of elements in eJe.

Conversely, we assume that the lattice of eR has the structure given in (11), (15) and (19). If j is in eJe, $e+j$ gives an isomorphism, and hence every submodule of eJ may be assumed standard except (15). Assume $f:(A_2 \rightarrow) A_2/E_2 \approx D_n$ such that (18) holds. Then $(e+j')A_2 \subset A_2(f)+E_2=A_2(f)$, since $f(E_2)=0$, and so $(e+j')A_2=A_2(f)$.

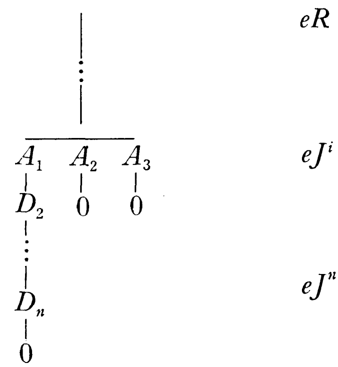
Summarizing the above we obtain

Theorem 17. *Let R be an algebra over a field satisfying Condition II″. Then R is a US-4 algebra with* $(*, 1)$ *if and only if, for each primitive idempotent e, eR has one of the following structures*:

1)

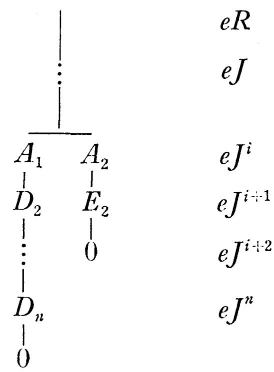


2)



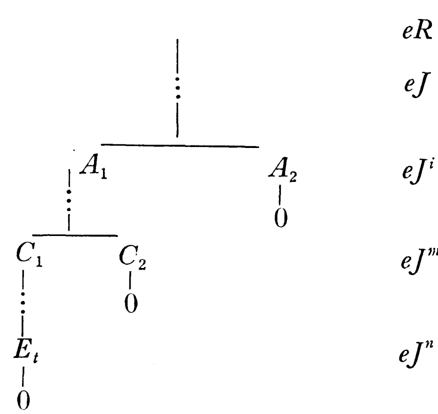
$D_i/D_{i+1}\not\approx A_j$ for $i=1, \cdots, n$; $j=2, 3$ $(D_1=A_1)$ and D_n is isomorphic to A_j via f if and only if $f=j_l$; $j\in eJe$.

3)



1) *Any g: $E_2\approx D_n$ is extendible to g': $A_2\approx D_{n-1}$.* 2) *Every submodule in eJ^i is isomorphic to a standard submodule in eJ^i via x_l; x is a unit in eRe. In this case no sub-factor modules of A_2 are isomorphic to any one of A_1/D_{n-1}.*

4)



A_2 (resp. C_2) is not isomorphic to any sub-factor module of A_1 except C_2 and E_t (resp. E_t). If they are isomorphic, those isomorphisms are j_l, where j_l is the left multiplication of an element j in eJe.

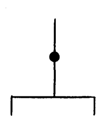
We give an example.

$$R=\begin{pmatrix} K & K & K & K & K & K & K \\ & K & K & K & K & K & K \\ & & K & 0 & 0 & 0 & 0 \\ & & & K & K & K & K \\ & 0 & & & K & 0 & 0 \\ & & & & & K & K \\ & & & & & & K \end{pmatrix}$$

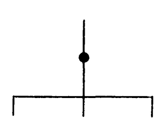
is of the type 4).

Theorem 18. *Let R be as in Theorem 17. Then R is a US-4 algebra with $(*, 2)$ if and only if, for each primitive idempotent e, eR has one of the structures 1)~3) replaced eJ^i with eJ in Theorem 17.*

Proof. Assume that R is a US-4 algebra with $(*, 2)$. Then $(*, 1)$ is fulfiled. First assume that $J^3=0$ and eR has the structure 4) replaced eJ^i with eJ in Theorem 17. Then C_1 and C_2 are simple. Since $A_1/A_1J \not\approx A_2/A_2J=A_2$, we can use the same argument in the first paragraph of [4], p. 87, and obtain $C_2=0$. Similary we can show that eR has one of the structures 1)~3) replaced eJ^i with eJ, provided $J^3=0$. In general case we assume that eR has one of the structures 2)~4) with $i \neq 1$. Taking $R/J^{i+1}=\tilde{R}$, we know that there exists an $\tilde{R}$-hollow module of the following structure:



or



They are also R/J^3-hollow modules. However, there do not exist those hollow modules as shown in the beginning. Hence $i=1$ (cf. the proof of Lemma 12 in [4]). Conversely assume the above structuie. Making use of Theorem 17, we can compute all types of maximal submodules of direct sum of two hollow modules, for example, non trivial maximal submodules of $(eR/(D_i \oplus A_2) \oplus eR/(D_j \oplus A_2))$ is isomorphic to $eR/D_j \oplus A_1/D_i$, where the A_i and D_k are in the structure 2) and $i \leqslant j$. Hence $(*, 2)$ is fulfiled.

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