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ON THE SPECTRAL DISTRIBUTION OF A DISORDERED SYSTEM AND THE RANGE OF A RANDOM WALK

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1. Introduction

Consider the ν-dimensional lattice Z^ν. We define a second order difference operator H^0 by

$$(H^0u)(a) = \frac{\sigma^2}{2}\sum_{i=1}^{\nu}\{u(a_1, \cdots, a_i-1, \cdots, a_\nu)-2u(a)+u(a_1, \cdots, a_i+1, \cdots, a_\nu)\}\ ,$$
$$a\in Z^\nu,\ u\in C_0(Z^\nu)\,,$$

where σ is a positive constant and $C_0(Z^\nu)$ is the space of functions on Z^ν with finite supports. Let $\{q(a)=q(a, \omega);\ a\in Z^\nu\}$ be a family of independent, identically distributed non-negative random variables defined on some probability space $(\Omega, \mathcal{B}, P)$. We are then concerned with the difference operator H^ω depending on the random parameter $\omega\in\Omega$:

$$(1)\qquad (H^\omega u)(a) = (H^0u)(a)-q(a, \omega)u(a)\,, \qquad a\in Z^\nu\,.$$

The operator $-H^\omega$, considered as a linear transform over $C_0(Z^\nu)$, is a non-negative definite symmetric operator on $L^2(Z^\nu)$ and has a unique self-adjoint extension $-\bar{H}^\omega$. Express $-\bar{H}^\omega$ as $-\bar{H}^\omega=\int_{[0,\infty)}xdE_x^\omega$ by the associated spectral family $\{E_x^\omega, -\infty<x<\infty\}$ and put $\rho^\omega(x)=(E_x^\omega I_0, I_0)$, where $(\ ,\ )$ is the L^2-inner product and $I_a(a')=\delta_{aa'}$, $a, a'\in Z^\nu$.

Denote by $\langle\ \rangle$ the expectation with respect to the probability measure P and set

$$(2)\qquad \rho(x) = \langle\rho^\cdot(x)\rangle, \ -\infty<x<\infty\,.$$

$\rho(x)$ is a probability distribution function vanishing on $(-\infty, 0)$. We call this the *spectral distribution function* associated with the ensemble of operators $\{H^\omega, \omega\in\Omega\}$ or rather with the disordered dynamical system governed by H^ω's (*e.g.* a tight binding electron model [4]).

Our main aim is to show in §4 the following asymptotic behaviours of $\rho(x)$ near the origin.

Theorem 1.1.

$$\overline{\lim_{x\downarrow 0}}\sqrt{x}\,\log\rho(x)\leqq-\frac{\nu\sigma\beta_1}{3\sqrt{6e}}$$

with
$$\beta_1=-\log\left\langle\frac{\nu\sigma^2}{\nu\sigma^2+2q(0)}\right\rangle\qquad(\geqq 0)\,.$$

Theorem 1.2. *When* $\nu=1$,

$$\underline{\lim}_{x\downarrow 0}\sqrt{x}\,\log\rho(x)\geqq-\frac{\sqrt{27}}{2\sqrt{2}}\pi\sigma\beta_2$$

with
$$\beta_2=-\log P(q(0)=0)\qquad(\leqq\infty)\,.$$

Theorem 1.3. *Suppose that the distribution of* $q(0)$ *obeys the one-sided stable law with exponent* α, $0<\alpha<1$; $\langle e^{-\lambda q(0)}\rangle=e^{-\lambda^{\alpha}}$, $\lambda>0$. *Then*

$$\overline{\lim_{x\downarrow 0}}\,x^{\alpha/(1-\alpha)}\log\rho(x)\leqq-(1-\alpha)\alpha^{\alpha/(1-\alpha)}\,.$$

If $\nu=1$ *in addition, then*

$$\underline{\lim}_{x\downarrow 0}\,x^{\{\alpha/(1-\alpha)\}+(1/2)}\log\rho(x)\geqq-\left(\frac{3-\alpha}{2}\right)^{(3-\alpha)/\{2(1-\alpha)\}}(1-\alpha)^{-1/2}\pi\sigma\,.$$

If $q(a)=q(a,\omega)$ is equal to a constant $c\geqq 0$ identically, then we see from the well known Tauberian theorem [2; vol. 2, XIII] and the asymptotic form (17) of the Laplace transform of $\rho(x)$, that $\rho(x)$ grows up from c in a polynomial order: $\rho(x)\sim\dfrac{1}{(2\pi\sigma^2)^{\nu/2}\Gamma\left(\frac{\nu}{2}+1\right)}(x-c)^{\nu/2}$, $x\downarrow c$. The exponential characters in Theorems 1.1$\sim$1.3 exhibit a sharp contrast to this state of the deterministic case.

Exponential characters of the spectral distribution of a disordered system near the end points of the spectrum were derived by I.M. Lifschitz [6] by some qualitative argument. M.M. Benderskii and L.A. Pastur [1] and L.A. Pastur [9] have proved those rigorously for the differential operator $\Delta-q(a)$ in the case of the ν-dimensional continuum R^{ν}. If $\nu=1$ and if $q(a)$, $a\in R^1$, is a stationary Markov process taking two values 0 and 1, then $\lim_{x\downarrow 0}\sqrt{x}\,\log\rho(x)$ exists and is negative finite ([1]). When $q(a)$ is a stationary Gaussian process with parameter $a\in R^{\nu}$, then under certain regularity conditions for the covariance $\lim_{x\downarrow-\infty}\dfrac{\log\rho(x)}{x^2}=-\dfrac{1}{2\langle q(0)^2\rangle}$ ([9]). Here $\rho(x)$ is understood to be defined as above but through the integral kernel of the operator E_x^{ω}.

Let us now put

$$(3)\qquad k(t)=\int_{[0,\infty)}e^{-tx}\rho(dx)\,,\qquad t>0\,.$$

Just as in [9], we will use the Kac formula for $k(t)$ and appeal to a Tauberian theorem of exponential type (Theorem 2.1) to prove Theorem 1.1~1.3.

We introduce the time-continuous Markov process $\dot{M}=(\dot{\Omega}, \dot{\mathcal{B}}, \dot{X}_t, \dot{P}_a)$ on Z^ν with the generator H^0: the function $T_t u(a)=\dot{E}_a(u(\dot{X}_t))$, $a\in Z^\nu$, $t>0$, $u\in C_0$, satisfies the Kolmogorov differential equation $\frac{dT_t u(a)}{dt}=H_0(T_t u)(a)$ and the initial condition $\lim_{t\downarrow 0} T_t u(a)=u(a)$, $a\in Z^\nu$. Here $\dot{E}_a$ stands for the expectation with respect to the probability measure $\dot{P}_a$ governing sample paths $\dot{X}_\cdot$ starting at a. We call the process $\dot{M}$ the *(time-continuous) simple random walk* on Z^ν. Let $\tilde{E}$ be the expectation with respect to the product measure $\tilde{P}=P\times \dot{P}_0$, then we have the Kac expression as follows (Lemma 4.1):

$$(4) \qquad k(t)=\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{X}_t=0)\,.$$

By making use of the formula (4) and in connection with the range of the sample path $\dot{X}_t$, we derive in §4 some asymptotic properties of $k(t)$. The range will be compared with the maximum of the absolute value of $\dot{X}_t$ whose behaviours are studied in §3. The results of §3 are valid for a large class of Markov processes including the present random walk and the Brownian motion.

Here we state two remarks on related matters.

A fine structure of the random spectral function $\rho^\omega(x)$ is known. Consider the operator (1) for the one dimensional semi-infinite lattice $\{0, 1, 2, \cdots\}$ under the fixed end boundary condition at -1. Suppose that the distribution of $q(0)$ is non-degenerate and that $\langle q(0)\rangle$ is finite. Then, for almost every fixed $\omega\in\Omega$, the one dimensional measure $\rho^\omega(dx)$ admits no absolutely continuous part (K. Ishii [4] and Y. Yoshioka [11]).

The second remark is that the averaged spectral distribution function $\rho(x)=\langle\rho^\cdot(x)\rangle$ can be obtained as the almost sure limit of the normalized ditribution of the eigenvalues of the operator $-H^\omega$ restricted to each of increasing bounded domains under some admissible boundary conditions. Thus $\rho(x)$ is a physically observable quantity of the disordered system. The ergodicity of this kind has been studied by L.A. Pastur systematically for continuous systems. The ergodicity for lattice systems will be formulated elsewhere.

Theorem 1.2 and 1.3 suggest some possibility to answer the question: Can we guess some properties of $q(0)$ from the knowledge of $\rho(x)$?

2. Tauberian theorems of exponential type

Consider a non-decreasing function ρ on $(-\infty, \infty)$ with $\rho(x)=0$, $x<0$. We assume that ρ is right continuous and that $k(t)=\int_{[0,\infty)} e^{-tx}\rho(dx)$ is finite for every $t>0$.

Theorem 2.1. *Fix a number* γ *such as* $0<\gamma<1$.

(i) *If* $\varliminf_{t\uparrow\infty}\frac{1}{t^{\gamma}}\log k(t)\geqq -A \quad (0\leqq A<\infty)$,

then $\varliminf_{x\downarrow 0}\log x^{\gamma/(1-\gamma)}\log\rho(x)\geqq -A^{1/(1-\gamma)}$.

(ii) *If* $\varlimsup_{t\uparrow\infty}\frac{1}{t^{\gamma}}\log k(t)\leqq -A \quad (0<A\leqq\infty)$,

then $\varlimsup_{x\downarrow 0}x^{\gamma/(1-\gamma)}\log\rho(x)\leqq -(1-\gamma)\gamma^{\gamma/(1-\gamma)}A^{1/(1-\gamma)}$.

Proof. We put

$$\kappa(t)=\int_0^\infty e^{-tx}\rho(x)dx\,. \tag{5}$$

Since $k(t)=t\kappa(t)$, it is enough to consider $\kappa(t)$ in place of $k(t)$ in order to show Theorem 2.1.

(i) Let us assume that $\varliminf_{t\uparrow\infty}\frac{1}{t^{\gamma}}\log\kappa(t)\geqq -A$. Then, for any $\varepsilon>0$, $\kappa(t)\geqq \exp\{-(A+\varepsilon)t^{\gamma}\}$ for sufficiently large t. Choose a constant $C>A$ and observe the equality $\int_0^{Ct^{\gamma-1}}e^{-tx}\rho(x)dx=\kappa(t)-\int_{Ct^{\gamma-1}}^{\infty}e^{-tx}\rho(x)dx$. The left hand side is not greater than $\rho(Ct^{\gamma-1})\frac{1-\exp(-Ct^{\gamma})}{t}$. The right hand side is not smaller than $\exp\{-(A+\varepsilon)t^{\gamma}\}-\kappa(\varepsilon t)\exp\{-(1-\varepsilon)Ct^{\gamma}\}=(1+o(1))\exp\{-(A+\varepsilon)t^{\gamma}\}$ when ε is small enough. Hence we have

$$\varliminf_{t\uparrow\infty}\frac{1}{t^{\gamma}}\log\rho(Ct^{\gamma-1})\geqq -(A+\varepsilon)\,. \tag{6}$$

Put $x=Ct^{\gamma-1}$, then $\varliminf_{x\downarrow 0}x^{\gamma/(1-\gamma)}\log\rho(x)\geqq -C^{\gamma/(1-\gamma)}(A+\varepsilon)$, from which we get our first assertion by letting $\varepsilon\downarrow 0$ and $C\downarrow A$.

(ii) Let us assume that $\varlimsup_{t\uparrow\infty}\frac{1}{t^{\gamma}}\log\kappa(t)\leqq -A$ with $0<A<\infty$. Then, for any $\varepsilon>0$, $\kappa(t)\leqq\exp\{-(A-\varepsilon)t^{\gamma}\}$ if t is sufficiently large. From $\kappa(t)\geqq \int_{Ct^{\gamma-1}}^{\infty}e^{-t\gamma}\rho(x)dx\geqq\rho(Ct^{\gamma-1})\frac{\exp(-Ct^{\gamma})}{t}$, we have $\rho(Ct^{\gamma-1})\leqq t\,\kappa(t)\exp(Ct^{\gamma})$. Hence $\varlimsup_{t\uparrow\infty}\frac{1}{t^{\gamma}}\log\rho(Ct^{\gamma-1})\leqq -(A-\varepsilon-C)$, from which follows

$$\varlimsup_{x\downarrow 0}x^{\gamma/(1-\gamma)}\log\rho(x)\leqq -(A-C)C^{\gamma/(1-\gamma)}\,. \tag{7}$$

Since $C>0$ is arbitrary and $\max_{0<C<A}(A-C)C^{\gamma/(1-\gamma)}=(1-\gamma)\gamma^{\gamma/(1-\gamma)}A^{1/(1-\gamma)}$, we arrive at (ii).

The second estimate of Theorem 2.1 is best possible. Indeed if we invoke the Minlos-Povzner Tauberian theorem [8], we can obtain the next theorem.

Theorem 2.2. *Fix γ with $0<\gamma<1$ and put $A_\gamma=(1-\gamma)\gamma^{\gamma/(1-\gamma)}A^{\gamma/(1-\gamma)}$.*

Then $\lim_{t\uparrow\infty}\frac{1}{t^\gamma}\log k(t)=-A$, $(0\leqq A\leqq\infty)$, *if and only if* $\lim_{x\downarrow 0} x^{\gamma/(1-\gamma)}\log\rho(x)=-A_\gamma$.

Proof. Rewriting the equality $\kappa(\beta t)=\int_0^\infty e^{-\beta t y}\rho(y)dy$ by the change of variable $y=t^{\gamma-1}y'$, we have

$$e^{-\alpha(t)\phi_t(\beta)}=\int_0^\infty e^{-\alpha(t)\{\beta y-\phi_t(y)\}}dy \tag{8}$$

with $\alpha(t)=t^\gamma$, $\phi_t(\beta)=-\frac{1}{t^\gamma}\log\kappa(\beta t)$ and $\phi_t(y)=\frac{1}{t^\gamma}\left\{\log\rho\left(\frac{y}{t^{1-\gamma}}\right)-(1-\gamma)\log t\right\}$. An obvious modification of [8; §5, Theorem 2] now implies "only if" part of Theorem 2.2 as follows; since $\lim_{t\uparrow\infty}\phi_t(\beta)=\beta^\gamma A$, $\lim_{t\uparrow\infty}\phi_t(1)=\lim_{x\downarrow 0}x^{\gamma/(1-\gamma)}\log\rho(x)$ exists and equals $\inf_{0<\beta<\infty}(\beta-\beta^\gamma A)=-A_\gamma$. "If" part of Theorem 2.2 follows from the analogous modification of [8; §5, Theorem 3].

Here is an immediate consequence of Theorem 2.2.

Corollary. *Let $G_\alpha(x)$, $0\leqq x<\infty$, be the distribution function of the one-sided stable law with exponent $\alpha(0<\alpha<1)$: $\int_0^\infty e^{-\lambda x}G_\alpha(dx)=\exp(-\lambda^\alpha)$, $\lambda>0$. Then* $\lim_{x\downarrow 0}x^{\alpha/(1-\alpha)}\log G_\alpha(x)=-(1-\alpha)\alpha^{\alpha/(1-\alpha)}$.

By appealing to the Tauberian theorem of Hardy-Littlewood type, we can only obtain $\lim_{x\downarrow 0}e^{x^{-\alpha}}G_\alpha(x)=0$ [2; vol. 2, XIII, 6, Theorem 1]. A much more precise formula of $G_\alpha(x)$ is known however [3; Theorem 2.4.6].

3. Maximum of the absolute value of the sample path

Let us start with a rather general setting. We consider a right continuous Markov process $M=(X_t, P_a)$ over Z^ν with a transition probability $p(t, a, b)$, $t>0$, $a, b\in Z^\nu$, and put

$$M_t=\sup_{0\leqq s\leqq t}|X_s|. \tag{9}$$

Here $|a|$ denotes $\sqrt{\sum_{k=1}^{\nu}|a_k|^2}$ for $a=(a_1, a_2, \cdots, a_\nu)$.

Theorem 3.1. *Suppose that there exists a constant $B>0$ such that*

$$p(t, a, b)\leqq Bt^{-\nu/2} \tag{10}$$

for sufficiently large t and for any $a, b\in Z^\nu$. Then, for each $\beta>0$, $\overline{\lim_{t\to\infty}}\, t^{-1/3}\log E_0(e^{-\beta M_t})\leqq-\left(\frac{\nu\beta B^{-1/\nu}}{\sqrt{2\pi e}}\right)^{2/3}$.

Proof. An integration by part leads us to

$$E_0(e^{-\beta M_t}) = \beta \int_0^\infty e^{-\beta x} F(x) dx\,, \tag{11}$$

where $F(x) = P_0(M_t \leqq x)$. We can further observe, for any large s $(0<s<t)$,

$$F(x) \leqq (x+\sqrt{\nu})^{\nu l} (C_\nu B)^l s^{-(\nu l/2)}\,. \tag{12}$$

Here $l = [t/s]$ and C_ν is the volume of the ν-dimensional unit ball. This is an easy consequence of the inequality $F(x) \leqq P_0(|X_s| \leqq x,\ |X_{2s}| \leqq x, \cdots, |X_{ls}| \leqq x)$, the Markovian transition mechanism and the bound (10).

Using then the bound $\int_0^\infty e^{-\beta x}(x+\sqrt{\nu})^{\nu l} dx \leqq e^{\beta\sqrt{\nu}} \dfrac{(\nu l)!}{\beta^{\nu l+1}}$ and the Stirling upper estimate $n! < \sqrt{2\pi n}\, n^n e^{-n} e^{1/12n}$ ([2; vol 1]), we see from (11) and (12) that

$$E_0(e^{-\beta M_t}) \leqq e^{\beta\sqrt{\nu}} \sqrt{2\pi\nu l} (D\nu s^{-1/2})^{\nu l} e^{-\nu l + (1/12\nu l)}\,, \tag{13}$$

with $D = \dfrac{\nu}{\beta}(C_\nu B)^{1/\nu}$.

Let us choose s such that $D\dfrac{t}{s} s^{-1/2} = 1$, namely, $s = D^{2/3} t^{2/3}$. Then $Dl \cdot s^{-1/2} \leqq 1$ and $l \geqq D^{-2/3} t^{1/3} - 1$. (13) now implies

$$\varlimsup_{t\to\infty} t^{-1/3} \log E_0(e^{-\beta M_t}) \leqq -\nu D^{-2/3}\,. \tag{14}$$

Furthr we have from the bound $C_\nu^{-1} \geqq \left(\dfrac{\nu}{2\pi e}\right)^{\nu/2}$ that $\nu D^{-2/3} = (\nu^{1/2} \beta B^{-1/\nu} C_\nu^{-1/\nu})^{2/3} \geqq \left(\dfrac{\nu\beta B^{-1/\nu}}{\sqrt{2\pi e}}\right)^{2/3}$, proving Theorem 3.1.

We return to the simple random walk $\dot{M} = (\dot{X}_t, \dot{P}_a)$ introduced in §1. $\dot{M}$ is spatially homogeneous and its transition function is expressed as

$$\dot{p}(t, 0, a) = \frac{1}{(2\pi)^\nu} \int_C e^{-i\theta\cdot a} e^{-2\sigma^2 t \sum_{k=1}^{\nu} \sin^2 \frac{\theta_k}{2}} d\theta \tag{15}$$

where C is the Euclidean cube $C = \{\theta;\ -\pi < \theta_i < \pi,\ i=1, 2, \cdots, \nu\}$. To see this, let us consider the characteristic function $\phi_t(\theta) = \dot{E}_0(e^{i\theta\cdot\dot{X}_t})$, then $\phi_t(\theta) = (\phi_{t/n}(\theta))^n$ and $\lim_{n\uparrow\infty} \dfrac{n}{t}(1-\phi_{t/n}(\theta)) = -H^0 I_0(0) - \sum_{a\neq 0} H^0 I_a(0) \cdot e^{i\theta\cdot a} = 2\sigma^2 \sum_{k=1}^{\nu} \sin^2 \dfrac{\theta_k}{2}$. Hence $\phi_t(\theta) = \exp\left(-2\sigma^2 t \sum_{k=1}^{\nu} \sin^2 \dfrac{\theta_k}{2}\right)$ and (15) is nothing but the inversion formula. (15) immediately implies

$$\dot{p}(t, 0, a) \leqq \dot{p}(t, 0, 0)\,, \qquad a \in Z^\nu. \tag{16}$$

Furthermore

(17) $$\dot{p}(t, 0, 0) \sim (2\pi\sigma^2 t)^{-\nu/2}, \qquad t \to \infty.$$

Since $\sin\frac{\theta}{2} \sim \frac{\theta}{2}$, $\theta \to 0$, the right hand side of (15) is of the same order as $\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\sigma^2 t\theta^2/2)} d\theta\right)^{\nu} = (2\pi t\sigma^2)^{-\nu/2}$, yielding (7).

Theorem 3.1 now applies and we have the following. $\dot{M}_t$ denotes $\sup_{0\leq s\leq t} |\dot{X}_s|$.

Theorem 3.2. *For the simple random walk $\dot{M}$ on Z^{ν},*

$$\overline{\lim_{t\to\infty}}\, t^{-1/3} \log \dot{E}_0(e^{-\beta \dot{M}_t}) \leq -\left(\frac{\nu\sigma\beta}{\sqrt{e}}\right)^{2/3}.$$

We continue to work with the simple random walk $\dot{M}$.

Theorem 3.3. *Fix α and β such that $0 \leq \alpha < 1$, $\beta > 0$. Then,*

$$\underline{\lim_{t\to\infty}}\, t^{-\{(1+\alpha)/(3-\alpha)\}} \log \dot{E}_0(e^{\beta t^{\alpha} \dot{M}_t^{1-\alpha}};\ \dot{X}_t = 0)$$
$$\geq -\frac{3-\alpha}{1-\alpha}\left(\frac{1-\alpha}{2}\right)^{2/(3-\alpha)}\left(\frac{\pi^2\nu^2\sigma^2}{8}\right)^{(1-\alpha)/(3-\alpha)}\beta^{2/(3-\alpha)}.$$

In particular ($\alpha = 0$)

$$\underline{\lim_{t\to\infty}}\, t^{-1/3} \log \dot{E}_0(e^{-\beta \dot{M}_t};\ \dot{X}_t = 0) \geq -\sqrt[3]{\frac{27}{32}}(\pi\nu\sigma\beta)^{2/3}.$$

We prepare two lemmas in order to prove Theorem 3.3. Denote by $\dot{X}_s^{(k)}$ the k-th coordinate of $\dot{X}_s$ and put $\dot{M}_t^{(k)} = \sup_{0\leq s\leq t} |\dot{X}_s^{(k)}|$, $k = 1, 2, \cdots, \nu$.

Lemma 3.1. *For each positive integer a, the function $\dot{p}^a(t, 0, 0) = \dot{P}_0(\dot{M}_t^{(1)} < a, \cdots, \dot{M}_t^{(\nu)} < a, \dot{X}_t = 0)$ has the following bound:*

$$\dot{p}^a(t, 0, 0) \geq e^{-(\pi^2\nu\sigma^2 t/8a^2)} \cdot \frac{1}{a^{\nu}}.$$

Proof. $\dot{p}^a(t, 0, 0)$ can be expanded in the form $\sum_{\substack{1\leq n_k\leq 2a-1\\ k=1,2,\cdots,\nu}} e^{-(\lambda_{n_1}+\cdots+\lambda_{n_\nu})t} v_{n_1}(0)^2 \cdots v_{n_\nu}(0)^2$, *where* λ_n, $1 \leq n \leq 2a-1$, is the eigenvalue of the one-dimensional problem

$$\begin{cases} \frac{\sigma^2}{2}\{u(x-1) - 2u(x) + u(x+1)\} = -\lambda u(x) & x \in Z^1,\ |x| < a, \\ u(-a) = u(a) = 0, \end{cases}$$

and v_n is the corresponding eigenfunction with $\sum_{x=-a}^{a} v_n(x)^2 = 1$. It is easy to see that $\lambda_n = 2\sigma^2 \sin^2\frac{n\pi}{4a}$, $v_{2k}(x) = \frac{1}{\sqrt{a}}\sin\frac{k\pi}{a}x$ $(1 \leq k \leq a-1)$ and $v_{2k-1}(x) =$

$\frac{1}{\sqrt{a}}\cos\frac{(2k-1)\pi}{2a}x$ $(1\leqq k\leqq a)$. Since $\left|\sin\frac{\pi}{4a}\right|\leqq\frac{\pi}{4a}$, the first term of the expansion gives rise to Lemma 3.1.

Lemma 3.2. *For α and β as in* Theorem 3.3. *and for fixed $\eta>0$ and $\zeta>0$, we put*

$$I(t)=\int_0^\infty\left[\exp\left(-\beta t^\alpha x^{1-\alpha}-\frac{\eta t}{x^2}\right)\right]\cdot(x+\zeta)^{-\nu-\alpha}dx\,.$$

Then

$$\varliminf_{t\to\infty} t^{(1+\alpha)/(3-\alpha)}\log I(t)\geqq-\frac{3-\alpha}{1-\alpha}\left(\frac{1-\alpha}{2}\right)^{1/(3-\alpha)}\eta^{(1-\alpha)/(3-\alpha)}\beta^{2/(3-\alpha)}\,. \tag{18}$$

Proof. For each $C>0$ and $\varepsilon>0$,

$$\begin{aligned}I(t)&\geqq\int_0^{Ct^\varepsilon}\left[\exp\left(-\beta t^\alpha x^{1-\alpha}-\frac{\eta t}{x^2}\right)\right]\cdot(x+\zeta)^{-\nu-\alpha}dx\\&\geqq(Ct^\varepsilon+\zeta)^{-\nu-\alpha}[\exp(-\beta C^{1-\alpha}t^{\alpha+\varepsilon-\varepsilon\alpha})]\int_0^{Ct^\varepsilon}e^{-(\eta t/x^2)}dx\,.\end{aligned}\tag{19}$$

On the other hand, integrating the inequality $e^{-h/x^2}\geqq\left(\frac{3x^2}{4h}+\frac{1}{2}\right)e^{-h/x^2}$, we see $\int_0^d e^{-h/x^2}dx\geqq\frac{d^3}{4h}e^{-h/d^2}$ provided that $\frac{d^2}{h}$ is sufficiently small.

Hence (19) leads us to

$$I(t)\geqq\frac{C^3}{4\eta t^{1-3\varepsilon}(Ct^\varepsilon+\zeta)^{\nu+\alpha}}\exp\left(-\beta C^{1-\alpha}t^{\alpha+\varepsilon-\varepsilon\alpha}-\frac{\eta t^{1-2\varepsilon}}{C^2}\right).$$

for $\varepsilon<\frac{1}{2}$ and sufficiently large t. We choose ε such that $\alpha+\varepsilon-\varepsilon\alpha=1-2\varepsilon$, namely, $\varepsilon=\frac{1-\alpha}{3-\alpha}\left(\leqq\frac{1}{3}\right)$. Then $1-2\varepsilon=\frac{1+\alpha}{3-\alpha}$ and

$$\varliminf_{t\to\infty} t^{-\{(1+\alpha)/(3-\alpha)\}}\log I(t)\geqq-\left(\beta C^{1-\alpha}+\frac{\eta}{C^2}\right). \tag{20}$$

The maximum of the right hand side of (20) for $C>0$ is just the right hand side of (18) and we have done.

Proof of Theorem 3.3. Let us put $G(x)=\dot{P}_0(\dot{M}_t\leqq x,\ \dot{X}_t=0)$ for every real x. Then

$$\begin{aligned}\dot{E}_0(e^{-\beta t^\alpha M_t^{1-\alpha}};\ \dot{X}_t=0)&=\int_{[0,\infty)}e^{-\beta t^\alpha x^{1-\alpha}}dG(x)\\&=(1-\alpha)\beta t^\alpha\int_0^\infty x^{-\alpha}e^{-\beta t^\alpha x^{1-\alpha}}G(x)dx\,.\end{aligned}\tag{21}$$

Since $G(x) \geqq \dot{p}^{[(x/\sqrt{\nu})+1]}(t, 0, 0)$, Lemma 3.1 implies that the left hand side of (21) is not smaller than

$$(1-\alpha)\beta t^{\alpha}\nu^{\nu/2}\int_0^{\infty}\left[\exp\left(-\beta t^{\alpha}x^{1-\alpha}-\frac{\pi^2\nu^2\sigma^2 t}{8x^2}\right)\right](x+\sqrt{\nu})^{-\nu-\alpha}dx\,.$$

Therefore Theorem 3.3 follows from Lemma 3.2 with $\eta=\dfrac{\pi^2\nu^2\sigma^2}{8}$.

REMARK. It is clear from our proof that Theorem 3.2 and 3.3 are also true for the "pinned" process obtained from the ν-dimensional Brownian motion with transition density $p(s, 0, a)=\dfrac{1}{(2\pi\sigma^2 s)^{\nu/2}}e^{-(|a|^2/2\sigma^2 s)}$. We have just the same estimate as in Lemma 3.1 for the pinned Brownian motion.

4. Proof of Theorem 1.1~1.3

In this section, we prove the theorems stated in §1. We first show the Kac formula (4).

Lemma 4.1.

$$k(t) = \tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{X}_t=0)\,.$$

Proof. By taking the expectation $\langle\ \rangle$ of the both hand sides of

$$\int_0^{\infty} e^{-tx}\rho^{\omega}(dx) = \dot{E}_0(\exp(-\int_0^t q(\dot{X}_s, \omega)ds);\ \dot{X}_t=0)\,, \tag{22}$$

we arrive at Lemma 4.1. In order to show (22), it suffices to prove

$$\int_0^{\infty}\frac{d_x(E_x^{\omega}I_0, I_a)}{\alpha+x} = \dot{E}_a(\int_0^{\infty} e^{-\alpha t}e^{-\int_0^t q(\dot{X}_s, \omega)ds}I_0(\dot{X}_t)dt)\,, \tag{23}$$

for every $\alpha>0$.

The right hand side of (23) is, as a function of $a\in Z^{\nu}$, a bounded solution of the next equation ([5]).

$$(\alpha-H^{\omega})v(a) = I_0(a)\,, \qquad a\in Z^{\nu}\,. \tag{24}$$

Here we regard $H^{\omega}v$ as the function on Z^{ν} defined at each point a by the explicit formula (1). The left hand side of (23) equals $G_{\alpha}^{\omega}I_0(a)$ with $G_{\alpha}^{\omega}=(\alpha-\bar{H}^{\omega})^{-1}$. Hence $(\alpha-H^{\omega})G_{\alpha}^{\omega}I_0(a)=((\alpha-H^{\omega})G_{\alpha}^{\omega}I_0, I_a)=(G_{\alpha}^{\omega}I_0, (\alpha-H^{\omega})I_a)=(G_{\alpha}^{\omega}I_0, (\alpha-\bar{H}^{\omega})I_a)$ $=((\alpha-\bar{H}^{\omega})G_{\alpha}^{\omega}I_0, I_a)=I_0(a)$.[1] Since moreover $|G_{\alpha}^{\omega}I_0(a)|^2\leqq(G_{\alpha}^{\omega}I_0, G_{\alpha}^{\omega}I_0)(I_a, I_a)$ $\leqq\alpha^{-2}$, the left hand side of (23) is also a bounded solution of (24).

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1) This point becomes trivial if one notices that the operator $\bar{H}^{\omega}$ is described explicitly as follows: $\mathcal{D}(\bar{H}^{\omega})=\{u\in L^2(Z^{\nu});\ H^{\omega}u\in L^2(Z^{\nu})\}$, $\bar{H}^{\omega}u=H^{\omega}u$, $u\in\mathcal{D}(\bar{H}^{\omega})$.

It is quite easy to show that the bounded solution of (24) is unique. To see this, consider a bounded solution v of $(\alpha-H^{\omega})v=0$ and put $M=\sup_{a\in Z^\nu}|v(a)|$. Then, from $v(a)=\frac{\sigma^2/2}{q(a)+\alpha+\nu\sigma^2}\sum_{|b-a|=1}v(b)$, we get $M\leqq\frac{\nu\sigma^2}{\alpha+\nu\sigma^2}M$. Therefore $v=0$, yielding the equality (23) and hence Lemma 4.1.

The sample path $\dot{X}_t$ of $\dot{M}$ behaves as follows: it stays at its starting point during an exponential holding time with expectation $1/\nu\sigma^2$ and then jumps to one of its 2ν neighbours with equal probability. Let us put

$$(25)\qquad \begin{cases} \xi_1(\dot{\omega})=\inf\{t>0;\ \dot{X}_t(\dot{\omega})\neq\dot{X}_0(\dot{\omega})\} \\ \xi_n(\dot{\omega})=\inf\{t>0;\ \dot{X}_t(\dot{\omega})\notin[\dot{X}_0(\dot{\omega}),\ \dot{X}_{\xi_1}(\dot{\omega}),\ \cdots,\ \dot{X}_{\xi_{n-1}}(\dot{\omega})]\} \\ \qquad n=2,3,\cdots. \end{cases}$$

We have then $P_a(\xi_1>t)=e^{-\nu\sigma^2t}$ for any $a\in Z^\nu$.

Denote by $\dot{R}_t(\dot{\omega})$ the number of different points visited by the sample path $\dot{X}_s(\omega)$ during the time interval $[0,t)$. The range $\dot{R}_t$ equals k if and only if $\xi_{k-1}(\dot{\omega})\leqq t<\xi_k(\dot{\omega})$. Since $q(a)$ and $q(b)$ are independent and have the same distribution as $q(0)$ when $a\neq b$, we can readily obtain the next lemma.

Lemma 4.2.

$$\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{R}_t=k,\ \dot{X}_t=0)$$
$$=\dot{E}_0(\langle e^{-J_0(t)q(0)}\rangle\langle e^{-J_{\dot{X}_{\xi_1}}(t)q(0)}\rangle\cdots\langle e^{-J_{\dot{X}_{\xi_{k-1}}}(t)q(0)}\rangle;\ \dot{R}_t=k,\ \dot{X}_t=0).$$

Here $J_a(t)=\int_0^t I_a(\dot{X}_s)ds,\ a\in Z^\nu$.

We are now in a position to give the proof of Theorem 1.2 and 1.3.

Proof of Theorem 1.2. From Lemma 4.2, $\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{R}_t=k,\ \dot{X}_t=0)$ $\geqq P(q(0)=0)^k\ \dot{P}_0(\dot{R}_t=k,\ \dot{X}_t=0)$. Putting $P(q(0)=0)=e^{-\beta_2}$ and using Lemma 4.1,

$$(26)\qquad k(t)\geqq\dot{E}_0(e^{-\beta_2\dot{R}_t};\ \dot{X}_t=0).$$

When $\nu=1$, $\dot{R}_t\leqq 2\dot{M}_t+1$ and hence $k(t)\geqq\dot{E}_0(e^{-\beta_2(2\dot{M}_t+1)};\ \dot{X}_t=0)$, which together with Theorem 3.3 implies

$$(27)\qquad \varliminf_{t\to\infty} t^{-1/3}\log k(t)\geqq-\sqrt[3]{\frac{27}{32}}(2\pi\sigma\beta_2)^{2/3}.$$

Theorem 1.2 is an immediate consequence of (27) and Theorem 2.1 (i).

Proof of Theorem 1.3. We assume that $q(0)$ is distributed according to the one-sided stable law with exponent α, $0<\alpha<1$. By Lemma 4.2,

$$\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{R}_t=k,\ \dot{X}_t=0)$$
$$=\dot{E}_0(\exp\{-(J_0(t)^\alpha+J_{\dot{X}_{\xi_1}}(t)^\alpha+\cdots+J_{\dot{X}_{\xi_{k-1}}}(t)^\alpha)\}\ ;\ \dot{R}_t=k,\ \dot{X}_t=0)\,.$$

Using then Lemma 4.1 and the inequality $t^\alpha k^{1-\alpha}\geqq s_1^\alpha+s_2^\alpha+\cdots+s_k^\alpha\geqq t^\alpha$, $(s_1, s_2, \cdots, s_k\geqq 0,\ s_1+s_2+\cdots+s_k=t)$, we are led to

$$\dot{E}_0(e^{-t^\alpha\dot{R}_t^{1-\alpha}};\ \dot{X}_t=0)\leqq k(t)\leqq e^{-t^\alpha}\dot{P}_0(\dot{X}_t=0)\,. \tag{28}$$

The second inequality of (28) and the asymptotic formula (17) imply

$$\overline{\lim_{t\to\infty}}\, t^{-\alpha}\log k(t)\leqq -1\,. \tag{29}$$

The first assertion of Theorem 1.3 follows from this and Theorem 2.1 (ii).

When $\nu=1$, we use the inequality $\dot{R}_t^{1-\alpha}\leqq(2\dot{M}_t+1)^{1-\alpha}\leqq(2\dot{M}_t)^{1-\alpha}+1$ to obtain from the first part of (28) and Theorem 3.3 that

$$\underline{\lim_{t\to\infty}}\, t^{-\{(1+\alpha)/(3-\alpha)\}}\log k(t)\geqq -A \tag{30}$$

with $A=\dfrac{3-\alpha}{1-\alpha}\left(\dfrac{1-\alpha}{2}\right)^{2/(3-\alpha)}\left(\dfrac{\pi^2\sigma^2}{8}\right)^{(1-\alpha)/(3-\alpha)}2^{(2-2\alpha)/(3-\alpha)}$.

By Theorem 2.1 (i), we then conclude $\underline{\lim}_{x\downarrow 0}\, x^{\alpha/(1-\alpha)+(1/2)}\log\rho(x)\geqq -A^{(1/2)[(3-\alpha)/(1-\alpha)]}$, which is the second assertion of Theorem 1.3.

It only remains to show Theorem 1.1. We need a lemma.

Lemma 4.3.

$$k(t)\leqq\{t\cdot\dot{E}_0(e^{-\beta_1\dot{R}_t})+te^{-\nu\sigma^2 t}\}^{1/2}+e^{-\{\beta_1(t-1)/2\}}\,,$$

with $\beta_1=-\log\left\langle\dfrac{\nu\sigma^2}{\nu\sigma^2+2q(0)}\right\rangle$.

Proof. Denote by $\dot{\omega}_u^+$ the shifted path: $\dot{X}_s(\dot{\omega}_u^+)=\dot{X}_{u+s}(\dot{\omega})$, $s\geqq 0$. Replacing $J_{\dot{X}_{\xi_l}}(t)$ by $\xi_1(\dot{\omega}_{\xi_l}^+)$, $l=0, 1, \cdots, k-1$, $(\xi_0=0)$, in the formula of Lemma 4.2 and applying the Schwarz inequality twice, we obtain

$$\begin{aligned}[\tilde{E}(\exp(-\int_0^t q(\dot{X}_t)ds);\ \dot{R}_t=k,\ \dot{X}_t=0)]^2\\ \leqq\dot{E}_0(\langle e^{-2\xi_1(\dot{\omega})q(0)}\rangle\langle e^{-2\xi_1(\dot{\omega}_{\xi_1}^+)q(0)}\rangle\cdots\langle e^{-2\xi_1(\dot{\omega}_{\xi_{k-1}}^+)q(0)}\rangle)\dot{P}_0(\dot{R}_t=k,\ \dot{X}_t=0)\,,\\ k=2, 3, \cdots\,.\end{aligned} \tag{31}$$

We next note the equality

$$\dot{E}_a(\langle e^{-2\xi_1 q(0)}\rangle)=e^{-\beta_1}\,,\qquad a\in Z^\nu\,. \tag{32}$$

In fact the left hand side equals

$$\langle \dot{E}_a(e^{-2\xi_1 q(0)})\rangle = \langle \nu\sigma^2 \int_0^\infty e^{-2q(0)s} e^{-\nu\sigma^2 s} ds\rangle = \left\langle \frac{\nu\sigma^2}{\nu\sigma^2+2q(0)} \right\rangle.$$

Furthermore we have

$$(33)\quad \dot{E}_0(\langle e^{-2\xi_1(\dot{\omega})q(0)}\rangle \langle e^{-2\xi_1(\dot{\omega}^+_{\xi_1})q(0)}\rangle \cdots \langle e^{-2\xi_1(\dot{\omega}^+_{\xi_{k-1}})q(0)}\rangle) = e^{-\beta_1 k}, \qquad k=2, 3, \cdots.$$

In order to see (33), notice that ξ_n is a stopping time (a Markov time in the terminology of [5]) and that the random variables $\xi_1(\dot{\omega}^+_{\xi_l})$, $l<n$, depend only on the events occurred up to time ξ_n. Therefore the strong Markov property ([5]) and (32) imply that the left hand side of (33) is equal to

$$\begin{aligned} &\dot{E}_0(\langle^{-2\xi_1(\dot{\omega})q(0)}\rangle \cdots \langle e^{-2\xi_1(\dot{\omega}^+_{\xi_{k-2}})q(0)}\rangle \dot{E}_{X_{\xi_{k-1}}}(\langle e^{-2\xi_1 q(0)}\rangle)) \\ &= \dot{E}_0(\langle e^{-2\xi_1(\dot{\omega})q(0)}\rangle \cdots \langle e^{-2\xi_1(\dot{\omega}^+_{\xi_{k-2}})q(0)}\rangle)e^{-\beta_1} \\ &= \cdots = e^{-\beta_1 k}. \end{aligned}$$

From (31) and (33), we have

$$\begin{aligned} &[\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{R}_t \leqq t,\ \dot{X}_t=0)]^2 \\ &\leqq t\sum_{k=1}^{[t]} [\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{R}_t=k,\ \dot{X}_t=0)]^2 \\ &\leqq t\sum_{k=1}^{[t]} e^{-\beta_1 k}\dot{P}_0(\dot{R}_t=k,\ \dot{X}_t=0)+t\dot{P}_0(\dot{R}_t=1) \\ &\leqq t\dot{E}_0(e^{-\beta_1 \dot{R}_t})+te^{-\nu\sigma^2 t}. \end{aligned}$$

On the other hand, $[\tilde{E}(\exp(-\int_0^t q(\dot{X}_s)ds);\ \dot{R}_t>t,\ \dot{X}_t=0)]^2$ is not greater than the left hand side of (33) with $k=[t]$.

Proof of Theorem 1.1. $\dot{X}=0$ implies $\dot{M}_t+1\leqq \dot{R}_t$. Hence by virtue of Lemma 4.3 and Theorem 3.2, we are led to

$$k(t)\leqq \sqrt{t}\left[\exp\left\{-t^{1/3}\left(\left(\frac{\nu\sigma\beta_1}{\sqrt{e}}\right)^{2/3}+o(1)\right)\right\}+\exp\{-\nu\sigma^2 t\}\right]^{1/2}+e^{-\{\beta_1(t-1)/2\}},$$

which implies

$$\overline{\lim_{t\to\infty}}\, t^{-1/3}\log k(t)\leqq -\frac{1}{2}\left(\frac{\nu\sigma\beta_1}{\sqrt{e}}\right)^{2/3}.$$

Theorem 1.1 follows from this and Theorem 2.1 (ii).

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