Wenpeng, Z. and Huaning, L.
Osaka J. Math.
42 (2005), 189–199

ON THE GENERAL GAUSS SUMS AND THEIR FOURTH POWER MEAN

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(Received June 27, 2003)

Abstract

The main purpose of this paper is to study the fourth power mean of the general Gauss sums, and give two exact calculating formulae.

1. Introduction

For any Dirichlet character χ mod q, the classical Gauss sums are defined by

$$G(n, \chi) = \sum_{b=1}^{q} \chi(b) \mathrm{e}\left(\frac{nb}{q}\right),$$

where $\mathrm{e}(y) = \mathrm{e}^{2\pi i y}$. The various properties of $G(n, \chi)$ appeared in many analytic number theory books (see references [1] and [2]). Perhaps the most famous properties of $G(n, \chi)$ are the following identities:

$$G(n, \chi^*) = \overline{\chi}^*(n)\tau(\chi^*) \qquad \text{and} \qquad |\tau(\chi^*)| = \sqrt{q},$$

where χ^* is a primitive character mod q, $\overline{\chi}^*$ is the conjugate character of χ^*, and $\tau(\chi^*) = G(1, \chi^*)$. If χ is a nonprimitive character modulo q, then the value distribution of $\tau(\chi)$ is much irregular, even more is zero!

Let $q \geq 3$ be a positive integer. For any integer n and positive integer k, we define the general k-th Gauss sums $G(n, k, \chi; q)$ as follows:

$$G(n, k, \chi; q) = \sum_{b=1}^{q} \chi(b) \mathrm{e}\left(\frac{nb^k}{q}\right).$$

The summation is very important, because it is a generalization of the classical Gauss sums $G(n, \chi)$. But about the properties of $G(n, k, \chi; q)$, we know very little at present. The value of $|G(n, k, \chi; q)|$ is irregular as χ varies. One can only get some upper bound estimates. For example, for any integer n with $(n, q) = 1$, from the gen-

This work is supported by N.S.F.(10271093) and P.N.S.F. of P.R.China.

eral result of Cochrane and Zheng [3] we can deduce

$$\left|G(n, 2, \chi; q)\right| \leq 2^{\omega(q)} q^{1/2},$$

where $\omega(q)$ denotes the number of distinct prime divisors of q. The case that q is prime is due to Weil [5].

However, it is surprising that $G(n, k, \chi; q)$ enjoys many good value distribution properties in some problems of weighted mean value. Also for $k = 2$, the first author studied the hybrid mean value of Dirichlet L-functions and the general quadratic Gauss sums, and obtained several interesting asymptotic formulae as follows (see references [6] and [7]):

Proposition 1. *For any integer n with $(n, p) = 1$, we have the asymptotic formulae*

$$\sum_{\chi \neq \chi_0} \left|G(n, 2, \chi; p)\right|^2 \cdot \left|L(1, \chi)\right| = C \cdot p^2 + O\left(p^{3/2} \ln^2 p\right)$$

and

$$\sum_{\chi \neq \chi_0} \left|G(n, 2, \chi; p)\right|^4 \cdot \left|L(1, \chi)\right| = 3 \cdot C \cdot p^3 + O\left(p^{5/2} \ln^2 p\right),$$

where $L(s, \chi)$ denotes the Dirichlet L-function corresponding to the character χ modulo p,

$$C = \prod_p \left[1 + \frac{\binom{2}{1}^2}{4^2 \cdot p^2} + \frac{\binom{4}{2}^2}{4^4 \cdot p^4} + \cdots + \frac{\binom{2m}{m}^2}{4^{2m} \cdot p^{2m}} + \cdots\right]$$

is a constant, $\sum_{\chi \neq \chi_0}$ denotes the summation over all nonprincipal characters modulo p, $\prod_p$ denotes the product over all primes, and $\binom{2m}{m} = (2m)!/(m!)^2$.

Proposition 2. *Let p be an odd prime with $p \equiv 3 \mod 4$. Then for any fixed positive integer n with $(n, p) = 1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} \left|G(n, 2, \chi; p)\right|^6 \cdot \left|L(1, \chi)\right| = 10 \cdot C \cdot p^4 + O\left(p^{7/2} \ln^2 p\right).$$

Let n be any integer with $(n, p) = 1$. The first author [5] also obtained the following two identities:

$$\sum_{\chi \bmod p} |G(n, 2, \chi; p)|^4 = \begin{cases} (p-1)\left[3p^2 - 6p - 1 + 4\left(\frac{n}{p}\right)\sqrt{p}\right], & \text{if } p \equiv 1 \bmod 4; \\ (p-1)(3p^2 - 6p - 1), & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

and

$$\sum_{\chi \bmod p} \left|G(n, 2, \chi; p)\right|^6 = (p-1)(10p^3 - 25p^2 - 4p - 1), \text{ if } p \equiv 3 \bmod 4,$$

where (n/p) is the Legendre symbol.

It is very natural to consider the calculating problem of the sum

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^{2m},$$

and try to give some exact calculating formulae. For $m = 1$, we easily get

$$\begin{aligned}\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^2 &= \sum_{\chi \bmod q} \sum_{a=1}^{q} \chi(a) \mathrm{e}\left(\frac{na^k}{q}\right) \sum_{b=1}^{q} \overline{\chi}(b) \mathrm{e}\left(-\frac{nb^k}{q}\right) \\ &= \phi(q) \sum_{a=1}^{q}{}' \mathrm{e}\left(\frac{na^k}{q} - \frac{na^k}{q}\right) = \phi^2(q),\end{aligned}$$

where $\sum'^{q}_{a=1}$ denotes the summation over all a such that $(a, q) = 1$. In this paper, we study the sum

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^4,$$

and give two exact calculating formulae. That is, we shall prove the following two main Theorems.

Theorem 1. *Let p be a prime with $3 \mid p-1$, then we have the identity*

$$\sum_{\chi \bmod p} \left|G(1, 3, \chi; p)\right|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U,$$

where $U = \sum_{a=1}^{p} e\left(a^3/p\right)$ is a real constant.

Theorem 2. *Let $q \geq 3$ be a square-full number (i.e. $p \mid q$ if and only if $p^2 \mid q$), n, k be integers with $(nk, q) = 1$ and $k \geq 1$. Then we have the identity*

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^4 = q\phi^2(q) \prod_{p \mid q} (k, p-1)^2 \prod_{\substack{p \mid q \\ (k, p-1)=1}} \frac{\phi(p-1)}{p-1},$$

where $\prod_{p \mid q}$ denotes the product over all prime divisors of q, and $\phi(q)$ is the Euler totient function.

For general integers $m, k \geq 3$, whether there exist some exact calculating formulae for

$$\sum_{\chi \bmod q} \left|G(n, k, \chi; q)\right|^{2m}$$

is an open problem.

2. Some Lemmas

To complete the proof of the Theorems, we need following several lemmas.

Lemma 1. *Let p be a prime with $3 \mid p-1$ and χ_1 be a cubic character mod p, then we have the identity*

$$\sum_{b=1}^{p-1} \chi_1(b^3 - 1) = \frac{\tau^3(\chi_1)}{p} - 2.$$

Proof. For any integer $1 \leq a \leq p-1$, it is easy to show that

$$1 + \chi_1(a) + \chi_1^2(a) = \begin{cases} 3, & \text{if } a \text{ is a cubic residue mod } p; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

So that

$$\sum_{b=1}^{p-1} \chi_1(b^3 - 1) = \sum_{b=1}^{p-1} \left(1 + \chi_1(b) + \chi_1^2(b)\right)\chi_1(b-1).$$

From the properties of cubic character we know that

$$\chi_1^2 = \overline{\chi}_1, \qquad \chi_1(-1) = 1 \qquad \text{and} \qquad \overline{\tau(\chi_1)} = \tau(\overline{\chi}_1), \tag{2}$$

therefore

$$\sum_{b=1}^{p-1} \chi_1^2(b)\chi_1(b-1) = \sum_{b=1}^{p-1} \overline{\chi}_1(b)\chi_1(b-1) = \sum_{b=1}^{p-1} \chi_1(1 - \overline{b}) = \sum_{b=1}^{p-1} \chi_1(b-1),$$

where $\overline{b}$ is the inverse of b defined by $b\overline{b} \equiv 1 \mod p$ and $1 \leq \overline{b} \leq p-1$. So we have

$$\sum_{b=1}^{p-1} \chi_1(b^3 - 1) = 2\sum_{b=1}^{p-1} \chi_1(b-1) + \sum_{b=1}^{p-1} \chi_1\big(b(b-1)\big) = \sum_{b=1}^{p-1} \chi_1\big(b(b-1)\big) - 2. \tag{3}$$

Note that

$$\tau^2(\chi_1)=\sum_{b=1}^{p-1}\chi_1(b)e\left(\frac{b}{p}\right)\sum_{c=1}^{p-1}\chi_1(-c)e\left(-\frac{c}{p}\right)=\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_1(bc)\chi_1(c)e\left(\frac{c(b-1)}{p}\right)$$
$$=\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi_1(b)\overline{\chi}_1(c)e\left(\frac{c(b-1)}{p}\right)=\overline{\tau(\chi_1)}\sum_{b=1}^{p-1}\chi_1\big(b(b-1)\big).$$

That is

$$\sum_{b=1}^{p-1}\chi_1\big(b(b-1)\big)=\frac{\tau^3(\chi_1)}{p}. \tag{4}$$

Now combining (3) and (4) we immediately get

$$\sum_{b=1}^{p-1}\chi_1(b^3-1)=\frac{\tau^3(\chi_1)}{p}-2.$$

This completes the proof of Lemma 1. □

Lemma 2. *Let p be a prime with $3\mid p-1$ and χ_1 be a cubic character* mod p, *then we have the following identities*

$$\begin{cases}\tau(\chi_1)+\overline{\tau(\chi_1)}=U;\\ \tau^2(\chi_1)+\overline{\tau^2(\chi_1)}=U^2-2p;\\ \tau^5(\chi_1)+\overline{\tau^5(\chi_1)}=U^5+5p^2U-5pU^3.\end{cases}$$

Proof. From formula (1) we have

$$\sum_{b=1}^{p-1}e\left(\frac{b^3}{p}\right)=-1+\tau(\chi_1)+\overline{\tau(\chi_1)},$$

therefore

$$\tau(\chi_1)+\overline{\tau(\chi_1)}=U.$$

Note that

$$U^2=\left(\tau(\chi_1)+\overline{\tau(\chi_1)}\right)^2=\tau^2(\chi_1)+\overline{\tau^2(\chi_1)}+2p,$$
$$U^3=\left(\tau(\chi_1)+\overline{\tau(\chi_1)}\right)^3=\tau^3(\chi_1)+\overline{\tau^3(\chi_1)}+3p\left(\tau(\chi_1)+\overline{\tau(\chi_1)}\right)$$

and

$$U^5=\left(\tau(\chi_1)+\overline{\tau(\chi_1)}\right)^5=\tau^5(\chi_1)+\overline{\tau^5(\chi_1)}+5p\left(\tau^3(\chi_1)+\overline{\tau^3(\chi_1)}\right)+10p^2\left(\tau(\chi_1)+\overline{\tau(\chi_1)}\right),$$

So we easily get

$$\tau^2(\chi_1) + \overline{\tau^2(\chi_1)} = U^2 - 2p \quad \text{and} \quad \tau^5(\chi_1) + \overline{\tau^5(\chi_1)} = U^5 + 5p^2U - 5pU^3.$$

This proves Lemma 2. $\square$

Lemma 3. *Let q be a square-full number. Then for any nonprimitive character χ modulo q, we have the identity*

$$\tau(\chi) = G(\chi, 1) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right) = 0.$$

Proof (see Theorem 7.2 of [4]). $\square$

Lemma 4. *Let p be a prime, k, α and β be positive integers with $(k, p) = 1$ and $\alpha \geq \beta \geq 2$, a be any integer with $(a, p) = 1$. Then we have the identity*

$$\sum_{c=1}^{p^\alpha}{}' e\left(\frac{ac^k}{p^\beta}\right) = 0.$$

Proof. Let $d = (k, p-1)$ and χ_2 be a dth-order character mod p. Then we have

$$\sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{ac^k}{p^\beta}\right) = p^{\alpha-\beta} \sum_{c=1}^{p^\beta}{}' \mathrm{e}\left(\frac{ac^k}{p^\beta}\right) = p^{\alpha-\beta} \sum_{c=1}^{p^\beta}{}' \left[1 + \chi_2(c) + \cdots + \chi_2^{d-1}(c)\right] \mathrm{e}\left(\frac{ac}{p^\beta}\right)$$
$$= p^{\alpha-\beta} \left(\sum_{c=1}^{p^\beta}{}' \mathrm{e}\left(\frac{c}{p^\beta}\right) + \overline{\chi_2}(a) \sum_{c=1}^{p^\beta} \chi_2(c) \mathrm{e}\left(\frac{c}{p^\beta}\right) + \cdots + \overline{\chi_2}^{d-1}(a) \sum_{c=1}^{p^\beta} \chi_2^{d-1}(c) \mathrm{e}\left(\frac{c}{p^\beta}\right) \right).$$

From the properties of Dirichlet characters and Lemma 3 we can get

$$\sum_{c=1}^{p^\alpha}{}' \mathrm{e}\left(\frac{ac^k}{p^\beta}\right) = 0.$$

This proves Lemma 4. $\square$

Lemma 5. *Let p be a prime, k and α be positive integers with $(k, p) = 1$ and $\alpha \geq 2$, n be any integer with $(n, p) = 1$. Let $d = (k, p-1)$, then we have the identity*

$$\sum_{\chi \bmod p^\alpha} \left|G(n, k, \chi, p^\alpha)\right|^4 = \begin{cases} p^\alpha \phi^2(p^\alpha) \cdot \dfrac{\phi(p-1)}{p-1}, & \text{if } d = 1; \\ d^2 p^\alpha \phi^2(p^\alpha), & \text{if } d > 1. \end{cases}$$

Proof. If $d = 1$, then from Lemma 3 we have

$$\sum_{\chi \bmod p^\alpha} \left|G(n,k,\chi,p^\alpha)\right|^4 = \sum_{\chi \bmod p^\alpha} \left|\sum_{b=1}^{p^\alpha} \chi^k(b) e\left(\frac{nb^k}{p^\alpha}\right)\right|^4 = \sum_{\chi \bmod p^\alpha} |\tau(\chi)|^4$$

$$= p^{2\alpha}\phi\big(\phi(p^\alpha)\big) = p^\alpha \phi^2(p^\alpha) \cdot \frac{\phi(p-1)}{p-1}.$$

On the other hand, if $d > 1$, then $p > 2$. From the properties of Dirichlet characters mod p^α we may get

$$\sum_{\chi \bmod p^\alpha} \left|G(n,k,\chi,p^\alpha)\right|^4 = \phi(p^\alpha) \sum_{b=1}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)}{p^\alpha}\right)\right|^2$$

$$= d\phi^3(p^\alpha) + \phi(p^\alpha) \sum_{\substack{b=1 \\ p^\alpha \nmid b^k-1}}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)}{p^\alpha}\right)\right|^2$$

(5) $$= d\phi^3(p^\alpha) + \phi(p^\alpha)\Psi.$$

By Lemma 4 we have

$$\Psi = \sum_{\beta=0}^{\alpha-1} \sum_{\substack{b=1 \\ (b^k-1,p^\alpha)=p^\beta}}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)/p^\beta}{p^{\alpha-\beta}}\right)\right|^2$$

$$= \sum_{\substack{b=1 \\ (b^k-1,p^\alpha)=p^{\alpha-1}}}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right)\right|^2$$

$$= \sum_{\substack{b=1 \\ p^{\alpha-1}|b^3-1}}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right)\right|^2 - \sum_{\substack{b=1 \\ p^\alpha|b^k-1}}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right)\right|^2$$

(6) $$= \Omega - d\phi^2(p^\alpha),$$

where

$$\Omega = \sum_{\substack{b=1 \\ p^{\alpha-1}|b^3-1}}^{p^\alpha}{}' \left|\sum_{c=1}^{p^\alpha}{}' e\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right)\right|^2.$$

Let g be a primitive root mod p^α, then we have

$$\Omega = p^{2(\alpha-1)} \sideset{}{'}\sum_{\substack{b=1\\ p^{\alpha-1}|b^k-1}}^{p^\alpha} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(b^k-1)/p^{\alpha-1}}{p}\right)\right|^2$$

$$= p^{2(\alpha-1)} \sum_{\substack{l=0\\ p^{\alpha-1}|g^{lk}-1}}^{\phi(p^\alpha)-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(g^{lk}-1)/p^{\alpha-1}}{p}\right)\right|^2$$

$$= p^{2(\alpha-1)} \sum_{\substack{l=0\\ \phi(p^{\alpha-1})|lk}}^{\phi(p^\alpha)-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(g^{lk}-1)/p^{\alpha-1}}{p}\right)\right|^2.$$

Let $lk = s\phi(p^{\alpha-1})$, where $0 \le s \le dp-1$. Note that $g^{\phi(p^{\alpha-1})} \equiv 1 \bmod p$,

$$g^{s\phi(p^{\alpha-1})} - 1 = \left(g^{\phi(p^{\alpha-1})} - 1\right)\left(g^{\phi(p^{\alpha-1})(s-1)} + \cdots + g^{\phi(p^{\alpha-1})} + 1\right)$$

and

$$g^{\phi(p^{\alpha-1})(s-1)} + \cdots + g^{\phi(p^{\alpha-1})} + 1 \equiv s \bmod p,$$

we have

$$\Omega = p^{2(\alpha-1)} \sum_{s=0}^{dp-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k(g^{s\phi(p^{\alpha-1})}-1)/p^{\alpha-1}}{p}\right)\right|^2$$

$$= p^{2(\alpha-1)}\cdot d(p-1)^2 + dp^{2(\alpha-1)} \sum_{s=1}^{p-1} \left|\sum_{c=1}^{p-1} e\left(\frac{nc^k\cdot s\cdot(g^{\phi(p^{\alpha-1})}-1)/p^{\alpha-1}}{p}\right)\right|^2$$

$$= d\phi^2(p^\alpha) + dp^{2(\alpha-1)} \sum_{s=1}^{p-1} \left|\sum_{c=1}^{p-1} e\left(\frac{c^k\cdot s}{p}\right)\right|^2 = dp^{2(\alpha-1)} \sum_{s=1}^{p} \left|\sum_{c=1}^{p-1} e\left(\frac{c^k\cdot s}{p}\right)\right|^2$$

$$= dp^{2(\alpha-1)} \sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{s=1}^{p} e\left(\frac{s(c^k-d^k)}{p}\right) = dp^{2(\alpha-1)} \sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{s=1}^{p} e\left(\frac{sd^k(c^k-1)}{p}\right)$$

$$= d^2p^{2(\alpha-1)}p(p-1) = d^2p^\alpha\phi(p^\alpha). \tag{7}$$

So for $d > 1$, from (5), (6) and (7) we get

$$\sum_{\chi \bmod p^\alpha} \left|G(n,k,\chi,p^\alpha)\right|^4 = d\phi^3(p^\alpha) + d^2p^\alpha\phi^2(p^\alpha) - d\phi^3(p^\alpha) = d^2p^\alpha\phi^2(p^\alpha).$$

This completes the proof of Lemma 5. □

Lemma 6. *Let n, k, q_1 and q_2 be integers with $(q_1, q_2) = 1$. Then for any character $\chi \bmod q_1q_2$, we have the identity*

$$\left|G(n,k,\chi;q_1q_2)\right| = \left|G(nq_2^{k-1},k,\chi_1;q_1)\right| \cdot \left|G(nq_1^{k-1},k,\chi_2;q_2)\right|,$$

where $\chi = \chi_1\chi_2$ with $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$.

Proof. Since $(q_1, q_2) = 1$, so if a and b pass through a complete residue system mod q_1 and q_2 respectively, then $aq_2 + bq_1$ passes through a complete residue system mod q_1q_2. Note that $\chi = \chi_1\chi_2$ with $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$ we have

$$\begin{aligned}\left|G(n,k,\chi;q_1q_2)\right| &= \left|\sum_{b=1}^{q_1q_2}\chi(b)e\left(\frac{nb^k}{q_1q_2}\right)\right|\\
&= \left|\sum_{a=1}^{q_1}\sum_{b=1}^{q_2}\chi_1(aq_2+bq_1)\chi_2(aq_2+bq_1)e\left(\frac{n(aq_2+bq_1)^k}{q_1q_2}\right)\right|\\
&= \left|\sum_{a=1}^{q_1}\chi_1(aq_2)e\left(\frac{n(aq_2)^k}{q_1q_2}\right)\right| \cdot \left|\sum_{b=1}^{q_2}\chi_2(bq_1)e\left(\frac{n(bq_1)^k}{q_1q_2}\right)\right|\\
&= \left|\sum_{a=1}^{q_1}\chi_1(a)e\left(\frac{nq_2^{k-1}a^k}{q_1}\right)\right| \cdot \left|\sum_{b=1}^{q_2}\chi_2(b)e\left(\frac{nq_1^{k-1}b^k}{q_2}\right)\right|,\end{aligned}$$

where we have used $|\chi_1(q_2)| = |\chi_2(q_1)| = 1$. This proves Lemma 6. □

3. Proof of the theorems

In this section, we complete the proof of the Theorems. Let p be a prime with $3 \mid p-1$ and χ_1 be a cubic character mod p, from (1) and (2) we have

$$\begin{aligned}\sum_{\chi \bmod p}\left|G(1,3,\chi;p)\right|^4 &= \sum_{\chi \bmod p}\left|\sum_{b=1}^{p-1}\chi(b)\mathrm{e}\left(\frac{b^3}{p}\right)\right|^4\\
&= (p-1)\sum_{b=1}^{p-1}\left|\sum_{c=1}^{p-1}\mathrm{e}\left(\frac{c^3(b^3-1)}{p}\right)\right|^2\\
&= (p-1)\sum_{b=1}^{p-1}\left|\sum_{c=1}^{p-1}(1+\chi_1(c)+\overline{\chi}_1(c))\mathrm{e}\left(\frac{c(b^3-1)}{p}\right)\right|^2\\
&= (p-1)\sum_{b=1}^{p-1}\left|\sum_{c=1}^{p-1}\mathrm{e}\left(\frac{c(b^3-1)}{p}\right)+\overline{\chi}_1(b^3-1)\tau(\chi_1)+\chi_1(b^3-1)\overline{\tau(\chi_1)}\right|^2\end{aligned}$$

$$
\begin{aligned}
&= 3(p-1)^3 + (p-1) \sum_{\substack{b=1 \\ p \nmid b^3-1}}^{p-1} \left| \overline{\chi}_1(b^3-1)\tau(\chi_1) + \chi_1(b^3-1)\overline{\tau(\chi_1)} - 1 \right|^2 \\
&= 3(p-1)^3 + (p-1)(p-4)(2p+1) + (p-1) \left[\tau^2(\chi_1) \sum_{b=1}^{p-1} \chi_1(b^3-1) \right. \\
&\quad \left. + \overline{\tau^2(\chi_1)} \sum_{b=1}^{p-1} \overline{\chi}_1(b^3-1) - 2\tau(\chi_1) \sum_{b=1}^{p-1} \overline{\chi}_1(b^3-1) - 2\overline{\tau(\chi_1)} \sum_{b=1}^{p-1} \chi_1(b^3-1) \right] \\
&= 3(p-1)^3 + (p-1)(p-4)(2p+1) + (p-1)\Psi,
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi \;=\; & \tau^2(\chi_1) \sum_{b=1}^{p-1} \chi_1(b^3-1) + \overline{\tau^2(\chi_1)} \sum_{b=1}^{p-1} \overline{\chi}_1(b^3-1) - 2\tau(\chi_1) \sum_{b=1}^{p-1} \overline{\chi}_1(b^3-1) \\
& - 2\overline{\tau(\chi_1)} \sum_{b=1}^{p-1} \chi_1(b^3-1).
\end{aligned}
$$

By Lemma 1 and Lemma 2 we get

$$
\begin{aligned}
\Psi &= \frac{\tau^5(\chi_1) + \overline{\tau^5(\chi_1)}}{p} - 4\left[\tau^2(\chi_1) + \overline{\tau^2(\chi_1)}\right] + 4\left[\tau(\chi_1) + \overline{\tau(\chi_1)}\right] \\
&= \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 8p + 4U.
\end{aligned}
$$

Therefore

$$
\sum_{\chi \bmod p} \left| G(1,3,\chi;p) \right|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U.
$$

This proves Theorem 1.

Let $q \geq 3$ be a square-full number, n, k be any integers with $(nk, q) = 1$ and $k \geq 1$. Let $q = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the factorization of q into prime powers and $\chi = \prod_{i=1}^{r} \chi_i$, where χ_i be a character mod $p_i^{\alpha_i}$. From Lemma 5 and Lemma 6 we have

$$
\begin{aligned}
\sum_{\chi \bmod q} \left| G(n,k,\chi;q) \right|^4 &= \prod_{i=1}^{r} \left[\sum_{\chi_i \bmod p_i^{\alpha_i}} \left| G\left(n \left(\frac{q}{p_i^{\alpha_i}} \right)^{k-1}, k, \chi_i; p_i^{\alpha_i} \right) \right|^4 \right] \\
&= \prod_{i=1}^{r} \left[(k, p_i-1)^2 p_i^{\alpha_i} \phi^2(p_i^{\alpha_i}) \right] \prod_{\substack{i=1 \\ (k,p_i-1)=1}}^{r} \frac{\phi(p_i-1)}{p_i-1}
\end{aligned}
$$

$$= q\phi^2(q)\prod_{p|q}(k,\, p-1)^2 \prod_{\substack{p|q\\(k,p-1)=1}} \frac{\phi(p-1)}{p-1}.$$

This completes the proof of Theorem 2.

ACKNOWLEDGEMENTS. The authors express their gratitude to the referee for his very helpful and detailed comments.

References

[1] T.M. Apostol: Introduction to analytic number theory, Springer-Verlag, New York, 1976.
[2] P. Chengdong and P. Chengbiao: Goldbach Conjecture, Science Press, Beijing, 1981.
[3] T. Cochrane and Z.Y. Zheng: *Pure and mixed exponential sums*, Acta Arithmetica **91** (1999), 249–278.
[4] L.K. Hua: Introduction to Number Theory, Science Press, Beijing, 1979, 175–176.
[5] A. Weil: *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 203–210.
[6] W.P. Zhang: *Moments of Generalized Quadratic Gauss Sums Weighted by L-Functions*, Journal of Number Theory **92** (2002), 304–314.
[7] W.P. Zhang and Y.P. Deng: *A hybrid mean value of the inversion of L-functions and general Quadratic Gauss sums*, Nagoya Math. Journal **167** (2002), 1–15.

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