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CAUCHY PROBLEM IN GEVREY CLASSES FOR SOME EVOLUTION EQUATIONS OF SCHRÖDINGER TYPE

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1. Introduction

In this paper the Cauchy problem in Gevrey classes is studied for some partial differential — or, more generally, pseudo-differential — equations of Schrödinger type, that is, for differential equations whose type of evolution is 2 and whose characteristic roots are real. Our aim is to determine some Gevrey index σ for which the well-posedness of the Cauchy problem holds in Gevrey classes of order σ. Such an index depends on the multiplicity of the characteristic roots and on the lower order terms. Our result was obtained in [2] in the special case of differential equations with constant leading coefficients.

2. Notation

Let us first introduce some notation about Gevrey spaces.

If $\sigma \geq 1$, then $\gamma^\sigma(\mathbb{R}^n)$ will denote the class of all the smooth functions f such that:

$$\sup_{\substack{x\in\mathbb{R}^n\\ \alpha\in\mathbb{N}^n}} |\partial_x^\alpha f(x)|.A^{-|\alpha|}\alpha!^{-\sigma} < +\infty$$

for some $A > 0$.

Now we define some Gevrey-Sobolev spaces (compare [4] and [5]). For $\varepsilon > 0$, $\sigma \geq 1$, $k > 0$, let $\mathcal{D}_{L^2}^{\sigma,\varepsilon,k}(\mathbb{R}^n)$ denote the space of all functions f such that $\|e^{\varepsilon\langle D_x\rangle^{1/\sigma}} f\|_k < +\infty$, where $\|.\|_k$ is the usual Sobolev norm in $H^k(\mathbb{R}^n)$. Note that, if $k' < k$ and $\varepsilon' > \varepsilon$, then $\mathcal{D}_{L^2}^{\sigma,\varepsilon,k}(\mathbb{R}^n) \subset \mathcal{D}_{L^2}^{\sigma,\varepsilon',k'}(\mathbb{R}^n)$. In this paper the space of the functions belonging to $\mathcal{D}_{L^2}^{\sigma,\varepsilon,0}(\mathbb{R}^n)$ for some ε, will be denoted by $\mathcal{D}_{L^2}^{\sigma}(\mathbb{R}^n)$. Let $\varepsilon(t)$ be a positive function of t, $t \in [-T, T]$. If $u(t,.) \in \mathcal{D}_{L^2}^{\sigma,\varepsilon(t),k}(\mathbb{R}^n)$, for every $t \in [-T, T]$, let us denote $\|e^{\varepsilon(t)\langle D_x\rangle^{1/\sigma}} u(t,x)\|_k$ by $|||u(t)|||_{\varepsilon(t),\sigma,k}$.

Let us now give some notation about pseudo-differential operators. We shall denote by S_σ^p the class of the pseudo-differential operators $s(x, D_x)$ whose symbol $s(x, \xi)$

satisfies the following condition:

$$\sup_{\alpha,\beta\in\mathbb{N}^n}\sup_{\substack{x,\xi\in\mathbb{R}^n\\|\xi|\geq B}}|\partial_\xi^\alpha D_x^\beta s(x,\xi)|.\langle\xi\rangle^{|\alpha|-p}A^{-|\alpha+\beta|}\alpha!^{-1}\beta!^{-\sigma}<\infty$$

for some $A>0,\ B\geq 0$.

Finally $s(x,\xi)$ is called a σ-regularizing symbol if:

$$\sup_{\beta\in\mathbb{N}^n}\sup_{\substack{x,\xi\in\mathbb{R}^n\\|\xi|\geq B}}|D_x^\beta s(x,\xi)|.\exp\big(h\langle\xi\rangle^{1/\sigma}\big)A^{-|\beta|}\beta!^{-\sigma}<\infty$$

for some $A,\ h>0,\ B\geq 0$. A σ-regularizing operator maps the dual space of $\mathcal{D}^\sigma_{L^2}(\mathbb{R}^n)$ to $\mathcal{D}^\sigma_{L^2}(\mathbb{R}^n)$.

3. The main result

Let us consider the following operator:

$$P=\pi_{2m}(t,x,D_t,D_x)+\sum_{j=1}^m a_j(t,x,D_x)D_t^{m-j} \tag{3.1}$$

where:

$$\pi_{2m}(t,x,D_t,D_x)=\prod_{j=1}^r\big(D_t-\lambda_j^1(t,x,D_x)\big)\cdots\big(D_t-\lambda_j^{s_j}(t,x,D_x)\big),$$

with $\sum_{j=1}^r s_j=m,\ s_r\geq s_{r-1}\geq\cdots\geq s_1$, and

$$\begin{aligned}&a_j(t,x,D_x)\in\mathcal{B}\big([-T_0,T_0];\ S_\sigma^{2j-q}(\mathbb{R}^n)\big)\qquad\text{for some } q,\ r<q\leq 2r,\\&\text{where } \sigma\in\left]1,\frac{2r}{2r-q}\right]\quad\text{if } q<2r \text{ and } \sigma\in]1,+\infty) \text{ if } q=2r.\end{aligned}\tag{3.2}$$

Moreover we assume that the $\lambda_j^i(t,x,\xi)'s$ are real-valued and satisfy the following properties:

$$\begin{aligned}&\text{(i)}\quad \lambda_j^i\in\mathcal{C}^{m-1}\big([-T_0,T_0];\ S_\sigma^2(\mathbb{R}^n)\big),\\&\text{(ii)}\quad \sum_{k=1,\ldots,n}\partial_{\xi_k}\partial_{x_k}\lambda_j^i\in S_\sigma^{1/\sigma}(\mathbb{R}^n),\ \nabla_x\lambda_j^i.\xi\in S_\sigma^2(\mathbb{R}^n)\\&\text{(iii)}\quad \text{if } i\neq h\ |\lambda_j^i(t,x,\xi)-\lambda_k^h(t,x,\xi)|\geq c_{jk}^{ih}|\xi|^2,\quad\text{for some } c_{jk}^{ih}>0\end{aligned}\tag{3.3}$$

REMARK. Assumptions of the type (3.3) (ii) are not unusual in the literature about Schrödinger equations: for example, compare (8) in [7].

Examples. 1) Assume that $P = \pi_{2m} + \sum_{h=1}^{2m} P_{2m-h}$ where $\pi_{2m}(t; D_t, D_x) = \prod_{i=1}^{k}(D_t - \lambda^i(t, D_x))^{r_i}$, with $\sum_{i=1}^{k} r_i = m$, $r = r_1 \geq \cdots \geq r_k$, λ^i are homogeneous of degree 2 in ξ and $\lambda^i(t,\xi) \neq \lambda^h(t,\xi)$ if $i \neq h$ and $\xi \neq 0$, and $P_{2m-h}(t, x, D_t, D_x) = \sum_{j=[(h+1)/2]}^{m} \sum_{|\alpha|=2j-h} a_{\alpha j}(t,x) D_x^\alpha D_t^{m-j}$, with $a_{\alpha j} \in \mathcal{B}([-T,T]; \gamma^\sigma(\mathbb{R}^n))$ for some $\sigma > 1$.

Then our result applies if we assume that P_{2m-h} vanishes for $h = 1, \ldots, 2r-1$.

2) Consider the operator in 1), but, more generally, assume that $\lambda^i(t, x, D_x)$ satisfy (3.3) (i) and are of the form $\lambda_0^i(t, D_x) + \mu^i(t, x, D_x)$, where λ_0^i is homogeneous of order 2 and μ^i is of order 1.

3) Let $P = \partial_t^2 + a_0(t,x)\partial_t + a_2(t,x,D_x) + a_3(t,x,D_x) + a_4(t,D_x)$ be a differential operators, where the subscripts denote the order of each operator a_i. We assume that $a_4(t,\xi) \geq \delta|\xi|^4 > 0$, $a_3(t,x,\xi)$ is real, that all the coefficients are in $\mathcal{C}([-T_0,T_0]; \gamma^\sigma(\mathbb{R}^n))$ for some $\sigma > 1$ (the coefficients of a_3 are in $\mathcal{C}^1([-T_0,T_0]; \gamma^\sigma(\mathbb{R}^n))$ and those of a_4 are in $\mathcal{C}^1([-T_0,T_0])$.

Then our theorem applies with any $\sigma > 1$, if we take $\lambda^1(t,x,\xi) = \sqrt{a_4(t,\xi) + a_3(t,x,\xi)}$, $\lambda^2(t,x,\xi) = -\sqrt{a_4(t,\xi) + a_3(t,x,\xi)}$. Note that if we had taken $\lambda^1(t,\xi) = \sqrt{a_4(t,\xi)}$, $\lambda^2(t,\xi) = -\sqrt{a_4(t,\xi)}$, then our theorem could not have been applied.

4) The pseudo-differential operators studied in [3] satisfy all our assumptions, if in the main Theorem in [3] we confine ourselves to the case $\sigma < 1/p$. Note that p in [3] is equal to $2 - q$, in the notation of this paper.

Theorem 3.1. *Let P be as in* (3.1), (3.2), (3.3). *If the initial data g_h are in $\mathcal{D}_{L^2}^\sigma(\mathbb{R}^n)$ and $f \in \mathcal{C}([-T_0,T_0]; \mathcal{D}_{L^2}^\sigma(\mathbb{R}^n))$, then there exists $T \in]0, T_0]$ such that the Cauchy problem*

$$\begin{cases} Pu(t) = f(t) \\ D_t^h u(0) = g_h \qquad h = 0, \ldots, m-1 \end{cases} \tag{3.4}$$

has a solution $u(t,.) \in \mathcal{D}_{L^2}^\sigma(\mathbb{R}^n)$, $\forall t \in [-T_0, T_0]$. More precisely, if M is an integer such that $M \leq m - q/2$, then there exists $\delta > 0$ such that $\partial_t^h u(t,.) \in \mathcal{D}_{L^2}^{\sigma,\delta(2T-t),2(m-h)}(\mathbb{R}^n)$ for every h, $h = 0, \ldots, M$, and the following energy inequality holds:

$$\begin{aligned} &\sum_{h=0}^{m} |||\partial_t^h u(t,.)|||_{\delta(2T-t),\sigma,2(m-h)-q} \\ &\leq C \left\{ \sum_{h=0}^{m-1} |||\partial_t^h u(0)|||_{2T\delta,\sigma,2-q/r} + \left| \int_0^t |||f(\tau)|||_{\delta(2T-\tau),\sigma,2-q/r}\, d\tau \right| \right\}. \end{aligned} \tag{3.5}$$

Proof. Let $\bar{s}_j = \sum_{h=1}^{j} s_h$. Denote λ_j^i by λ_i if $j = 1$ and by $\lambda_{\bar{s}_{j-1}+1}$ if $j > 1$. Let ∂_i denote $D_t - \lambda_i(t,x,D_x)$. If $J = (j_1, \ldots, j_k)$ set $\{J\} = \{j_1, \ldots, j_k\}$, $|J| = k$, $\partial_J = \partial_{j_1} \cdots \partial_{j_k}$.

Let $\mathcal{I}_h^{(1)} = \{J = (j_1, \ldots, j_h);\ j_1 < \cdots < j_h,\ \{J\} \subset \{1, \ldots, s_1\}\}$ and, for $k = 2, \ldots, r$, $\mathcal{I}_h^{(k)} = \{J = (j_1, \ldots, j_h);\ j_1 < \cdots < j_h,\ \{J\} \subset \{\overline{s}_{k-1}, \ldots, \overline{s}_k\}\}$. Thus π_{2m} can be written in the form $\partial_{J_1} \cdots \partial_{J_r}$, with $J_k \in \mathcal{I}_{s_k}^{(k)}$.

First of all, by using Proposition 4.1, we write P in the following form (modulo σ-regularizers):

$$\pi_{2m} + \sum_{J_1 \in \mathcal{I}^{(1)}_{s_1-1}, \ldots, J_r \in \mathcal{I}^{(r)}_{s_r-1}} \tilde{a}_{J_1,\ldots,J_r}(t, x, D_x)\partial_{J_1} \cdots \partial_{J_r} + \sum_{\substack{h_i=0,\ldots,s_i-1 \\ i=1,\ldots,r}} \sum_{J_i \in \mathcal{I}^{(i)}_{h_i}} \nu_{J_1,\ldots,J_r} \partial_{J_1} \cdots \partial_{J_r} \tag{3.6}$$

where $\tilde{a}_{J_1,\ldots,J_r} \in \mathcal{B}([-T, T];\ S^0_\sigma)$ and $\nu_{J_1,\ldots,J_r} \in \mathcal{B}([-T, T];\ S^{-N}_\sigma)$.

Now we reduce the Cauchy problem for P to a first-order system with diagonal principal part. Set $\rho = 2 - q/r$. Let us introduce the new unknown $\mathcal{U} = \{U_J\}_{|J| \le m-1}$, as follows:

$$\begin{cases} U_0 = \langle D_x \rangle^{\rho(r-1)} u \\ U_J = \langle D_x \rangle^{\rho(r-1)} \partial_J u & \text{if } |J| \le m - r \\ U_J = \langle D_x \rangle^{\rho(m-|J|-1)} \partial_J u & \text{if } m - r < |J| \le m - 1 \end{cases}.$$

Then we have a system of the form:

$$D_t\mathcal{U} - \mathcal{L}(t, x, D_x)\mathcal{U} - \mathcal{B}(t, x, D_x)\mathcal{U} = \mathcal{F}(t, x) \tag{3.7}$$

where $\mathcal{L}$ is a diagonal matrix of the form $\begin{pmatrix} \lambda_{j_1} & & \\ & \ddots & \\ & & \lambda_{j_{2m-1}} \end{pmatrix}$ with $\lambda_{j_i} \in \{\lambda_1, \ldots, \lambda_m\}$, the entries of $\mathcal{B}$ belong to $\mathcal{B}([-T_0, T_0]; S^\rho_\sigma)$, and the entries of $\mathcal{F}(t, .)$ belong to $\mathcal{D}^\sigma_{L^2}$.

The initial values of $\mathcal{U}$ are determined as follows:

$$\begin{aligned} U_0(t = 0) &= \langle D_x \rangle^{\rho(r-1)} g_0 = \psi_0 \\ U_j(t = 0) &= \langle D_x \rangle^{\rho\mu(j)} (-i)^{|J|} \sum_{\substack{k \le |J| \\ j_1,\ldots,k_k \in \{J\} \\ j_1 < \cdots < j_k}} i^k (\lambda_{j_1} \circ \cdots \circ \lambda_{j_k}) g_{|j|-k} = \psi_j, \end{aligned}$$

where $\mu(J) = r - 1$ if $|J| \le m - r$ and $\mu(J) = m - |J| - 1$ if $m - r < |J| \le m - 1$.

Then the initial conditions are:

$$\mathcal{U}(0, x) = \Psi(x) \tag{3.8}$$

where the entries of $\Psi = (\psi_j)_{|j| \le m-1}$ belong to $\mathcal{D}^\sigma_{L^2}$. For any $\delta > 0$, we can write:

$$\frac{d}{dt} |||\mathcal{U}(t)|||^2_{\delta(2T-t),\sigma,0} = 2\,\mathrm{Re}\ \left\langle \frac{d}{dt} \left(e^{\delta(2T-t)\langle D \rangle^{1/\sigma}} \mathcal{U}(t) \right), e^{\delta(2T-t)\langle D \rangle^{1/\sigma}} \mathcal{U}(t) \right\rangle$$

where $\langle\ ,\ \rangle$ denotes the inner product in the cartesian product $\times L^2$ and $T\ \in]0, T_0]$ will be chosen suitably in what follows.

First note that, in view of Lemma 4.3 and its proof, we have:

$$e^{\delta(2T-t)\langle D\rangle^{1/\sigma}}\mathcal{L}(t)e^{-\delta(2T-t)\langle D\rangle^{1/\sigma}} - e^{-\delta(2T-t)\langle D\rangle^{1/\sigma}}\mathcal{L}^*(t)e^{\delta(2T-t)\langle D\rangle^{1/\sigma}}$$
$$= \mathcal{L}(t) - \mathcal{L}^*(t) + R(t),$$

where $R(t) \in S^{1/\sigma}_\sigma(\mathbb{R}^n)$ and $|R(t)|_l = \sup_{|\alpha+\beta|\le l} |\partial^\alpha_\xi \partial^\beta_x R(t,x,\xi)|\langle\xi\rangle^{|\alpha|-1/\sigma}$ is a non-decreasing function of $\delta(2T-t)$, for any l.

Thus, if, say, $2\delta T < 1$, we have:

$$2\,\mathrm{Re}\left\langle e^{\delta(2T-t)\langle D\rangle^{1/\sigma}} i\mathcal{L}(t)\mathcal{U}(t), e^{\delta(2T-t)\langle D\rangle^{1/\sigma}}\mathcal{U}(t)\right\rangle \le \tilde{C}|||\mathcal{U}(t)|||^2_{\delta(2T-t),\sigma,1/(2\sigma)}$$

where $\tilde{C}$ is independent of δ.

Then we can write:

$$\begin{aligned}\frac{d}{dt}|||\mathcal{U}(t)|||^2_{\delta(2T-t),\sigma,0} \le\ & -2\delta|||\mathcal{U}(t)|||^2_{\delta(2T-t),\sigma,1/(2\sigma)} + \tilde{C}_0|||\mathcal{U}(t)|||^2_{\delta(2T-t),\sigma,1/(2\sigma)}\\ &+ C_0|||\mathcal{U}(t)|||^2_{\delta(2T-t),\sigma,\rho/2}\\ &+ 2|||\mathcal{F}(t)|||_{\delta(2T-t),\sigma,0}|||\mathcal{U}(t)|||_{\delta(2T-t),\sigma,0},\end{aligned}$$

where $\tilde{C}$ and C_0 depend only on $\mathcal{L}$ and $\mathcal{B}$, respectively. Now we fix $\delta \ge (\tilde{C} + C_0)/2$. Hence, in view of $\rho \le 1/\sigma$, we obtain:

$$\frac{d}{dt}|||\mathcal{U}(t)|||_{\delta(2T-t),\sigma,0} \le |||\mathcal{F}(t)|||_{\delta(2T-t),\sigma,0}$$

and finally:

$$|||\mathcal{U}(t)|||_{\delta(2T-t),\sigma,0} \le |||\mathcal{U}(0)|||_{2\delta T,\sigma,0} + \left|\int_0^t |||\mathcal{F}(\tau)|||_{\delta(2T-\tau),\sigma,0}\,d\tau\right| \tag{3.9}$$

$\forall t \in [-T, T]$. Now, applying Lemma 4.2, we obtain

$$\begin{aligned}&|||\partial^h_t u(t,.)|||_{\delta(2T-t),\sigma,2(m-h)-q}\\ &\le \sum_{|J|\le m-r} c'_J|||\partial_J u(t)|||_{\delta(2T-t),\sigma,2r-q} = \sum_{|J|\le m-r} c'_J|||U_J(t)|||_{\delta(2T-t),\sigma,\rho}\end{aligned} \tag{3.10}$$

If $g_h \in \mathcal{D}^{\sigma,\varepsilon_h,0}_{L^2}(\mathbb{R}^n)$ for some $\varepsilon_h > 0$, we choose T such that $2\delta T < \varepsilon_h$, $h = 0,\dots,m-1$. Then plugging (3.10) into (3.9), we get the energy inequality (3.5). □

4. Preliminary results

In this section the notation is the same as in §3. For brevity's sake, σ-regularizers are not mentioned explicitly in the identities involving pseudo-differential operators.

Lemma 4.1. *Assume that* λ_i *and* λ_j *satisfy* (3.3) (i) *and are distinct in the sense that there exists* $c_{ij} > 0$ *such that*

$$|\lambda_i(t,x,\xi) - \lambda_j(t,x,\xi)| \geq c_{ij}|\xi|^2. \tag{4.1}$$

Then, for any positive integer N, *we can write the identity in the following way*:

$$\text{Id.} = d_{ij}^{(N)}(t,x,D_x)\partial_j + d_{ji}^{(N)}(t,x,D_x)\partial_i + r^{(N)}(t,x,D_x), \tag{2.4}$$

where $d_{ij}^{(N)}$, $d_{ji}^{(N)} \in S_\sigma^{-2}$ *and* $r^{(N)} \in S_\sigma^{-N}$.

Proof. Let us denote $(\lambda_i(t,x,\xi) - \lambda_j(t,x,\xi))^{-1}$ by $d_{ij}(t,x,\xi)$. Then we can write the identity as follows:

Id. $= d_{ij}(t,x,D_x)\partial_j + d_{ji}(t,x,D_x)\partial_i + r^{(1)}(t,x,D_x)$, where $r^{(1)} \in S_\sigma^{-1}$. Finally the required identity follows by induction. □

Lemma 4.2. *Assume that the* λ_j*'s satisfy* (3.3) (i) *and are distinct in the sense of* (4.1) *for* $j \in \{1,\dots,s\}$. *For* $h = 1,\dots,s$, *let* $\mathcal{I}_h$ *be* $\{J = (j_1,\dots,j_h); j_1 < \cdots < j_h, \{J\} \subset \{1,\dots,s\}\}$. *Then for all* $k = 0,\dots,s-1$ *and for any positive integer* N *we can write*:

$$D_t^{s-1-k} = \sum_{J \in \mathcal{I}_{s-1}} c_J^{(k)}(t,x,D_x)\partial_J + \sum_{h=0}^{s-2} \sum_{J \in \mathcal{I}_h} r_J(t,x,D_x)\partial_J$$

for some $c_J^{(k)}$ *and* r_J *depending on* N *and belonging to* S_σ^{-2k} *and* S_σ^{-N} *respectively.*

Proof. To prove this Lemma we refer to the proof of Lemma 2.4 in [1]. The only change is that Lemma 2.1 in [1] is to be replaced throughout by the Lemma 4.1 above. Moreover, we just have to observe that, if $i \neq j$, then D_t can be written as $c_{ij}(t,x,D_x)\partial_i + c_{ji}(t,x,D_x)\partial_j + \tilde{r}_1$, where $c_{ij}(t,x,\xi) = \lambda_j(t,x,\xi)/(\lambda_j(t,x,\xi) - \lambda_i(t,x,\xi)) \in S_{\sigma,1}^0$ and $\tilde{r}_1 \in S_\sigma^{-1}$. □

Finally, arguing as in the proof of Proposition 2.1 in [1] and applying Lemma 4.1 and Lemma 4.2, we obtain the following:

Proposition 4.1. *If the operator* P *satisfies* (3.1), (3.2), (3.3), *then, for any positive integer* N, P *can be written in the following form* (*modulo* σ*-regularizers*):

$$\pi_{2m} + \sum_{J_1 \in \mathcal{I}_{s_1-1}^{(1)},\dots,J_r \in \mathcal{I}_{s_r-1}^{(r)}} \tilde{a}_{J_1,\dots,J_r}(t,x,D_x)\partial_{J_1}\cdots\partial_{J_r}$$
$$+ \sum_{\substack{h_i=0,\dots,s_i-1 \\ i=1,\dots,r}} \sum_{J_i \in \mathcal{I}_{h_i}^{(i)}} \nu_{J_1,\dots,J_r}\partial_{J_1}\cdots\partial_{J_r}$$

where $\tilde{a}_{J_1,\dots,J_r} \in \mathcal{B}([-T,T]; S^0_\sigma)$ *and* $\nu_{J_1,\dots,J_r} \in \mathcal{B}([-T,T]; S^{-N}_\sigma)$.

Lemma 4.3. *If* λ *satisfies* (3.3) (i), (ii), *then, for any* $\varepsilon > 0$, *the operator*

$$e^{\varepsilon\langle D\rangle^{1/\sigma}}\lambda(t)e^{-\varepsilon\langle D\rangle^{1/\sigma}} - e^{-\varepsilon\langle D\rangle^{1/\sigma}}\lambda^*(t)e^{\varepsilon\langle D\rangle^{1/\sigma}} \tag{4.2}$$

is $\lambda(t,x,D_x) - \lambda^*(t,x,D_x) + r(t,x,D_x)$, *where* $r(t) \in S^{1/\sigma}_\sigma(\mathbb{R}^n)$.

Proof. The symbol of (4.2) has an expansion $\sum_N s_N(t,x,\xi)$, where

$$s_N(t,x,\xi) = \sum_{|\gamma|=N} \frac{1}{\gamma!}\Big\{D^\gamma_x\lambda(t,x,\xi)\partial^\gamma_\eta\big(e^{\varepsilon\langle\xi+\eta\rangle^{1/\sigma}-\varepsilon\langle\xi\rangle^{1/\sigma}}\big) - D^\gamma_x\lambda^*(t,x,\xi). \\ \partial^\gamma_\eta\big(e^{-\varepsilon\langle\xi+\eta\rangle^{1/\sigma}+\varepsilon\langle\xi\rangle^{1/\sigma}}\big)\Big\}_{\eta=0}$$

Note that $s_0(t,x,\xi)$ is $\lambda(t,x,\xi) - \lambda^*(t,x,\xi)$, which is in $S^{1/\sigma}_\sigma(\mathbb{R}^n)$ because of the first assumption in (3.3) (ii). The symbol $s_1(t,x,\xi)$ is $-i(\varepsilon/\sigma)\langle\xi\rangle^{1/\sigma-2}\nabla_x(\lambda(t,x,\xi) + \lambda^*(t,x,\xi)).\xi$, which is in $S^{1/\sigma}_\sigma$ in view of (3.3) (ii). More generally, the terms multiplying $D^\gamma_x\lambda$ or $D^\gamma_x\lambda^*$ in $s_N(t,x,\xi)$ for $N \geq 2$, are of the form:

$$\varepsilon g^{(1)}_\gamma(\xi) + \varepsilon^2 g^{(2)}_\gamma(\xi) + \cdots + \varepsilon^N g^{(N)}_\gamma(\xi), \text{ where } g^{(h)}_\gamma \in S^{h/\sigma-N}.$$

When N is even, then $\varepsilon^N g^{(N)}_\gamma(\xi)$ actually multiplies $D^\gamma_x(\lambda - \lambda^*)$ which is in $S^{1/\sigma}_\sigma$; thus $s_N(t)$ is in $S^{1/\sigma+(N-2)(1/\sigma-1)}_\sigma$ which is a subset of $S^{1/\sigma}_\sigma$. If N is odd, then we can see that $g^{(N)}_\gamma(\xi)$ is of the form $\tilde{g}^{(N)}_{\gamma_k}(\xi)\xi_k$, where $|\gamma_k| = N-1$ and $\tilde{g}^{(N)}_{\gamma_k} \in S^{(N-1)/\sigma-N}$. Thus, writing $D^\gamma_x(\lambda + \lambda^*) = D^\gamma_x(\lambda^* - \lambda) + 2D^{\gamma_k}_x D_{x_k}\lambda$ and arguing as in the case $|\gamma| = 1$, we prove again that $s_N(t)$ is in $S^{1/\sigma+(N-2)(1/\sigma-1)}_\sigma$. □

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