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ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES III

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We have studied, in [3], a right artinian ring R satisfying Condition I (see below) as a generalization of right artinian serial rings. However, there, we have restricted ourselves to the case that $J^2=0$, where J is the Jacobson radical of R.

In this paper, instead of removing the restriction $J^2=0$, we shall add one more condition (Condition II′: *every hollow module is quasi-projective* or Condition II″: *R is an algebra of finite dimension over an algebraically closed field*). We shall give a characterization of a right artinian ring satisfying Condition I and Condition II′ (resp. II″), and show that such a ring is closely related to an algebra of right local type studied by H. Tachikawa [8] (see also [7]). Actually, if the assumption "*left serial*" is removed in [8], the situation is very similar to that in this paper.

Further, under Condition I, we shall consider Condition II: $|eJ/eJ^2|\leq 2$ *for each primitive idempotent e*, which is weaker than Conditions II′ and II″. We shall give the structure of a ring satisfying Conditions I and II, and show that the structure gives a characterization of such a ring provided $J^3=0$.

1 Conditions and Theorems. In this paper, we shall study a right artinian ring R with identity, and every R-module is assumed to be a unitary right R-module. We denote the Jacobson radical and the socle of an R-module M by $\mathrm{J}(M)$ and $\mathrm{Soc}(M)$, respectively. Occasionally, we write $J=\mathrm{J}(R)$. $|M|$ means the length of a composition series of M. If eR is a right uniserial module for each primitive idempotent e, R is called a *right serial* (*generalized uniserial*) *ring*. If R is a right serial ring then the following conditions are satisfied:

Condition I: *every submodule in any finite direct sum of hollow modules is also a direct sum of hollow modules* [3]

and

1) Conditions II and II-a are added in the revise.

Condition II′: *every hollow module is quasi-projective* [6].
In [3], we have studied rings R satisfying Condition I, under the extra hypothesis $J^2=0$, and further known that, among them, there exist some rings which fail to satisfy Condition II′ (e.g., algebras over an algebraically closed field).

Let R be an algebra of finite dimension over a field K such that $R/J=\sum\oplus K_{n_i}$, where K_{n_i} is the $n_i\times n_i$ full matrix ring over K (e.g., K is an algebraically closed field). We have thus the following condition:

Condition II″: *$eRe/eJe=\bar{e}K$ for each primitive idempotent e, where K is a subfield contained in the center of R.*

As is shown in Corollary 2 below, if R satisfies Conditions I and either II′ or II″ then $|eJ/eJ^2|\leq 2$ for each e. From the study of rings satisfying Condition I and $J^2=0$ (see [3]), it seems to the author that $|eJ/eJ^2|\leq 2$ holds without assuming $J^2=0$. (For the present, he has no counter-examples.) Several conditions under which $|eJ/eJ^2|\leq 2$ holds are given in [3]. On the other hand, since eJ/eJ^2 is semisimple, $eJ/eJ^2=\sum_i\oplus S_i^{(n_i)}$, where S_i are simple ($S_i\not\approx S_j$ provided $i\neq j$) and $S_i^{(n_i)}$ means the direct sum of n_i copies of S_i. If R satisfies Condition I and $J^2=0$, then $n_i\leq 2$ for all i if and only if $|eJ/eJ^2|\leq 2$ (see [3]). From this point of view, we consider the following conditions:

Condition II: *for each primitive idempotent e, $|eJ/eJ^2|\leq 2$,*
and

Condition II-a: *for each primitive idempotent e, $n_i\leq 2$ for all i.*

Lemma 1. *Let P be a two-sided ideal of R. If R satisfies any one of the conditions above, then so does R/P.*

Proof. Assume that R satisfies Condition I. Put $\bar{\bar{R}}=R/P$. Let D be a finite direct sum of hollow $\bar{\bar{R}}$-modules N_i, and M an $\bar{\bar{R}}$-submodule of D. Then N_i are hollow R-modules. Hence, from Condition I, $M=\sum_i\oplus M_i$ with hollow R-modules M_i. Since $MP=0$, M_i is also a hollow $\bar{\bar{R}}$-module. Hence Condition I holds for $\bar{\bar{R}}$. It is clear that $J(R/P)=(J+P)/P$. Let e be a primitive idempotent in R not contained in $J+P$. Then $\bar{\bar{e}}$ is a primitive idempotent in $\bar{\bar{R}}$. Since $\overline{\overline{eR}}$ is a homomorphic image of eR, the remainder is also clear.

Corollary 2. *Assume that R satisfies the following condition:*

Condition I*: *every submodule in any direct sum of three hollow modules is also a direct sum of hollow modules.*[2)]

Then Conditions II *and* II-a *are equivalent, and each of* Conditions II′ *and* II″ *implies* Condition II.

Proof. By Lemma 1, R/J^2 satisfies Condition I. Hence the corollary

2) We needed only Condition I* in the proof in [3].

is clear from [3], Lemmas 9 and 14.

As is easily seen, the conditions above except II′ are invariant for Morita equivalence. Let $R_0=e_0Re_0$ be the basic ring of R. If x is an element in the center of R, e_0x is in the center of R_0. Hence, in order to study the structure of rings which satisfy those conditions, we may assume that R is a basic ring.

Let M be a hollow module. Then $M\approx eR/A$ with a primitive idempotent e and a right ideal A in eR. Put $\Delta=eRe/eJe$ and $\Delta(A)=\{\bar{x}\in\Delta\,|\,x\in eRe,\ xA\subset A\}$, where $\bar{x}$ is the coset of x in Δ. It is clear that $\Delta(A)$ is a subdivision ring of Δ. We regard Δ as a right $\Delta(A)$-module (see [3] and [4]), so $[\Delta\colon\Delta(A)]$ means the dimension of Δ over $\Delta(A)$ as a right $\Delta(A)$-module.

Let $M_1\supset N_1$ and $M_2\supset N_2$ be R-modules. A submodule $N_1\oplus N_2$ in $M_1\oplus M_2$ is called a *trivial submodule* of $M_1\oplus M_2$. For $N_3\subset N_1$, N_1/N_3 ($\subset M_1/N_3$) is called a *sub-factor module* of M_1.

We shall give the following theorems.

Theorem 1. *Let R be a right artinian ring. If R satisfies* Conditions I* *and* II, *then for each primitive idempotent e in R we have the following properties*:

1) *$eJ=A_1\oplus B_1$, where A_1 and B_1 are uniserial modules. Further, if $A_1/J(A_1)\approx B_1/J(B_1)$, $\alpha A_1=B_1$ for some unit α in eRe.*

2) *For every submodule N in eJ, there exists a trivial submodule $A_i\oplus B_j$ of eJ and a unit β in eRe such that $N=\beta(A_i\oplus B_j)$, where $A_i=A_1J^{i-1}\subset A_1$ and $B_j=B_1J^{j-1}\subset B_1$.*

3) *If $A_1\approx B_1$, then $\Delta(A_i\oplus B_i)=\Delta$ and $[\Delta\colon\Delta(A_i\oplus B_j)]=2$ provided $i\neq j$; further $\Delta(A_1)=\Delta(A_i)=\Delta(A_i\oplus B_j)$ $(i<j)$ and $\Delta(B_1)=\Delta(B_j)=\Delta(A_i\oplus B_j)$ $(i>j)$. If $A_1\not\approx B_1$, then $\Delta(N)=\Delta$ for any submodule N in eJ.*

Theorem 2. *Let R be a right artinian ring. Then the following are equivalent*:

1) *R satisfies* Conditions I *and* II′.

2) *R satisfies* Conditions I* *and* II′.

3) *For each primitive idempotent e, eJ is a direct sum of two uniserial modules A_1 and B_1, and no sub-factor module of A_1 is isomorphic to any sub-factor module of B_1, and hence every submodule in eJ is trivial.*

Theorem 2′. *Let R be a right artinian ring. If R satisfies* Condition II″, *then the following are equivalent*:

1) *R satisfies* Condition I.

2) *R satisfies* Condition I*.

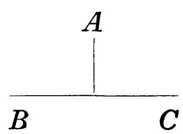
3) *For each primitive idempotent e, eJ is a direct sum of two uniserial modules A_1 and B_1 and every submodule in eJ is isomorphic to a trivial submodule via the left-sided multiplication of a unit element in eRe.*

2 Proof of Theorem 1. We always assume that R is a right artinian ring with identity, and J is the Jacobson radical of R, unless otherwise stated. Further, we may assume that R is basic in the proof. In advance of giving the proof, we state the following proposition.

Proposition 3. *If $J^2=0$, then every submodule of a direct sum of two hollow modules is also a direct sum of hollow modules.*

Proof. As is shown in [3], §3, it suffices to consider a direct sum of hollow modules eR/A for a fixed primitive idempotent e and show that every maximal submodule M of $D=eR/A_1\oplus eR/A_2$ is a direct sum of hollow modules. Let π_i be the projection of D onto eR/A_i. If $\pi_1(M)\subset eJ/A_1$ then $M=eJ/A_1\oplus eR/A_2$. Since eJ/A_1 is semisimple by the assumption $J^2=0$, M is a direct sum of hollow modules. Assume that π_i is an epimorphism for $i=1, 2$. Put $\bar{D}=D/\mathrm{J}(D)$ and $\bar{M}=M/\mathrm{J}(D)$. Then $\bar{M}$ has a basis of the form $(\bar{e}+\overline{er})$ over $\Delta=eRe/eJe$, where $\bar{e}$ is the coset of e in eR/eJ (note that R is assumed to be basic). We have the natural mapping φ of eR to D by setting $\varphi(e)=(e+A_1)+(er+A_2)$. Then $D\supset\varphi(eR)\approx eR/C$, where $C=\ker\varphi$. Since $\varphi(eR)=\bar{M}$, $M=\varphi(eR)+\mathrm{J}(D)$. Noting that $\mathrm{J}(D)$ is semisimple, we obtain that $M=\varphi(eR)+(\sum_i\oplus M_i)$, where M_i are simple. Hence M is a direct sum of hollow modules.

From Proposition 3, we see that Condition I has a meaning for direct sums of at least three hollow modules. In what follows, we shall use a diagram



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which means that A, B, and C are hollow modules and $\mathrm{J}(A)=B\oplus C$.

Proof of Theorem 1. We always assume that R satisfies Conditions I* and II, and that R is a basic ring, unless otherwise stated.

Lemma 3. *Assume that R satisfies* Condition I*. *Let E_1 and E_2 be submodules in eJ with $JE_2=0$. Put $D=eR/E_1\oplus eR/E_2$. For each unit α in eRe, D contains a maximal submodule with a direct summand isomorphic to $eR/(E_1\cap\alpha E_2)$ via the mapping: $x+(E_1\cap\alpha E_2)\rightarrow(x+E_1)+(\alpha^{-1}x+E_2)$.*

Proof. Let $\Delta=eRe/eJe$. Then $\bar{D}=D/\mathrm{J}(D)$ is a right Δ-module, because R is basic. Now, let M be the maximal submodule of D such that $\bar{M}=M/\mathrm{J}(D)$ $=(\overline{(e+E_1)}+\overline{(\alpha^{-1}x+E_2)})\Delta\subset\bar{D}$. By assumption, M contains a hollow submodule M_1 with $M_1/\mathrm{J}(M_1)\approx(M_1+\mathrm{J}(D))/\mathrm{J}(D)=\bar{M}$. Let m_1 be a generator of M_1 such that $\bar{m}_1=\overline{(e+\alpha^{-1})}$ in $\bar{D}$. We denote by π_i the projection of D onto eR/E_i. Then we obtain a homomorphism f_i: $eR\rightarrow M_1\rightarrow eR/E_i$ by setting $f_i(e)=$

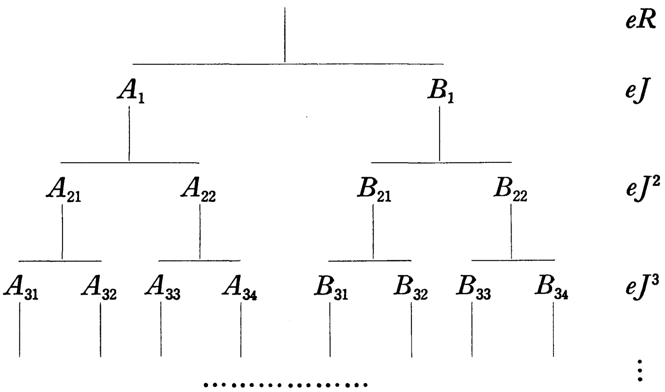
$\pi_i(m_1)$, which is given by the left multiplication of a unit β_i in *eRe*. Hence $M_1\approx eR/(\beta_1^{-1}E_1\cap\beta_2^{-1}E_2)$ and $m_1=\pi_1(m_1)+\pi_2(m_1)=f_1(e)+f_2(e)=(\beta_1+E_1)+(\beta_2+E_2)$. Since $\bar{m}_1=\overline{(e+E_1)}+\overline{(\alpha^{-1}+E_2)}$, $\bar{\beta}_1=\bar{e}$ and $\bar{\beta}_2=\alpha^{-1}$ in Δ. Now $eR/(\beta_1^{-1}E_1\cap\beta_2^{-1}E_2)\approx eR/(E_1\cap\beta_1\beta_2^{-1}E_2)$. On the other hand, $\beta_1\beta_2^{-1}=\alpha+j$ for some j in *eJe*. Hence $\beta_1\beta_2^{-1}E_2=\alpha E_2$ by assumption. The mapping is clear.

REMARK 4. If we drop the assumption $JE_2=0$ in Lemma 3, we obtain $eR/(E_1\cap(\alpha+j)E_2)$ instead of $eR/(E_1\cap\alpha E_2)$, from the proof.

Lemma 5. *Let A be a right ideal in eR with $JA=0$. If $\alpha A=A$ for every unit α in eRe, then A is a two-sided ideal, provided R is a basic ring.*

Proof. Since $eJeA=0$, A is a characteristic submodule of *eR*. Since R is basic, $RA=\sum_i e_iReA=\sum_{e_i\neq e} e_iJeA+eReA=A$, where $1=\sum_i e_i$ and e_i are orthogonal primitive idempotents.

Lemma 6. *eR has the structure given in the following diagram:*



(1)

where A_{ij} and B_{ij} are hollow modules and $eJ^i=\sum_j\oplus A_{ij}\oplus\sum_{j'}\oplus B_{ij'}$.

Proof. Let A be a hollow module contained in *eR* and $A\approx fR/B$, where f is a primitive idempotent in R. Then $\mathrm{J}(A)=AJ=\sum_{i=1}^{t}\oplus A_i$ with hollow modules A_i, and $AJ^2=\sum_i\oplus A_iJ$. On the other hand, $AJ/AJ^2\approx fJ/(fJ^2+B)$. Since $fJ/(fJ^2+B)$ is a homomorphic image of fJ/fJ^2, $|AJ/AJ^2|\leq 2$ by Condition II. Hence $t\leq 2$, which proves the lemma.

Lemma 7. *Let $eJ=A_1\oplus B_1$ be as in Lemma 6. If $A_1/A_1J\approx B_1/B_1J$ then $|A_1|=|B_1|$. If $A_1/A_1J\not\approx B_1/B_1J$, $\pi_2\alpha(A_1)\subsetneqq B_1$ for any unit α in eRe, where π_2 is the projection of eJ onto B_1.*

Proof. Put $\bar{\bar{R}}=R/J^2$, $\bar{\bar{A}}_1=(A_1+eJ^2)/eJ^2$ and $\bar{\bar{B}}_1=(B_1+eJ^2)/eJ^2\subset\overline{\overline{eJ}}$. Since $\bar{\bar{A}}_1\approx A_1/A_1J\approx B_1/B_1J\approx\bar{\bar{B}}_1$, by [3], Theorem 12 there exists a unit $\bar{\bar{\beta}}$ in eRe/eJ^2e (and hence a unit β in *eRe*) such that $\overline{\overline{\beta A_1}}=\bar{\bar{B}}_1$, i.e., $\beta(A_1+eJ^2)=B_1+eJ^2$. As

B_1 is hollow, there holds $\pi_2\beta(A_1)=B_1$, so $|A_1|\geq|B_1|$. By symmetry, $|A_1|\leq|B_1|$, and hence $|A_1|=|B_1|$. Since A_1 and B_1 are hollow, there exist no epimorphisms of A_1 onto B_1, provided $A_1/A_1J\not\approx B_1/B_1J$. Therefore $\pi_2\alpha(A_1)\neq B_1$ if $A_1/A_1J\not\approx B_1/B_1J$.

Now let N be a submodule of $eJ=A_1\oplus B_1$, and π_1 (resp. π_2) the projection of eJ onto A_1 (resp. B_1). Put $N_1=N\cap A_1$, $N_2=N\cap B_1$, $N^1=\pi_1(N)$ and $N^2=\pi_2(N)$. Then, as is well known, $N^1/N_1\approx N^2/N_2$. Further, if $N_2=0$ then $N=N^1(f)=\{x+f(x)\,|\,x\in N^1, f(x)=\pi_2\,\pi_1^{-1}(x)\}$.

First we shall study R with $J^3=0$ and satisfying Conditions I* and II. Then, by assumption, $eJ=A_1\oplus B_1$ where A_1 and B_1 are hollow. Since A_1 is a hollow R/J^2-module, $\mathrm{J}(A_1)=C_1\oplus C_2$ by Lemma 1, where C_i are simple or zero. Similarly $\mathrm{J}(B_1)=D_1\oplus D_2$ with D_i simple or zero.

Lemma 8. *Assume that* $C_i\neq 0$ *and* $D_i\neq 0$ $(i=1, 2)$, *and that* C_1 *is isomorphic to* D_1 *via* f. *Put* $C_1'=C_1(f)=\{c_1+f(c_1)\,|\,c_1\in C_1\}\subset C_1\oplus D_1$. *Then there holds the following*:

1) $\mathrm{Soc}(eR/C_1')=(C_1+C_1')/C_1'\oplus(C_2+C_1')/C_1'\oplus(D_2\oplus C_1')/C_1'$ *and* $(C_i+C_1')/C_1'\approx C_i$, $(D_2+C_1')/C_1'\approx D_2$.

2) *If* N/C_1' *is uniform in* eR/C_1' *for a submodule* N *in* eR *then* $|N|\leq 3$.

Proof. 1) Let N^* be a submodule of eR such that $N^*\supset C_1'$ and $|N^*/C_1'|=1$. Since $N^*\subset eJ$ and $|N^*|=2$, $|N^{*1}|\leq|N^*|=2$. Hence $N^{*1}\subset\mathrm{J}(A_1)\subset\mathrm{Soc}(eR)$. Similarly, $N^{*2}\subset\mathrm{Soc}(eR)$, and therefore $N^*\subset\mathrm{Soc}(eR)$.

2) It is clear that $N\subset eJ$. If $N_1=0$ then $|N|\leq 3$. We assume henceforth $N_1\neq 0$. If $N_1\supset C_1$ then $N\supset C_1\oplus D_1$. Since $\mathrm{Soc}(eJ/(C_1\oplus D_1))\approx C_2\oplus D_2$, N contains a non-zero element x in $C_2\oplus D_2$, provided $N\neq C_1\oplus D_1$. Then $N/C_1'\supset(C_1+xR+C_1')/C_1'\approx C_1\oplus xR$, so N/C_1' is not uniform, and hence $N=C_1\oplus D_1$, so that $|N|\leq 2$. On the other hand, if $N_1\not\supset C_1$ then N contains an element $z=x+y\in N_1$; $x\in C_1$, $y\neq 0\in C_2$. Hence $N/C_1'\supset(C_1+zR+C_1')/C_1'\approx C_1\oplus zR$, a contradiction.

Lemma 9. *If* $J^3=0$, *then both* A_1 *and* B_1 *in Theorem* 1 *are uniserial.*

Proof. 1) Assume that eJ is hollow. Since eJ is an R/J^2-module, $eJ^2=C_1\oplus C_2$ by Lemma 1, where C_i are simple. Assume $C_i\neq 0$ for $i=1, 2$, and put $D=eR/C_1\oplus eR/C_2$. Since $JC_2=0$, D contains a maximal submodule M with a direct summand M_1 isomorphic to eR, by Lemma 3 (take $\alpha=e$). Then $|\mathrm{Soc}(D)|=|\mathrm{Soc}(eR)|=2$, and therefore $M=M_1$. On the other hand, $|D|=6$ and $|M|=|eR|=4$, which is a contradiction. Hence, if eJ is hollow then eR is uniserial.

2) Assume that $eJ=A_1\oplus B_1$ and $A_1\neq 0$, $B_1\neq 0$. Let $\mathrm{J}(A_1)=C_1\oplus C_2$ and $\mathrm{J}(B_1)=D_1\oplus D_2$ as before (see Lemma 8). We shall show that $C_2=D_2=0$.

i) Assume that $C_2=0$, $D_1\neq 0$ and $D_2\neq 0$. Then $C_1\neq 0$ or A_1 is simple by assumption. First assume that $C_1 \neq 0$. Since $|A_1| < |B_1|$, $A_1/A_1J \not\approx B_1/B_1J$ by Lemma 7. Let a_1 be a generator of A_1, and α a unit in eRe. Then, by Lemma 7, $\alpha a_1 = a_1' + b_2$, where $a_1' \in A_1 - A_1J$ and $b_2 \in B_1J$. Hence $\alpha C_1 = \alpha a_1 J \subset a_1'J + b_2J \subset A_1J = C_1$. Therefore C_1 is a two-sided ideal of R, by Lemma 5. Considering R/C_1, in view of Lemma 1, we may assume that A_1 is simple in either case. Put $D = eR/D_1 \oplus eR/D_2$. Then, by taking $\alpha = e$ in Lemma 3, we see that D contains a maximal submodule M with a direct summand M_1 isomorphic to $eR/(D_1 \cap D_2) = eR$. Now, $|D| = 8$, $|M_1| = 5$, $|\mathrm{Soc}(D)| = 4$ and $|\mathrm{Soc}(M_1)| = 3$. Hence $M = M_1 \oplus M_2$ with M_2 uniform. Since the uniform module M_2 is isomorphic to a submodule of eJ/D_i ($i = 1, 2$), we get $|M_2| \leq |eJ/D_i|$. Therefore $M_2 \subset eJ/D_1 \oplus eJ/D_2 = (A_1 \oplus B_1)/D_1 \oplus (A_1 \oplus B_1)/D_2$. On the other hand, $\mathrm{Soc}(M_1) = (e, e+j)\mathrm{Soc}(eR) \subset (D_2 \oplus D_1)/D_1 \oplus (D_1 \oplus D_2)/D_2$ for some $j \in eJe$, where $(e, e+j)\colon eR \to D$ is the mapping given in Lemma 3. Hence M_2 is monomorphic to $(A_1 \oplus D_1)/D_1 \oplus (A_2 \oplus D_2)/D_2 \approx A_1 \oplus A_1$, and so to A_1, for M_2 is uniform. But, $|M_2| = |M| - |M_1| = |D| - 1 - |M_1| = 2$, which is a contradiction. Hence, if $C_2 = 0$ then $D_1 = 0$ or $D_2 = 0$.

ii) Assume $C_i \neq 0$ and $D_i \neq 0$ for $i = 1, 2$.

α) Assume that there exists a unit α in eRe such that $(C_1 \oplus D_1) \cap \alpha(C_1 \oplus D_1) = 0$. Put $D = eR/(C_1 \oplus D_1) \oplus eR/(C_1 \oplus D_1)$. Then, by Lemma 3, D contains a maximal submodule M with a direct summand M_1 isomorphic to eR. Since $|\mathrm{Soc}(D)| = 4$ and $|\mathrm{Soc}(M_1)| = 4$, we have $M = M_1$. But, $|D| = 10 > 7 = |M|$, which is a contradiction.

β) Assume that $(C_1 \oplus D_1) \cap \alpha(C_1 \oplus D_1)$ is simple. Then this module is of the form C_1' (or D_1'), and $M_1 \approx eR/C_1'$. Since $|\mathrm{Soc}(M_1)| = 3$ by Lemma 8, $M = M_1 \oplus M_2$ with M_2 uniform. Hence $|M_2| \leq 2$ by Lemma 8, and so $|M| \leq 8$, which is a contradiction.

Thus, we have shown that $(C_1 \oplus D_1) = \alpha(C_1 \oplus D_1)$ for every unit α in eRe. Then $C_1 \oplus D_1$ is a characteristic submodule by Lemma 5, since $J(C_1 \oplus D_1) = 0$. By making use of arguments similar to those employed in α) and β), we may assume that $C_1 \oplus D_2$ is also characteristic. Hence $C_1 = (C_1 \oplus D_1) \cap (C_1 \oplus D_2)$ is characteristic, so that C_1 is a two-sided ideal of R. Consider the factor ring R/C_1. Then $e(R/C_1)$ is of the form considered in i), which is a contradiction.

Summarizing all above, we see that $C_i = 0$ and $D_j = 0$ for some $i, j \in \{1, 2\}$. We have thus shown the lemma.

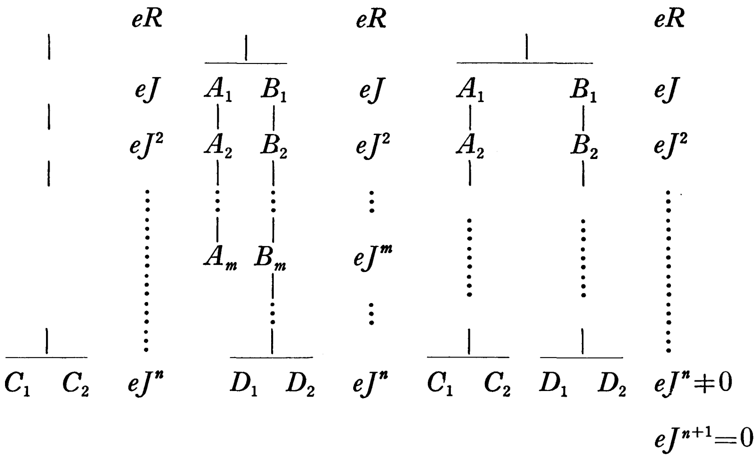
Lemma 10. *If $J^3 = 0$, then every hollow module is isomorphic to one of the following*: 1) *uniform,* 2) eR, 3) eR/A_2 *and* eR/B_2; $A_2 \subsetneqq A_1$ *and* $B_2 \subsetneqq B_1$, *and* 4) $eR/(A_2 \oplus B_2)$ (see the diagram (2) below).

Proof. This is immediate from Lemma 9 and the proof of Lemma 8. (Note that for any $f\colon A_1 \to B_1$, $eJ = A_1 \oplus B_1 = A_1(f) \oplus B_1$.)

Lemma 11. *Let R be a right artinian ring satisfying* Conditions I* *and*

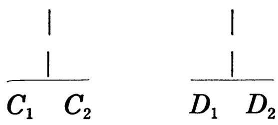
II. *Then $eJ=A_1\oplus B_1$ and both A_1 and B_1 are uniserial.*

Proof. Let $J^{n+1}=0$. If $J^3=0$, then the lemma is true by Lemma 9. We proceed by induction on n (≥ 2). By Lemma 6 and the induction hypothesis, we have the following cases:



(2)

where $A_1\supset A_2\supset\cdots\supset A_m$(resp. $B_1\supset B_2\supset\cdots\supset B_n$) is the unique composition series between A_1 and A_m (resp. B_1 and B_n) and $m\leq n$. Since $n\geq 3$, we can find a hollow R/J^3-module as follows:



Hence $C_1=0$ or $C_2=0$ (resp. $D_1=0$ or $D_2=0$) by Lemma 9. Thus we have shown the lemma.

Next we shall show the second part of 1) in Theorem 1. We always assume that $m\leq n$.

Lemma 12. *Assume that $A_1/A_2\approx B_1/B_2$, where $A_2=\mathrm{J}(A_1)$ and $B_2=\mathrm{J}(B_1)$. Then there exists a unit α in eRe such that $\alpha A_1=B_1$.*

Proof. From the proof of Lemma 7, we obtain a unit α in eRe such that $\alpha(A_1+eJ^2)=(B_1+eJ^2)$. Let π_1 be the projection of eJ onto A_1, and put $f=\pi_1\alpha\,|\,A_1$. Then $K=\ker f\neq 0$ by $\alpha(A_1+eJ^2)=(B_1+eJ^2)$, and $\alpha K\subset B_1$. Accordingly $\alpha A_1\cap B_1\neq 0$. Let j be an arbitrary element of eJe. Since $eJ(A_2\oplus B_2)\subset eJ^2$ and $\alpha+j$ is a unit, $(\alpha+j)$ $(A_1\oplus B_2)=(A_2\oplus B_1)$. Hence, replacing α by $\alpha+j$ in the above, we get $(\alpha+j)A_1\cap B_1\neq 0$. Put $D=eR/A_1\oplus eR/B_1$. Then D contains a maximal submodule M with a direct summand M_1 isomorphic to $eR/((\alpha+j_0)A_1\cap$

B_1) for some j_0 in eJe, by Remark 4. If $(\alpha+j_0)A_1\cap B_1\neq B_1$ then $|\mathrm{Soc}(M_1)|=2$, so $M=M_1$. However, $|D|=2(|A_1|+1)$ by Lemma 7 and $|M|\lneqq 2|A_1|+1$ by $(\alpha+j_0)A_1\cap B_1\neq 0$. This is a contradiction. Hence $(\alpha+j_0)A_1\cap B_1=B_1$, and therefore $(\alpha+j_0)A_1=B_1$.

Lemma 13. *Let A_i, B_j be as in the diagram* (2). *Let f be an element in* $\mathrm{Hom}_R(A_i, B_j)$. *If f is not extendible to any element in* $\mathrm{Hom}_R(A_{i-1}, B_{j-1})$ *then $eR/A_i(f)$ is uniform. In particular, $eR/A_1(f)$ is uniform.*

Proof. Since $eJ=A_1\oplus B_1=A_1(f)\oplus B_1$, $eR/A_1(f)\supset eJ/A_1(f)=B_1$, and so $eR/A_1(f)$ is uniform. Assume $i<1$, and put $\ker f=A_k$. Then $k<i$. Let N be a submodule of eJ such that $N\supset A_i(f)$ and $|N/A_i(f)|=1$. If $N\supset A_{k-1}$ then $N\supset A_{k-1}+A_i(f)\supsetneqq A_i(f)$, and hence $N=A_{k-1}+A_i(f)$. On the other hand, if $N\not\supset A_{k-1}$ then $N_1=A_k$, and $N_2=0$ by $N\supset A_i(f)$. Accordingly, $N=A_{i-1}(g)$ by the remark stated just before Lemma 8, where g: $A_{i-1}\to B_{j-1}$. Then g is an extension of f, which is a contradiction. Hence $\mathrm{Soc}(eR/A_i(f))=(A_{k-1}+A_i(f))/A_i(f)$.

Lemma 14. *Let A_i be as in Lemma* 13 ($i\geq 2$). *Let N be a submodule of eR containing A_{i-1}. If N/A_i is uniform then $|N/A_i|\leq i-1$.*

Proof. Since N/A_i is uniform, N is contained in eJ. Now, considering the projection of eJ/A_i onto A_1/A_i, we can easily see the lemma.

Lemma 15. *Let f be an arbitrary element of* $\mathrm{Hom}_R(A_1, B_1)$. *Then there exists a unit α in eRe such that $A_1(f)=\alpha A_1$.*

Proof. If f is an isomorphism then $eJ=A_1\oplus A_1(f)$ and $A_1\approx A_1(f)$, so $A_1(f)=\alpha A_1$ for some α by Lemma 12. Next, assume that f is not an isomorphism. If $A_1\approx B_1$ then $eJ=A_1(f)\oplus B_1$ and $A_1(f)\approx A_1\approx B_1$. Hence there exists a unit β in eRe such that $A_1(f)=\beta B_1$ by Lemma 12, and so $A_1(f)=\beta\alpha A_1$ with some α. Assume $A_1\not\approx B_1$, and put $D=eR/A_1\oplus eR/A_1(f)$ ($f\neq 0$). Let M be such a maximal submodule of D as in the proof of Lemma 3. Then M contains a direct summand isomorphic to $eR/(A_1\cap\alpha A_1(f))$, where α is a unit in eRe (see Remark 4). Now, assume that $K=A_1\cap\alpha A_1(f)\subsetneqq A_1$. Then $|\mathrm{Soc}(eR/K)|=2$. On the other hand, $|\mathrm{Soc}(D)|=2$ by Lemma 13. Hence $M\approx eR/K$. Since f is not an isomorphism, $A_1(f)\supset A_m$ and $\pi_2\alpha A_1\neq B_1$ by Lemma 7. If $n=m$ then $\pi_2\alpha|A_1$ is not a monomorphism, and $\pi_2\alpha A_m=0$. Hence $\alpha A_m=A_m$, so that $K\supset A_m$. But, then, $|M|\leq n+n-1+1=2n$ and $|D|=2n+2$, which is a contradiction. Also, if $n<m$, $|M|\leq m+n+1\leq 2n<2n+1=|D|-1$, a contradiction. We have thus seen that $\alpha A_1(f)=A_1$.

Corollary 16. *If $A_1\not\approx B_1$ then B_n is a two-sided ideal provided R is a basic ring.*

Proof. We have known, from the proof of Lemma 15, that $\alpha B_n = B_n$ for any unit α in eRe. Hence B_n is a two-sided ideal of R, by Lemma 5.

Lemma 17. *Let A_i be as in Lemma 13. Given f: $A_i \to B_j$, there exists a unit α in eRe such that $A_i(f) = \alpha A_i$.*

Proof. We proceed by induction on i. If $i=1$, we are done by Lemma 15. If f is extendible to $g \in \mathrm{Hom}_R(A_{i-1}, B_1)$ then, by induction hypothesis, $A_{i-1}(g) = \beta A_{i-1}$ with some unit β in eRe. Since A_{i-1} is uniserial, we get $A_i(g) = A_i(f) = \beta A_i$. Henceforth, we assume that f is not extendible. Then $eR/A_i(f)$ is uniform by Lemma 13. Put $D = eR/A_i \oplus eR/A_i(f)$, and take such a maximal submodule M as in the proof of Lemma 3. Then M contains a direct summand M_1 isomorphic to eR/K, where $K = A_i \cap \alpha A_i(f)$ and α is a unit in eRe such that $\bar{\alpha} = \bar{e}$. Since $|\mathrm{Soc}(M_1)| = 2$ and $|\mathrm{Soc}(D)| = 3 < |\mathrm{Soc}(M)|$ by Lemma 13, $M = M_1 \oplus M_2$ and M_2 is uniform. Assume now that $A_i \subsetneqq K$. Then $\mathrm{Soc}(M_1) \approx A_{i'}/K \oplus B_n$ for some $i' \geq i$. Considering the mapping in Lemma 3, we see that M_2 is monomorphic to eR/A_i. Hence $|M_2| \leq i-1$ by Lemma 14. Accordingly, $|M| \leq |eR| + i - 1 = n + m + i$ and $|D| = 2n + 2i$. But, as $i \geq 2$ and $n \geq m$, we have a contradiction: $|M| + 1 < |D|$. Hence $K = A_i$, so that $A_i(f) = \alpha A_i$.

Lemma 18. *Let B_j be as in Lemma 13, and let g be in* $\mathrm{Hom}_R(B_j, A_1)$. *Then $B_j(g) = \beta B_j$, provided g is not a monomorphism, and $B_j(g) = A_j(g^{-1})$, so $B_j(g) = \beta A_j$, provided g is a monomorphism, where β is a unit in eRe and $A_j = g(B_j) \subset A_1$.*

Proof. In case $n=m$, we are done by Lemma 17. We assume henceforth $m<n$. Since the second assertion is clear from Lemma 17, we may further assume that g is not a monomorphism. Then ker $g \supset B_n$ and g induces $\bar{\bar{g}}$: $B_j/B_n \to A_1$. By Corollary 16 and induction on the nilpotency index of J, we can see that there exists a unit $\bar{\bar{\beta}}$ in $eRe/eJ^n e$ such that $(B_j/B_n)\ (\bar{\bar{g}}) = \bar{\bar{\beta}}(B_j/B_n)$. This together with $B_i(g) \cap \beta B_j \supset B_n$ gives $B_j(g) = \beta B_j$.

Thus we have completed the proof of Theorem 1 2), by the induction on n. Next, we shall show Theorem 1 3). In view of Theorem 1 2), we may assume that N is a trivial submodule $A_i \oplus B_j$ of eJ.

Lemma 19. *If N contains eJ^t for some t, $\Delta(N) = (\Delta(N/eJ^t)$ in $R/J^t)$.*

Proof. This is clear.

Here, we quote the following condition in [3]:

(**) *every maximal submodule of any finite direct sum D of hollow modules contains a non-zero direct summand of D.*

Lemma 20. $[\Delta : \Delta(A_1)] \leq 2$.

Proof. In view of [4], Theorem 2, it suffices to show that (**) is satisfied for $D=eR/A_1\oplus eR/A_1\oplus eR/A_1$. Let M be a maximal submodule in D. Then, by Condition I*, there exists a direct summand M_1 of M with $M_1\not\subset J(D)$, where $M_1\approx eR/K$. Let ρ be the natural epimorphism of eR to M_1, and π_i the projection of D onto the i-th component. Then $\pi_i\rho$ is given by the left multiplication of an element α_i in eRe. Since $M_1\not\subset J(D)$, we may assume that α_1 is a unit. Further, $\alpha_1 K$ being contained in A_1, we may assume that $K\subset A_1$.

i) If $K=A_1$ then $|M_1|=|eR/A_1|$. Hence $\pi_1\rho$ is an isomorphism, so M_1 is a direct summand of D.

ii) If $K\subsetneq A_1$ then $|M_1|=n+m-k$ and $|D|=3(n+1)$, where $k=|K|$. On the other hand, $|\mathrm{Soc}(D)|=3$ and $|\mathrm{Soc}(M_1)|=2$. Hence $M=M_1\oplus M_2$ with a uniform M_2. Since M_2 is monomorphic to eR/A_1, we have $n+1\geq|M_2|$ $=3n+2-(n+m+1-k)=(2n-m)+1+k$, which implies $n=m$ and $k=0$. Then $|M_2|=n+1$ and M_2 is isomorphic to eR/A_1 via some π_i. Therefore M_2 is a direct summand of D.

Lemma 21. $[\Delta\colon \Delta(B_1)]\leq 2$.

Proof. In case $m=n$, we are done by Lemma 20. If $m<n$ then $B_1\supset eJ^{m+1}$ $\neq 0$, and so $\Delta(B_1)=\Delta(B_1/eJ^{m+1})$ by Lemma 19. On the other hand, by [3], Theorem 12 and induction on the nilpotency index of J, we can show that $[\Delta\colon \Delta(B_1/eJ^{m+1})]\leq 2$.

Lemma 22. $[\Delta\colon \Delta(N)]\leq 2$ *for every submodule N of eJ.*

Proof. We may assume that $N=A_i\oplus B_j$. Then $\Delta(N)\supset\Delta(A_j)$ $(i\leq j)$ or $\Delta(N)\supset\Delta(B_j)$ $(i\geq j)$. Further, since A_1 and B_1 are uniserial, $\Delta(A_1)\subset\Delta(A_i)$ and $\Delta(B_1)\subset\Delta(B_j)$. Hence $[\Delta\colon \Delta(N)]\leq[\Delta\colon \Delta(A_1)]\leq 2$ or $[\Delta\colon \Delta(N)]\leq[\Delta\colon \Delta(B_1)]\leq 2$ by Lemma 20 or Lemma 21.

Lemma 23. *Let A_i and B_i be as in Lemma* 13. *If $\beta A_i=B_i$ for some i with a unit β in eRe, then $\bar{\beta}\notin\Delta(A_i)$ and* $[\Delta\colon \Delta(A_i)]=2$.

Proof. Let j be an arbitrary element in eJe. Then $(\beta+j)A_i\subset B_i+jA_i$. Since $jA_i\subset eJ^{i+1}$, we have $\pi_1(B_i+jA_i)\subset A_i$, where π_1 is the projection of eJ onto A_1. Hence $(\beta+j)A_i\neq A_i$, so that $\Delta\neq\Delta(A_i)$, and therefore $[\Delta\colon \Delta(A_i)]=2$ by Lemma 22.

Lemma 24. *Assume that $\beta A_1=B_1$ with a unit β in eRe. If δ is a unit in eRe such that $\bar{\delta}\notin\Delta(A_i)$ for some j, then $\pi_2\delta\colon A_i\to B_i$ is an isomorphism, where π_2 is the projection of eJ onto B_1.*

Proof. Since $\bar{e}$, $\bar{\beta}$ are independent over $\Delta(A_i)$, $\bar{\delta}=\bar{x}+\bar{\beta}\bar{y}$ for some $\bar{x}$, $\bar{y}\in$ $\Delta(A_i)$ with $xA_i=A_i$ and $yA_i=A_i$. Let a_i be a generator of A_i. Then βya_i is a generator of B_i. Hence $\pi_2\delta|A_i$ is an epimorphism of A_i onto B_i, and hence an isomorphism (cf. the proof of Lemma 23).

Lemma 25. *If $A_1 \not\approx B_1$ then $\Delta=\Delta(N)$ for every submodule N of eJ. If $A_1\approx B_1$, then $[\Delta\colon \Delta(A_i\oplus B_j)]=2$ provided $i\neq j$, and $\Delta(A_i\oplus B_i)=\Delta$; $\Delta(A_i\oplus B_j)=\Delta(A_i)=\Delta(A_1)$ for $i<j$.*

Proof. Put $D=eR/A_1\oplus eR/A_1\oplus eR/B_1$. Then $|\mathrm{Soc}(D)|=3$. Let M be a maximal submodule of D, and $M=M_1\oplus M_2\oplus\cdots$ with hollow modules $M_l\approx eR/E_l$. Since $|\bar{M}|=2$, we may assume that $\bar{M}_1\neq 0$ and $\bar{M}_2\neq 0$. Let π_h be the projection of D onto the h-th component. Then $\pi_k|M_1$ is an epimorphism for some k. Hence $E_1\subset\alpha A_1$ or αB_1, where α is a unit in eRe. If $E_1\neq\alpha A_1$ (or αB_1) then $|\mathrm{Soc}(eR/E_1)|=2$. Therefore either E_1 or E_2 coincides with αA_1 or αB_1, and so $\pi_k|M_1$ is an isomorphism. Accordingly, (**) is satisfied for D. Hence, if $[\Delta\colon \Delta(A_1)]=2$, there exists a unit α' such that $\alpha' A_1\subset B_1$ (or $\alpha' B_1\subset A_1$), by [5], Proposition 1. Since $A_1\not\subset eJ^2$, we get $\alpha' A_1=B_1$ (or $\alpha' B_1=A_1$). Conversely, assume that $A_1\approx B_1$. Then, by Lemma 12, there exists a unit β such that $\beta A_1=B_1$. Hence $[\Delta\colon \Delta(A_1)]=2=[\Delta\colon \Delta(B_1)]$ by Lemma 23, and $\Delta(A_i)=\Delta(A_1)$ by $\Delta(A_1)\subset\Delta(A_i)$. Now, let N be an arbitrary submodule of eJ. Then we may assume that $N=A_i\oplus B_j$. If $i=j$ then $N=eJ^i$, and so $\Delta=\Delta(N)$. If $i>j$ then $\beta\notin\Delta(N)$ by Lemma 24 (cf. the proof of Lemma 23), and hence $[\Delta\colon \Delta(N)]=2$, and either $\Delta(N)=\Delta(A_1)$ or $\Delta(B_1)$. Finally, if $\Delta=\Delta(A_1)$ then $\Delta=\Delta(B_1)$ from the above. Hence $\Delta=\Delta(N)$.

3 Proof of Theorems 2 and 2′. We assume that R satisfies Condition I* and either Condition II′ or Condition II″. Then, by Corollary 2, R satisfies Condition II. Hence the assumptions of Theorem 1 are fulfilled. Further, it is clear that $\Delta=\Delta(N)$ for every submodule N of eJ by Condition II′ or II″. It suffices therefore to show the equivalence of 1) and 3) in Theorems 2 and 2′.

Let $A_i\supset A_j$ and $B_{i'}\supset B_{j'}$ be as in Theorem 1, and assume that there exists $f\colon A_i/A_j\approx B_{i'}/B_{j'}$. Put $N=\{x+y\in eJ\mid x\in A_i,\ y\in B_{i'},\ f(x+A_j)=y+B_{j'}\}$. Then N is a submodule of eJ containing $A_j\oplus B_{j'}$. On the other hand, since every submodule in eJ is characteristic by Condition II′, N is a trivial submodule by Theorem 1 2). Hence $f=0$, which shows the "*only if*" part of Theorems 2 and 2′. We shall show the "*if*" part. We shall show, by induction on the nilpotency index n of J, that if the condition 3) in Theorem 2 (resp. Theorem 2′) is satisfied then R satisfies Conditions I and II′ (resp. Condition I). In order to show that R satisfies Condition I, it suffices to show the following:

(*) *every maximal submodule in any finite direct sum of hollow modules is also a direct sum of hollow modules* (cf. [3]).

Further, as was shown in [3], §3, we may consider a direct sum of hollow modules which are homomorphic to eR for a fixed e.

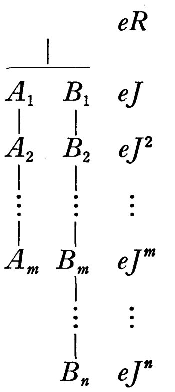
Lemma 26. *If the condition* 3) *in Theorem* 2 *is satisfied, then R satisfies* Condition II$'$.

Proof. This is clear (cf. [6]).

Lemma 27. *Assume that R satisfies* Condition II$'$ (*or* Condition II$''$). *Let $\{eR/D_h\}_{h=1}^{t}$ be a family of hollow modules. If $D_i \subset D_j$ for some i and j, then* (**) *is satisfied for $D=\sum_{h=1}^{t}\oplus eR/D_h$.*

Proof. Let π_h be the projection of D onto eR/D_h. Take a maximal submodule M in D. If $\pi_l(M)\neq eR/D_l$ for some l then $M=eJ/D\oplus\sum_{h\neq l}\oplus eR/D_h$. Hence we may assume that $\pi_h(M)=eR/D_h$ for all h. Setting $\bar{D}=D/J(D)$, we may regard $\bar{D}$ as a t dimensional vector space over $\Delta=eRe/eJe$ (note that R is basic). Further we may assume that $D_1\subset D_2$. Since $\pi_h(M)=eR/D_h$ for all h, $\bar{M}=M/J(D)$ contains a subspace $S=(\bar{e}, \overline{ek}, \bar{0}, \cdots, \bar{0})\Delta$ (note that k is a central element of R for the case of Theorem 2$'$). Since $D_1\subset D_2$, this simple subspace S is lifted to a direct summand M_1 of D by [1], Theorem 2 and its proof. Then $M_1\subset M$, proving (**) for D.

In view of Lemma 27, it remains to show that (*) is satisfied for $D=\sum_{h=1}^{t}\oplus eR/D_h$ provided $D_i\not\subset D_j$ for all distinct i, j. Let M be a maximal submodule in D, and let π_h be as above. As was claimed in the proof of Lemma 27, we may restrict ourselves to the case where $\pi_h(M)=eR/D_h$ for all h. Then we can take such a basis of $\bar{M}=M/J(D)$ as $\{\alpha_i=(\bar{0}, \cdots, \overset{h}{\check{\bar{e}}}, \overline{ek_h}, \bar{0}, \cdots, \bar{0})\}_{h=1}^{t-1}$, where $k_h\in eRe$ (central elements of R for the case of Theorem 2$'$). We assume that eR has the structure given in Theorem 2 (resp. Theorem 2$'$), i.e.,



In the case of Theorem 2$'$, $D_h=\alpha(A_r\oplus B_s)$. Hence $eR/D_h\approx eR/(A_r\oplus B_s)$. Accordingly, we may assume that all D_h are trivial submodules. If all D_h contain B_n (resp. $A_n\oplus B_n$ for the case $m=n$), all eR/D_h are hollow R/J^n-modules. Hence, by induction hypothesis, (*) is satisfied for D. Thus, in what follows, we consider the case where some D_h is equal to A_i; $1\leq i\leq m$ (resp. B_j; $1\leq j\leq n$).

Therefore we should check the following cases:

1) $D_1=A_i$, $D_k=A_{j_k}\oplus B_{j_k}$; $i<i_1<i_2<\cdots<i_p, j_1>j_2>\cdots>j_p$.

2) $D_1=A_i$, $D_2=B_j$, $D_k=A_{i_k}\oplus B_{j_k}$; $i<i_1<i_2<\cdots<i_p$, $j_1>j_2>\cdots>j_p$.

However, 2) is a special case of 1) obtained by putting $i_p=n+1$ and $j_p=j$. So, we consider the case 1): $D=eR/A_i\oplus eR/(A_{i_1}\oplus B_{j_1})\oplus\cdots\oplus eR/(A_{i_p}\oplus B_{j_p})$. Then $|D|=(n+i)+\sum_{s=1}^{p}(i_s+j_s-1)$. Set $M^*=A_1/A_i\oplus\sum_{s=1}^{p}\oplus eR/(A_{i_s}\oplus B_{j_{s-1}})\oplus B_1/B_{j_p}$ $(B_{j_0}=0)$. Then $|M^*|=|D|-1$. Define a homomorphism φ of M^* to D by setting

$$\varphi((x+A_i)+\textstyle\sum_{s=1}^{p}(y_s+(A_{i_s}\oplus B_{j_{s-1}}))+(z+B_{j_p})=(x+y_1+A_i)$$
$$+(ek_1y_1+y_2+(A_{i_1}\oplus B_{j_1}))+(ek_2y_2+y_3+(A_{i_2}\oplus B_{j_2})+\cdots$$
$$+(ek_py_p+z+(A_{i_p}\oplus B_{j_p}))\,,$$

where $x\in A_1$, $y_s\in eR$, $z\in B_1$ and k_s are central elements for the case of Theorem 2'. Now, let $(x+A_i)+\sum_{s=1}^{p}(y_s+(A_{i_s}\oplus B_{j_{s-1}}))+(z+B_{j_p})$ be in ker φ. Since x and z are in eJ, y_s are all in eJ. Set $y_s=y_{s1}+y_{s2}$ $(y_{s1}\in A_1, y_{s2}\in B_1)$. Since $x+y_1=x+y_{11}+y_{12}\in A_i$, we have $y_{12}=0$. Then $ek_1(y_{11}+y_{12})+(y_{21}+y_{22})\in A_{i_1}\oplus B_{j_1}$ implies that $y_{22}\in B_{j_1}$ and $ek_1y_{11}+y_{21}\in A_{i_1}$, and $ek_2(y_{21}+y_{22})+(y_{31}+y_{32})\in A_{i_2}\oplus B_{j_2}$ implies that $y_{32}\in B_{j_2}$ (note that $B_{j_1}\subset B_{j_2}$). Repeating this procedure, we see that $y_{s2}\in B_{j_{s-1}}$ and $z\in B_{j_p}$. Similarly, from the fact that $ek_p(y_{p1}+y_{p2})+z\in A_{i_p}\oplus B_{j_p}$, it follows that $y_{p1}\in A_{i_p}$, $\cdots$, $y_{s1}\in A_{i_s}$, $\cdots$, $y_{11}\in A_i$, and $x\in A_i$ (note that k_s are central elements for the case of Theorem 2'). Hence φ is a monomorphism and $\overline{\text{im}\,\varphi}=\bar{M}$. Therefore, noting that $M\supset \mathrm{J}(D)$, we see that $M\approx M^*$.

4 Rings with $J^3=0$. We have shown in [3] that the converse of Theorem 1 is true provided $J^2=0$. In this section, we shall show that the same is still true for the case $J^3=0$, namely if $J^3=0$ then 1)~3) in Theorem 1 imply Condition I.

Lemma 28. *Assume that $\beta A_1=B_1$. If α is a unit in eRe such that $\bar{\alpha}\notin\Delta(A_i)$ for some, $A_i\cap\alpha A_i=0$.*

Proof. Let a_i be a generator of A_i. Then, by Lemma 24, $\alpha a_i=a_i'+b_i$, where $a_i'\in A_i$, $b_i\in B_i$ and $\notin B_{i+1}$. Hence $\alpha A_n=\alpha a_iJ^{n-i}=(a_i'+b_i)J^{n-1}\not\subset A_n$, and therefore $\alpha A_i\cap A_i=0$.

In order to show that R satisfies Condition I, it suffices to show that R satisfies (*). Further, as is claimed in §3, we may restrict ourselves to the case that hollow direct summands in (*) are isomorphic to eR/E for a fixed primitive idempotent e. We shall divide the proof into two cases: 1) $A_1\not\approx B_1$, and 2) $A_1\approx B_1$.

1) $A_1\not\approx B_1$. By 3) in Theorem 1, $\Delta=\Delta(N)$ for every submodule N of eJ. This situation is very similar to that in [3], and we can apply the argument

employed in [3] to see that R satisfies Condition I.

2) $A_1 \approx B_1$. There holds $\Delta(A_1)=\Delta(A_2)=\Delta(A_1 \oplus B_2)$. We shall give the explicit form of a maximal submodule M in $D=\sum_{h=1}^{t} \oplus N_h$, where N_h are hollow modules isomorphic to eR/E_h. Now, by 1) in Theorem 1, $B_1=\alpha A_1$ for some unit α. If $i>j$ then $\alpha(A_i \oplus B_j) \subset A_j \oplus B_i$, for $\alpha A_i = B_i$. Hence $\alpha(A_i \oplus B_j)=A_j \oplus B_i$. Therefore $eR/(A_i \oplus B_j) \approx eR/(A_j \oplus B_i)$. Consequently, we may assume that $N_h \approx eR/(A_i \oplus B_j)$ for some $i \leq j$.

i) $t=2$.

(1) $D=eR/A_2 \oplus eR/A_2$. Then we may assume that $\bar{M}=\overline{(e+A_2)}\Delta \oplus \overline{(\alpha+A_2)}\Delta$, where α is a unit in eRe.

α) $\alpha A_2 = A_2$. Then M contains a direct summand of D by [1], Theorem 2. Hence M is a direct sum of hollow modules.

β) $\alpha A_2 \cap A_2 = 0$. Put $M^*=eR \oplus A_1/A_2 \oplus A_1/A_2$, and define a homomorphism φ of M^* into D by setting

$$\varphi(z_1+(z_2+A_2)+(z_3+A_2)) = (z_1+z_2+A_2)+(\alpha z_1+z_3+A_2)\,,$$

where $z_1 \in eR$ and z_2, $z_3 \in A_1$. Suppose $z_1+z_2+z_3$ be in ker φ. Then z_i are in eJ. Set $z_1=x_1+y_1$ ($x_1 \in A_1, y_1 \in B_1$). Since $z_1+z_2 \in A_2$, we have $y_1=0$. If $x_1 \neq 0$ then $\pi_2 \alpha z_1 = \pi_2 \alpha x_1 \neq 0$ by Lemma 24, where π_2 is a projection of eJ onto B_1 However, $0=\pi_2(\alpha z_1+z_3)=\pi_2 \alpha z_1$. This contradiction shows that $x_1=0$, and so $z_1=0$. Now, it is clear that $\varphi(M^*)=M$ by $|M^*|=|M|$.

(2) $D=eR/A_1 \oplus eR/A_1$. Let M be as above. If $(\alpha+j)A_1=A_1$ for some $j \in eJe$ then we are done by [1], Theorem 2. On the other hand, if $(\alpha+j)A_1 \neq A_1$ for every $j \in eJe$ then $\bar{\alpha} \notin \Delta(A_1)$, and so $\alpha A_1 \cap A_1=0$ by Lemma 29. Hence $M \approx eR$.

(3) $D=eR/A_1 \oplus eR/A_2$. If $\alpha A_2 \subset A_1$, we are done. Next, if $\alpha A_2 \cap A_1=0$ then $M \approx eR \oplus A_1/A_2$ via $\varphi(z_1+(z_2+A_2))=(z_1+A_1)+(\alpha z_1+z_2+A_2)$ (note that $\alpha A_1 \neq A_1$).

(4) $D=eR/A_2 \oplus eR/(A_1 \oplus B_2)$, $eR/A_2 \oplus eR/(A_2 \oplus B_2)$ or $eR/A_2 \oplus eR/(A_2 \oplus B_1)$ ($eR/A_1 \oplus eR/(A_1 \oplus B_1)$). Since $\alpha A_2 \subset A_2 \oplus B_2$ ($\alpha A_1 \subset A_1 \oplus B_1$), M contains a direct summand of D, by [1], Theorem 2.

(5) $D=eR/A_1 \oplus eR/(A_2 \oplus B_2)$. Note that either $\rho_1=\pi_1\alpha | A_1$ or $\rho_2=\pi_2\alpha | A_1$ is an isomorphism. If ρ_1 (resp. ρ_2) is an isomorphism, then $M \approx eR/A_2 \oplus B_1/B_2$ (resp. $eR/A_2 \oplus A_1/A_2$).

(6) $D=eR/A_1 \oplus eR/(A_1 \oplus B_2)$. Since $\alpha A_1 \cap (A_1 \oplus B_2)$ is either a simple module B_2' or A_1, $M \approx eR/\alpha^{-1}(B_2') \approx eR/A_2$ or M is a direct summand of D.

(7) Other cases can be reduced to the case $J^2=0$ [3].

ii) $t=3$. If N_1, N_2 and N_3 are linearly ordered by inclusion, then M contains a direct summand of D, by [5], Corollary 1. Hence, it suffices to consider the following two cases:

(1) $D=eR/A_1 \oplus eR/A_2 \oplus eR/(A_2 \oplus B_2)$. Since $eReA_2 \subset A_2 \oplus B_2$, M contains

a direct summand of D.

(2) $D=eR/A_1\oplus eR/A_1\oplus eR/(A_2\oplus B_2)$. We may assume that $\bar{M}$ has a basis $\{\xi=\overline{(e+A_1)}+\bar{0}+\overline{(\delta_2+(A_2\oplus B_2))},\ \eta=(\bar{0}+\overline{(e+A_1)})+\overline{(\delta_2+(A_2\oplus B_2))}\}$, where δ_1, δ_2 are units in eRe (see [3], §3).

α) Assume that there exists a unit x such that $\bar{x}\in\Delta(A_1)$ and $\bar{\delta}_2=\bar{\delta}_1\bar{x}$. Then $\xi+\eta\bar{x}=(\bar{e},\ \overline{ex},\ \bar{0})\in\bar{M}$. Since $\bar{x}\in\Delta(A_1)$, M contains a direct summand of D, by [1], Theorem 2.

β) Assume that $\bar{\delta}_2^{-1}\bar{\delta}_1\notin\Delta(A_1)$. Put $M^*=eR/A_2\oplus eR/A_2$, and define a homomorphism φ of M^* to D by setting $\varphi((z_1+A_2)+(z_2+A_2))=(z_1+A_1)+(z_2+A_1)+(\delta_1 z_1+\delta_2 z_2+(A_2\oplus B_2))$, where $z_i\in eR$. Suppose that $(z_1+A_2)+(z_2+A_2)$ is in ker φ. Then $z_i\in A_1$. If $z_1\notin A_2$ then $\delta_1 A_1=\delta_1 z_1 R\subset\delta_2 A_1+(A_2\oplus B_2)$. Hence $\delta_2^{-1}\delta_1 A_1\subset A_1+\delta_2^{-1}(A_2\oplus B_2)=A_1\oplus B_2$, which contradicts Lemma 24. Hence $z_1\in A_2$, and similarly $z_2\in A_2$. Therefore $M\approx M^*$.

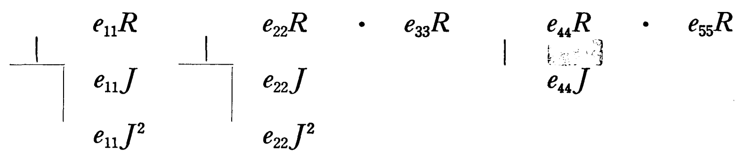
iii) $t\geq 4$. In view of [4], Lemma 1 and Theorem 1 and [5], Corollary 1, this case can be reduced to the cases i) and ii).

Thus we have shown that R satisfies Condition I provided $J^3=0$.

5 Examples. 1. Let $L_1\subset L_2\subset L_3\subset\cdots$ be fields. Set

$$R=\begin{pmatrix} L_1 & 0 & L_3 & L_4 & L_5 \\ 0 & L_2 & L_3 & L_4 & L_5 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & L_4 & L_5 \\ 0 & 0 & 0 & 0 & L_5 \end{pmatrix}$$

Then



Hence R satisfies Conditions I and II$'$. However, R is not left serial (cf. [7] and [8]). If $[L_3: L_1]\geq 2$ then R does not satisfy Condition I for hollow left R-modules.

2. Let K be a field and let R be a vector space over K with basis $\{e_1, x_{11}, y_{12}, x_{12}, e_2, x_{22}, y_{21}, x_{21}\}$. Define $e_i e_j=e_i\delta_{ij}$, $e_i x_{jk} e_p=x_{jk}\delta_{ij}\delta_{kp}$, $e_i y_{jk} e_p=y_{ik}\delta_{ij}\delta_{kp}$, $x_{11}x_{12}=y_{12}$ and $x_{22}x_{21}=y_{21}$. Putting other multiplications to be zero, we see that R is a ring with $J^3=0$. We can easily see that R satisfies the conditions in Theorem 2$'$ as both left and right R-modules. Further R satisfies Condition II$'$ as a left R-module. Next, let $R_1=\langle e_1, x_{11}, y_{13}, x_{13}, e_2, x_{22}, y_{21}, x_{21}, e_3, x_{31}\rangle$. Define $x_{11}x_{13}=y_{13}$ and the same as above for others. Then R_1 satisfies

the conditions in Theorem 2′ as a right R-module, but does not as a left R-module, since $Je_1=Rx_{11}\oplus Rx_{21}\oplus Rx_{31}$.

3. Let $K\subset L$ be fields with $[L: K]=2$, say $L=K\oplus uK$. Put

$$R=\begin{pmatrix} L & L & L \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

Then $e_{11}J=(0, K, K)\oplus(0, uK, uK)$. Hence R satisfies 1)～3) in Theorem 1 and $A_1\approx B_1$. Therefore R satisfies neither Condition II′ nor Condition II″. But, by [8], R is of right local type.

4. Let k be a field, and x an indeterminate. Put $L=k(x)$ and $K=k(x^2)$. Take a left L-vector space $V=Lu$ of one dimension. Putting $ux=x^2u$ and $uk=ku$ for all $k\in K$, we make V a right L-vector space (see [3], Example 2). Put

$$R=\begin{pmatrix} L & 0 & L & L & L \\ 0 & L & V & 0 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & L & L \\ 0 & 0 & 0 & 0 & L \end{pmatrix}$$

Then $e_{11}J=A_1\oplus B_1$ with $A_1\not\approx B_1$, $e_{22}J=A_1'\oplus B_1'$ with $A_1'\approx B_1'$. Further, $Je_{33}=A_1''\oplus B_1''$ with $A_1''\not\approx B_1''$ as left R-modules. Hence R satisfies Condition I for both left and right hollow R-modules.

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