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ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES III

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We have studied, in [3], a right artinian ring R satisfying Condition I (see below) as a generalization of right artinian serial rings. However, there, we have restricted ourselves to the case that $J^2=0$, where J is the Jacobson radical of R.

In this paper, instead of removing the restriction $J^2=0$, we shall add one more condition (Condition II': every hollow module is quasi-projective or Condition II'': R is an algebra of finite dimension over an algebraically closed field). We shall give a characterization of a right artinian ring satisfying Condition I and Condition II' (resp. II''), and show that such a ring is closely related to an algebra of right local type studied by H. Tachikawa [8] (see also [7]). Actually, if the assumption "left serial" is removed in [8], the situation is very similar to that in this paper.

Further, under Condition I, we shall consider Condition II: $|eJ/eJ^2| \leq 2$ for each primitive idempotent e, which is weaker than Conditions II' and II". We shall give the structure of a ring satisfying Conditions I and II, and show that the structure gives a characterization of such a ring provided $J^3=0$.

1 Conditions and Theorems. In this paper, we shall study a right artinian ring R with identity, and every R-module is assumed to be a unitary right R-module. We denote the Jacobson radical and the socle of an R-module M by J(M) and Soc(M), respectively. Occasionally, we write J=J(R). |M| means the length of a composition series of M. If eR is a right uniserial module for each primitive idempotent e, R is called a *right serial (generalized uniserial)* ring. If R is a right serial ring then the following conditions are satisfied:

Condition I: every submodule in any finite direct sum of hollow modules is also a direct sum of hollow modules [3]

and

¹⁾ Conditions II and II-a are added in the revise.

Condition II': every hollow module is quasi-projective [6]. In [3], we have studied rings R satisfying Condition I, under the extra hypothesis $J^2=0$, and further known that, among them, there exist some rings which fail to satisfy Condition II' (e.g., algebras over an algebraically closed field).

Let R be an algebra of finite dimension over a field K such that $R/J=\Sigma \oplus K_{n_i}$, where K_{n_i} is the $n_i \times n_i$ full matrix ring over K (e.g., K is an algebraically closed field). We have thus the following condition:

Condition II'': $eRe/eJe = \overline{e}K$ for each primitive idempotent e, where K is a subfield contained in the center of R.

As is shown in Corollary 2 below, if R satisfies Conditions I and either II' or II'' then $|eJ/eJ^2| \leq 2$ for each e. From the study of rings satisfying Condition I and $J^2=0$ (see [3]), it seems to the author that $|eJ/eJ^2| \leq 2$ holds without assuming $J^2=0$. (For the present, he has no counter-examples.) Several conditions under which $|eJ/eJ^2| \leq 2$ holds are given in [3]. On the other hand, since eJ/eJ^2 is semisimple, $eJ/eJ^2 = \sum_i \bigoplus S_i^{(n_i)}$, where S_i are simple $(S_i \approx S_j$ provided $i \neq j$) and $S_i^{(n_i)}$ means the direct sum of n_i copies of S_i . If R satisfies Condition I and $J^2=0$, then $n_i \leq 2$ for all i if and only if $|eJ/eJ^2| \leq 2$ (see [3]). From this point of view, we consider the following conditions:

Condition II: for each primitive idempotent e, $|eJ|eJ^2| \leq 2$, and

Condition II-a: for each primitive idempotent e, $n_i \leq 2$ for all i.

Lemma 1. Let P be a two-sided ideal of R. If R satisfies any one of the conditions above, then so does R/P.

Proof. Assume that R satisfies Condition I. Put $\overline{R}=R/P$. Let D be a finite direct sum of hollow \overline{R} -modules N_i , and M an \overline{R} -submodule of D. Then N_i are hollow R-modules. Hence, from Condition I, $M=\sum_i \oplus M_i$ with hollow R-modules M_i . Since MP=0, M_i is also a hollow \overline{R} -module. Hence Condition I holds for \overline{R} . It is clear that J(R/P)=(J+P)/P. Let ebe a primitive idempotent in R not contained in J+P. Then \overline{e} is a primitive idempotent in \overline{R} . Since \overline{eR} is a homomorphic image of eR, the remainder is also clear.

Corollary 2. Assume that R satisfies the following condition:

Condition I*: every submodule in any direct sum of three hollow modules is also a direct sum of hollow modules.²⁾

Then Conditions II and II-a are equivalent, and each of Conditions II' and II'' implies Condition II.

Proof. By Lemma 1, R/J^2 satisfies Condition I. Hence the corollary

²⁾ We needed only Condition I* in the proof in [3].

is clear from [3], Lemmas 9 and 14.

As is easily seen, the conditions above except II' are invariant for Morita equivalence. Let $R_0 = e_0 R e_0$ be the basic ring of R. If x is an element in the center of R, $e_0 x$ is in the center of R_0 . Hence, in order to study the structure of rings which satisfy those conditions, we may assume that R is a basic ring.

Let M be a hollow module. Then $M \approx eR/A$ with a primitive idempotent e and a right ideal A in eR. Put $\Delta = eRe/eJe$ and $\Delta(A) = \{\bar{x} \in \Delta \mid x \in eRe, xA \subset A\}$, where \bar{x} is the coset of x in Δ . It is clear that $\Delta(A)$ is a subdivision ring of Δ . We regard Δ as a right $\Delta(A)$ -module (see [3] and [4]), so $[\Delta: \Delta(A)]$ means the dimension of Δ over $\Delta(A)$ as a right $\Delta(A)$ -module.

Let $M_1 \supset N_1$ and $M_2 \supset N_2$ be R-modules. A submodule $N_1 \oplus N_2$ in $M_1 \oplus M_2$ is called a *trivial submodule* of $M_1 \oplus M_2$. For $N_3 \subset N_1$, N_1/N_3 ($\subset M_1/N_3$) is called a *sub-factor module* of M_1 .

We shall give the following theorems.

Theorem 1. Let R be a right artinian ring. If R satisfies Conditions I* and II, then for each primitive idempotent e in R we have the following properties: 1) $eJ = A_1 \oplus B_1$, where A_1 and B_1 are uniserial modules. Further, if $A_1/J(A_1)$

1) $e_{J} = A_{1} \oplus B_{1}$, where A_{1} and B_{1} are uniserial modules. Further, if $A_{1/J}(A_{1}) \approx B_{1/J}(B_{1})$, $\alpha A_{1} = B_{1}$ for some unit α in eRe.

2) For every submodule N in eJ, there exists a trivial submodule $A_i \oplus B_j$ of eJ and a unit β in eRe such that $N = \beta(A_i \oplus B_j)$, where $A_i = A_1 J^{i-1} \subset A_1$ and $B_j = B_1 J^{j-1} \subset B_1$.

3) If $A_1 \approx B_1$, then $\Delta(A_i \oplus B_i) = \Delta$ and $[\Delta: \Delta(A_i \oplus B_j)] = 2$ provided $i \neq j$; further $\Delta(A_1) = \Delta(A_i) = \Delta(A_i \oplus B_j)$ (i < j) and $\Delta(B_1) = \Delta(B_j) = \Delta(A_i \oplus B_j)$ (i > j). If $A_1 \approx B_1$, then $\Delta(N) = \Delta$ for any submodule N in eJ.

Theorem 2. Let R be a right artinian ring. Then the following are equivalent:

- 1) R satisfies Conditions I and II'.
- 2) R satisfies Conditions I* and II'.

3) For each primitive idempotent e, eJ is a direct sum of two uniserial modules A_1 and B_1 , and no sub-factor module of A_1 is isomorphic to any sub-factor module of B_1 , and hence every submodule in eJ is trivial.

Theorem 2'. Let R be a right artinian ring. If R satisfies Condition II'', then the following are equivalent:

- 1) R satisfies Condition I.
- 2) R satisfies Condition I*.

3) For each primitive idempotent e, eJ is a direct sum of two uniserial modules A_1 and B_1 and every submodule in eJ is isomorphic to a trivial submodule via the left-sided multiplication of a unit element in eRe.

2 Proof of Theorem 1. We always assume that R is a right artinian ring with identity, and J is the Jacobson radical of R, unless otherwise stated. Further, we may assume that R is basic in the proof. In advance of giving the proof, we state the following proposition.

Proposition 3. If $J^2=0$, then every submodule of a direct sum of two hollow modules is also a direct sum of hollow modules.

Proof. As is shown in [3], §3, it suffices to consider a direct sum of hollow modules eR/A for a fixed primitive idempotent e and show that every maximal submodule M of $D=eR/A_1\oplus eR/A_2$ is a direct sum of hollow modules. Let π_i be the projection of D onto eR/A_i . If $\pi_1(M) \subset eJ/A_1$ then $M=eJ/A_1\oplus$ eR/A_2 . Since eJ/A_1 is semisimple by the assumption $J^2=0$, M is a direct sum of hollow modules. Assume that π_i is an epimorphism for i=1, 2. Put $\overline{D}=$ D/J(D) and $\overline{M}=M/J(D)$. Then \overline{M} has a basis of the form $(\overline{e}+\overline{er})$ over $\Delta=$ eRe/eJe, where \overline{e} is the coset of e in eR/eJ (note that R is assumed to be basic). We have the natural mapping φ of eR to D by setting $\varphi(e)=(e+A_1)+(er+$ $A_2)$. Then $D\supset \varphi(eR)\approx eR/C$, where $C=\ker \varphi$. Since $\varphi(eR)=\overline{M}, M=\varphi(eR)+$ J(D). Noting that J(D) is semisimple, we obtain that $M=\varphi(eR)+(\sum_i \oplus M_i)$, where M_i are simple. Hence M is a direct sum of hollow modules.

From Proposition 3, we see that Condition I has a meaning for direct sums of at least three hollow modules. In what follows, we shall use a diagram



which means that A, B, and C are hollow modules and $J(A) = B \oplus C$.

Proof of Theorem 1. We always assume that R satisfies Conditions I^{*} and II, and that R is a basic ring, unless otherwise stated.

Lemma 3. Assume that R satisfies Condition I*. Let E_1 and E_2 be submodules in eJ with $JE_2=0$. Put $D=eR/E_1\oplus eR/E_2$. For each unit α in eRe, D contains a maximal submodule with a direct summand isomorphic to $eR/(E_1 \cap \alpha E_2)$ via the mapping: $x+(E_1 \cap \alpha E_2) \rightarrow (x+E_1)+(\alpha^{-1}x+E_2)$.

Proof. Let $\Delta = eRe/eJe$. Then $\overline{D} = D/J(D)$ is a right Δ -module, because R is basic. Now, let M be the maximal submodule of D such that $\overline{M} = M/J(D) = (\overline{(e+E_1)} + (\overline{\alpha^{-1} x + E_2}))\Delta \subset \overline{D}$. By assumption, M contains a hollow submodule M_1 with $M_1/J(M_1) \approx (M_1 + J(D))/J(D) = \overline{M}$. Let m_1 be a generator of M_1 such that $\overline{m}_1 = \overline{(e+\alpha^{-1})}$ in \overline{D} . We denote by π_i the projection of D onto eR/E_i . Then we obtain a homomorphism $f_i: eR \to M_1 \to eR/E_i$ by setting $f_i(e) =$

 $\pi_i(m_1)$, which is given by the left multiplication of a unit β_i in eRe. Hence $M_1 \approx eR/(\beta_1^{-1}E_1 \cap \beta_2^{-1}E_2)$ and $m_1 = \pi_1(m_1) + \pi_2(m_1) = f_1(e) + f_2(e) = (\beta_1 + E_1) + (\beta_2 + E_2)$. Since $\overline{m}_1 = (\overline{e+E_1}) + (\overline{\alpha^{-1}+E_2})$, $\overline{\beta}_1 = \overline{e}$ and $\overline{\beta}_2 = \alpha^{-1}$ in Δ . Now $eR/(\beta_1^{-1}E_1 \cap \beta_2^{-1}E_2) \approx eR/(E_1 \cap \beta_1\beta_2^{-1}E_2)$. On the other hand, $\beta_1\beta_2^{-1} = \alpha + j$ for some j in eJe. Hence $\beta_1\beta_2^{-1}E_2 = \alpha E_2$ by assumption. The mapping is clear.

REMARK 4. If we drop the assumption $JE_2=0$ in Lemma 3, we obtain $eR/(E_1 \cap (\alpha+j)E_2)$ instead of $eR/(E_1 \cap \alpha E_2)$, from the proof.

Lemma 5. Let A be a right ideal in eR with JA=0. If $\alpha A=A$ for every unit α in eRe, then A is a two-sided ideal, provided R is a basic ring.

Proof. Since eJeA=0, A is a characteristic submodule of eR. Since R is basic, $RA=\sum_i e_i ReA=\sum_{e_i\neq e} e_i JeA+eReA=A$, where $1=\sum_i e_i$ and e_i are orthogonal primitive idempotents.

Lemma 6. eR has the structure given in the following diagram:



where A_{ij} and B_{ij} are hollow modules and $eJ^i = \sum_j \oplus A_{ij} \oplus \sum_{j'} \oplus B_{ij'}$.

Proof. Let A be a hollow module contained in eR and $A \approx fR/B$, where f is a primitive idempotent in R. Then $J(A) = AJ = \sum_{i=1}^{t} \oplus A_i$ with hollow modules A_i , and $AJ^2 = \sum_i \oplus A_i J$. On the other hand, $AJ/AJ^2 \approx fJ/(fJ^2+B)$. Since $fJ/(fJ^2+B)$ is a homomorphic image of fJ/fJ^2 , $|AJ/AJ^2| \leq 2$ by Condition II. Hence $t \leq 2$, which proves the lemma.

Lemma 7. Let $eJ = A_1 \oplus B_1$ be as in Lemma 6. If $A_1/A_1 J \approx B_1/B_1 J$ then $|A_1| = |B_1|$. If $A_1/A_1 J \approx B_1/B_1 J$, $\pi_2 \alpha(A_1) \subseteq B_1$ for any unit α in eRe, where π_2 is the projection of eJ onto B_1 .

Proof. Put $\overline{\overline{R}} = R/J^2$, $\overline{\overline{A}}_1 = (A_1 + eJ^2)/eJ^2$ and $\overline{\overline{B}}_1 = (B_1 + eJ^2)/eJ^2 \subset \overline{eJ}$. Since $\overline{\overline{A}}_1 \approx A_1/A_1 J \approx B_1/B_1 J \approx \overline{\overline{B}}_1$, by [3], Theorem 12 there exists a unit $\overline{\overline{\beta}}$ in eRe/eJ^2e (and hence a unit β in eRe) such that $\overline{\overline{\beta}\overline{A}_1} = \overline{\overline{B}}_1$, i.e., $\beta(A_1 + eJ^2) = B_1 + eJ^2$. As

 B_1 is hollow, there holds $\pi_2\beta(A_1) = B_1$, so $|A_1| \ge |B_1|$. By symmetry, $|A_1| \le |B_1|$, and hence $|A_1| = |B_1|$. Since A_1 and B_1 are hollow, there exist no epimorphisms of A_1 onto B_1 , provided $A_1/A_1J \approx B_1/B_1J$. Therefore $\pi_2\alpha(A_1) \neq B_1$ if $A_1/A_1J \approx B_1/B_1J$.

Now let N be a submodule of $eJ = A_1 \oplus B_1$, and π_1 (resp. π_2) the projection of eJ onto A_1 (resp. B_1). Put $N_1 = N \cap A_1$, $N_2 = N \cap B_1$, $N^1 = \pi_1(N)$ and $N^2 = \pi_2(N)$. Then, as is well known, $N^1/N_1 \approx N^2/N_2$. Further, if $N_2 = 0$ then $N = N^1(f) = \{x + f(x) \mid x \in N^1, f(x) = \pi_2 \pi_1^{-1}(x)\}$.

First we shall study R with $J^3=0$ and satisfying Conditions I^{*} and II. Then, by assumption, $eJ=A_1\oplus B_1$ where A_1 and B_1 are hollow. Since A_1 is a hollow R/J^2 -module, $J(A_1)=C_1\oplus C_2$ by Lemma 1, where C_i are simple or zero. Similarly $J(B_1)=D_1\oplus D_2$ with D_i simple or zero.

Lemma 8. Assume that $C_i \neq 0$ and $D_i \neq 0$ (i=1, 2), and that C_1 is isomorphic to D_1 via f. Put $C'_1 = C_1(f) = \{c_1 + f(c_1) | c_1 \in C_1\} \subset C_1 \oplus D_1$. Then there holds the following:

1) Soc $(eR/C_1) = (C_1 + C_1')/C_1' \oplus (C_2 + C_1')/C_1' \oplus (D_2 \oplus C_1')/C_1'$ and $(C_i + C_1')/C_1' \approx C_i$, $(D_2 + C_1')/C_1' \approx D_2$.

2) If N/C'_1 is uniform in eR/C'_1 for a submodule N in eR then $|N| \leq 3$.

Proof. 1) Let N^* be a submodule of eR such that $N^* \supset C'_1$ and $|N^*/C'_1| = 1$. Since $N^* \subset eJ$ and $|N^*| = 2$, $|N^{*1}| \le |N^*| = 2$. Hence $N^{*1} \subset J(A_1) \subset$ Soc(eR). Similarly, $N^{*2} \subset$ Soc(eR), and therefore $N^* \subset$ Soc(eR).

2) It is clear that $N \subset eJ$. If $N_1 = 0$ then $|N| \leq 3$. We assume henceforth $N_1 \neq 0$. If $N_1 \supset C_1$ then $N \supset C_1 \oplus D_1$. Since $\operatorname{Soc}(eJ/(C_1 \oplus D_1)) \approx C_2 \oplus D_2$, N contains a non-zero element x in $C_2 \oplus D_2$, provided $N \neq C_1 \oplus D_1$. Then $N/C'_1 \supset (C_1 + xR + C'_1)/C'_1 \approx C_1 \oplus xR$, so N/C'_1 is not uniform, and hence $N = C_1 \oplus$ D_1 , so that $|N| \leq 2$. On the other hand, if $N_1 \oplus C_1$ then N contains an element $z = x + y \in N_1$; $x \in C_1$, $y \neq 0 \in C_2$. Hence $N/C'_1 \supset (C_1 + xR + C'_1)/C'_1 \approx C_1 \oplus xR$, a contradiction.

Lemma 9. If $J^3=0$, then both A_1 and B_1 in Theorem 1 are uniserial.

Proof. 1) Assume that eJ is hollow. Since eJ is an R/J^2 -module, $eJ^2 = C_1 \oplus C_2$ by Lemma 1, where C_i are simple. Assume $C_i \neq 0$ for i=1, 2, and put $D = eR/C_1 \oplus eR/C_2$. Since $JC_2=0$, D contains a maximal submodule M with a direct summand M_1 isomorphic to eR, by Lemma 3 (take $\alpha = e$). Then $|\operatorname{Soc}(D)| = |\operatorname{Soc}(eR)| = 2$, and therefore $M = M_1$. On the other hand, |D| = 6 and |M| = |eR| = 4, which is a contradiction. Hence, if eJ is hollow then eR is uniserial.

2) Assume that $eJ = A_1 \oplus B_1$ and $A_1 \neq 0$, $B_1 \neq 0$. Let $J(A_1) = C_1 \oplus C_2$ and $J(B_1) = D_1 \oplus D_2$ as before (see Lemma 8). We shall show that $C_2 = D_2 = 0$.

i) Assume that $C_2=0$, $D_1\neq 0$ and $D_2\neq 0$. Then $C_1\neq 0$ or A_1 is simple

by assumption. First assume that $C_1 \neq 0$. Since $|A_1| < |B_1|$, $A_1/A_1J \approx B_1/B_1J$ by Lemma 7. Let a_1 be a generator of A_1 , and α a unit in *eRe*. Then, by Lemma 7, $\alpha a_1 = a'_1 + b_2$, where $a'_1 \in A_1 - A_1 J$ and $b_2 \in B_1 J$. Hence $\alpha C_1 = \alpha a_1 J \subset A_1 - A_1 J$ $a'_1J+b_2J \subset A_1J=C_1$. Therefore C_1 is a two-sided ideal of R, by Lemma 5. Considering R/C_1 , in view of Lemma 1, we may assume that A_1 is simple in either case. Put $D=eR/D_1\oplus eR/D_2$. Then, by taking $\alpha=e$ in Lemma 3, we see that D contains a maximal submodule M with a direct summand M_1 isomorphic to $eR/(D_1 \cap D_2) = eR$. Now, |D| = 8, $|M_1| = 5$, $|\operatorname{Soc}(D)| = 4$ and $|\operatorname{Soc}(M_1)|=3$. Hence $M=M_1\oplus M_2$ with M_2 uniform. Since the uniform module M_2 is isomorphic to a submodule of eJ/D_i (i=1, 2), we get $|M_2| \leq$ $|eI/D_i|$. Therefore $M_2 \subseteq eI/D_1 \oplus eI/D_2 = (A_1 \oplus B_1)/D_1 \oplus (A_1 \oplus B_1)/D_2$. On the other hand, $\operatorname{Soc}(M_1) = (e, e+j) \operatorname{Soc}(eR) \subset (D_2 \oplus D_1) / D_1 \oplus (D_1 \oplus D_2) / D_2$ for some $j \in e J e$, where (e, e+j): $e R \rightarrow D$ is the mapping given in Lemma 3. Hence M_2 is monomorphic to $(A_1 \oplus D_1)/D_1 \oplus (A_2 \oplus D_2)/D_2 \approx A_1 \oplus A_1$, and so to A_1 , for M_2 is uniform. But, $|M_2| = |M| - |M_1| = |D| - 1 - |M_1| = 2$, which is a contradiction. Hence, if $C_2=0$ then $D_1=0$ or $D_2=0$.

ii) Assume $C_i \neq 0$ and $D_i \neq 0$ for i=1, 2.

 α) Assume that there exists a unit α in *eRe* such that $(C_1 \oplus D_1) \cap \alpha(C_1 \oplus D_1) = 0$. Put $D = eR/(C_1 \oplus D_1) \oplus eR/(C_1 \oplus D_1)$. Then, by Lemma 3, *D* contains a maximal submodule *M* with a direct summand M_1 isomorphic to *eR*. Since $|\operatorname{Soc}(D)| = 4$ and $|\operatorname{Soc}(M_1)| = 4$, we have $M = M_1$. But, |D| = 10 > 7 = |M|, which is a contradiction.

 β) Assume that $(C_1 \oplus D_1) \cap \alpha(C_1 \oplus D_1)$ is simple. Then this module is of the form C'_1 (or D'_1), and $M_1 \approx eR/C'_1$. Since $|\operatorname{Soc}(M_1)| = 3$ by Lemma 8, $M = M_1 \oplus M_2$ with M_2 uniform. Hence $|M_2| \leq 2$ by Lemma 8, and so $|M| \leq 8$, which is a contradiction.

Thus, we have shown that $(C_1 \oplus D_1) = \alpha(C_1 \oplus D_1)$ for every unit α in *eRe*. Then $C_1 \oplus D_1$ is a characteristic submodule by Lemma 5, since $J(C_1 \oplus D_1) = 0$. By making use of arguments similar to those employed in α) and β), we may assume that $C_1 \oplus D_2$ is also characteristic. Hence $C_1 = (C_1 \oplus D_1) \cap (C_1 \oplus D_2)$ is characteristic, so that C_1 is a two-sided ideal of R. Consider the factor ring R/C_1 . Then $e(R/C_1)$ is of the form considered in i), which is a contradiction.

Summarizing all above, we see that $C_i=0$ and $D_j=0$ for some $i, j \in \{1, 2\}$. We have thus shown the lemma.

Lemma 10. If $J^3=0$, then every hollow module is isomorphic to one of the following: 1) uniform, 2) eR, 3) eR/A_2 and eR/B_2 ; $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$, and 4) $eR/(A_2 \oplus B_2)$ (see the diagram (2) below).

Proof. This is immediate from Lemma 9 and the proof of Lemma 8. (Note that for any $f: A_1 \rightarrow B_1$, $eJ = A_1 \oplus B_1 = A_1(f) \oplus B_1$.)

Lemma 11. Let R be a right artinian ring satisfying Conditions I^* and

II. Then $eJ = A_1 \oplus B_1$ and both A_1 and B_1 are uniserial.

Proof. Let $J^{n+1}=0$. If $J^3=0$, then the lemma is true by Lemma 9. We proceed by induction on $n \ (\geq 2)$. By Lemma 6 and the induction hypothesis, we have the following cases:

where $A_1 \supset A_2 \supset \cdots \supset A_m$ (resp. $B_1 \supset B_2 \supset \cdots \supset B_n$) is the unique composition series between A_1 and A_m (resp. B_1 and B_n) and $m \le n$. Since $n \ge 3$, we can find a hollow R/J^3 -module as follows:



Hence $C_1=0$ or $C_2=0$ (resp. $D_1=0$ or $D_2=0$) by Lemma 9. Thus we have shown the lemma.

Next we shall show the second part of 1) in Theorem 1. We always assume that $m \leq n$.

Lemma 12. Assume that $A_1/A_2 \approx B_1/B_2$, where $A_2 = J(A_1)$ and $B_2 = J(B_1)$. Then there exists a unit α in eRe such that $\alpha A_1 = B_1$.

Proof. From the proof of Lemma 7, we obtain a unit α in eRe such that $\alpha(A_1+eJ^2)=(B_1+eJ^2)$. Let π_1 be the projection of eJ onto A_1 , and put $f=\pi_1\alpha|A_1$. Then $K=\ker f \neq 0$ by $\alpha(A_1+eJ^2)=(B_1+eJ^2)$, and $\alpha K \subset B_1$. Accordingly $\alpha A_1 \cap B_1 \neq 0$. Let j be an arbitrary element of eJe. Since $eJ(A_2 \oplus B_2) \subset eJ^2$ and $\alpha+j$ is a unit, $(\alpha+j)$ $(A_1 \oplus B_2)=(A_2 \oplus B_1)$. Hence, replacing α by $\alpha+j$ in the above, we get $(\alpha+j)A_1 \cap B_1 \neq 0$. Put $D=eR/A_1 \oplus eR/B_1$. Then D contains a maximal submodule M with a direct summand M_1 isomorphic to $eR/((\alpha+j_0)A_1 \cap A_1)$.

 B_1) for some j_0 in eJe, by Remark 4. If $(\alpha+j_0)A_1 \cap B_1 \neq B_1$ then $|\operatorname{Soc}(M_1)| = 2$, so $M = M_1$. However, $|D| = 2(|A_1|+1)$ by Lemma 7 and $|M| \leq 2|A_1|+1$ by $(\alpha+j_0)A_1 \cap B_1 \neq 0$. This is a contradiction. Hence $(\alpha+j_0)A_1 \cap B_1 = B_1$, and therefore $(\alpha+j_0)A_1 = B_1$.

Lemma 13. Let A_i , B_j be as in the diagram (2). Let f be an element in $\operatorname{Hom}_{\mathbb{R}}(A_i, B_j)$. If f is not extendible to any element in $\operatorname{Hom}_{\mathbb{R}}(A_{i-1}, B_{j-1})$ then $e\mathbb{R}/A_i(f)$ is uniform. In particular, $e\mathbb{R}/A_1(f)$ is uniform.

Proof. Since $eJ = A_1 \oplus B_1 = A_1(f) \oplus B_1$, $eR/A_1(f) \supset eJ/A_1(f) = B_1$, and so $eR/A_1(f)$ is uniform. Assume i < 1, and put ker $f = A_k$. Then k < i. Let N be a submodule of eJ such that $N \supset A_i(f)$ and $|N/A_i(f)| = 1$. If $N \supset A_{k-1}$ then $N \supset A_{k-1} + A_i(f) \supseteq A_i(f)$, and hence $N = A_{k-1} + A_i(f)$. On the other hand, if $N \not$ A_{k-1} then $N_1 = A_k$, and $N_2 = 0$ by $N \supset A_i(f)$. Accordingly, $N = A_{i-1}(g)$ by the remark stated just before Lemma 8, where $g: A_{i-1} \rightarrow B_{j-1}$. Then g is an extension of f, which is a contradiction. Hence $\operatorname{Soc}(eR/A_i(f)) = (A_{k-1} + A_i(f))/A_i(f)$.

Lemma 14. Let A_i be as in Lemma 13 ($i \ge 2$). Let N be a submodule of eR containing A_{i-1} . If N/A_i is uniform then $|N/A_i| \le i-1$.

Proof. Since N/A_i is uniform, N is contained in eJ. Now, considering the projection of eJ/A_i onto A_1/A_i , we can easily see the lemma.

Lemma 15. Let f be an arbitrary element of $\operatorname{Hom}_{\mathbb{R}}(A_1, B_1)$. Then there exists a unit α in eRe such that $A_1(f) = \alpha A_1$.

Proof. If f is an isomorphism then $eJ=A_1\oplus A_1(f)$ and $A_1\approx A_1(f)$, so $A_1(f)=\alpha A_1$ for some α by Lemma 12. Next, assume that f is not an isomorphism. If $A_1\approx B_1$ then $eJ=A_1(f)\oplus B_1$ and $A_1(f)\approx A_1\approx B_1$. Hence there exists a unit β in eRe such that $A_1(f)=\beta B_1$ by Lemma 12, and so $A_1(f)=\beta \alpha A_1$ with some α . Assume $A_1 \approx B_1$, and put $D=eR/A_1\oplus eR/A_1(f)$ (f=0). Let M be such a maximal submodule of D as in the proof of Lemma 3. Then M contains a direct summand isomorphic to $eR/(A_1\cap\alpha A_1(f))$, where α is a unit in eRe (see Remark 4). Now, assume that $K=A_1\cap\alpha A_1(f)\subseteq A_1$. Then $|\operatorname{Soc}(eR/K)|=2$. On the other hand, $|\operatorname{Soc}(D)|=2$ by Lemma 13. Hence $M\approx eR/K$. Since f is not an isomorphism, $A_1(f)\supset A_m$ and $\pi_2\alpha A_1 \equiv B_1$ by Lemma 7. If n=m then $\pi_2\alpha |A_1|$ is not a monomorphism, and $\pi_2\alpha A_m=0$. Hence $\alpha A_m=A_m$, so that $K\supset A_m$. But, then, $|M|\leq n+n-1+1=2n$ and |D|=2n+2, which is a contradiction. Also, if n < m, $|M|\leq m+n+1\leq 2n<2n+1=|D|-1$, a contradiction. We have thus seen that $\alpha A_1(f)=A_1$.

Corollary 16. If $A_1 \approx B_1$ then B_n is a two-sided ideal provided R is a basic ring.

Proof. We have known, from the proof of Lemma 15, that $\alpha B_n = B_n$ for any unit α in *eRe*. Hence B_n is a two-sided ideal of *R*, by Lemma 5.

Lemma 17. Let A_i be as in Lemma 13. Given $f: A_i \rightarrow B_j$, there exists a unit α in eRe such that $A_i(f) = \alpha A_i$.

Proof. We proceed by induction on *i*. If i=1, we are done by Lemma 15. If *f* is extendible to $g \in \operatorname{Hom}_{\mathbb{R}}(A_{i-1}, B_1)$ then, by induction hypothesis, $A_{i-1}(g) = \beta A_{i-1}$ with some unit β in *eRe*. Since A_{i-1} is uniserial, we get $A_i(g) = A_i(f) = \beta A_i$. Henceforth, we assume that *f* is not extendible. Then $eR/A_i(f)$ is uniform by Lemma 13. Put $D = eR/A_i \oplus eR/A_i(f)$, and take such a maximal submodule *M* as in the proof of Lemma 3. Then *M* contains a direct summand M_1 isomorphic to eR/K, where $K = A_i \cap \alpha A_i(f)$ and α is a unit in *eRe* such that $\overline{\alpha} = \overline{e}$. Since $|\operatorname{Soc}(M_1)| = 2$ and $|\operatorname{Soc}(D)| = 3 < |\operatorname{Soc}(M)|$ by Lemma 13, $M = M_1 \oplus M_2$ and M_2 is uniform. Assume now that $A_i \subseteq K$. Then $\operatorname{Soc}(M_1) \approx A_{i'}/K \oplus B_n$ for some $i' \ge i$. Considering the mapping in Lemma 3, we see that M_2 is monomorphic to eR/A_i . Hence $|M_2| \le i-1$ by Lemma 14. Accordingly, $|M| \le |eR| + i - 1 = n + m + i$ and |D| = 2n + 2i. But, as $i \ge 2$ and $n \ge m$, we have a contradiction: |M| + 1 < |D|. Hence $K = A_i$, so that $A_i(f) = \alpha A_i$.

Lemma 18. Let B_j be as in Lemma 13, and let g be in $\operatorname{Hom}_R(B_j, A_1)$. Then $B_j(g) = \beta B_j$, provided g is not a monomorphism, and $B_j(g) = A_j(g^{-1})$, so $B_j(g) = \beta A_j$, provided g is a monomorphism, where β is a unit in eRe and $A_j = g(B_j) \subset A_1$.

Proof. In case n=m, we are done by Lemma 17. We assume henceforth m < n. Since the second assertion is clear from Lemma 17, we may further assume that g is not a monomorphism. Then ker $g \supset B_n$ and g induces \overline{g} : $B_j/B_n \rightarrow A_1$. By Corollary 16 and induction on the nilpotency index of J, we can see that there exists a unit $\overline{\beta}$ in $eRe/eJ^n e$ such that $(B_j/B_n)(\overline{g}) = \overline{\beta}(B_j/B_n)$. This together with $B_i(g) \cap \beta B_j \supset B_n$ gives $B_j(g) = \beta B_j$.

Thus we have completed the proof of Theorem 1 2), by the induction on *n*. Next, we shall show Theorem 1 3). In view of Theorem 1 2), we may assume that N is a trivial submodule $A_i \oplus B_j$ of eJ.

Lemma 19. If N contains eJ^t for some t, $\Delta(N) = (\Delta(N/eJ^t) \text{ in } R/J^t)$.

Proof. This is clear.

Here, we quote the following condition in [3]:

(**) every maximal submodule of any finite direct sum D of hollow modules contains a non-zero direct summand of D.

Lemma 20. $[\Delta: \Delta(A_1)] \leq 2.$

Proof. In view of [4], Theorem 2, it suffices to show that (**) is satisfied for $D=eR/A_1\oplus eR/A_1\oplus eR/A_1$. Let M be a maximal submodule in D. Then, by Condition I*, there exists a direct summand M_1 of M with $M_1 \oplus J(D)$, where $M_1 \approx eR/K$. Let ρ be the natural epimorphism of eR to M_1 , and π_i the projection of D onto the *i*-th component. Then $\pi_i \rho$ is given by the left multiplication of an element α_i in eRe. Since $M_1 \oplus J(D)$, we may assume that α_1 is a unit. Further, $\alpha_1 K$ being contained in A_1 , we may assume that $K \subset A_1$.

i) If $K=A_1$ then $|M_1|=|eR/A_1|$. Hence $\pi_1\rho$ is an isomorphism, so M_1 is a direct summand of D.

ii) If $K \subseteq A_1$ then $|M_1| = n + m - k$ and |D| = 3(n+1), where k = |K|. On the other hand, $|\operatorname{Soc}(D)| = 3$ and $|\operatorname{Soc}(M_1)| = 2$. Hence $M = M_1 \oplus M_2$ with a uniform M_2 . Since M_2 is monomorphic to eR/A_1 , we have $n+1 \ge |M_2| = 3n+2-(n+m+1-k)=(2n-m)+1+k$, which implies n=m and k=0. Then $|M_2| = n+1$ and M_2 is isomorphic to eR/A_1 via some π_i . Therefore M_2 is a direct summand of D.

Lemma 21. $[\Delta: \Delta(B_1)] \leq 2.$

Proof. In case m=n, we are done by Lemma 20. If m < n then $B_1 \supset eJ^{m+1} = 0$, and so $\Delta(B_1) = \Delta(B_1/eJ^{m+1})$ by Lemma 19. On the other hand, by [3], Theorem 12 and induction on the nilpotency index of J, we can show that $[\Delta: \Delta(B_1/eJ^{m+1})] \le 2$.

Lemma 22. $[\Delta: \Delta(N)] \leq 2$ for every submodule N of eJ.

Proof. We may assume that $N = A_i \oplus B_j$. Then $\Delta(N) \supset \Delta(A_j)$ $(i \le j)$ or $\Delta(N) \supset \Delta(B_j)$ $(i \ge j)$. Further, since A_1 and B_1 are uniserial, $\Delta(A_1) \subset \Delta(A_j)$ and $\Delta(B_1) \subset \Delta(B_j)$. Hence $[\Delta: \Delta(N)] \le [\Delta: \Delta(A_1)] \le 2$ or $[\Delta: \Delta(N)] \le [\Delta: \Delta(B_1)] \le 2$ by Lemma 20 or Lemma 21.

Lemma 23. Let A_i and B_i be as in Lemma 13. If $\beta A_i = B_i$ for some *i* with a unit β in eRe, then $\overline{\beta} \notin \Delta(A_i)$ and $[\Delta: \Delta(A_i)] = 2$.

Proof. Let j be an arbitrary element in eJe. Then $(\beta+j)A_i \subset B_i+jA_i$. Since $jA_i \subset eJ^{i+1}$, we have $\pi_1(B_i+jA_i) \subset A_i$, where π_1 is the projection of eJ onto A_1 . Hence $(\beta+j)A_i \neq A_i$, so that $\Delta \neq \Delta(A_i)$, and therefore $[\Delta: \Delta(A_i)]=2$ by Lemma 22.

Lemma 24. Assume that $\beta A_1 = B_1$ with a unit β in eRe. If δ is a unit in eRe such that $\delta \oplus \Delta(A_i)$ for some j, then $\pi_2 \delta$: $A_i \rightarrow B_i$ is an isomorphism, where π_2 is the projection of eJ onto B_1 .

Proof. Since \bar{e} , $\bar{\beta}$ are independent over $\Delta(A_i)$, $\bar{\delta}=\bar{x}+\bar{\beta}\bar{y}$ for some \bar{x} , $\bar{y} \in \Delta(A_i)$ with $xA_i=A_i$ and $yA_i=A_i$. Let a_i be a generator of A_i . Then βya_i is a generator of B_i . Hence $\pi_2 \delta | A_i$ is an epimorphism of A_i onto B_i , and hence

an isomorphism (cf. the proof of Lemma 23).

Lemma 25. If $A_1 \not\approx B_1$ then $\Delta = \Delta(N)$ for every submodule N of eJ. If $A_1 \approx B_1$, then $[\Delta: \Delta(A_i \oplus B_j)] = 2$ provided $i \neq j$, and $\Delta(A_i \oplus B_i) = \Delta; \Delta(A_i \oplus B_j) = \Delta(A_i) = \Delta(A_i)$ for i < j.

Proof. Put $D = eR/A_1 \oplus eR/A_1 \oplus eR/B_1$. Then |Soc(D)| = 3. Let M be a maximal submodule of D, and $M=M_1\oplus M_2\oplus \cdots$ with hollow modules $M_1\approx$ eR/E_l . Since $|\overline{M}|=2$, we may assume that $\overline{M}_1 \neq 0$ and $\overline{M}_2 \neq 0$. Let π_h be the projection of D onto the h-th component. Then $\pi_k | M_1$ is an epimorphism for some k. Hence $E_1 \subset \alpha A_1$ or αB_1 , where α is a unit in eRe. If $E_1 \neq \alpha A_1$ (or αB_1) then $|\operatorname{Soc}(eR/E_1)|=2$. Therefore either E_1 or E_2 coincides with αA_1 or αB_1 , and so $\pi_k | M_1$ is an isomorphism. Accordingly, (**) is satisfied for D. Hence, if $[\Delta: \Delta(A_1)]=2$, there exists a unit α' such that $\alpha'A_1 \subset B_1$ (or $\alpha'B_1 \subset A_1$) A_1), by [5], Proposition 1. Since $A_1 \oplus eJ^2$, we get $\alpha' A_1 = B_1$ (or $\alpha' B_1 = A_1$). Conversely, assume that $A_1 \approx B_1$. Then, by Lemma 12, there exists a unit β such that $\beta A_1 = B_1$. Hence $[\Delta: \Delta(A_1)] = 2 = [\Delta: \Delta(B_1)]$ by Lemma 23, and $\Delta(A_i) = \Delta(A_1)$ by $\Delta(A_1) \subset \Delta(A_i)$. Now, let N be an arbitrary submodule of eJ. Then we may assume that $N=A_i\oplus B_j$. If i=j then $N=eJ^i$, and so $\Delta = \Delta(N)$. If i > j then $\overline{\beta} \notin \Delta(N)$ by Lemma 24 (cf. the proof of Lemma 23), and hence $[\Delta: \Delta(N)]=2$, and either $\Delta(N)=\Delta(A_1)$ or $\Delta(B_1)$. Finally, if $\Delta=$ $\Delta(A_1)$ then $\Delta = \Delta(B_1)$ from the above. Hence $\Delta = \Delta(N)$.

3 Proof of Theorems 2 and 2'. We assume that R satisfies Condition I* and either Condition II' or Condition II''. Then, by Corollary 2, R satisfies Condition II. Hence the assumptions of Theorem 1 are fulfilled. Further, it is clear that $\Delta = \Delta(N)$ for every submodule N of *eJ* by Condition II' or II''. It suffices therefore to show the equivalence of 1) and 3) in Theorems 2 and 2'.

Let $A_i \supset A_j$ and $B_{i'} \supset B_{j'}$ be as in Theorem 1, and assume that there exists $f: A_i | A_j \approx B_{i'} | B_{j'}$. Put $N = \{x + y \in eJ \mid x \in A_i, y \in B_{i'}, f(x + A_j) = y + B_{j'}\}$. Then N is a submodule of eJ containing $A_j \oplus B_{j'}$. On the other hand, since every submodule in eJ is characteristic by Condition II', N is a trivial submodule by Theorem 1 2). Hence f=0, which shows the "only if" part of Theorems 2 and 2'. We shall show the "if" part. We shall show, by induction on the nilpotency index n of J, that if the condition 3) in Theorem 2 (resp. Theorem 2') is satisfied then R satisfies Condition I and II' (resp. Condition I). In order to show that R satisfies Condition I, it suffices to show the following:

(*) every maximal submodule in any finite direct sum of hollow modules is also a direct sum of hollow modules (cf. [3]).

Further, as was shown in [3], \$3, we may consider a direct sum of hollow modules which are homomorphic to eR for a fixed e.

Lemma 26. If the condition 3) in Theorem 2 is satisfied, then R satisfies Condition II'.

Proof. This is clear (cf. [6]).

Lemma 27. Assume that R satisfies Condition II' (or Condition II''). Let $\{eR/D_h\}_{h=1}^t$ be a family of hollow modules. If $D_i \subset D_j$ for some *i* and *j*, then (**) is satisfied for $D = \sum_{h=1}^t \bigoplus eR/D_h$.

Proof. Let π_h be the projection of D onto eR/D_h . Take a maximal submodule M in D. If $\pi_l(M) \neq eR/D_l$ for some l then $M = eJ/D \oplus \sum_{h \neq l} \oplus eR/D_h$. Hence we may assume that $\pi_h(M) = \epsilon R/D_h$ for all h. Setting $\overline{D} = D/J(D)$, we may regard \overline{D} as a t dimensional vector space over $\Delta = eRe/eJe$ (note that R is basic). Further we may assume that $D_1 \subset D_2$. Since $\pi_h(M) = eR/D_h$ for all h, $\overline{M} = M/J(D)$ contains a subspace $S = (\overline{e}, \overline{ek}, \overline{0}, \dots, \overline{0})\Delta$ (note that k is a central element of R for the case of Theorem 2'). Since $D_1 \subset D_2$, this simple subspace S is lifted to a direct summand M_1 of D by [1], Theorem 2 and its proof. Then $M_1 \subset M$, proving (**) for D.

In view of Lemma 27, it remains to show that (*) is satisfied for $D = \sum_{h=1}^{t} \bigoplus eR/D_h$ provided $D_i \oplus D_j$ for all distinct *i*, *j*. Let *M* be a maximal submodule in *D*, and let π_h be as above. As was claimed in the proof of Lemma 27, we may restrict ourselves to the case where $\pi_h(M) = eR/D_h$ for all *h*. Then we can take such a basis of $\overline{M} = M/J(D)$ as $\{\alpha_i = (\overline{0}, \dots, \overline{e}, \overline{eR}_h, \overline{0}, \dots, \overline{0}\}_{h=1}^{t-1}$, where $k_h \in eRe$ (central elements of *R* for the case of Theorem 2'). We assume that *eR* has the structure given in Theorem 2 (resp. Theorem 2'), i.e.,

In the case of Theorem 2', $D_h = \alpha(A_r \oplus B_s)$. Hence $eR/D_h \approx eR/(A_r \oplus B_s)$. Accordingly, we may assume that all D_h are trivial submodules. If all D_h contain B_n (resp. $A_n \oplus B_n$ for the case m=n), all eR/D_h are hollow R/J^n -modules. Hence, by induction hypothesis. (*) is satisfied for D. Thus, in what follows, we consider the case where some D_h is equal to A_i ; $1 \le i \le m$ (resp. B_j ; $1 \le j \le n$).

Therefore we should check the following cases:

1) $D_1 = A_i, D_k = A_{j_k} \oplus B_{j_k}; i < i_1 < i_2 < \cdots < i_p, j_1 > j_2 > \cdots > j_p.$

2) $D_1 = A_i$, $D_2 = B_j$, $D_k = A_{i_k} \oplus B_{j_k}$; $i < i_1 < i_2 < \cdots < i_p$, $j_1 > j_2 > \cdots > j_p$. However, 2) is a special case of 1) obtained by putting $i_p = n+1$ and $j_p = j$. So, we consider the case 1): $D = eR/A_i \oplus eR/(A_{i_1} \oplus B_{j_1}) \oplus \cdots \oplus eR/(A_{i_p} \oplus B_{j_p})$.

So, we consider the case 1): $D = eR/A_i \oplus eR/(A_{i_1} \oplus B_{j_1}) \oplus \cdots \oplus eR/(A_{i_p} \oplus B_{j_p})$. Then $|D| = (n+i) + \sum_{s=1}^{p} (i_s+j_s-1)$. Set $M^* = A_1/A_i \oplus \sum_{s=1}^{p} \oplus eR/(A_{i_s} \oplus B_{j_{s-1}}) \oplus B_1/B_{j_p}$ $(B_{j_0} = 0)$. Then $|M^*| = |D| - 1$. Define a homomorphism φ of M^* to D by setting

$$\begin{aligned} \varphi((x+A_i) + \sum_{s=1}^{b} (y_s + (A_{i_s} \oplus B_{j_{s-1}})) + (z+B_{j_p}) &= (x+y_1 + A_i) \\ + (ek_1 y_1 + y_2 + (A_{i_1} \oplus B_{j_1})) + (ek_2 y_2 + y_3 + (A_{i_2} \oplus B_{j_2}) + \cdots \\ + (ek_p y_p + z + (A_{i_p} \oplus B_{j_p})), \end{aligned}$$

where $x \in A_1$, $y_s \in eR$, $z \in B_1$ and k_s are central elements for the case of Theorem 2'. Now, let $(x+A_i)+\sum_{s=1}^{p}(y_s+(A_{i_s}\oplus B_{j_{s-1}}))+(z+B_{j_p})$ be in ker φ . Since x and z are in eJ, y_s are all in eJ. Set $y_s=y_{s1}+y_{s2}$ $(y_{s1}\in A_1, y_{s2}\in B_1)$. Since $x+y_1=x+y_{11}+y_{12}\in A_i$, we have $y_{12}=0$. Then $ek_1(y_{11}+y_{12})+(y_{21}+y_{22})\in A_{i_1}\oplus B_{j_1}$ implies that $y_{22}\in B_{j_1}$ and $ek_1y_{11}+y_{21}\in A_{i_1}$, and $ek_2(y_{21}+y_{22})+(y_{31}+y_{32})\in A_{i_2}\oplus B_{j_2}$ implies that $y_{32}\in B_{j_2}$ (note that $B_{j_1}\subset B_{j_2}$). Repeating this procedure, we see that $y_{s2}\in B_{j_{s-1}}$ and $z\in B_{j_p}$. Similarly, from the fact that $ek_p(y_{p1}+y_{p2})+z\in A_{i_p}\oplus B_{j_p}$, it follows that $y_{p1}\in A_{i_p}$, \cdots , $y_{s1}\in A_{i_s}$, \cdots , $y_{11}\in A_i$, and $x\in A_i$ (note that k_s are central elements for the case of Theorem 2'). Hence φ is a monomorphism and $\operatorname{im} \varphi = \overline{M}$. Therefore, noting that $M \supset J(D)$, we see that $M \approx M^*$.

4 Rings with $J^3=0$. We have shown in [3] that the converse of Theorem 1 is true provided $J^2=0$. In this section, we shall show that the same is still true for the case $J^3=0$, namely if $J^3=0$ then 1)~3) in Theorem 1 imply Condition I.

Lemma 28. Assume that $\beta A_1 = B_1$. If α is a unit in eRe such that $\overline{\alpha} \notin \Delta(A_i)$ for some, $A_i \cap \alpha A_i = 0$.

Proof. Let a_i be a generator of A_i . Then, by Lemma 24, $\alpha a_i = a'_1 + b_i$, where $a'_1 \in A_i$, $b_i \in B_i$ and $\notin B_{i+1}$. Hence $\alpha A_n = \alpha a_i J^{n-i} = (a'_i + b_i) J^{n-1} \oplus A_n$, and therefore $\alpha A_i \cap A_i = 0$.

In order to show that R satisfies Condition I, it suffices to show that R satisfies (*). Further, as is claimed in §3, we may restrict ourselves to the case that hollow direct summands in (*) are isomorphic to eR/E for a fixed primitive idempotent e. We shall divide the proof into two cases: 1) $A_1 \approx B_1$, and 2) $A_1 \approx B_1$.

1) $A_1 \approx B_1$. By 3) in Theorem 1, $\Delta = \Delta(N)$ for every submodule N of eJ. This situation is very similar to that in [3], and we can apply the argument

employed in [3] to see that R satisfies Condition I.

2) $A_1 \approx B_1$. There holds $\Delta(A_1) = \Delta(A_2) = \Delta(A_1 \oplus B_2)$. We shall give the explicit form of a maximal submodule M in $D = \sum_{k=1}^{t} \oplus N_k$, where N_k are hollow modules isomorphic to eR/E_k . Now, by 1) in Theorem 1, $B_1 = \alpha A_1$ for some unit α . If i > j then $\alpha(A_i \oplus B_j) \subset A_j \oplus B_i$, for $\alpha A_i = B_i$. Hence $\alpha(A_i \oplus B_j) = A_j \oplus B_i$. Therefore $eR/(A_i \oplus B_j) \approx eR/(A_j \oplus B_i)$. Consequently, we may assume that $N_k \approx eR/(A_i \oplus B_j)$ for some $i \le j$.

i) t=2.

(1) $D=eR/A_2 \oplus eR/A_2$. Then we may assume that $\overline{M}=(\overline{e+A_2})\Delta \oplus (\overline{\alpha+A_2})\Delta$, where α is a unit in eRe.

 α) $\alpha A_2 = A_2$. Then *M* contains a direct summand of *D* by [1], Theorem 2. Hence *M* is a direct sum of hollow modules.

 β) $\alpha A_2 \cap A_2 = 0$. Put $M^* = eR \oplus A_1/A_2 \oplus A_1/A_2$, and define a homomorphism φ of M^* into D by setting

$$\varphi(z_1+(z_2+A_2)+(z_3+A_2))=(z_1+z_2+A_2)+(lpha z_1+z_3+A_2)$$
 ,

where $z_1 \in eR$ and z_2 , $z_3 \in A_1$. Suppose $z_1 + z_2 + z_3$ be in ker φ . Then z_i are in eJ. Set $z_1 = x_1 + y_1$ ($x_1 \in A_1$, $y_1 \in B_1$). Since $z_1 + z_2 \in A_2$, we have $y_1 = 0$. If $x_1 \neq 0$ then $\pi_2 \alpha z_1 = \pi_2 \alpha x_1 \neq 0$ by Lemma 24, where π_2 is a projection of eJ onto B_1 However, $0 = \pi_2(\alpha z_1 + z_3) = \pi_2 \alpha z_1$. This contradiction shows that $x_1 = 0$, and so $z_1 = 0$. Now, it is clear that $\varphi(M^*) = M$ by $|M^*| = |M|$.

(2) $D=eR/A_1\oplus eR/A_1$. Let M be as above. If $(\alpha+j)A_1=A_1$ for some $j\in eJe$ then we are done by [1], Theorem 2. On the other hand, if $(\alpha+j)A_1 = A_1$ for every $j\in eJe$ then $\overline{\alpha} \notin \Delta(A_1)$, and so $\alpha A_1 \cap A_1=0$ by Lemma 29. Hence $M \approx eR$.

(3) $D=eR/A_1\oplus eR/A_2$. If $\alpha A_2 \subset A_1$, we are done. Next, if $\alpha A_2 \cap A_1=0$ then $M \approx eR \oplus A_1/A_2$ via $\varphi(z_1+(z_2+A_2))=(z_1+A_1)+(\alpha z_1+z_2+A_2)$ (note that $\alpha A_1 \neq A_1$).

(4) $D = eR/A_2 \oplus eR/(A_1 \oplus B_2)$, $eR/A_2 \oplus eR/(A_2 \oplus B_2)$ or $eR/A_2 \oplus eR/(A_2 \oplus B_1)$ $(eR/A_1 \oplus eR/(A_1 \oplus B_1))$. Since $\alpha A_2 \subset A_2 \oplus B_2$ ($\alpha A_1 \subset A_1 \oplus B_1$), *M* contains a direct summand of *D*, by [1], Theorem 2.

(5) $D = eR/A_1 \oplus eR/(A_2 \oplus B_2)$. Note that either $\rho_1 = \pi_1 \alpha | A_1$ or $\rho_2 = \pi_2 \alpha | A_1$ is an isomorphism. If ρ_1 (resp. ρ_2) is an isomorphism, then $M \approx eR/A_2 \oplus B_1/B_2$ (resp. $eR/A_2 \oplus A_1/A_2$).

(6) $D = eR/A_1 \oplus eR/(A_1 \oplus B_2)$. Since $\alpha A_1 \cap (A_1 \oplus B_2)$ is either a simple module B'_2 or A_1 , $M \approx eR/\alpha^{-1}(B'_2) \approx eR/A_2$ or M is a direct summand of D.

(7) Other cases can be reduced to the case $J^2=0$ [3].

ii) t=3. If N_1 , N_2 and N_3 are linearly ordered by inclusion, then M contains a direct summand of D, by [5], Corollary 1. Hence, it suffices to consider the following two cases:

(1) $D = eR/A_1 \oplus eR/A_2 \oplus eR/(A_2 \oplus B_2)$. Since $eReA_2 \subset A_2 \oplus B_2$, M contains

a direct summand of D.

(2) $D = eR/A_1 \oplus eR/A_1 \oplus eR/(A_2 \oplus B_2)$. We may assume that \overline{M} has a basis $\{\xi = (\overline{e+A_1}) + \overline{0} + (\overline{\delta_2 + (A_2 \oplus B_2)}), \eta = (\overline{0} + (\overline{e+A_1}) + (\overline{\delta_2 + (A_2 \oplus B_2)})\}$, where δ_1, δ_2 are units in eRe (see [3], §3).

 α) Assume that there exists a unit x such that $\bar{x} \in \Delta(A_1)$ and $\bar{\delta}_2 = \bar{\delta}_1 \bar{x}$. Then $\xi + \eta \bar{x} = (\bar{e}, \bar{e}x, \bar{0}) \in \bar{M}$. Since $\bar{x} \in \Delta(A_1)$, M contains a direct summand of D, by [1], Theorem 2.

 $\beta) \text{ Assume that } \overline{\delta_2}^{-1} \overline{\delta_1} \oplus \Delta(A_1). \text{ Put } M^* = eR/A_2 \oplus eR/A_2, \text{ and define a homomorphism } \varphi \text{ of } M^* \text{ to } D \text{ by setting } \varphi((z_1 + A_2) + (z_2 + A_2)) = (z_1 + A_1) + (z_2 + A_1) + (\delta_1 z_1 + \delta_2 z_2 + (A_2 \oplus B_2)), \text{ where } z_i \in eR. \text{ Suppose that } (z_1 + A_2) + (z_2 + A_2) \text{ is in ker } \varphi. \text{ Then } z_i \in A_1. \text{ If } z_1 \oplus A_2 \text{ then } \delta_1 A_1 = \delta_1 z_1 R \subset \delta_2 A_1 + (A_2 \oplus B_2). \text{ Hence } \delta_2^{-1} \delta_1 A_1 \subset A_1 + \delta_2^{-1} (A_2 \oplus B_2) = A_1 \oplus B_2, \text{ which contradicts Lemma 24. Hence } z_1 \in A_2, \text{ and similarly } z_2 \in A_2. \text{ Therefore } M \approx M^*.$

iii) $t \ge 4$. In view of [4], Lemma 1 and Theorem 1 and [5], Corollary 1, this case can be reduced to the cases i) and ii).

Thus we have shown that R satisfies Condition I provided $J^3=0$.

5 Examples. 1. Let $L_1 \subset L_2 \subset L_3 \subset \cdots$ be fields. Set

| 1 | L_1 | 0 | L_3 | L_4 | L_5 \ |
|-----|-------|-------|---------|---------|---------|
| | 0 | L_2 | L_{3} | L_4 | L_5 |
| R = | 0 | 0 | L_{3} | 0 | 0 |
| | 0 | 0 | 0 | L_{4} | L_5 |
| (| 0 | 0 | 0 | 0 | L_5 |

Then

Hence R satisfies Conditions I and II'. However, R is not left serial (cf. [7] and [8]). If $[L_3: L_1] \ge 2$ then R does not satisfy Condition I for hollow left R-modules.

2. Let K be a field and let R be a vector space over K with basis $\{e_1, x_{11}, y_{12}, x_{12}, e_2, x_{22}, y_{21}, x_{21}\}$. Define $e_i e_j = e_i \delta_{ij}$, $e_i x_{jk} e_j = x_{jk} \delta_{ij} \delta_{kp}$, $e_i y_{jk} e_p = y_{ik} \delta_{ij} \delta_{kp}$, $x_{11} x_{12} = y_{12}$ and $x_{22} x_{21} = y_{21}$. Putting other multiplications to be zero, we see that R is a ring with $J^3 = 0$. We can easily see that R satisfies the conditions in Theorem 2' as both left and right R-modules. Further R satisfies Condition II' as a left R-module. Next, let $R_1 = \langle e_1, x_{11}, y_{13}, x_{13}, e_2, x_{22}, y_{21}, x_{21}, e_3, x_{31}\rangle$. Define $x_{11} x_{13} = y_{13}$ and the same as above for others. Then R_1 satisfies

the conditions in Theorem 2' as a right R-module, but does not as a left R-module, since $Je_1 = Rx_{11} \oplus Rx_{21} \oplus Rx_{31}$.

3. Let $K \subset L$ be fields with [L: K] = 2, say $L = K \oplus uK$. Put

$$R = \left(\begin{array}{rrrr} L & L & L \\ 0 & K & K \\ 0 & 0 & K \end{array}\right)$$

Then $e_{11} J = (0, K, K) \oplus (0, uK, uK)$. Hence R satisfies 1 > 3 in Theorem 1 and $A_1 \approx B_1$. Therefore R satisfies neither Condition II' nor Condition II''. But, by [8], R is of right local type.

4. Let k be a field, and x an indeterminate. Put L=k(x) and $K=k(x^2)$. Take a left L-vector space V=Lu of one dimension. Putting $ux=x^2u$ and uk=ku for all $k \in K$, we make V a right L-vector space (see [3], Example 2). Put

$$R = \begin{pmatrix} L & 0 & L & L & L \\ 0 & L & V & 0 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & L & L \\ 0 & 0 & 0 & 0 & L \end{pmatrix}$$

Then $e_{11}J=A_1\oplus B_1$ with $A_1 \approx B_1$, $e_{22}J=A_1'\oplus B_1'$ with $A_1'\approx B_1'$. Further, $Je_{33}=A_1''\oplus B_1''$ with $A_1''\approx B_1''$ as left *R*-modules. Hence *R* satisfies Condition I for both left and right hollow *R*-modules.

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