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PSEUDO-ANOSOV HOMEOMORPHISMS WHICH EXTEND TO ORIENTATION REVERSING HOMEOMORPHISMS OF S^3

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1. Introduction

Let F be the orientable closed surface of genus $g(>1)$, and $\{l_1, \dots, l_g\}$ be a system of mutually disjoint, non-parallel essential loops on F such that $\cup l_i$ cuts F into a $2g$ punctured sphere. Let f be a self-homeomorphism of F . Then \bar{M}_f denotes the 3-manifold which is obtained from $F \times [0, 1]$ by attaching 2-handles along the simple loops $l_1 \times \{0\}, \dots, l_g \times \{0\}, f(l_1) \times \{1\}, \dots, f(l_g) \times \{1\}$. We note that $\partial \bar{M}_f$ consists of two 2-spheres. Then M_f denotes the closed 3-manifold which is obtained from \bar{M}_f by capping off the boundary by 3-cells. It is easy to see that if g is isotopic to f , then M_g is homeomorphic to M_f . Then, in [7], T. Yoshida posed the following question.

Question (Yoshida). Suppose that f is a pseudo-Anosov homeomorphism. Does there exist a constant n_f such that $\pi_1(M_{f^n}) \neq \{1\}$ for all $n > n_f$?

In this note we will give a negative answer to this question.

Theorem. *For each $g(>1)$ there exist infinitely many pseudo-Anosov homeomorphisms f such that*

$$M_{f^{2n}} = \#_{i=1}^g (S^2 \times S^1)_i, \quad \text{and}$$

$$M_{f^{2n+1}} = S^3, \quad \text{where } S^m \text{ denotes the } m\text{-dimensional sphere.}$$

Actually we will show that there are infinitely many pseudo-Anosov homeomorphisms of Heegaard surfaces of S^3 which have the property described in the title of this note (Theorem 2.1).

In the following, we assume that the reader is familiar with [2]. For the definitions of standard terms in the 3-dimensional topology, we refer to [4].

2. Proof of Theorem

In this section we will give the proof of Theorem.

Theorem 2.1. *Let $(V_1, V_2; F)$ be a genus $g(>1)$ Heegaard splitting of the 3-sphere S^3 (Figure 1). Then there exist infinitely many orientation reversing self-homeomorphisms g' of S^3 such that $g'(V_1)=V_2$, $g'(V_2)=V_1$, and $g'|_F: F \rightarrow F$ has mutually distinct classes of pseudo-Anosov homeomorphisms.*

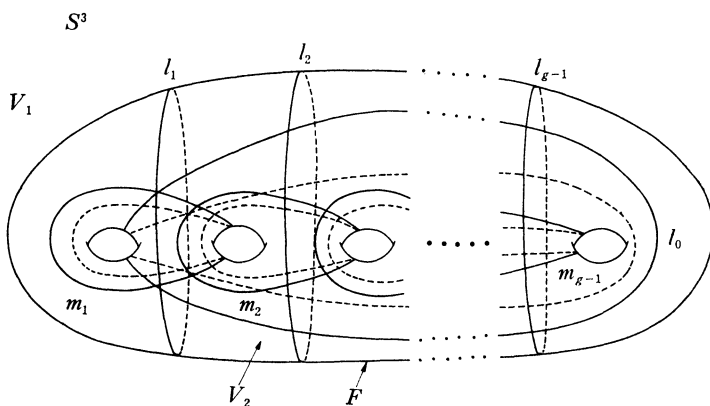


Figure 1

The key of the proof of Theorem 2.1 is Lemma 2.2, which is a generalization of a result of I. Aitchison [1].

Lemma 2.2. *There exist infinitely many ambient isotopies f_t , $(0 \leq t \leq 1)$ of S^3 such that $f_1(V_1)=V_1$, $f_1(V_2)=V_2$, and $f_1|_F: F \rightarrow F$ has mutually distinct classes of pseudo-Anosov homeomorphisms.*

Proof. Let $l_0, l_1, \dots, l_{g-1}, m_1, \dots, m_{g-1}$ be the system of simple loops on F as in Figure 1. We note that each simple loop bounds a disk in $V_i (i=1, 2)$. If p is a simple loop on F , then T_p denotes a right hand Dehn twist along p . Let m be the image of m_{g-1} by the homeomorphism $T_{m_1} \circ T_{m_2} \circ \dots \circ T_{m_{g-2}}: F \rightarrow F$, and l be the image of l_0 by the homeomorphism $T_{l_1} \circ T_{l_2} \circ \dots \circ T_{l_{g-1}}: F \rightarrow F$. Then we may suppose that l, m intersect transversely, and the number of the intersections is minimal.

Assertion. l, m fill up F i.e. each component of $F-(l \cup m)$ is an open disk.

Proof. By [2, Theorem 5.13], we see that m is carried by the train track τ as in Figure 2, where the numbers in Figure 2 denote the weights which represent m . On the other hand, we draw a picture of l as in Figure 3. Then we directly see that $\tau \cup l$ cuts F into a system of disks. Moreover, by seeing the weights on τ , we easily verify that the assertion holds. We show pictures of $l \cup m$ in the case of $g=2, 3$ to convince the reader that the assertion holds (Figure 4).

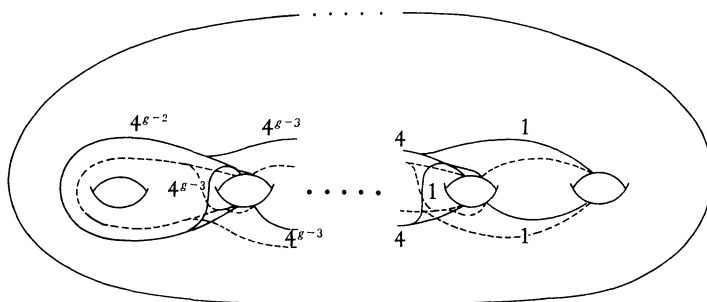


Figure 2

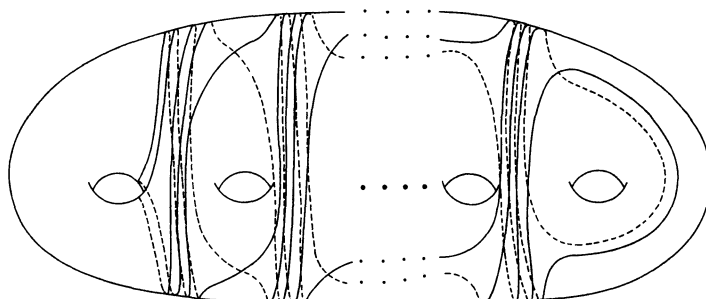


Figure 3

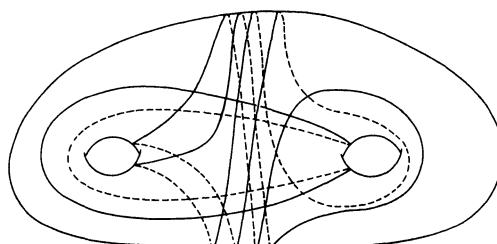


Figure 4 (i)

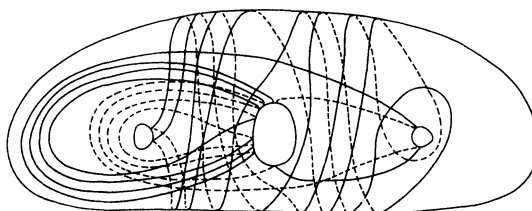


Figure 4 (ii)

We note that l, m bound disks in $V_i (i=1, 2)$. Hence, there are homeomorphisms $f_l, f_m: S^3 \rightarrow S^3$ such that f_l, f_m are ambient isotopic to the identity map, $f_l(V_i) = f_m(V_i) = V_i$ and $f_l|_F = T_l, f_m|_F = T_m$. By [3, Expose 13], we see that

$T_l \circ T_m^{-n}$ is a pseudo-Anosov class of F provided $n > 0$. Moreover $T_l \circ T_m^{-n}$ and $T_l \circ T_m^{-n'}$ have pairwise distinct invariant laminations if $n \neq n'$. Hence, we have infinitely many ambient isotopies of S^3 which satisfies the conclusions of Lemma 2.2.

We need two more lemmas for the proof of Theorem 2.1.

Lemma 2.3 ([5, Lemma 2.4]). *Let $f: F \rightarrow F$ be a pseudo-Anosov homeomorphism with invariant laminations $L^+, L^- \in PL(F)$. Suppose that $R: F \rightarrow F$ is an infinite order reducible map. Then $R(L^+) \neq L^+, L^-$.*

Lemma 2.4 ([6, Theorem A]). *Let f, L^+, L^- be as in Lemma 2.3. Let $g: F \rightarrow F$ be a homeomorphism such that $g(L^-) \neq L^+$. Then there exists a number k_0 such that $f^k \circ g$ is isotopic to a pseudo-Anosov homeomorphism for all $k \geq k_0$.*

Proof of Theorem 2.1. Let $a_1, \dots, a_g, b_1, \dots, b_g, c$ be a system of oriented simple loops on F as in Figure 5. Then there exists an ambient isotopy h_t ($0 \leq t \leq 1$) of S^3 such that $h_1(V_1) = V_2, h_1(V_2) = V_1$, and $h_1(a_i) = b_i, h_1(b_i) = a_i$ ($i = 1, \dots, g$), $h_1(c) = \bar{c}$ (Figure 6), where $\bar{}$ means the specified loop with the opposite orientation.

On the other hand, there exists an orientation reversing involution $g: S^3 \rightarrow S^3$ such that $g(V_i) = V_i$, and $g(a_j) = \bar{a}_j, g(b_j) = \bar{b}_j$ ($j = 1, \dots, g$). If needed, by adding twists along c to h_1 , we may suppose that $h_1 \circ g|_F$ is an infinite order re-

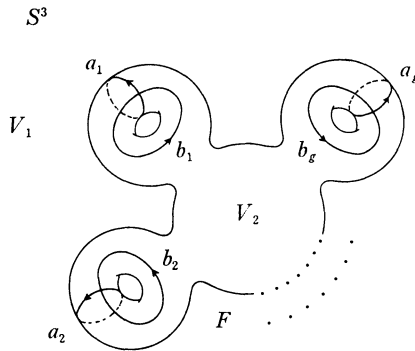


Figure 5

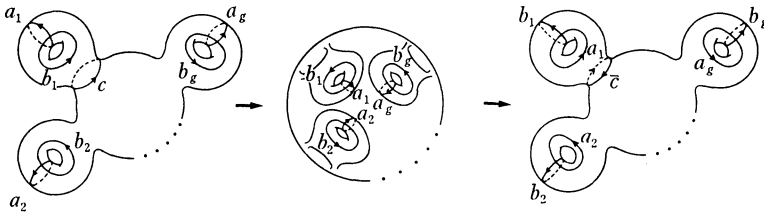


Figure 6

ducible map. Then, by Lemmas 2.3, 2.4, we see that $g' = (f_1)^n \circ h_1 \circ g$ satisfies the conclusion of Theorem 2.1 for sufficiently large n , where f_1 is a homeomorphism obtained in Lemma 2.2. Moreover, if n grows larger, then the stable lamination of $g'|_F$ tends to that of $f_1|_F$. Hence we have infinitely many g' 's with mutually distinct invariant laminations.

Proof of Theorem. Let g' be a homeomorphism obtained in Theorem 2.1. Then clearly the homeomorphism $g'|_F$ with the system of loops $\{a_1, \dots, a_g\}$ satisfies the conclusion of Theorem.

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