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PSEUDO-ANOSOV HOMEOMORPHISMS WHICH EXTEND TO ORIENTATION REVERSING HOMEOMORPHISMS OF S^3

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1. Introduction

Let F be the orientable closed surface of genus $g(>1)$, and $\{l_1, \cdots, l_g\}$ be a system of mutually disjoint, non-parallel essential loops on F suth that $\cup l_i$ cuts F into a $2g$ punctured sphere. Let f be a self-homeomorphism of F. Then $\bar{M}_f$ denotes the 3-manifold which is obtained from $F\times[0, 1]$ by attaching 2-handles along the simple loops $l_1\times\{0\}, \cdots, l_g\times\{0\}, f(l_1)\times\{1\}, \cdots, f(l_g)\times\{1\}$. We note that $\partial\bar{M}_f$ consists of two 2-spheres. Then M_f denotes the closed 3-manifold which is obtained from $\bar{M}_f$ by capping off the boundary by 3-cells. It is easy to see that if g is isotopic to f, then M_g is homeomorphic to M_f. Then, in [7], T. Yoshida posed the following question.

Question (Yoshida). *Suppose that f is a pseudo-Anosov homeomorphism. Does there exist a constant n_f such that $\pi_1(M_{f^n})\neq\{1\}$ for all $n>n_f$?*

In this note we will give a negative answer to this question.

Theorem. *For each $g(>1)$ there exist infinitely many pseudo-Anosov homeomorphisms f such that*

$$M_{f^{2n}} = \mathop{\#}_{i=1}^{g} (S^2\times S^1)_i\,, \quad \textit{and}$$

$$M_{f^{2n+1}} = S^3, \quad \textit{where } S^m \textit{ denotes the m-dimensional sphere.}$$

Actually we will show that there are infinitely many pseudo-Anosov homeomorphisms of Heegaard surfaces of S^3 which have the property described in the title of this note (Theorem 2.1).

In the following, we assume that the reader is familiar with [2]. For the definitions of standard terms in the 3-dimensional topology, we refer to [4].

2. Proof of Theorem

In this section we will give the proof of Theorem.

Theorem 2.1. *Let $(V_1, V_2; F)$ be a genus $g(>1)$ Heegaard splitting of the 3-sphere S^3 (Figure 1). Then there exist infinitely many orientation reversing self-homeomorphisms g' of S^3 such that $g'(V_1)=V_2$, $g'(V_2)=V_1$, and $g'|_F: F\to F$ has mutually distinct classes of pseudo-Anosov homeomorphisms.*

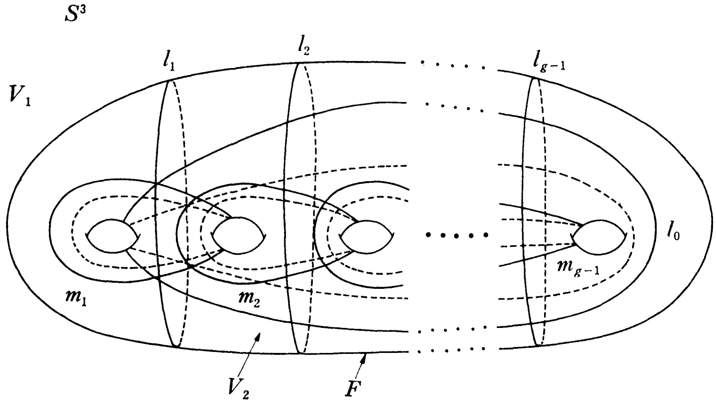

Figure 1

The key of the proof of Theorem 2.1 is Lemma 2.2, which is a generalization of a result of I. Aitchison [1].

Lemma 2.2. *There exist infinitely many ambient isotopies f_t $(0\leqq t\leqq 1)$ of S^3 such that $f_1(V_1)=V_1$, $f_1(V_2)=V_2$, and $f_1|_F: F\to F$ has mutually distinct classes of pseudo-Anosov homeomorphisms.*

Proof. Let $l_0, l_1, \cdots, l_{g-1}, m_1, \cdots, m_{g-1}$ be the system of simple loops on F as in Figure 1. We note that each simple loop bounds a disk in $V_i (i=1, 2)$. If p is a simple loop on F, then T_p denotes a right hand Dehn twist along p. Let m be the image of m_{g-1} by the homeomorphism $T_{m_1}\circ T_{m_2}\circ\cdots\circ T_{m_{g-2}}: F\to F$, and l be the image of l_0 by the homeomorphism $T_{l_1}\circ T_{l_2}\circ\cdots\circ T_{l_{g-1}}: F\to F$. Then we may suppose that l, m intersect transversely, and the number of the intersections is minimal.

Assertion. l, m fill up F i.e. each component of F-$(l\cup m)$ is an open disk.

Proof. By [2, Theorem 5.13], we see that m is carried by the train track τ as in Figure 2, where the numbers in Figure 2 denote the weights which represent m. On the other hand, we draw a picture of l as in Figure 3. Then we directly see that $\tau\cup l$ cuts F into a system of disks. Moreover, by seeing the weights on τ, we easily verify that the assertion holds. We show pictures of $l\cup m$ in the case of $g=2$, 3 to convince the reader that the assertion holds (Figure 4).

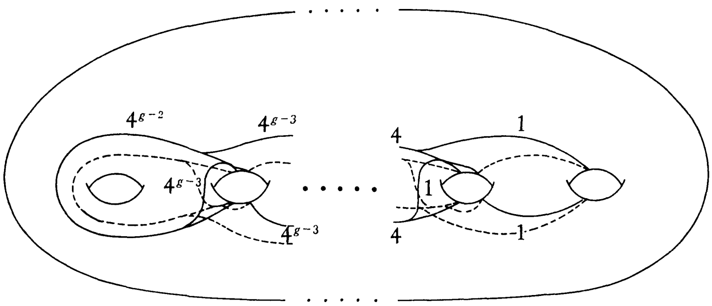

Figure 2

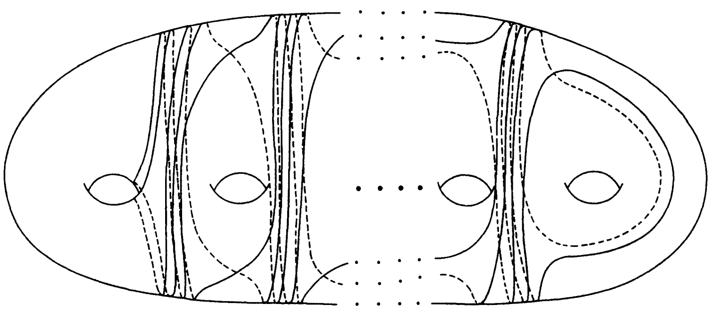

Figure 3

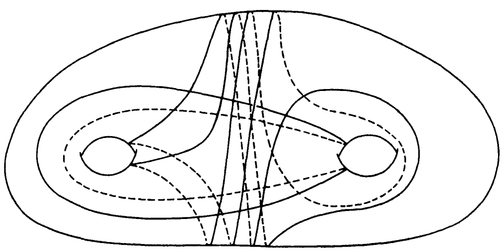

Figure 4 (i)

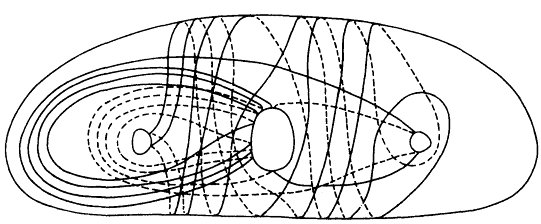

Figure 4 (ii)

We note that l,m bound disks in $V_i(i=1, 2)$. Hence, there are homeomorphisms $f_l, f_m: S^3 \rightarrow S^3$ such that f_l, f_m are ambient isotopic to the identitiy map, $f_l(V_i)=f_m(V_i)=V_i$ and $f_l|_F=T_l$, $f_m|_F=T_m$. By [3, Expose 13], we see that

$T_l \circ T_m^{-n}$ is a pseudo-Anosov class of F provided $n>0$. Moreover $T_l \circ T_m^{-n}$ and $T_l \circ T_m^{-n'}$ have pairwise distinct invariant laminations if $n \neq n'$. Hence, we have infinitely many ambient isotopies of S^3 which satisfies the conclusions of Lemma 2.2.

We need two more lemmas for the proof of Theorem 2.1.

Lemma 2.3 ([5, Lemma 2.4]). *Let $f\colon F \to F$ be a pseudo-Anosov homeomorphism with invariant laminations L^+, $L^- \in PL(F)$. Suppose that $R\colon F \to F$ is an infinite order reducible map. Then $R(L^+) \neq L^+$, L^-.*

Lemma 2.4 ([6, Theorem A]). *Let f, L^+, L^- be as in Lemma 2.3. Let $g\colon F \to F$ be a homeomorphism such that $g(L^-) \neq L^+$. Then there exists a number k_0 such that $f^k \circ g$ is isotopic to a pseudo-Anosov homeomorphsim for all $k \geq k_0$.*

Proof of Theorem 2.1. Let $a_1, \cdots, a_g, b_1, \cdots, b_g, c$ be a system of oriented simple loops on F as in Figure 5. Then there exists an ambient isotopy h_t $(0 \leq t \leq 1)$ of S^3 such that $h_1(V_1) = V_2$, $h_1(V_2) = V_1$, and $h_1(a_i) = b_i$, $h_1(b_i) = a_i$ $(i = 1, \cdots, g)$, $h(c) = \bar{c}$ (Figure 6), where $^-$ means the specified loop with the opposite orientation.

On the other hand, there exists an orientation reversing involution $g\colon S^3 \to S^3$ such that $g(V_i) = V_i$, and $g(a_j) = \bar{a}_j$, $g(b_j) = b_j (j = 1, \cdots, g)$. If needed, by adding twists along c to h_1, we may suppose that $h_1 \circ g|_F$ is an infinite order re-

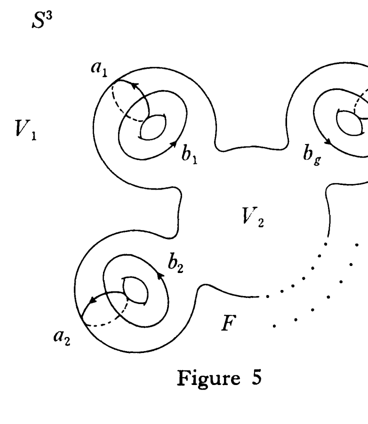

Figure 5

Figure 6

ducible map. Then, by Lemmas 2.3, 2.4, we see that $g'=(f_1)^n \circ h_1 \circ g$ satisfies the conclusion of Theorem 2.1 for sufficiently large n, where f_1 is a homeomorphism obtained in Lemma 2.2. Moreover, if n grows larger, then the stable lamination of $g'|_F$ tends to that of $f_1|_F$. Hence we have infinitely many g′'s with mutually distinct invariant laminations.

Proof of Theorem. Let g' be a homeomorphism obtained in Theorem 2.1. Then clearly the homeomorphism $g'|_F$ with the system of loops $\{a_1, \cdots, a_g\}$ satisfies the conclusion of Theorem.

References

[1] I. Aitchson: Isotoping and twisting knots and ribbons, thesis, University of California, Berkeley, 1985.

[2] A. Casson: Automorphisms of surfaces after Nielsen and Thurston, Lecture Note, University of Texas at Austin.

[3] A. Fathi, F. Laudenbach, & V. Poenaru: *Travaux de Thurston sur les surfaces,* Astérisque, **66–67** (1979).

[4] J. Hempel: 3-manifolds, Ann. of Math. Studies No. 86, Princeton University Press, Princeton N.J., 1976.

[5] D. Long: *A note on the normal subgroups of mapping class groups,* Math. Proc. Cambridge Philos. Soc. **99** (1986), 79–87.

[6] A. Papadopoulos: *Reseaux ferroviaires, diffeomorphismes pseudo-Anosov et automorphismes symplectiques de l'homologie d'une surface,* Publ. Math. Orsay, 1983.

[7] T. Yoshida: in Problems, Proc. Sympos. Res. Inst. Math. Sci. **568** (1985).

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