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ON COMPLEX COBORDISM GROUPS OF CLASSIFYING SPACES FOR DIHEDRAL GROUPS

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1. Introduction

Let $G=H\cdot\Gamma$ be a semi-direct product of a finite group H by a finite group Γ, X a compact G-manifold which induces by restriction a principal H-manifold and Y a principal Γ-manifold. Then we have a principal G-space $X\times Y$ with a G-action defined by $h\gamma(x, y)=(h\gamma x, \gamma y)$, $h\gamma\in H\cdot\Gamma$. The equivariant map $i\colon X\to X\times Y$ defined by $i(x)=(x, y_0)$, induces a homomorphism

$$i^*\colon U^*((X\times Y)/G)\to U^*(X/H)\,.$$

We can define a Γ-action over $U^*(X/H)$ corresponding to a Γ-action over the complex bordism group of unitary G-manifolds defined by (1.3) of [7]. The action is denoted by x^γ, $x\in U^*(X/H)$, $\gamma\in\Gamma$.

In this paper, we define a homomorphism

$$i_*\colon U^*(X/H)\to U^*((X\times Y)/G)$$

and obtain the following.

Theorem 1.1. *For* $x\in U^*(X/H)$, $i^*i_*(x)=\sum_{\gamma\in\Gamma}x^\gamma$.

Let $D_p(m, n)$ be the orbit manifold of $S^{2m+1}\times S^n$ by the dihedral group D_p whose action is given in [7]. Making use of Theorem 1.1 and the Atiyah-Hirzebruch spectral sequence of the complex cobordism group, we have the following.

Theorem 1.2. *Suppose that p is an odd prime. There exists an isomorphism*

$$\tilde{U}^{2m}(D_p(2k+1, 4k+3))\cong\tilde{U}^{2m}(L^{2k+1}(p))^{Z_2}\oplus\tilde{U}^{2m}(RP^{4k+3})\oplus U^{2m-8k-6}\,,$$

where $L^l(p)=S^{2l+1}/Z_p$ *is a* $(2l+1)$*-dimensional lens space,* RP^s *is an s-dimensional real projective space and* $U^*(\ )^{Z_2}$ *is the subgroup consisting of the elements which are fixed under the* Z_2*-action.*

Let BZ_p be a classifying space for Z_p. There exists an isomorphism $U^{ev}(BZ_p)\cong U^*[[X]]/([p]_F(X))$, $U^{ev}(\ )=\sum U^{2i}(\ )$ [8]. Consider the Z_2-action on $U^{ev}(BZ_p)$ defined by

$$f(X)^t = f([-1]_F(X)),$$

where t is a generator of Z_2. We use Milnor's short exact sequence [10] and Theorem 1.2 to compute the complex cobordism group of a classifying space for the dihedral group D_p.

Theorem 1.3. *Suppose that p is an odd prime. There exist isomorphisms*

$$\tilde{U}^{2m}(BD_p) \cong \tilde{U}^{2m}(BZ_p)^{Z_2} \oplus \tilde{U}^{2m}(BZ_2)$$

and

$$\tilde{U}^{2m+1}(BD_p) \cong 0 .$$

Making use of the Conner and Floyd isomorphism

$$\tilde{K}(X) \cong \tilde{U}^{ev}(X) \otimes_{U^*} Z$$

and Theorem 1.2, we can deduce the structure of the K-group of $D_p(2k+1, 4k+3)$ which is also obtained in [5] and [6].

2. The homomorphism i^*: $U^*(X/H) \to U^*((X \times Y)/G)$

By a G-manifold we mean a C^∞-manifold which can be embedded equivariantly in some Euclidean G-space [11]. Let M and X be G-manifolds. By a complex orientation of a G-map f: $M \to X$ we mean an equivalence class of factorizations

$$Z \xrightarrow{i} E \xrightarrow{p} X$$

where p: $E \to X$ is a complex G-vector bundle over X and where i is an equivariant G-embedding endowed with a complex structure compatible with the G-action on its normal bundle ν_i. As Quillen [12] we can define equivariantly a cobordant relation joining such proper complex oriented G-maps for a G-manifold X. We denote by $U_G^m(X)$ the set of cobordism classes of proper complex oriented G-maps of dimension $-m$. Assume that X is a principal G-manifold which is a G-manifold such that no element of the group other than the identity has a fixed point [2]. Then the complex cobordism group $U_G^m(X)$ is isomorphic to $U^m(X/G)$ by sending the equivariant cobordism class $[Z \xrightarrow{i} E \xrightarrow{p} X]_G$ to $[Z/G \xrightarrow{i'} E/G \xrightarrow{p'} X/G]$, where i' and p' are quotient maps.

From now on, we suppose that G is a semi-direct product $H \cdot \Gamma$ of a finite group H by a finite group Γ and that X is a G-manifold whose action restricted to H is free and Y is a principal Γ-manifold. The element γ of Γ acts on the group H by the inner automorphisms $h^\gamma = \gamma^{-1} h \gamma$ and the group operation of $H \cdot \Gamma$ is given by

$$(h_1\gamma_1)(h_2\gamma_2) = h_1 h_2^{\gamma_1^{-1}}\gamma_1\gamma_2 .$$

The map i: $X \to X \times Y$, $i(x)=(x, y_0)$, is an equivariant map. Then, there exists a composition homomorphism

$$i^*: U^*((X\times Y)/G) \xrightarrow{r^*} U^*((X\times Y)/H) \xrightarrow{i_H^*} U^*(X/H)$$

where r^* sends an equivariant cobordism class $[Z\to E\to X]_G$ to the class $[Z\to E\to X]_H$ obtained by restriction of the group action and i_H is the quotient map of i. Suppose that X is a compact principal G-manifold, $G=H\cdot\Gamma$. Let $[Z \xrightarrow{i} E \xrightarrow{p} X]_H$ be an element of $U_H^m(X)$ represented by an H-equivariant factorization. Since q: $X\to X/H$ is a principal bundle, a functor q^* from the category of vector bundles and homomorphisms over X/H to the category of H-vector bundles and H-homomorphisms over X is an equivalence [1]. There exists an H-complex vector bundle F over X such that $E\oplus F=X\times C^n$ where H acts on $X\times C^n$ by the rule $h(x, z)=(hx, z)$. Therefore,

$$[Z \xrightarrow{i} E \xrightarrow{p} X]_H = [Z \xrightarrow{\hat{i}} X\times C^n \xrightarrow{\tilde{p}} X]_H$$

as equivariant cobordism classes, where $\hat{i}(z)=(i(z), 0)$ and $\tilde{p}(x, z)=x$. We form the quotient space $G\times_H Z$. The group G acts on $G\times_H Z$ by $\hat{g}(g\times_H x)=(\hat{g}g\times_H x)$. We have then the equivariant embedding

$$i_1\colon G\times_H Z\times Y \to X\times C^n\times Y\times V$$
$$i_1(h\gamma\times_H z, y) = (h\gamma\hat{i}(z), y, e(\gamma))$$

where $G\times_H Z\times Y$ is a G-space by $h\gamma(g\times_H z, y)=(h\gamma g\times_H z, \gamma y)$, V is a complex Euclidean Γ-space, for example a regular representation space of Γ, $X\times C^n\times Y\times V$ is a G-space by $h\gamma(x, z, y, v)=(h\gamma x, z, \gamma y, \gamma v)$ and e: $\Gamma\to V$ is a Γ-equivariant embedding.

Lemma 2.1. *If the normal bundle ν of i: $Z\to X\times C^n$ has a complex structure compatible with the H-action, then the normal bundle ν_1 of i_1: $G\times_H Z\times Y\to X\times C^n\times Y\times V$ has a complex structure compatible with the G-action.*

Proof. Let J: $\nu\to\nu$ be a complex structure compatible with H-action, that is, $hJ=Jh$. We may consider that X and Y are embedded in a Euclidean G-space V_x and a Euclidean Γ-space V_y respectively and that each element of G operates on $V_x\times C^n\times V_y\times V$ as an orthogonal linear transformation. The total space of the normal bundle ν_1 is described as follows:

$$E(\nu_1) = \{(i_1(h\gamma\times_H z, y), (h\gamma w, v))\colon w \text{ is a vector of a fiber of } \nu \text{ over } i(z) \text{ and } v\in V\}.$$

We put

$$\tilde{J}(i_1(h\gamma\times_H z, y), (w, v)) = (i_1(h\gamma\times_H z, y), (\gamma J\gamma^{-1}w, \sqrt{-1}v)).$$

The homomorphism $\tilde{J}$ is a complex structure of the bundle ν_1 q.e.d.

From Lemma 2.1, we have a factorization

$$G\times_H Z\times Y \xrightarrow{i_1} X\times C^n\times Y\times V \xrightarrow{p_1} X\times Y,$$

$p_1(x, z, y, v)=(x, y)$, which is a complex orientation of a map $p_1\cdot i_1$. We set

$$i_*[Z \xrightarrow{i} E \xrightarrow{p} X]_H = [G\times_H Z\times Y \xrightarrow{i_1} X\times C^n\times Y\times V \xrightarrow{p_1} X\times Y]_G.$$

This defines a U^*-module homomorphism

$$i_*: U^*(X/H) \to U^*((X\times Y)/G)$$

of degree 0.

We define a Γ-action on $U^*(X/H)$: We take an equivariant cobordism class $[Z\xrightarrow{i} X\times C^n \xrightarrow{p} X]_H \in U_H^*(X)=U^*(X/H)$, with an H-action ϕ: $H\times Z \to Z$. Let Z^γ be a copy of Z whose action ϕ^γ: $H\times Z\to Z$ is given by

$$\phi^\gamma(h, z) = \phi(h^\gamma, z)$$

and i^γ: $Z^\gamma\to X\times C^n$ be an equivariant H-map given by

$$i^\gamma(z) = \gamma i(z).$$

Denote by ν the normal bundle of i: $Z\to X\times C^n$ and ν_x the fiber over x. The total space E of the normal bundle ν^γ of i^γ: $Z^\gamma\to X\times C^n$ is

$$E = \{(i^\gamma(z), \gamma v): v \text{ is a vector in the fiber } \nu_{i(z)}\}.$$

Let J: $\nu\to\nu$ be a complex structure compatible with the H-action. Then, a bundle map J^γ: $E\to E$, $J^\gamma(i^\gamma(z), w)=(i^\gamma(z), \gamma J\gamma^{-1}w)$, is a complex structure of ν^γ compatible with the H-action. We set

$$[Z\xrightarrow{i} X\times C^n \xrightarrow{p} X]_H^\gamma = [Z^\gamma \xrightarrow{i^\gamma} X\times C^n \xrightarrow{p} X]_H.$$

Proof of Theorem 1.1.

We recall that $i_*[Z\xrightarrow{\hat{i}} X\times C^n \xrightarrow{\tilde{p}} X]_H=[G\times_H Z\times Y\xrightarrow{i_1} X\times C^n\times Y\times V \xrightarrow{p_1} X\times Y]_G$. Consider the map j: $X\times C^n\times V\to X\times C^n\times Y\times V$, $j(x, z, v)=(x, z, y_0, v)$. The map j is an H-map and transversally regular on $i_1(G\times_H Z\times Y)$. Let Γ be the set consisting of $\gamma_1, \gamma_2, \cdots \gamma_m$. It follows that

$$j^{-1}(i_1(G\times_H Z\times Y)) = \bigcup_k Z_k$$

where $Z_k=\{(h\gamma_k i(z), e(\gamma_k)) : h\in H, z\in Z\}\subset X\times C^n\times V$. Clearly, Z_k is equivariantly diffeomorphic to Z^{γ_k} and $[Z_k\xrightarrow{i_k} X\times C^n\times V\xrightarrow{\tilde{p}} X]_H=[Z\xrightarrow{\hat{\imath}} X\times C^n\xrightarrow{\tilde{p}} X]_H^{\gamma_k}$, where i_k is an inclusion. Therefore, we have $i^*i_*[Z\xrightarrow{\hat{\imath}} X\times C^n\xrightarrow{\tilde{p}} X]_H=\Sigma[Z\xrightarrow{\hat{\imath}} X\times C^n\xrightarrow{\tilde{p}} X]_H^{\gamma_k}$. q.e.d.

3. The structure of $\tilde{U}^{2m}(D_p(2k+1, 4k+3))$

In [7], the manifold $D_p(l, n)=(S^{2l+1}\times S^n)/D_p$ was useful to determine the structure of complex bordism group of principal dihedral group D_p-actions. In this section, we determine the additive structure of $\tilde{U}^{2m}(D_p(2k+1, 4k+3))$. Consider an action of the dihedral group $D_p=Z_p\cdot Z_2$ over $S^{2l+1}\times S^n$ given by

$$(1)\qquad (g^i t^j)(z, x) = (\rho^i c^j(z), (-1)^j x),\quad \rho = \exp 2\pi\sqrt{-1}/p$$

where g is a generator of order p and t is the generator of order 2 and $c(z)$ is the conjugation operator. The manifold $D_p(l, n)$ is the orbit space. This manifold is an example of manifolds described in §2. We take a Z_p-space S^{2l+1} with $g\cdot z=\rho z$ ($z\in S^{2l+1}$, g is a generator of Z_p), a Z_2-space S^n with $t\cdot x=(-1)x$ ($x\in S^n$, t is the generator of Z_2) and a D_p-space $S^{2l+1}\times S^n$ with the D_p-action given by (1). Then, there are equivariant maps

$$i\colon S^{2l+1}\to S^{2l+1}\times S^n \quad i(z) = (z, (1, 0, \cdots, 0))$$
$$j\colon S^n\to S^{2l+1}\times S^n \quad j(x) = ((1, 0, \cdots, 0), x)$$

and

$$p\colon S^{2l+1}\times S^n\to S^n \quad p(z, x) = x$$

with respect to inclusions $i\colon Z_p\to D_p$, $j\colon Z_2\to D_p$ and a projection $p\colon D_p\to Z_2$ respectively. Denote by $U^*(S^{2l+1}/Z_p)^{Z_2}$ the subgroup consisting of elements fixed under the Z_2-action over $U^*(S^{2l+1}/Z_p)$ described in §2. Then we have the following.

Proposition 3.1. *If p is an odd prime, the homomorphism Φ: $\tilde{U}^{2m}(S^{2l+1}/Z_p)^{Z_2}\oplus\tilde{U}^{2m}(S^n/Z_2)\to\tilde{U}^{2m}(D_p(l, n))$ given by $\Phi(x, y)=i_*(x)+p^*(y)$ is injective.*

Proof. We remark that $\tilde{U}^{2m}(S^{2l+1}/Z_p)$ is a p-group and $\tilde{U}^{2m}(S^n/Z_2)$ is a 2-group. Hence, $i^*p^*=0$. Since $j^*p^*=1$ and from Theorem 1.1 $i^*i_*(x)=2x$, Φ is injective. q.e.d.

Denote by $L^l(p)$ a $(2l+1)$-dimensional lens space. The manifold $D_p(l, n)$ is homeomorphic to the orbit space of $L^l(p)\times S^n$ by a Z_2-action $t([z], x)=([cz], -x)$, $t\in Z_2$ the generator. Let C_i and D_j be the standard cells of $L^l(p)$ and S^n respectively. The images (C_i, D_j) of the $C_i\times D_j$ by the quotient map $L^l(p)\times S^n\to D_p(l, n)$ give a cellular decomposition of $D_p(l, n)$. Denote by (c^i, d^j) the dual cochain element to (C_i, D_j). Then we have the following coboundary relations

$$\delta(c^{2i+1}, d^j) = \{(-1)^i+(-1)^j\}(c^{2i+1}, d^{j+1})+p(c^{2i+2}, d^j)$$
$$\delta(c^{2i}, d^j) = \{(-1)^i+(-1)^{j+1}\}(c^{2i}, d^{j+1}) .$$

Therefore, we have the following.

Proposition 3.2. *The integral cohomology group $\tilde{H}^*(D_p(l, n); Z)$ is a direct sum of the following groups*

(i) *case l: even and n: even*
a free group generated by (c^{2l+1}, d^n), torsion groups generated by the (c^0, d^{2j}) and the (c^{2l+1}, d^{2j-1}) whose orders are 2 *and torsion groups generated by the (c^{4i}, d^0) and the (c^{4i-2}, d^n) whose orders are p,*

(ii) *case l: even and n: odd*
a free group generated by (c^0, d^n), torsion groups generated by the (c^0, d^{2j}) and the (c^{2l+1}, d^{2j+1}) whose orders are 2 *and torsion groups generated by the (c^{4i}, d^0) and the (c^{4i}, d^n) whose orders are p,*

(iii) *case l: odd and n: even*
a free group generated by (c^{2l+1}, d^0), torsion groups generated by the (c^0, d^{2j}) and the (c^{2l+1}, d^{2j}) whose orders are 2 *and torsion groups generated by the (c^{4i}, d^0) and the (c^{4i-2}, d^n) whose orders are p,*

(iv) *case l: odd and n: odd*
free groups generated by (c^0, d^n), (c^{2l+1}, d^0) and (c^{2l+1}, d^n), torsion groups generated by the (c^0, d^{2j}) and the (c^{2l+1}, d^{2j}) whose orders are 2 *and tosion groups generated by the (c^{4i}, d^0) and the (c^{4i}, d^n) whose orders are p,*

where $0 \leqq 2j \leqq n$ and $0 \leqq 2i \leqq l$.

Let Y_k be the $(8k+5)$-skeleton of $D_p(2k+1, 4k+3)$. Denote by $(E_r^{s,t}(X), d_r^{s,t})$ the Atiyah-Hirzebruch spectral sequence for $U^*(X)$.

Lemma 3.3. *If $s \neq 8k+6$ then an inclusion ι: $Y_k \to D_p(2k+1, 4k+3)$ induces the isomorphism for any r*

$$E_r^{s,t}(Y_k) \cong E_r^{s,t}(D_p(2k+1, 4k+3)) .$$

Proof. Using Proposition 3.2, it follows that ι^*: $E_2^{s,t}(D_p(2k+1, 4k+3)) \to E_2^{s,t}(Y_k)$ is isomorphic if $s \neq 8k+6$. We note that the images of the differentials $d_r^{s,t}$ for any r are torsion groups [4]. By induction on r we have the lemma. q.e.d.

Proposition 3.4. *There exists a short exact sequence*

$$0 \to U^{2m-8k-6} \to \tilde{U}^{2m}(D_p(2k+1, 4k+3)) \to \tilde{U}^{2m}(Y_k) \to 0 .$$

Proof. Consider the exact sequence of complex cobordism groups for a pair $(D_p(2k+1, 4k+3), Y_k)$:

$$\cdots \to \tilde{U}^*(D_p(2k+1, 4k+3)) \to \tilde{U}^*(Y_k) \to \tilde{U}^{*+1}(D_p(2k+1, 4k+3)/Y_k) \to$$

From Lemma 3.3 ι^*: $\tilde{U}^i(D_p(2k+1, 4k+3)) \to \tilde{U}^i(Y_k)$ is isomorphic for i odd. Since $\tilde{H}^i(D_p(2k+1, 4k+3)/Y_k; Z)=0$ if $i \neq 8k+6$ and $\tilde{H}^{8k+6}(D_p(2k+1, 4k+3)/Y_k; Z) \cong Z$, we have that $\tilde{U}^{2m}(D_p(2k+1, 4k+3)/Y_k) \cong U^{2m-8k-6}$. q.e.d.

We investigate the Thom homomorphism μ: $U^*(X) \to H^*(X)$ which is the edge homomorphism of the spectral sequence associated with $U^*(X)$. Let X be an orientable manifold. We take an element $[M \xrightarrow{i} X \xrightarrow{id} X] \in U^*(X)$ which is represented by an inclusion map $M \xrightarrow{i} X$ with the normal bundle ν equipped with a complex structure. Denote by $N(\nu)$ the tubular neighborhood of M, and we have a canonical map j: $(X, \phi) \to (X, \{\text{Int}\, N(\nu)\}^c)$. Then, we can describe the Thom homomorphism as $\mu[M \xrightarrow{i} X \xrightarrow{id} X] = j^*\tau(\nu)$, $\tau(\nu)$ is the Thom class of ν, and

$$(2) \qquad \mu[M \xrightarrow{i} X \xrightarrow{id} X] = Di_*\sigma(M)$$

where D is the Poincaré duality isomorphism $H_*(M) \cong H^*(M)$ and $\sigma(M)$ is a fundamental class of M.

We put

$$L_{k-m} = [S^{4m+3} \xrightarrow{i} S^{4k+3} \xrightarrow{id} S^{4k+3}]_{Z_p} \in U_{Z_p}^{4(k-m)}(S^{4k+3}),$$

where S^{4k+3} and S^{4m+3} are Z_p-spaces with canonical action $g \cdot z = \rho z$ and i is the canonical inclusion, and

$$R_{2k+1-n} = [S^{2n+1} \xrightarrow{i} S^{4k+3} \xrightarrow{id} S^{4k+3}]_{Z_2} \in U_{Z_2}^{4k+2-2m}(S^{4k+3})$$

where S^{2n+1} and S^{4k+3} are Z_2-spaces with the canonical action $t \cdot x = (-1)x$, and i is the canonical inclusion.

Proposition 3.5. *Suppose that p is an odd prime, then*

$$\mu i_*(L_{k-m} + L_{k-m}^t) = a(c^{4(k-m)}, d^0), \quad a \not\equiv 0 \text{ modulo } p$$

and

$$\mu p^*(R_{2k+1-n}) = (c^0, d^{4k+2-2m}).$$

Proof. The manifold $D_p(2k+1, 4k+3)$ is orientable. Using Theorem 1.1 and (2), we have the proposition. q.e.d.

Proof of Theorem 1.2.

Proposition 3.5 shows that in the Atiyah-Hirzebruch spectral sequence for $\tilde{U}^*(D_p(2k+1, 4k+3))$, the (c^{4i}, d^0) and the (c^0, d^{2j}) are parmanent cycles. It is easy to prove that the spectral sequence is trivial. Therefore it follows from Propositions 3.1 and 3.5 that there exists an isomorphism

$$\lambda^*+i_*+p^*\colon \tilde{U}^{2m}(D_p(2k+1,\ 4k+3)/Y_k)\oplus\tilde{U}^{2m}(S^{4k+3}/Z_p)^{Z_2}\oplus\tilde{U}^{2m}(S^{4k+3}/Z_2)$$
$$\rightarrow \tilde{U}^{2m}(D_p(2k+1,\ 4k+3))$$

where $\lambda\colon D_p(2k+1,\ 4k+3)\rightarrow D_p(2k+1,\ 4k+3)/Y_k$ is the projection map. q.e.d.

4. $\tilde{U}^*(BZ_p)$, p an odd prime

The complex cobordism group $\tilde{U}^{ev}(L^n(p))\cong\tilde{U}^{ev}(S^{2n+1}/Z_p)$ is a U^*-module with a generating set $\{[S^{2k+1}\xrightarrow{i}S^{2n+1}\xrightarrow{id}S^{2n+1}]_{Z_p}$; Z_p-equivariant cobordism classes which are represented by the canonical equivariant inclusion map $i(z_0,\cdots,z_k)=(z_0,\cdots,z_k,0,\cdots,0)$, $0\leqq k\leqq n-1\}$.

Lemma 4.1. $\{\iota_n^*([S^{2k+1}\xrightarrow{i}S^{2n+1}\xrightarrow{id}S^{2n+1}]_{Z_p})\}^t$

$$=\iota_n^*([S^{2k+1}\xrightarrow{i}S^{2n+1}\xrightarrow{id}S^{2n+1}]_{Z_p}^t),$$

where $\iota_n\colon L^{n-1}(p)\rightarrow L^n(p)$ *is the inclusion map* $\iota_n(z_0,\cdots,z_{n-1})=(z_0,\cdots,z_{n-1},0)$.

Proof. By the definition of the Z_2-action, $[S^{2k+1}\xrightarrow{i}S^{2n+1}\xrightarrow{id}S^{2n+1}]_{Z_p}^t=[(S^{2k+1})^t\xrightarrow{i^t}S^{2n+1}\xrightarrow{id}S^{2n+1}]_{Z_p}$ with $i^t(z)=ci(z)$. Let $H_n\colon S^{2n-1}\times I\rightarrow S^{2n+1}$ be a map defined by

$$H_n(z_0,\cdots,z_{n-1},t)=\frac{1}{A}(tz_0,\ tz_1+(1-t)z_0,\cdots,tz_{n-1}+(1-t)z_{n-2},\ (1-t)z_{n-1})$$

where A is the norm of $(tz_0,\ tz_1+(1-t)z_0,\cdots,(1-t)z_{n-1})$. H_n is an equivariant Z_p-map. Put

$$j_n(z)=H_n(z,0),$$

then we have that $j_n^*=\iota_n^*$. Moreover $j_n\colon S^{2n-1}\rightarrow S^{2n+1}$ is transverse regular on $i^t(S^{2k+1})$. Therefore, we have

$$j_n^*[(S^{2k+1})^t\xrightarrow{i^t}S^{2n+1}\xrightarrow{id}S^{2n+1}]_{Z_p}=[(S^{2k-1})^t\xrightarrow{i^t}S^{2n-1}\xrightarrow{id}S^{2n-1}]_{Z_p}.$$

q.e.d.

Let $F(X,Y)$ be the formal group of the complex cobordism theory. Denote by $[-1]_F(X)$ the element of $U^*[[X]]$ satisfying $F(X,[-1]_F(X))=0$ and by $[k]_F(X)$ the element of $U^*[[X]]$ defined by the following formulae

$$\begin{cases}[1](X)_F=X\\ F(X,[k]_F(X))=[k+1]_F(X).\end{cases}$$

We define a Z_2-action on $U^*[[X]]$ by

$$f(X)^t = f([-1]_F(X)).$$

By the definition of the formal group law, it follows immediately that $\{[p]_F(X)\}^t$ and $(X^{n+1})^t$ belong to the ideal $([p]_F(X), X^{n+1})$ generated by $[p]_F(X)$ and X^{n+1} in $U^*[[X]]$ and thus Z_2 acts on $U^*[[X]]/([p]_F(X), X^{n+1})$. We can see that the element $[S^{2n-1} \xrightarrow{i} S^{2n+1} \xrightarrow{id} S^{2n+1}]_{Z_p}$ corresponds to the cobordism 1-st Chern class $c_1(\xi_n)$ of the canonical line bundle ξ_n over $L^n(p)$ and that $[S^{2n-1} \xrightarrow{i} S^{2n+1} \xrightarrow{id} S^{2n+1}]^t_{Z_p}$ is the cobordism 1-st Chern class $c_1(\bar{\xi}_n)$ of the conjugate bundle $\bar{\xi}_n$. Therefore, we have the following.

Lemma 4.2. $U^{ev}(L^n(p))^{Z_2} \cong \{U^*[[X]]/([p]_F(X), X^{n+1})\}^{Z_2}$.

Proof. From the definition of the multiplication in $U^{ev}(L^n(p))$ we have that for $0 \leqq k, l \leqq n$

$$[S^{2k+1} \xrightarrow{i} S^{2n+1} \xrightarrow{id} S^{2n+1}]_{Z_p}[S^{2l+1} \xrightarrow{i} S^{2n+1} \xrightarrow{id} S^{2n+1}]_{Z_p}$$
$$= \begin{cases} [S^{2(-n+k+l)+1} \xrightarrow{i} S^{2n+1} \xrightarrow{id} S^{2n+1}]_{Z_p} \\ 0 \quad \text{if } n-k-l>0 . \end{cases}$$

Then, it follows immediately that the Z_2-action on $U^*(L^n(p))$ is multiplicative. There exists an isomorphism $U^{ev}(L^n(p)) \cong U^*[[X]]/([p]_F(X), X^{n+1})$ which maps $c_1(\xi_n)$ to X [13]. Since $F(c_1(\xi_n), c_1(\bar{\xi}_n)) = c_1(\xi_n \otimes \bar{\xi}_n) = 0$, the lemma follows. q.e.d.

Denote by j_k: $D_p(2k-1, 4k-1) \to D_p(2k+1, 4k+3)$ and $\hat{j}_k$: $L^{2k-1}(p) \to L^{2k+1}(p)$ respectively, the maps induced by the inclusions $S^{4k-1} \times S^{4k-1} \subset S^{4k+3} \times S^{4k+3}$ and $S^{4k-1} \subset S^{4k+3}$. The following diagram is commutative

$$\begin{array}{ccc} \tilde{U}^{2m}(L^{2k+1}(p)) & \xrightarrow{i_*} & \tilde{U}^{2m}(D_p(2k+1, 4k+3)) \\ \downarrow \hat{j}^*_k & & \downarrow j^*_k \\ \tilde{U}^{2m}(L^{2k-1}(p)) & \xrightarrow{i_*} & \tilde{U}^{2m}(D_p(2k-1, 4k-1)). \end{array}$$

Since the Z_2-action on $U^*(L^n(p))$ and $\hat{j}^*_k$ are U^*-homomorphisms, it follows from Lemma 4.1 that i_* induces a homomorphism of inverse systems

$$i_*: \{\tilde{U}^{2m}(L^{2k+1}(p))^{Z_2}, \hat{j}^*_k\} \to \{\tilde{U}^{2m}(D_p(2k+1, 4k+3)), j^*_k\} .$$

Consider the quotient map of j_k

$$\tilde{j}_k;\ D_p(2k-1, 4k-1)/Y_{k-1} \to D_p(2k+1, 4k+3)/Y_k ,$$

where Y_k is a $(8k+5)$-skeleton of $D_p(2k+1, 4k+3)$. Maps λ: $D_p(2k+1, 4k+3)$

$\to D_p(2k+1, 4k+3)/Y_k$ and p; $D_p(2k+1, 4k+3) \to RP^{4k+3}$ induce homomorphisms of inverse systems

$$\lambda^*: \{\tilde{U}^{2m}(D_p(2k+1, 4k+3)/Y_k) \cong U^{2m-8k-6}, \tilde{j}_k^*\} \\ \to \{\tilde{U}^{2m}(D_p(2k+1, 4k+3)), j_k^*\}$$

and

$$p^*: \{\tilde{U}^{2m}(RP^{4k+3}), \hat{\hat{j}}_k^*\} \to \{\tilde{U}^{2m}(D_p(2k+1, 4k+3)), j_k^*\},$$

where $\hat{\hat{j}}_k: RP^{4k-1} \to RP^{4k+3}$ is the inclusion map. From Theorem 1.2, we have an isomorphism

$$(4.1) \qquad i_*+p^*: \varprojlim \tilde{U}^{2m}(L^{2k+1}(p))^{Z_2} \oplus \varprojlim \tilde{U}^{2m}(RP^{4k+3}) \\ \to \varprojlim \tilde{U}^{2m}(D_p(2k+1, 4k+3)),$$

because $\tilde{j}_k^*: \tilde{U}^{2m}(D_p(2k+1, 4k+3)/Y_k) \to \tilde{U}^{2m}(D_p(2k-1, 4k-1)/Y_{k-1})$ is a zero homomorphism.

Lemma 4.3. *$j_k^*: \tilde{U}^{2m+1}(D_p(2k+1, 4k+3)) \to \tilde{U}^{2m+1}(D_p(2k-1, 4k-1))$ is a zero homomorphism.*

Proof. Let $\tilde{Y}_k$ be a $(8k+2)$-skeleton of $D_p(2k, 4k+2)$. We consider the map j_k as a composition map $j_k: D_p(2k-1, 4k-1) \to \tilde{Y}_k \to D_p(2k, 4k+2) \to D_p(2k+1, 4k+3)$. By Proposition 3.2 case (i), it follows that $\tilde{H}^{odd}(Y_k; Z) \cong 0$ and $\tilde{U}^{2m+1}(\tilde{Y}_k) \cong 0$. Therefore, j_k^* is the zero homomorphism. q.e.d.

Lemma 4.4. $\varprojlim^1 \tilde{U}^{2m}(D_p(2k+1, 4k+3))=0$.

Proof. From Proposition 3.5 and Theorem 1.2 it follows that $\{L_{k-m} + L_{k-m}^t\}$ is a generating set for U^*-module $\tilde{U}^{ev}(S^{4k+3}/Z_p)^{Z_2}$. By Lemma 4.1, $\hat{j}_k^*: \tilde{U}^{2m}(L^{4k+3}(p))^{Z_2} \to \tilde{U}^{2m}(L^{4k-1}(p))^{Z_2}$ is surjective. Therefore, it follows that an inverse system $\{\tilde{U}^{2m}(D_p(2k+1, 4k+3)), j_k^*\}$ satisfies the Mittag-Leffler condition and the lemma follows. q.e.d.

Proof of Theorem 1.3.

There exists Milnor's short exact sequence

$$(4.2) \qquad 0 \to \varprojlim^1 \tilde{U}^{*-1}(D_p(2k+1, 4k+3)) \to \tilde{U}^*(BD_p) \\ \to \varprojlim \tilde{U}^*(D_p(2k+1, 4k+3)) \to 0 \ [10].$$

Using Lemma 4.3 and 4.4, we have $\tilde{U}^{2m+1}(BD_p)=0$.

Lemma 4.3 implies that the inverse system $\{\tilde{U}^{2m+1}(D_p(2k+1, 4k+3)), j_k^*\}$ satisfies the Mittag-Leffler condition. Therefore we have that

$$\tilde{U}^{2m}(BD_p) \cong \varprojlim \tilde{U}^{2m}(D_p(2k+1, 4k+3)).$$

Using Theorem 1.2 and Lemma 4.2 we complete the proof.

5. The structure of $\tilde{K}(D_p(2k+1, 4k+3))$

In [3], Conner and Floyd gave the isomorphism

$$(5.1) \qquad c\colon \tilde{K}(X) \simeq \tilde{U}^{ev}(X)\otimes_{U^*}Z,$$

which maps $\eta_n - n$ to $c_1(\eta_n)\times 1$. Consider a Z_2-action on $K(L^n(p))$ defined by $\eta^t = \bar{\eta}$, t a generator of Z_2. Since Z_2-action on $U^*(L^n(p))$ is multiplicative, we have the commutative diagram

$$(5.2) \qquad \begin{array}{ccc} \tilde{K}(L^n(p)) & \xrightarrow{c} & \tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z \\ \downarrow t & & \downarrow {}^t\otimes_{U^*}id \\ \tilde{K}(L^n(p)) & \xrightarrow{c} & \tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z \end{array}$$

Lemma 5.1. *$\tilde{U}^{ev}((L^n(p))\otimes_{U^*}Z)^{Z_2} = \tilde{U}^{ev}(L^n(p))^{Z_2}\otimes_{U^*}Z$, where $(\tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z)^{Z_2}$ is an invariant subgroup of $\tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z$ under the Z_2-action $\cdot^t\times_{U^*}id$.*

Proof. By the definition of Z_2-action of $\tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z$, it follows that $\tilde{U}^{ev}(L^n(p))^{Z_2}\otimes_{U^*}Z\subset(\tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z)^{Z_2}$. Suppose that $x\otimes_{U^*}m\in\tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z$ and $x^t\otimes_{U^*}m = x\otimes_{U^*}m$. Since c is isomorphic, there exists an element $\eta\in\tilde{K}(L^n(p))$ with $c(\eta)=x\otimes_{U^*}m$. By the commutative diagram (5.2),

$$c(\eta) = c(\eta)^t = c(\eta^t) \quad \text{and} \quad \eta = \eta^t .$$

N. Mahammed [9] proved that $\tilde{K}(L^n(p)) = Z[\xi_n]/(\xi_n^p - 1, (\xi_n - 1)^{n+1})$, ξ_n is the canonical line bundle over $L^n(p)$. Put $X = c_1(\xi_n)$. Then, the element $c_1(\eta)$ is described as a polynomial $f(X)$ with the coefficient in U^*. We can see that $c_1(\bar{\eta}) = f([-1]_F(X))$. By the observation in Lemma 4.2, it follows that $c_1(\eta)\in\tilde{U}^{ev}(L^n(p))^{Z_2}$. Therefore, we have that if $x\otimes_{U^*}m\in(\tilde{U}^{ev}(L^n(p))\otimes_{U^*}Z)^{Z_2}$, then there exists an element $\eta\in\tilde{K}(L^n(p))$ such that

$$x\otimes_{U^*}m = c_1(\eta)\otimes_{U^*}1 , \quad c_1(\eta)\in\tilde{U}^{ev}(L^n(p))^{Z_2} .$$

q.e.d.

From the isomorphism (5.1), Lemma 5.1 and Theorem 1.2, we have the following.

Theorem 5.2 ([5] and [6]).

$$\tilde{K}(D_p(2k+1, 4k+3)) \simeq Z\oplus\tilde{K}(L^{2k+1}(p))^{Z_2}\oplus\tilde{K}(RP^{4k+3}) .$$

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