OPTIMAL INVENTORY CONTROL POLICIES FOR PERISHABLE COMMODITIES

（品質が劣化する場合の最適在庫管理）

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CONTENTS

CHAPTER I

INTRODUCTION

Since the presentation of inventory model which was proposed by Wilson in 1915, a number of papers have been published concerning mathematical models and analysis of inventory. In these models, it was assumed that the inventory was nonperishable in nature, that is, once they are stocked, it could be used to satisfy demand at any time in the future. Little consideration was given to the case of ordering stock where the commodity could be used to satisfy demand only for a limited amount of time after it was received into storage.

Recently, however there has been considerable interest in developing mathematical models of inventory for describing optimal inventory policies with respect to perishing or decaying commodities. Commodities can be classified into two types taking account of their quality level ; one is the commodity with a maximum usable lifetime (perishable items) and the other is the commodity whose lifetime is decreasing with a fixed decay constant (decaying items).

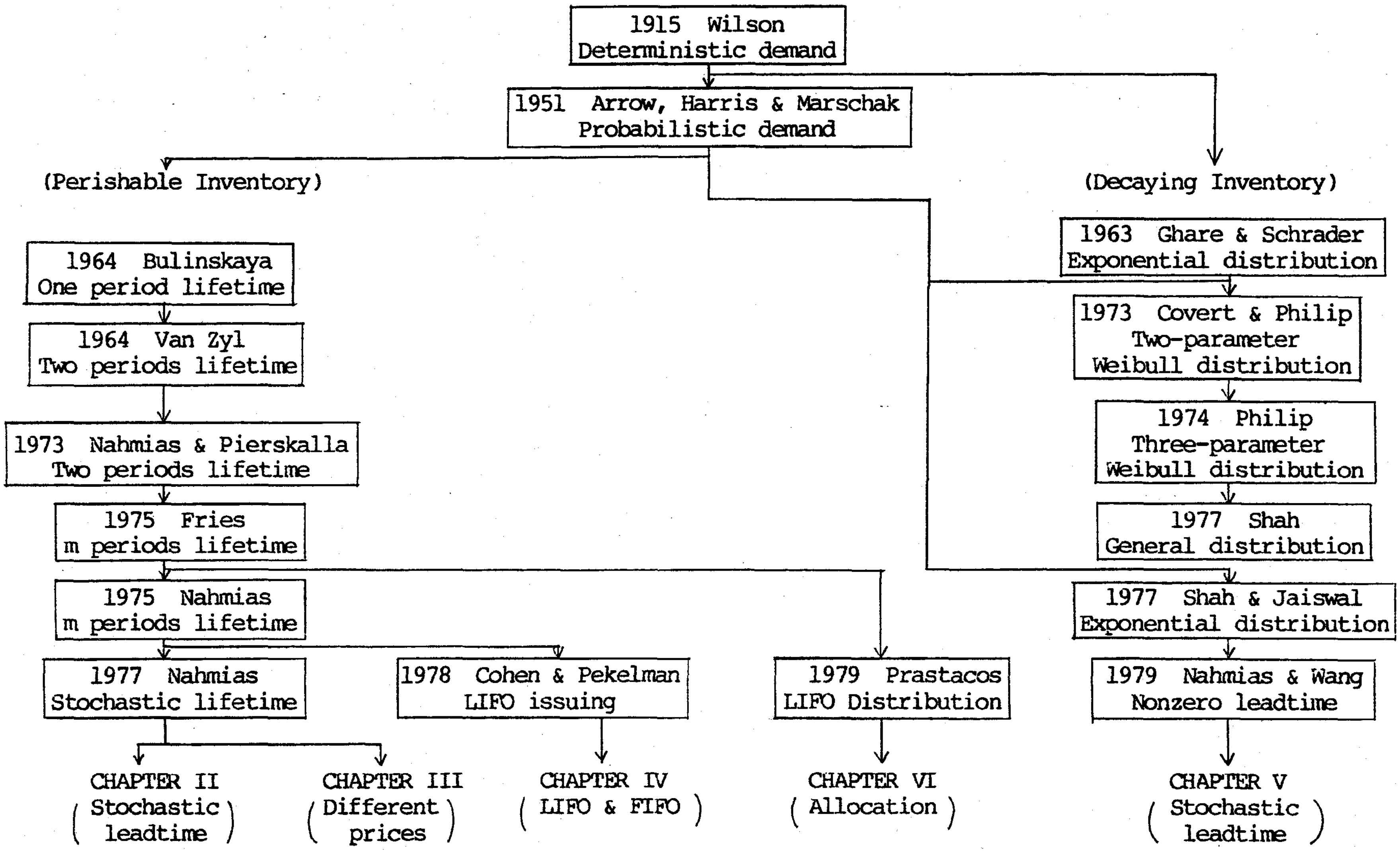

Figure 1.1 Historical Stream of Inventory Problem for Perishable and Decaying Commodities

Ghare and Schrader [6] developed an EOQ(Economic Ordering Quantity) model. They assumed the commodity whose lifetime is decreasing with a fixed decay constant. Covert and Philip [4] obtained an EOQ model for items with a variable rate of deterioration by assuming the two-parameter Weibull distribution for the progress of the item deterioration. Philip [16] discussed a model by assuming the three-parameter Weibull distribution for the progress of the item deterioration. Shah [20] developed a model in which shortages are permissible. They are generalization of the works of Ghare and Schrader [6] and of Covert and Philip [4]. Cohen [2] has considered the problem of joint ordering and pricing decisions for decaying inventory under known demand. In above models the demand rate is assumed to be deterministic. Shah and Jaiswal [21, 22] discussed periodic review inventory models for deteriorating items with stochastic demand. Nahmias and Wang [10] considered the leadtime with respect to decaying items. Nose et al. [11] discusses a (Q, r) inventory control system with finite varying stochastic procurement leadtime for exponentially decaying commodities.

Recently, remarkable researches have been worked out for obtaining optimal ordering policies in case of perishable commodities. In such a case, when demand is deterministic, the problem has a trivial solution, that is, place an order so that no item perishes. However, the solution is much more complex when demand is random. In this case it is not possible to order so that no item will perish. Hence outdating becomes an important factor for placing an order. Bulinskaya [1] considered a case where the lifetime of an item was exactly one period. Though, his work was concerned with an extension of the previous analysis of nonperishable commodity's inventory model to the analysis of a simple inventory model for perishable commodity. This analysis is a simple extension of the inventory models for nonperishable commodities to perishable ones.

Analysis of a single item inventory model with a lifetime of exactly two periods was pioneered by Van Zyl [23]. He restricted his attention to only shortage and ordering costs. His model does not include a perishing cost as my work do, so that the perishability feature is included only through the transfer function. Nahmias and Pierskalla [8] have developed the model with an outdating cost which is incurred at the time when the order is placed instead of the time the lifetime of the ordered commodity reaches to the specified lifetime.

The earlier models with a lifetime of only two periods have been extended for an item with predetermined maximum lifetime of m periods by Fries [5] and Nahmias [9]. In the single period model of Fries [5] the expected outdating cost is irrelevant to whether one places an order or not, so that the optimal ordering policy is the same as the case of nonperishable commodity's inventory problem. Therefore, it is no good as the optimal ordering policy for perishable commodity's inventory problem. On the other hand, Nahmias [9] includes in an explicit way the effect of outdating . He noted that the number of units of the current order that will outdate without meeting any demand is a random variable that depends upon both the age distribution of inventories on hand and the realizations of the demands over the next m periods.

Almost of those models mentioned above assume that their remaining lifetime is predetermined fixed when the reordered perishable commodities are received. However this assumption is not realistic in many actual circumstances. Therefore it may be reasonable to assume that if the procurement leadtime varies, the value of commodities may also be changeable. Ishii et al. [7] discussed the model for a perishable commodity with the stochastic procurement leadtime. Furthermore, Nose et al. [12] derived some properties of optimal ordering policies for a perishable commodity with the stochastic procurement leadtime. Nose et al. [13] is a generalized model of [7] and [12] incorporating the discriminating selling prices.

On the other hand, most of the previous works have been discussed with customers who had no control over the depletion of inventories. That is FIFO (First-In-First-Out) issuing which is assumed that the older inventory is always used first to meet demand. One of the most interesting studies is that of Cohen and Pekelman [3] in which they give attention to customer controlled depletion. Given commodity information such as an expiration date, it may be reasonable to assume that customers will select the newest available item on the shelf if all items are equally priced and if no search cost is involved. Also, whenever high reliability is needed, the newest first issuing may be desirable. The newest first issuing doctorine is equivalent to a LIFO (Last-In-First-Out) issuing sequence. Nose et al. [14] discusses an inventory control for perishable commodity subject to stochastic procurement leadtime on LIFO and FIFO issuing sequences respectively.

Optimal allocation policies for perishable commodities were analyzed first by Prastacos [17] [18] [19]. These models were discussed under charging cost only for shortage and outdating. But, if we discuss the allocation problem, transportation cost should be also an important factor. Nose et al. [15] discussed a single period allocation problem of perishable commodities based on a LIFO issuing and a rotation allocation policies considering shortage, outdating and transportation costs.

Chapter II constructs a model for perishable commodities subject to stochastic procurement leadtime and the optimal ordering policies are derived. And the chance constraint for shortage is added and the optimal ordering policy is again derived. Furthermore, the influences of leadtime, rate of excess perishability and on-hand stock upon the obtained optimal ordering policies are discussed.

Chapter III discusses a model for a perishable product with the stochastic procurement leadtime and discriminating selling prices. And the optimal ordering policy and its properties are analyzed.

In Chapter IV, an inventory control for perishable commodity subject to stochastic procurement leadtime on both LIFO and FIFO issuing policies is considered under zero or 1 unit leadtime. Optimal ordering policies on both LIFO and FIFO are derived and the influences of the change of probability of occurring leadtime 1, on-hand inventory and costs of unit holding, shortage and outdating upon the optimal order-up-to-level are investigated on both issuing policies, respectively.

Chapter V describes a (Q,r) inventory control system for decaying commodities with finite varying stochastic procurement leadtime for exponentially decaying commodities. Characteristics of optimal ordering decision, i.e., optimal ordering quantity and reorder point are derived under given probability of stockout occurrence and their sensitivities of changes in decay are also clarified.

Chapter VI discusses a single period allocation problem of perishable commodities based on a LIFO issuing and a rotation allocation policies considering shortage, outdating and transportation costs. First, the existence of the optimal allocation policy is clarified. Furthermore, an algorithm of obtaining the optimal allocation policy for a perishable commodity is offered. Finally, we note Figure 1-1 summarized this chapter visually.

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CHAPTER II

INVENTORY CONTROL SUBJECT TO STOCHASTIC LEADTIME FOR PERISHABLE COMMODITIES

2.1 Introduction

There exist so many commodities whose value does not remain constant over time during transportation, holding in stock, etc.. Bulinskaya [1], Van Zyl [8], etc., treated the cases where lifetimes are one or two periods of time. But they did not refer to the perishing cost for the determination of optimal ordering policy. Nahmias and Pierskalla [7] introduced perishing cost into the determination of optimal ordering policy for the two periods lifetime dynamic inventory model. Later, Fries [2] and Nahmias [5] generalized their model to the m periods lifetime model.

But most of those models assume that the maximum lifetime of the product is fixed when the reordered perishable commodities are received. However this assumption is not realistic in many actual circumstances.

Therefore it is reasonable to assume that if the procurement leadtime varies, the value of commodities may also be changeable. Taking the above view point into consideration, this chapter discusses the inventory control for perishable commodities under a certain maximum lifetime and the stochastic leadtime. Particularly the case of 0 or 1 period leadtime is considered and its optimal ordering policy is derived. Further the influences for optimal ordering policy of some important factors, i.e., the probability of leadtime 0 or 1, the rate of excess perishablity under the occurrence of leadtime 1 and the status of inventory on hand (newer or older), are analyzed.

Next, considering the fixed ordering cost, some other characteristics of the ordering policy are obtained. Note that the model in this chapter is a generalization of the one period horizon model of Nahmias [4].

Section 2.2 states the assumptions and the notations which are used throughout this chapter. Subsection 2.3.1 formulates the model and the optimal ordering policy is obtained.

Furthermore the proposed model is compared with the Nahmias' model.

In subsection 2.3.2, the chance constraint for shortage is added and the optimal ordering policy is again derived. In subsection 2.3.3, we give an example in order to illustrate the results of subsections 2.3.1 and 2.3.2.

Subsection 2.4.1 discusses the influences of leadtime, rate of excess perishability and on-hand stock upon the obtained optimal ordering policy. Subsection 2.4.2 gives a numerical example in order to illustrate the results of the subsection 2.4.1.

Section 2.5 concludes this chapter and discusses further research problems.

2.2 Assumptions of inventory model for perishable commodities

A periodic review inventory model is considered for one planning period horizon and single item. That is, ordering takes place at the start of a period and costs are incurred during a period, rather than continuously. The period length is predetermined fixed constant with unit length L . And the followings are assumed throughout this chapter.

(1) Maximum lifetime of the perishable commodity discussed in this chapter is m finite periods. If the commodity has not been depleted by demand until the period it reaches age m, then it perishes and must be discarded at a specified per-unit cost r .

(2) Demands D_j in succesive periods $j=1,2,...$ are independent nonnegative random variables with known distribution function $F_j(\cdot)$ and probability density $f_j(\cdot)$.

(3) After commodities are placed into stock, deterioration proceeds monotonically one stage in each period and inventory is depleted by demand at the start of each period according to a FIFO policy .

(4) When procurement leadtime ℓ is 0, the stock arrives new, that is, maximum lifetime m, and when $\ell=1$, the stock with lifetime $m-1$ or $m-2$ arrives. Leadtime 0 occurs with probability ℓ_0 and 1 with ℓ_1, where $\ell_0+\ell_1=1$, $\ell_0>0$, $\ell_1\geq 0$, and when $\ell=1$, the stock with lifetime $m-1$ arrives at a constant rate α $(0\leq\alpha\leq 1)$ and $m-2$ at $1-\alpha$. Here, $1-\alpha$ corresponds to the rate of excess perishability under the occurrence of leadtime 1.

In stocking perishable commodities, it is necessary to keep track of the amount of inventory on hand at each lifetime level. Concentrating on our model, we define the following notations;

x_i ;the amount of commodity on hand with i periods of usable lifetime left,

X_p ;inventory level in stock, i.e., $X_p \triangleq (x_p, x_{p-1}, \ldots, x_1)$, $1 \leq p \leq m-1$,

D_j ;demand during j th period

B_j ;the total unsatisfied demand until the end of the j th period after depleting all the commodities $x_j, x_{j-1}, \ldots, x_1$, i.e.,

$$B_j = [D_j + B_{j-1} - x_j]^+ , \quad 1 \leq j \leq m-1 \tag{2.1}$$

where $B_0 = 0$ and $[b]^+ = max\ (b, 0)$.

Figure 2.1 illustrates this model.

$Q_n(u | X_{n-1})$;the probability that $D_n + B_{n-1}$ is less than a real number u, i.e.,

$$Q_n(u | X_{n-1}) = Pr\{D_n + B_{n-1} \leq u\}, \quad 1 \leq n \leq m \tag{2.2}$$

where, if $u \leq 0$, then $Q_n(u | X_{n-1}) \triangleq 0$,

$A(X_{m-1}, y)$;the total expected cost when y is ordered under the current stock level X_{m-1} , i.e., including the fixed ordering, purchasing, holding, shortage and outdating costs, i.e.,

K ;fixed ordering cost/order,

c ;purchasing cost/unit, h ;holding cost/unit,

p ;shortage cost/unit, r ;outdating cost/unit,

x ;total amount of inventory on hand, i.e., $x \triangleq \sum_{i=1}^{m-1} x_i$,

$\mathbf{e}_i$;i-th unit vector, i.e., $\mathbf{e}_i \triangleq (0, \ldots, 1, 0, \ldots, 0)$, $i = 1, 2, \ldots, m-1$.

$L(X_{m-1}, y)$;the total expected cost excluding the fixed ordering cost from $A(X_{m-1}, y)$,

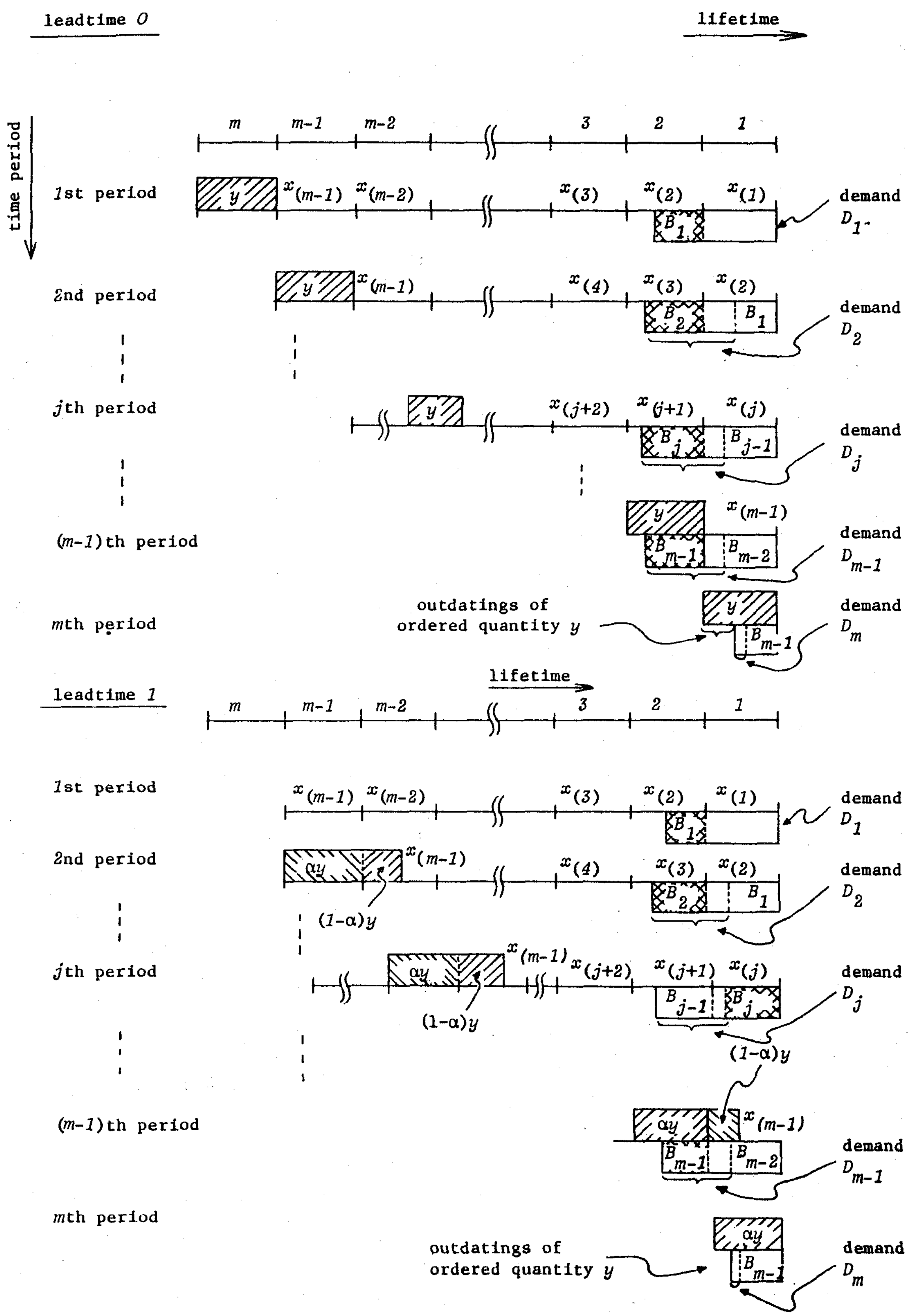

Figure 2.1 Inventory model for perishable commodities

2.3 Optimal ordering policy subject to stochastic leadtime and stockout constraint

2.3.1 Inventory model for perishable commodities

From the definitions (2.1) and (2.2), the distribution function of the amount of new order y that perishes is

$$Pr\{[y-(D_m+B_{m-1})]^+ \leq u\} = 1-Q_m(y-u\,|\,X_{m-1}). \tag{2.3}$$

Note that y is a decision variable which represents the amount of ordered commodity. Then, among the incoming amount y of new commodity, the expected amount remaining until the end of m units of period and thus to be abandoned eventually is described as in the following formula [4]; from the definitions (2.2) and (2.3).

$$\int_0^\infty u\,d\{1-Q_m(y-u\,|\,X_{m-1})\} = \int_0^y Q_m(u\,|\,X_{m-1})\,du. \tag{2.4}$$

Then the total expected cost $A(X_{m-1}, y)$ is established as the equations (2.5) and (2.6):

$$A(X_{m-1}, y) = K\cdot\delta(y)+L(X_{m-1}, y), \tag{2.5}$$

where

$$\delta(y) = \begin{cases} 1\ ; & y>0. \\ 0\ ; & y=0. \end{cases}$$

And,

$$L(X_{m-1}, y)=c\cdot y+\ell_0\Big[h\int_0^{x+y}(x+y-u)\cdot f_1(u)du$$

$$+p\int_{x+y}^{\infty}(u-x-y)\cdot f_1(u)du+r\cdot\int_0^{y}Q_m(u|X_{m-1})du\Big]$$

$$+\ell_1\Big[h\int_0^{x}(x-u)\cdot f_1(u)du+p\int_x^{\infty}(u-x)\cdot f_1(u)du$$

$$+r\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}}Q_{m-1}(u|X_{m-2})du$$

$$+r\int_0^{\alpha y}Q_m(u|X_{m-1}+(1-\alpha)y.e_{m-1})du\Big]. \qquad (2.6)$$

Now, we show the convexity of the cost function $L(X_{m-1},y)$, described above, utilizing the lemma due to Nahmias in the literature [4], where $Q_n(u:X_{n-1})$ is derived recursively by using the convolution of Q_{n-1} with $f_n(\cdot)$, namely

$$Q_n(u|X_{n-1})=\int_0^u Q_{n-1}(v+x_{n-1}|X_{n-2})f_n(u-v)dv,\ 1\leq n\leq m \qquad (2.7)$$

and $Q_0(u)=1.$

Especially, the integrand of last term of the equation (2.6) is derived by the following lemma, i.e.,

Lemma 2.1

$$Q_m(u|X_{m-1}+(1-\alpha)\ y\cdot e_{m-1})=\int_0^u Q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(u-v)dv \qquad (2.8)$$

and $Q_0(u)=1.$

Lemma 2.2 ([4]) Assume that each demand distribution function $F_k(d)$ possesses density function $f_k(d)$ which is continuous in $d\geq 0$. Then the functions $\partial Q_n(X_n)/\partial x_i$ are continuous over n dimensional real space R^n. If f_n has a jump at 0, then $\partial Q_n(X_n)/\partial x_i$ are all continuous over R^n for $i\leq n-1$ and $\partial Q_n(X_n)/\partial x_n$ is continuous in all its arguments except a jump at $x_n=0$.

Applying Lemma 2.1 to the last integral in equation (2.6) and differentiating $L(X_{m-1}, y)$ with respect to y,

$$\frac{\partial L(X_{m-1}, y)}{\partial y} = c + \ell_0 \left[(h+p)F_1(x+y) - p + rQ_m(y \mid X_{m-1})\right]$$
$$+ \ell_1 \left[(1-\alpha)rQ_{m-1}(x_{m-1} + (1-\alpha)y \mid X_{m-2})\{1 - F_m(\alpha y)\}\right.$$
$$\left. + r\int_0^{\alpha y} Q_{m-1}(y + x_{m-1} - u \mid X_{m-2}) f_m(u)du\right] = 0, \tag{2.9}$$

is obtained.

Next theorem demonstrates the existence of an optimal ordering policy $y^*(X_{m-1})$ under inventory level X_{m-1}.

Theorem 2.1

$L(X_{m-1}, y)$ is a convex function of y and there exists $y^*(X_{m-1})$ in $(-\infty, \infty)$ which minimizes the value of $L(X_{m-1}, y)$ i.e.,

$$L(X_{m-1}, y^*(X_{m-1})) = \min[L(X_{m-1}, y)]. \tag{2.10}$$

Proof.

First, note that by setting $t = v, s = u - v$,

$$\int_0^{\alpha y}\int_0^u q_{m-1}(v + x_{m-1} + (1-\alpha)y \mid X_{m-2}) f_m(u-v)dvdu$$
$$= \int_0^{\alpha y}\int_0^{\alpha y - s} q_{m-1}(t + x_{m-1} + (1-\alpha)y \mid X_{m-2}) f_m(s)dtds$$
$$= \int_0^{\alpha y} f_m(s) Q_{m-1}(y + x_{m-1} - s \mid X_{m-2})ds - Q_{m-1}(x_{m-1} + (1-\alpha)y \mid X_{m-2})F_m(\alpha y)$$
$$\tag{2.11}$$

and by setting $t = \alpha y - v$,

$$\int_0^{\alpha y} Q_{m-1}(v + x_{m-1} + (1-\alpha)y \mid X_{m-2}) f_m(\alpha y - v)\mathrm{d}v = \int_0^{\alpha y} Q_{m-1}(y + x_{m-1} - t \mid X_{m-2}) f_m(t)\mathrm{d}t \quad .$$

Then

$$\frac{\partial^2 L(X_{m-1}, y)}{\partial y^2} = \ell_0[(h+p)f_1(x+y) + rq_m(y \mid X_{m-1})]$$
$$+\ell_1\Big[(1-\alpha)^2 rq_{m-1}((1-\alpha)y + x_{m-1} \mid X_{m-2})\{1 - F_m(\alpha y)\}$$
$$+\alpha^2 rQ_{m-1}((1-\alpha)y + x_{m-1} \mid X_{m-2}) f_m(\alpha y)$$
$$+r\int_0^{\alpha y} q_{m-1}(y + x_{m-1} - t \mid X_{m-2}) f_m(t)\mathrm{d}t\Big] \geq 0. \tag{2.12}$$

Since all the terms are composed of non-negative values. This implies the convexity of $L(X_{m-1}, y)$,[3]. Moreover,

$$\lim_{y\to\infty} \frac{\partial L(X_{m-1}, y)}{\partial y} = c + \ell_0(h+r) + \ell_1\Big[(1-\alpha)r$$
$$+\lim_{y\to\infty}(1-\alpha)r\int_0^{\alpha y}\int_0^{u} q_{m-1}(v + x_{m-1} + (1-\alpha)y \mid X_{m-2}) f_m(u - v)\mathrm{d}u\mathrm{d}v\Big] > 0, \tag{2.13}$$

$$\lim_{y\to-\infty} \frac{\partial L(X_{m-1}, y)}{\partial y} = -p\ell_0 < 0. \tag{2.14}$$

Inequalities (2.12)-(2.14) together prove Theorem 2.1. □

The equation (2.10) guarantees the existence of an optimal ordering quantity $y(X_{m-1})$ under inventory on-hand stock level X_{m-1} .

Theorem 2.2

When $x_{m-1} = 0$, the following critical order policy is optimal:

$$\begin{cases} \text{order, if } & x < x^* \\ \text{not order,} & \text{otherwise,} \end{cases} \tag{2.15}$$

where the critical order point, x^*, is obtained uniquely as follows:

$$x^*=\begin{cases}\hat{x}, & \text{if } F_1(\hat{x})=\dfrac{p\ell_0-c}{(h+p)\ell_0} \text{ and } c<pl_0,\\ 0, & \text{otherwise}.\end{cases} \tag{2.16}$$

When $x_{m-1}>0$,

$$\begin{cases}\text{order}, & x_{m-1}<x^*_{m-1}(X_{m-2}),\\ \text{not order}, & \text{otherwise},\end{cases} \tag{2.17}$$

is optimal where the critical order point $x^*_{m-1}(X_{m-2})\geqslant 0$ is derived based on the solution $\hat{x}_{m-1}(X_{m-2})$ of the equation (2.19) :

$$x^*_{m-1}(X_{m-2})=\begin{cases}\hat{x}_{m-1}(X_{m-2}), & \text{if } x_{m-1}(X_{m-2}) \text{ satisfying}(2,19)\text{exists},\\ 0, & \text{otherwise}\end{cases} \tag{2.18}$$

$$\begin{aligned}c+\ell_0\Bigg[hF_1\Bigg(\sum_{i=1}^{m-2}x_i+\hat{x}_{m-1}(X_{m-2})\Bigg)-p\Bigg\{1-F_1\Bigg(\sum_{i=1}^{m-2}x_i+\hat{x}_{m-1}(X_{m-2})\Bigg)\Bigg\}\Bigg]\\ +(1-\alpha)r\ell_1Q_{m-1}(\hat{x}_{m-1}(X_{m-2})\mid X_{m-2})=0.\end{aligned} \tag{2.19}$$

Proof.

(Case $x_{m-1}=0$): If $p\ell_0\leqslant c$,then for all X_{m-1}, $\partial L(X_{m-1},y)/\partial y|_{y=0}>0$ holds, that is, for all X_{m-1}, $y=0$ is optimal giving $\min_y L(X_{m-1},y)$, because of the monotonicity property of $\partial L/\partial y$ with respect to y. Therefore x^* must be 0. In case of $p\ell_0>c$, then there exists x^* satisfying

$$\partial L(X_{m-1},y)/\partial y|_{y=0} \quad =c+\ell_0\big[hF_1(x^*)-p\{1-F_1(x^*)\}\big]=0,$$

that is, x^* is equal to $F^{-1}((p\ell_0-c)/(p+h)\ell_0)$. Since $\partial L(X_{m-1},y)/\partial y|_{y=0}<0$ for all X_{m-1} such that $x<x^*_1$, again by the monotonicity property of $\partial L/\partial y$, y satisfying $\partial L(X_{m-1},y)/\partial y=0$ must be positive, that is, some amount of the product must be ordered.

On the other hand, if $x\geqslant x^*_1$ then $\partial L(X_{m-1},y)/\partial y|_{y=0}\geqslant 0$, that is, not to order is optimal.

(Case $x_{m-1} > 0$): If $\hat{x}_{m-1}(X_{m-2})$ satisfying (2.19) does not exist, then $\partial L(X_{m-1}, y)/\partial y|_{y=0} > 0$ for all X_{m-1} By the similar argument to the case $x_{m-1} = 0$, $x^*_{m-1}(X_{m-2}) = 0$. In case that $\hat{x}_{m-1}(X_{m-2})$ satisfying (2.19) exists, then $\partial L(X_{m-1}, y)/\partial y|_{y=0}$ =0 for X_{m-1} such that $x_{m-1} = x^*_{m-1}(X_{m-2})$. Again by the similar argument to the case x_{m-1} =0 , for X_{m-1} such that $x_{m-1} < x^*_{m-1}(X_{m-2})$, some amount of the product must be ordered and for $x_{m-1} \leqslant x^*_{m-1}(X_{m-2})$, not to order is optimal. □

Equation (2.19) would be able to be solved with respect to the distribution function F_1 . Then the following relation is obtained:

$$F_1\left(\sum_{i=1}^{m-2} x_i + x^*_{m-1}(X_{m-2})\right) = \frac{p\ell_0 - c - \mathcal{U}_1(1-\alpha)rQ_{m-1}(x^*_{m-1}(X_{m-2}) \mid X_{m-2})}{(h+p)\ell_0} \leqslant \frac{p-c}{p+h}.$$

Further, the following inequality is hold.

$$\sum_{i=1}^{m-2} x_i + x^*_{m-1}(X_{m-2}) \leqslant F^{-1}\left(\frac{p-c}{p+h}\right). \tag{2.20}$$

The right-hand side of the above inequality represents the critical order point of Nahmias [4] which does not take account of stochastic leadtime.

We conclude this section by discussing the effects of α . It is easily shown that $\hat{x}_{m-1}(X_{m-2})$ is a monotonically increasing function of α in (2.19).

Proposition 2.1

$L(X_{m-1},y)$ is a nondecreasing function of α.

Proof.

The partial derivative of $L(X_{m-1},y)$ with respect to α is as follows.

$$\frac{\partial L(X_{m-1},y)}{\partial \alpha}=-\ell_1 ry\Big\{Q_{m-1}((1-\alpha)y+x_{m-1}\mid X_{m-2})-Q_m(\alpha y\mid X_{m-1}+(1-\alpha)ye_{m-1})$$
$$+\int_0^{\alpha y}\int_0^u q_{m-1}(v+x_{m-1}+(1-\alpha)y\mid X_{m-2})f_m(u-v)\,dv\,du\Big\}$$

(by applying (2.11) to the third tenn in the above brace)

$$=-\ell_1 ry\Big\{Q_{m-1}((1-\alpha)y+x_{m-1}\mid X_{m-2})$$
$$-\int_0^{\alpha y}Q_{m-1}(v+x_{m-1}+(1-\alpha)y\mid X_{m-2})f_m(\alpha y-v)\,dv$$
$$+\int_0^{\alpha y}f_m(s)Q_{m-1}(y+x_{m-1}-s\mid X_{m-2})\,ds-Q_{m-1}((1-\alpha)y+x_{m-1}\mid X_{m-2})F_m(\alpha y)\Big\}$$
$$=-\ell_1 ryQ_{m-1}((1-\alpha)y+x_{m-2}\mid X_{m-2})(1-F_m(\alpha y))\leqslant 0$$

(by equation (2.12)). □

2.3.2 The optimal ordering policy under an added chance constraint

In order to compensate shortage under varying procurement leadtime, optimal ordering quantity should be determined taking account of the delay of product's arrival ordered. For this purpose, the following chance constraint is added so as to keep the occurrence probability of shortage less than a given level, $1-\beta$, [7].

$$\Pr\{x-x_1+y-(D_1-x_1)^+-D_2\leqslant 0\}\leqslant 1-\beta. \tag{2.21}$$

In other words, the above inequality means that the probability of which the second period's demand plus the remains of first period's demand exceeds $(x-x_1)+y$ is less than $1-\beta$.

First note that inequality (2.21) is equivalent to the following inequality of service level:

$$\Pr\{x+y-(D_1-x_1)^+-D_2\geqslant 0\}\geqslant\beta. \tag{2.22}$$

Further (2.22) is equivalent to the condition that y satisfies at least one pair of inequalities (2.23) and (2.24) when β_1 changes within the range $0\leqslant\beta_1\leqslant\beta$.

$$\Pr\{D_2\leqslant x+y-x_1\}\cdot\Pr\{D_1\leqslant x_1\}\geqslant\beta_1, \tag{2.23}$$

$$\Pr\{D_2\leqslant x+y-D_1,\ D_1>x_1\}\geqslant\beta-\beta_1. \tag{2.24}$$

(2.23) and (2.24) are transformed as follows:

$$y\geqslant F_2^{-1}(\beta_1/F_1(x_1))-x+x_1, \tag{2.25}$$

$$\int_{x_1}^{x+y}F_2(x+y-d_1)f_1(d_1)dd_1\triangleq P(y)\geqslant\beta-\beta_1. \tag{2.26}$$

In (2.25), as β_1 increases, lower bound of y increases. For $P(y)$ in (2.26),

$$\frac{\partial P(y)}{\partial y}=\int_{x_1}^{x+y}f_2(x+y-d_1)f_1(d_1)dd_1\geqslant 0\ . \tag{2.27}$$

From (2.27), in (2.26), lower bound of y decreases as β_1 increases. ((2.25) and (2.26) are illustrated in Figure 2.2.)

Thus ,we obtain

$$y \geqslant z(\beta), \tag{2.28}$$

where

$$z(\beta) = \min_{\beta_1 \in S} \max\{F^{-1}(\beta_1/F_1(x_1)) - x + x_1, P^{-1}(\beta - \beta_1)\}$$

and

$$S \triangleq \{\beta_1 \mid 0 \leqslant \beta_1 \leqslant \max(\beta, F_1(x_1))\}.$$

Inequality (2.28) implies that in order to maintain the service level β, at least the quantity $z(\beta)$ must be ordered. Therefore, the optimal ordering policy under above constraint (2.28) is summarized as the following Theorem 2.3.

Theorem 2.3

Optimal ordering quantity y_0^* under the consideration of chance constraint is given in (2.29):

$$y_0^* = \max\{y^*(X_{m-1}), z(\beta)\}. \tag{2.29}$$

Following propositions clarify the influence of x_1 and β on $z(\beta)$ respectively.

Proposition 2.2

$z(\beta)$ is a monotonically decreasing function of x_1 .

Proof. Since $x-x_1=\Sigma_{i=2}^{m-1}x_i$, we have

$$\frac{\partial\left(P^{-1}(\beta-\beta_1)\right)}{\partial x_1}=0.$$

Further,

$$\frac{\partial\left(F_2^{-1}(\beta_1/F_1(x_1))-\sum_{i=2}^{m-1}x_i\right)}{\partial x_1}=\frac{\partial\left(F_2^{-1}(\beta_1/F_1(x_1))\right)}{\partial x_1}=-\frac{\beta_1 f_1(x_1)}{(F_1(x_1))^2}\frac{1}{f_2\left(F_2^{-1}(\beta_1/F_1(x_1))\right)}\leqslant 0.$$

And equality holds if and only if $x_1=0$. This implies $z(\beta)$ is monotonically decreasing function of x_1 . □

Proposition 2.2 implies optimal ordering quantity decreases as increase of the oldest inventory x_1 .

Proposition 2.3.

$z(\beta)$ is a nondecreasing function of β .

Proof. Letting $\kappa=P^{-1}(\beta-\beta_1)$,

$$\frac{\partial\kappa}{\partial\beta}=\frac{1}{\frac{dP(\kappa)}{d\kappa}}=f_1(x+\kappa-x_1)\,F_2(x_1)+\int_0^{x+\kappa-x_1}f_1(d_2)f_2(x+\kappa-d_2)\mathrm{d}d_2\geqslant 0$$

and $F_2^{-1}(\beta_1/F_1(x_1))$ does not contain β . This proves Proposition 2.3. □

Proposition 2.3 means y_0^* is nondecreasing function of service level β.

Theorem 2.3 and following Proposition 2.3 together show that if the service level β becomes high, more quantity than that of minimizing the total cost must be ordered.

2.3.3 Numerical example

This subsection gives an example in order to illustrate results of subsection 2.3.1 and 2.3.2. In equation (2.5), an experimental datum was chosen; the cost parameters (c , h , p , r)=(40,10,200,40), the leadtime probability ℓ_0 at four levels (ℓ_0=0.4, 0.6, 0.8 and 1.0) and the rate α at three levels (α =0.0, 0.5 and 1.0). The demand distribution function is assumed to be exponential with mean 20.0. For this datum, each $L(X_{m-1}, y)$ is illustrated in Figure 2.2 (m=3 in this case). Table 2.1 shows obtained results and followings are observed:

(1)The optimal ordering quantity with taking the influence of stochastic leadtime into consideration (ℓ_0=0.4, 0.6, and 0.8) is more than that of Nahmias [4], i.e., ℓ_0 =1.0. (Inequality (2.20) indicates this inclination.)

(2)The total expected cost $L(X_{m-1}, y)$ increases as ℓ_0 decreases.

(3)At each level of ℓ_0 (ℓ_0 =0.4, 0.6 and 0.8), $L(X_{m-1}, y)$ increases as α decreases, but the significant differences do not appear among the optimal ordering quantities.

Furthermore, under the chance constraint (2.21), regions of feasible solutions are obtained with respect to β at three levels (β=0.85, 0.90 and 0.95). Table 2.2 shows optimal ordering quantity under the above chance constraint. (For example, in case of β=0.85 and ℓ_0 =0.8, using Theorem 2.3, the optimal ordering quantity is 35.36 and the total expected cost is about 2750 in case $x_1 = x_2$ = 5.0.) (See Figure 2.3.)

Note that using the chance constraint (2.21), remarkably more quantities are to be ordered than the optimal ordering quantity obtained in the equations (2.15) to (2.18).

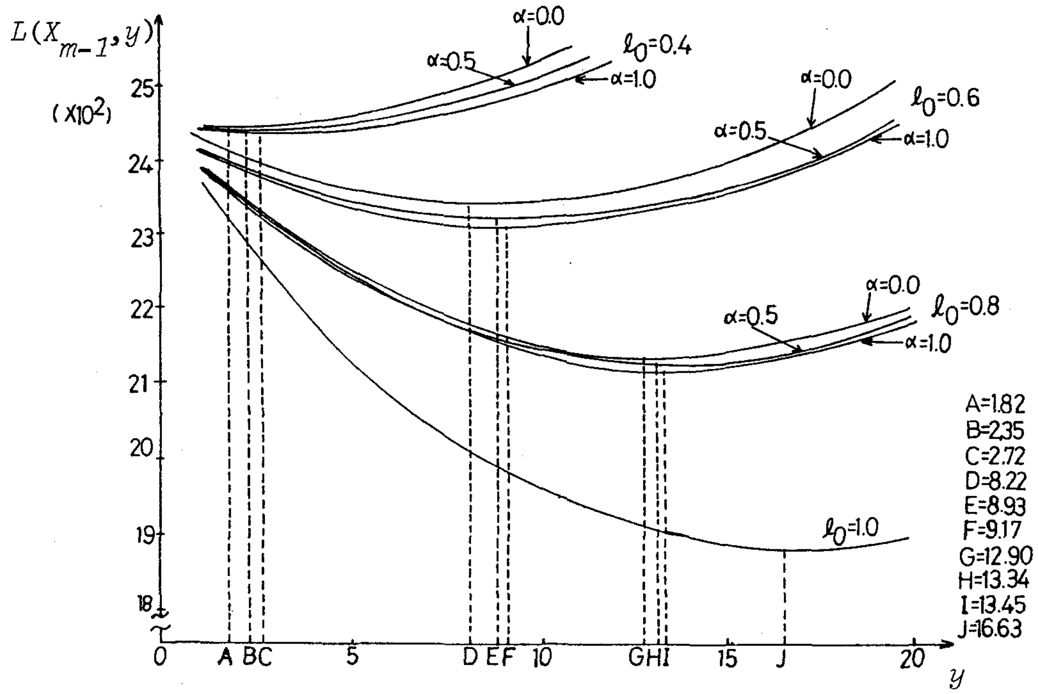

Figure 2.2. Total expected cost $(x_1=5.00, x_2=5.00)$

Table 2.1 Optimal ordering quantity

	$\ell_0 = 0.4$				$\ell_0 = 0.6$			
	$x_1 = 0.00$ $x_2 = 0.00$	$x_1 = 0.00$ $x_2 = 5.00$	$x_1 = 5.00$ $x_2 = 0.00$	$x_1 = 5.00$ $x_2 = 5.00$	$x_1 = 0.00$ $x_2 = 0.00$	$x_1 = 0.00$ $x_2 = 5.00$	$x_1 = 5.00$ $x_2 = 0.00$	$x_1 = 5.00$ $x_2 = 5.00$
$\alpha = 0.0$	11.57	6.58	6.78	1.82	18.03	13.03	13.18	8.22
$\alpha = 0.5$	12.36	7.28	7.44	2.35	18.85	13.81	13.96	8.93
$\alpha = 1.0$	12.47	7.50	7.61	2.72	18.96	13.97	14.10	9.17
	$\ell_0 = 0.8$				$\ell_0 = 1.0$			
	$x_1 = 0.00$ $x_2 = 0.00$	$x_1 = 0.00$ $x_2 = 5.00$	$x_1 = 5.00$ $x_2 = 0.00$	$x_1 = 5.00$ $x_2 = 5.00$	$x_1 = 0.00$ $x_2 = 0.00$	$x_1 = 0.00$ $x_2 = 5.00$	$x_1 = 5.00$ $x_2 = 0.00$	$x_1 = 5.00$ $x_2 = 5.00$
$\alpha = 0.0$	22.72	17.73	17.86	12.90	26.46	21.47	21.59	16.63
$\alpha = 0.5$	23.21	18.19	18.33	13.34	26.46	21.47	21.59	16.63
$\alpha = 1.0$	23.26	18.27	18.40	13.45	26.46	21.47	21.59	16.63

2.4 Some properties of perishable inventory control for perishable commodities

2.4.1 Properties on the optimal ordering policy

First, $Q_m^{(i)}(y|X_{m-1})$ is defined as follows:

$$Q_m^{(i)}(y|X_{m-1}) = \begin{cases} \dfrac{\partial Q_m(y|X_{m-1})}{\partial y} \; ; & i = 1, \\ \dfrac{\partial Q_m(y|X_{m-1})}{\partial x_{m-i+1}} \; ; & i = 2, \ldots, m. \end{cases}$$

Next inequality (2.30) which is presented by Nahmias [4] will be used in order to prove the followings :

$$Q_m^{(i)}(y|X_{m-1}) \geq Q_m^{(i+1)}(y|X_{m-1}) \tag{2.30}$$

This inequality (2.30) and the preceding Lemma 2.1 together show the following relations.

Proposition 2.4

If $\alpha = 0$ or 1, then

$$-1 \leq y^{(i)}(X_{m-1}) \leq y^{(i+1)}(X_{m-1}) \leq 0 \; ; \; i = 2, \ldots, m-2, \tag{2.31}$$

holds. If $0 < \alpha \leq \frac{1}{2}$,

$$-2 \leq y^{(i)}(X_{m-1}) \leq y^{(i+1)}(X_{m-1}) \leq 0 \; ; \; i = 1, \ldots, m-2, \tag{2.32}$$

is obtained, where $y^{(i)}(X_{m-1})$ is defined as follows:

$$y^{(i)}(X_{m-1}) = \frac{\partial y(X_{m-1})}{\partial x_{m-i}}, \; i = 1, 2, \ldots, m-1.$$

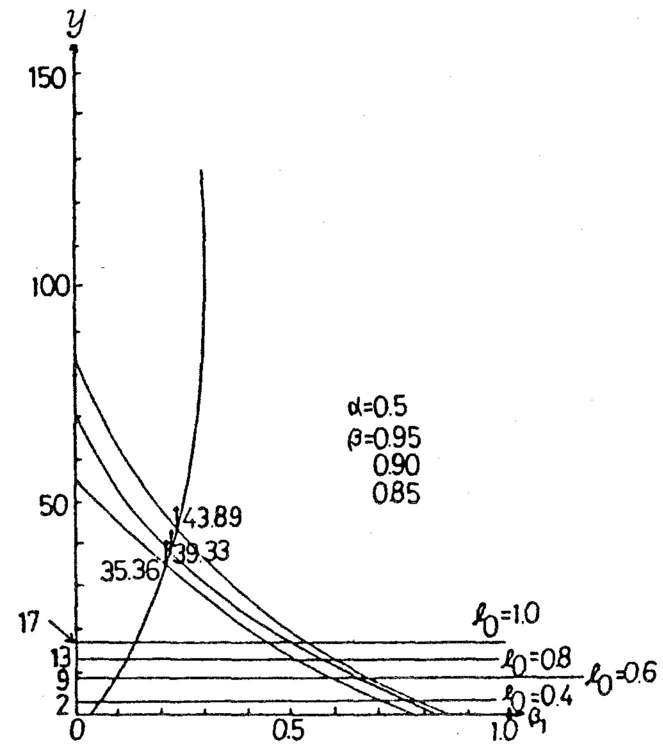

Figure 2.3 Feasible region of y under (x_1=5.0, x_2=5.0)

Table 2.2 Optimal ordering quantity under chance constraint (α=0.5)

	x_1 = 0.00 x_2 = 0.00	x_1 = 0.00 x_2 = 5.00	x_1 = 5.00 x_2 = 0.00	x_1 = 5.00 x_2 = 5.00
β = 0.85	67.44	62.45	40.36	35.36
β = 0.90	77.79	72.79	44.33	39.33
β = 0.95	94.88	89.88	48.89	43.89

Proof:

When $y(X_{m-1})$ is optimal quantity obtained from the equation (2.9), the partial differentiation of the equation (2.9) with respect to x_{m-i} ($i=2, \ldots, m-1$) becomes as follows:

$$\frac{\partial^2 L(X_{m-1}, y(X_{m-1}))}{\partial y \cdot \partial x_{m-i}} = \ell_0 \{ (h+p) f_1 (x+y(X_{m-1}))(1+y^{(i)}(X_{m-1}))$$
$$+rQ_m^{(i+1)}(y(X_{m-1})|X_{m-1}) + rQ_m^{(1)}(y(X_{m-1})|X_{m-1}) y^{(i)}(X_{m-1})$$
$$+\ell_1 \{ (1-\alpha) r [Q_{m-1}^{(i)}((1-\alpha)y(X_{m-1}) + x_{m-1} \mid X_{m-2})$$
$$+q_{m-1}((1-\alpha)y(X_{m-1}) + x_{m-1} | X_{m-2}) y^{(i)}(X_{m-1})][1$$
$$-F_m(\alpha y(X_{m-1}))] - (1-\alpha)\alpha r f_m(\alpha y(X_{m-1})) y^{(i)}(X_{m-1}) Q_{m-1}((1$$
$$-\alpha)y(X_{m-1}) + x_{m-1} \mid X_{m-2})\} + l_1 \{ r\alpha Q_{m-1}((1-\alpha)y(X_{m-1})$$
$$+x_{m-1} \mid X_{m-2}) f_m(\alpha y(X_{m-1})) y^{(i)}(X_{m-1})$$
$$+r\int_0^{\alpha y(X_{m-1})} \{ q_{m-1}(y(X_{m-1}) + x_{m-1} - u \mid X_{m-2}) y^{(i)}(X_{m-1}) f_m(u)$$
$$+Q_{m-1}^{(i)}(y(X_{m-1}) + x_{m-1} - u \mid X_{m-2}) f_m(u) \; du = 0 \; . \qquad (2.33)$$

Solving the equation (2.33) with respect to $y^{(i)}(X_{m-1})$,

$$y^{(i)}(X_{m-1}) = -\frac{E}{Z} \qquad (2.34)$$

where

$$Z = \ell_0\{(h+p)f_1(x+y(X_{m-1}))+rq_m(y(X_{m-1})\mid X_{m-1})\}$$
$$+\ell_1\{(1-\alpha)rq_{m-1}((1-\alpha)y(X_{m-1})+x_{m-1}\mid X_{m-2})[1-F_m(\alpha y(X_{m-1}))]$$
$$+\alpha^2 rf_m(\alpha y(X_{m-1}))Q_{m-1}((1-\alpha)y(X_{m-1})+x_{m-1}\mid X_{m-2})$$
$$+r\int_0^{\alpha y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{m-1}-u\mid X_{m-2})f_m(u)du > 0 \ , \tag{2.35}$$

$$E = \ell_0\{(h+p)f_1(x+y(X_{m-1}))+rQ_m^{(i+1)}(y(X_{m-1})\mid X_{m-1})\}$$
$$+\ell_1\{(1-\alpha)rQ_{m-1}^{(i)}((1-\alpha)y(X_{m-1})+x_{m-1}\mid X_{m-2})[1-F(\alpha y(X_{m-1}))]$$
$$+r\int_0^{\alpha y(X_{m-1})} Q_{m-1}^{(i)}(y(X_{m-1})+x_{m-1}-u\mid X_{m-2})f_m(u)du\} > 0 \ . \tag{2.36}$$

From (2.34), (2.35) and (2.36), the inequality

$$y^{(i)}(X_{m-1}) \leq 0 \tag{2.37}$$

is easily obtained.

Besides, using inequality (2.30) for each term of the equations (2.35) and (2.36), the inequality

$$y^{(i)}(X_{m-1}) \geq -1 \tag{2.38}$$

is also obtained.

Representing the numerator of $y^{(i)}(X_{m-1})$ and $y^{(i+1)}(X_{m-1})$ by $E_{(i)}$ and $E_{(i+1)}$ respectively and using inequality (2.20), the relation between $y^{(i)}(X_{m-1})$ and $y^{(i+1)}(X_{m-1})$ is obtained as follows:

$$y^{(i)}(X_{m-1})-y^{(i+1)}(X_{m-1})= -\frac{E_{(i)}-E_{(i+1)}}{Z} \leqq 0 \tag{2.39}$$

where,

$$\begin{aligned}E_{(i)}-E_{(i+1)}= {} & \ell_0 r[Q_m^{(i+1)}(y(X_{m-1})\mid X_{m-1})-Q_m^{(i+2)}(y(X_{m-1})\mid X_{m-1})] \\ & +\ell_1\{(1-\alpha)r[1-F_m(\alpha y(X_{m-1}))][Q_{m-1}^{(i)}((1-\alpha)y(X_{m-1})+x_{m-1}\mid X_{m-2}) \\ & -Q_{m-1}^{(i+1)}((1-\alpha)y(X_{m-1})+x_{m-1}\mid X_{m-2})] \\ & +r[\int_0^{\alpha y(X_{m-1})} Q_{m-1}^{(i)}(y+x_{m-1}-u\mid X_{m-2})f_m(u)du \\ & -\int_0^{\alpha y(X_{m-1})} Q_{m-1}^{(i+1)}(y(X_{m-1})+x_{m-1}-u\mid X_{m-2})f_m(u)du]\} \geqq 0\end{aligned}$$

(by inequality (2.30)).

From (2.37), (2.38) and (2.39), (2.31) can be derived.

When $i=1$, the partial derivative, $\partial^2 L(X_{m-1}, y(X_{m-1}))/(\partial y.\partial x_{m-i})$, of the equation (2.9) with respect to x_{m-1}, is obtained similarly. Then $y^{(1)}(X_{m-1})$ becomes as follows:

$$y^{(1)}(X_{m-1})= -\frac{E'}{Z'} \tag{2.40}$$

where,

$$
\begin{aligned}
Z' = \ell_0\{&(h+p)f_1(x+y(X_{m-1}))+rq_m(y(X_{m-1})\mid X_{m-1})\} \\
&+\ell_1\{(1-\alpha)^2 rq_{m-1}(x_{m-1}+(1-\alpha)y(X_{m-1})\mid X_{m-2})[1-F_m(\alpha y(X_{m-1}))] \\
&+\alpha^2 rQ_{m-1}(x_{m-1}+(1-\alpha)y(X_{m-1})\mid X_{m-2})f_m(\alpha y(X_{m-1})) \\
&+r\int_0^{\alpha y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{m-1}-u\mid X_{m-2})f_m(u)du\} \geqq 0\,, \qquad (2.41)
\end{aligned}
$$

$$
\begin{aligned}
E' = \ell_0\{&(h+p)f_1(x+y(X_{m-1}))+rQ_m^{(2)}(y(X_{m-1})\mid X_{m-1})\} \\
&+\ell_1\{(1-\alpha)rq_{m-1}(x_{m-1}+(1-\alpha)y(X_{m-1})\mid X_{m-2})[1-F_m(\alpha y(X_{m-1}))] \\
&+r\int_0^{\alpha y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{m-1}-u\mid X_{m-2})f_m(u)du\} \geqq 0\,. \qquad (2.42)
\end{aligned}
$$

From (2.40), (2.41) and (2.42), the same inequality as the case for $i=2,3,\ldots,m-1$, is obtained, i.e.,

$$y^{(1)}(X_{m-1}) \leqq 0\,. \qquad (2.43)$$

But the similar relation to (2.38) could not be obtained for $i=1$. Only for the case $\alpha=0,1$ and $0 < \alpha \leqq 1/2$, (2.31) and (2.32) hold respectively. □

This proposition implies that the optimal ordering quantity is more sensitive to the increase of newer on-hand inventory than that of the older on-hand inventory, and also implies that the increase of on-hand inventory by one unit induces the decrease of order quantity by less than one unit as shown in inequality (2.31), or by less than two units as shown in inequality (2.32).

Proposition 2.5

$$\frac{\partial}{\partial \ell_1}\left(\frac{L(X_{m-1}, y)}{\partial y}\right)\Bigg|_{y=y\ (X_{m-1})} > 0 \tag{2.44}$$

Proof: Let $y\ (X_{m-1})$ satisfies

$$\frac{\partial L(X_{m-1}, y)}{\partial y}\Bigg|_{y=y\ (X_{m-1})} = 0,$$

then the following is obtained by the use of equation (2.9).

$$\frac{\partial}{\partial \ell_0}\left(\frac{\partial L(X_{m-1}, y)}{\partial y}\right)\Bigg|_{y=y\ (X_{m-1})} = -[(h+p)F_1(x+y\ (X_{m-1}))-p+rQ_m(y\ (X_{m-1})|X_{m-1})]$$

$$+[(1-\alpha)rQ_{m-1}(x_{m-1}+(1-\alpha)y\ (X_{m-1})|X_{m-2})\{1$$

$$-F_m(\alpha y\ (X_{m-1}))\}+r\int_0^{\alpha y\ (X_{m-1})} Q_{m-1}(y\ (X_{m-1})$$

$$+x_{m-1}-u|X_{m-2})f_m(u)du]$$

$$= \frac{1}{\ell_0}\{c+\ell_0[(1-\alpha)rQ_{m-1}(x_{m-1}+(1-\alpha)y(X_{m-1})|X_{m-2})\{1-F_m(\alpha y(X_{m-1}))\}$$

$$+r\int_0^{\alpha y(X_{m-1})} Q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f\ (u)du]\}$$

$$+[(1-\alpha)rQ_{m-1}(x_{m-1}+(1-\alpha)y(X_{m-1})|X_{m-2})\{1-F_m(\alpha y(X_{m-1}))\}$$

$$+r\int_0^{\alpha y(X_{m-1})} Q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u)du]$$

$$= \frac{1}{\ell_0}\{c+[(1-\alpha)rQ_{m-1}(x_{m-1}+(1-\alpha)y(X_{m-1})|X_{m-2})\{1-F_m(\alpha y$$

$$(X_{m-1}))\}+\int_0^{\alpha y(X_{m-1})} Q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u)du]\} > 0$$

□

It may be interpreted that the increase of the probability of occurring procurement leadtime 1 enlarges the partial derivative of $L(X_{m-1}, y)$ with respect to the optimal ordering quantity. By direct calculation, we have inequality (2.45).

Proposition 2.6

$$\frac{\partial L(X_{m-1}, y)}{\partial (1-\alpha)} > 0 \tag{2.45}$$

This proposition indicates that when $(1-\alpha)$, i.e., the rate of excess perishability under the occurrence of leadtime 1, becomes closer to 1, the total expected cost increases.

Lemma 2.3. ([4])

$$\int_0^y Q_m^{(i)}(u \mid X_{m-1})du = Q_m(y \mid X_{m-1}) - \Sigma_{j=1}^{i-1} Q_{m-j}(X_{m-j}) H_j(y \mid X_{m-j}^*), \quad i=2,3,\ldots,m. \tag{2.46}$$

where $X_{m-j}^* = (x_{m-1}, x_{m-2}, \ldots, x_{m-j+1})$, and

$$H_1(y) = F_1(y)$$

$$\vdots$$

$$H_j(y \mid X_{m-j}^*) = \int_0^y F_m(y - v_{m-1}) \int_0^{v_{m-1}+x_{m-1}} f_{m-1}(v_{m-1} + x_{m-1} - v_{m-2})$$

$$\cdots \int_0^{v_{m-j+2}+x_{m-j+2}} f_{m-j+2}(v_{m-j+2} + x_{m-j+2} - v_{m-j+1})$$

$$f_{m-j+1}(v_{m-j+1} + x_{m-j+1}) dv_{m-j+1} \cdots dv_{m-1}.$$

Now we define $L^{(i)}(X_{m-1}, y)$ as follows.

$$L^{(i)}(X_{m-1}, y) = \begin{cases} \dfrac{\partial L(X_{m-1}, y)}{\partial x_{m-i}} & ; i=1,2, \ldots, m-1 \\ \dfrac{\partial L(X_{m-1}, y)}{\partial y} & ; i=m \end{cases}$$

Lemma 2.4.

$$L^{(i)}(X_{m-1}, y)=L^{(m)}(X_{m-1}, y)-c-\ell_0 r\Sigma_{j=1}^{i} Q_{m-j}(X_{m-j})H_j(y|X^*_{m-j})$$
$$+\ell_1\{(h+p)F_m(x)-p-r[\Sigma_{j=1}^{i-1} Q_{m-j-1}(X_{m-j-1})H_j((1-\alpha)y+x_{m-1}|X^*_{m-j-1})$$
$$+\int_0^{x_{m-1}} Q^{(i)}_{m-1}(u|X_{m-2})du-\alpha Q_{m-1}(x_{m-1}+(1-\alpha)y|X_{m-2})$$
$$+\Sigma_{j=2}^{i} Q_{m-j}(\check{X}_{m-j})H_j(\alpha y|\check{X}^*_{m-j})+\alpha Q_{m-1}(x_{m-1}+(1-\alpha)y|X_{m-2})F_m(\alpha y)]\} \qquad (2.47)$$

where $\check{X}_{m-j}$; rewriting form of X_{m-j} when X_{m-1} is replaced by $X_{m-1}+(1-\alpha)y.e_{m-1}$,
$\check{X}^*_{m-j}$; rewriting form of X^*_{m-j} when X_{m-1} is replaced by $X_{m-1}+(1-\alpha)y.e_{m-1}$,

Proof: For $i=2,3, \ldots, m-1$, the partrial derivative of $L(X_{m-1}, y)$ with respect to x_{m-i} is obtained as follows:

$$L^{(i)}(X_{m-1}, y)=\ell_0[(h+p)F_1(x+y)-p+r\int_0^y Q_m^{(i+1)}(u|X_{m-1})du$$
$$+\ell_1[(h+p)F_m(x)-p+r\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q^{(i)}(u|X_{m-2})du$$
$$+r\int_0^{\alpha y} Q_m^{(i+1)}(u|X_{m-1}+(1-\alpha)y\, e_{m-1})du] \; . \qquad (2.48)$$

And substituting the partial derivative $L^{(m)}(X_{m-1}, y)$ of $L(X_{m-1}, y)$ with respect to y, i.e., (2.9) into (2.48), the equation (2.48) is rewritten as follows:

$$L^{(i)}(X_{m-1}, y) = L^{(m)}(X_{m-1}, y) - c + \ell_0 r\left[\int_0^y Q_m^{(i+1)}(u|X_{m-1})\,du - Q_m(y|X_{m-1})\right]$$

$$+\ell_1\left[(h+p)F_m(x) - p + r\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q_{m-1}^{(i)}(u|X_{m-2})\,du\right.$$

$$+r\int_0^{\alpha y} Q_m^{(i+1)}(u|X_{m-1}+(1-\alpha)y.e_{m-1})\,du$$

$$-(1-\alpha)rQ_{m-1}(x_{m-1}+(1-\alpha)y|X_{m-2})\{1-F_m(\alpha y)\}$$

$$-r\int_0^{\alpha y} Q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(\alpha y - v)\,dv$$

$$= L^{(m)}(X_{m-1}, y) - c - \ell_0 r\Sigma_{j=1}^{i} Q_{m-j}(X_{m-j})H_j(y|X^*_{m-j})$$

$$+\ell_1\left[(h+p)F_m(x) - p - r\{\Sigma_{j=1}^{i-1} Q_{m-j-1}(X_{m-j-1})H_j((1-\alpha)y+x_{m-1}|X^*_{m-j-1})\right.$$

$$+\int_0^{x_{m-1}} Q_{m-1}^{(i)}(u:X_{m-2})\,du - \alpha Q_{m-1}(x_{m-1}+(1-\alpha)y:X_{m-2})$$

$$\left.+\Sigma_{j=2}^{i} Q_{m-j}(X_{m-j})H_j(\alpha y: X^*_{m-j}) + \alpha Q_{m-1}(x_{m-1}+(1-\alpha)y:X_{m-2})F_m(\alpha y)\}\right]. \quad (2.49)$$

(by using Lemma 2.3) □

Lemma 2.5.

$$L^{(i,m)}(X_{m-1}, y) \geq 0, \quad i=1,2, \ldots, m.$$

where

$$L^{(i,m)}(X_{m-1}, y) = \frac{\partial L^{(i)}(X_{m-1}, y)}{\partial y}$$

Proof:

For $i=1,2\ \dots,\ m-1$,

$$\int_0^{\alpha y} \frac{\partial^2}{\partial x_{m-i}\ \partial y} Q_m(u|X_{m-1}+(1-\alpha)y.e_{m-1})du$$

$$=\int_0^{\alpha y} \frac{\partial^2}{\partial x_{m-i}\ \partial y} \int_0^u Q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(u-v)\mathrm{d}v\mathrm{d}u$$

(by the equation (2.7))

$$=(1-\alpha)\int_0^{\alpha y} \frac{\partial}{\partial x_{m-i}} \int_0^u q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(u-v)dvdu$$

$$=(1-\alpha)\int_0^{y} \frac{\partial}{\partial x_{m-i}} \int_0^{\alpha-t} q_{m-1}(s+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(t)dsdt$$

$$=(1-\alpha)\int_0^{\alpha y} Q_{m-1}^{(i)}(y+x_{m-1}-t|X_{m-2})f_m(t)dt-(1-\alpha)Q_{m-1}^{(i)}(x_{m-1}+(1-\alpha)y|X_{m-2})F_m(\alpha y) \tag{2.50}$$

(setting $t=u-v$, $s=v$).

Differentiating, both sides of the equation (2.49) with respect to y ,

$$L^{(i,m)}(X_{m-1},\ y) = \ell_0[hf_1(x+y)+pf_1(x+y)+rQ_m^{(i+1)}(y|X_{m-1})]$$
$$+\ell_1[(1-\alpha)rQ_{m-1}^{(i)}((1-\alpha)y+x_{m-1}|X_{m-2})$$
$$+\alpha rQ_m^{(i+1)}(\alpha y|X_{m-1}+(1-\alpha)y.e_{m-1})$$
$$+r\int_0^{\alpha y} \frac{\partial^2}{\partial x_{m-i}\ \partial y} Q_m(u|X_{m-1}+(1-\alpha)y.X_{m-1})du] \tag{2.51}$$

Substituting the equation (2.50) into the last term of the equation (2.51),

$$L^{(i,m)}(X_{m-1}, y) = \ell_0[hf_1(x+y)+pf_1(x+y)+rQ_m^{(i+1)}(y|X_{m-1})]$$
$$+\ell_1[(1-\alpha)rQ_{m-1}^{(i)}((1-\alpha)y+x_{m-1}|X_{m-2})\{1-F_m(\alpha y)\}$$
$$+\alpha rQ_m^{(i+1)}(\alpha y|X_{m-1}+(1-\alpha)y.e_{m-1})$$
$$+(1-\alpha)r\int_0^{\alpha y} Q_{m-1}^{(i)}(y+x_{m-1}-t|X_{m-2})f_m(t)dt] \geq 0 .$$

For $i=m$, differentiating both sides of the equation (2.9) with respect to y,

$$L^{(m,m)}(X_{m-1},y) = \ell_0[(h-p)f_1(x+y)+rQ_m^{(1)}(y|X_{m-1})]$$
$$+\ell_1[(1-\alpha)^2 rQ_{m-1}^{(1)}(x_{m-1}+(1-\alpha)y|X_{m-2})\{1-F_m(\alpha y)\}$$
$$+\alpha^2 rQ_{m-1}(x_{m-1}+(1-\alpha)y|X_{m-2})f_m(\alpha y)$$
$$+r\int_0^{\alpha y} Q_{m-1}^{(1)}(y+x_{m-1}-u|X_{m-2})f_m(u)du] \geq 0 \qquad \square$$

Given the fixed ordering cost K and $y(X_{m-1})$, there exists $s(X_{m-1})$ $(\leq y(X_{m-1}))$ satisfying the equation (2.50).

$$L(X_{m-1},s(X_{m-1})) = L(X_{m-1},y(X_{m-1}))+K \qquad (2.52)$$

Here, if $s(X_{m-1}) > 0$, $y(X_{m-1})$ should be ordered and if $s(X_{m-1}) \leq 0$, the opitmal policy is not to order .

Differentiating both sides of the equation (2.52) with respect to x_{m-i} and noting

$L^{(m)}(X_{m-1}, y(X_{m-1}))= 0$ from equation (2.9), the following equation (2.53) is obtained.

$$s^{(i)}(X_{m-1})= \frac{L^{(i)}(X_{m-1}, y(X_{m-1}))-L^{(i)}(X_{m-1}, s(X_{m-1}))}{L^{(m)}(X_{m-1}, s(X_{m-1}))}, \tag{2.53}$$

where $s^{(i)}(X_{m-1})$ is defined as follows:

$$s^{(i)}(X_{m-1}) = \frac{\partial s(X_{m-1})}{\partial x_{m-i}}, \quad i=1,2, \ldots, m-1.$$

Proposition 2.7

$$s^{(i)}(X_{m-1}) < s^{(i+1)}(X_{m-1}) \leq 0, \quad i=1,2, \ldots, m-1. \tag{2.54}$$

The two cases of lower bounds for $s^{(1)}(X_{m-1})$ are given as follows:

$$s^{(1)}(X_{m-1}) \geq -1 \quad ; \alpha = 0,1.$$
$$s^{(1)}(X_{m-1}) \geq -2 \quad ; 0 < \alpha \leq 1/2. \tag{2.55}$$

Proof:

Since $y(X_{m-1}) > s(X_{m-1})$ from the equation (2.52) and $L^{(i)}(X_{m-1}, y)$ is increasing function from Lemma 2.3, the numerator of the equation (2.53) is larger than or equal to 0. Since $L^{(m)}(X_{m-1}, y)$ is also increasing function from Lemma 2.5, considering $L^{(m)}(X_{m-1}, y(X_{m-1}))=0$,the denominator of the equation (2.53) is less than or equal to 0. Thus,

$$s^{(i)}(X_{m-1}) \leq 0 \tag{2.56}$$

holds.

Further the relation between $s^{(i)}(X_{m-1})$ and $s^{(i-1)}(X_{m-1})$ is obtained as followes:

$$s^{(i)}(X_{m-1})-s^{(i-1)}(X_{m-1})=\frac{A}{L^{(m)}(X_{m-1},\ s(X_{m-1}))}, \tag{2.57}$$

where

$$\begin{aligned}
A &= L^{(i)}(X_{m-1},\ y(X_{m-1}))-L^{(i-1)}(X_{m-1},\ y(X_{m-1}))-L^{(i)}(X_{m-1},\ s(X_{m-1})) \\
&\quad +L^{(i-1)}(X_{m-1},\ s(X_{m-1})) \\
&= \ell_0 rQ_{m-i}(X_{m-i})\{H_i(s(X_{m-1})\,|\,X^*_{m-i})-H_i(y(X_{m-1})\,|\,X^*_{m-i})\} \\
&\quad +\ell_1 rQ_{m-i}(X_{m-i})\{H_{i-1}((1-\alpha)s(X_{m-1})+x_{(m-1)}\,|\,X^*_{m-i}) \\
&\quad -H_{i-1}((1-\alpha)y(X_{m-1})+x_{m-1}\,|\,X^*_{m-i})\} \\
&\quad +\ell_1 rQ_{m-i}(X_{m-1})\{H_i(\alpha s(X_{m-1})\,|\,X^*_{m-i})-H_i(\alpha y(X_{m-1})\,|\,X^*_{m-i})\} < 0
\end{aligned}$$

(using Lemma 2.3 and Lemma 2.4).

Taking the fact that $L^{(m)}(X_{m-1},\ s(X_{m-1})) \leq 0$ into consideration, the following relations are obtained:

$$s^{(i-1)}(X_{m-1}) < s^{(i)}(X_{m-1}) \ . \tag{2.58}$$

In order to obtain the lower bound of $s^{(i)}(X_{m-1})$, it is sufficient to examine the lower bound of $s^{(1)}(X_{m-1})$.

$$s^{(1)}(X_{m-1}) = \frac{V}{W} \tag{2.59}$$

where

$$\begin{aligned}
V = & -L^{(m)}(X_{m-1}, s(X_{m-1})) - \ell_0 r\, Q_{m-1}(X_{m-1})\{F_1(y(X_{m-1}) - F(s(X_{m-1}))\} \\
& + \ell_1 r\alpha\{(1-F_m(\alpha y(X_{m-1}))Q_{m-1}(x_{m-1} + (1-\alpha)y(X_{m-1}) | X_{m-2}) \\
& - (1-F_m(\alpha s(X_{m-1}))))Q_{m-1}(x_{m-1} + (1-\alpha)s(X_{m-1}) | X_{m-2})\} , \\
W = & L^{(m)}(X_{m-1}, s(X_{m-1})) \\
= & \ell_0[(p+h)\{F_1(x+s(X_{m-1})) - F_1(x+y(X_{m-1}))\} \\
& + r\{Q_{m-1}(X_{m-1})F_m(s(X_{m-1})) - Q_{m-1}(X_{m-1})F_m(y(X_{m-1}))\} \\
& + r\{\int_0^{s(X_{m-1})} Q_{m-1}^{(1)}(v+x_{m-1} | X_{m-2})F_m(s(X_{m-1}) - v)dv \\
& - \int_0^{y(X_{m-1})} Q_{m-1}^{(1)}(v+x_{m-1} | X_{m-2})F_m(y(X_{m-1}) - v)dv\}] \\
& + \ell_1[r(1-\alpha)\{Q_{m-1}(x_{m-1} + (1-\alpha)s(X_{m-1}) | X_{m-2})(1-F_m(\alpha s(X_{m-1}))) \\
& - Q_{m-1}(x_{m-1} + (1-\alpha)y(X_{m-1}) | X_{m-2})(1-F_m(\alpha y(X_{m-1})))\}].
\end{aligned}$$

It is easily shown that in equation (2.59),

$$s^{(i)}(X_{m-1}) > -1 \quad , \text{ if } \alpha = 0 \text{ or } 1, \tag{2.60}$$

and

$$s^{(i)}(X_{m-1}) > -2 \ , \quad \text{if } \alpha \leqq \frac{1}{2} \ . \tag{2.61}$$

But for $\frac{1}{2} < \alpha < 1$, the lower bound of $s^{(i)}(X_{m-1})$ could not be obtained. Inequalities (2.56), (2.58), (2.60) and (2.61) prove Proposition 2.7.

This proposition means that the increase of newer on-hand inventory reduces the optimal ordering quantity more than that of the older one. This proposition is very similar to Proposition 2.4 with respect to $y^{(i)}(X_{m-1})$

and implies that the increase of on-hand inventory by one unit leads to the decrease of $s(X_{m-1})$ by less than one unit for $\alpha=0,1$, and by less than two units for $0 < \alpha \leqq 1/2$. □

Proposition 2.8

For $s(X_{m-1}) \geqq 0$,

$$y(X_{m-1})-s(X_{m-1}) \geqq \frac{-K}{c+\ell_0\{(h+p)F_1(x)-p\}+\ell_1\{(1-\alpha)rQ_{m-1}(X_{m-1})\}} \ . \tag{2.62}$$

Proof: From the convexity of $L(X_{m-1},y)$, it is clear that the following inequality holds.

$$\frac{-K}{y(X_{m-1})-s(X_{m-1})} \geqq L^{(m)}(X_{m-1}, s(X_{m-1})) . \tag{2.63}$$

Since an order is placed when $s(X_{m-1}) \geqq 0$, the following relationship is established:

$$L(X_{m-1},0) > L(X_{m-1},y(X_{m-1}))+K \quad . \tag{2.64}$$

So an interesting relation between $y(X_{m-1})-s(X_{m-1})$ and $1-\alpha$ is derived as follows:

$$y(X_{m-1})-s(X_{m-1}) \geqq \frac{-K}{L^{(m)}(X_{m-1},s(X_{m-1}))} \geqq \frac{-K}{L^{(m)}(X_{m-1},0)}$$

$$= \frac{-K}{c+l_0\{(h+p)F_1(x)-p\}+l_1\{(1-\alpha)rQ_{m-1}(X_{m-1})\}} \quad .$$

The last equality comes from Lemma 2.3.

Inequality (2.62) clarifies the following relation between $y(X_{m-1})-s(X_{m-1})$ and $1-\alpha$;
if the rate of excess perishability, $1-\alpha$, increases, the lower bound of $y(X_{m-1})-s(X_{m-1})$ also increases.

2.4.2 Numerical example

This subsection provides an example in order to illustrate the results of subsection 2.4.1. The cost parameters (c, h, p, r) =(40,10,200,40) and the maximum lifetime m =3. The demand distribution function is assumed to be gamma with parameters λ and γ in equation (2.65).

$$f(x) = \frac{\gamma^{\lambda}}{\Gamma(\beta)} x^{\lambda-1} exp(-\gamma x) \tag{2.65}$$

Table 2.3 represents the relations between on-hand inventory and optimal ordering quantity when $\alpha=0.5$ and $\ell_1=0.4$. Figure 2.4 illustrates the total expected cost for on-hand inventory $x=0$, 2 and 4 when $\lambda=1$ and $\gamma=0.05$. Figure 2.5 illustrates the relation between the age distribution of inventory on hand x and optimal ordering quantity $y(X_2)$ for $x=4$, i.e., $(x_2, x_1)=(0,4.0),(1.0,3.0),(2.0,2.0),(3.0,1.0),(4.0,0)$ when $\lambda=1$ and $\gamma=0.05$. Result asserts the increases of newer inventory on hand makes more influences upon the decrease of the optimal ordering quantity than that of older one.

Table 2.4 represents the relation between the leadtime probability and optimal ordering quantity for $\alpha=0.5$ and $(x_2, x_1)=(2.0,2.0)$. Figure 2.6 illustrates the case of $\lambda=1, \gamma=0.05$ in Table 2.4. As ℓ_1 increases, the total expected cost increases and optimal ordering quantity reduces.

Table 2.5 represents the relation between the rate of excess perishability $(1-\alpha)$ and the optimal ordering quantity for $\ell_1=0.4$ and $(x_2, x_1)=(2.0,2.0)$. Figure 2.7 illustrates the case of $\lambda=1$, $\gamma=0.05$ in Table 2.5. When $(1-\alpha)$ increases, total expected cost increases and optimal ordering quantity decreases, but the significant differences are not observed when $(1-\alpha)$ approaches to 0.

Table2.3 The relation between the age distribution of inventory on hand (x_1, x_2) and optimal ordering quantity $y(X_2)$

x	(x_1, x_2)	$y(X_2)$ $\lambda=5$ $\gamma=0.25$ ($\mu=20$ $\sigma=9$)	$y(X_2)$ $\lambda=1$ $\gamma=0.05$ ($\mu=20$ $\sigma=20$)
0	(0, 0)	21.793	18.855
1	(1, 0)	20.789	17.861
	(0, 1)	20.784	17.839
2	(2, 0)	19.806	16.858
	(1, 1)	19.795	16.846
	(0, 2)	19.790	16.838
3	(3, 0)	18.804	15.884
	(2, 1)	18.800	15.844
	(1, 2)	18.788	15.834
	(0, 3)	18.780	15.826
4	(4, 0)	17.810	14.914
	(3, 1)	17.804	14.873
	(2, 2)	17.804	14.845
	(1, 3)	17.796	14.833
	(0, 4)	17.796	14.812

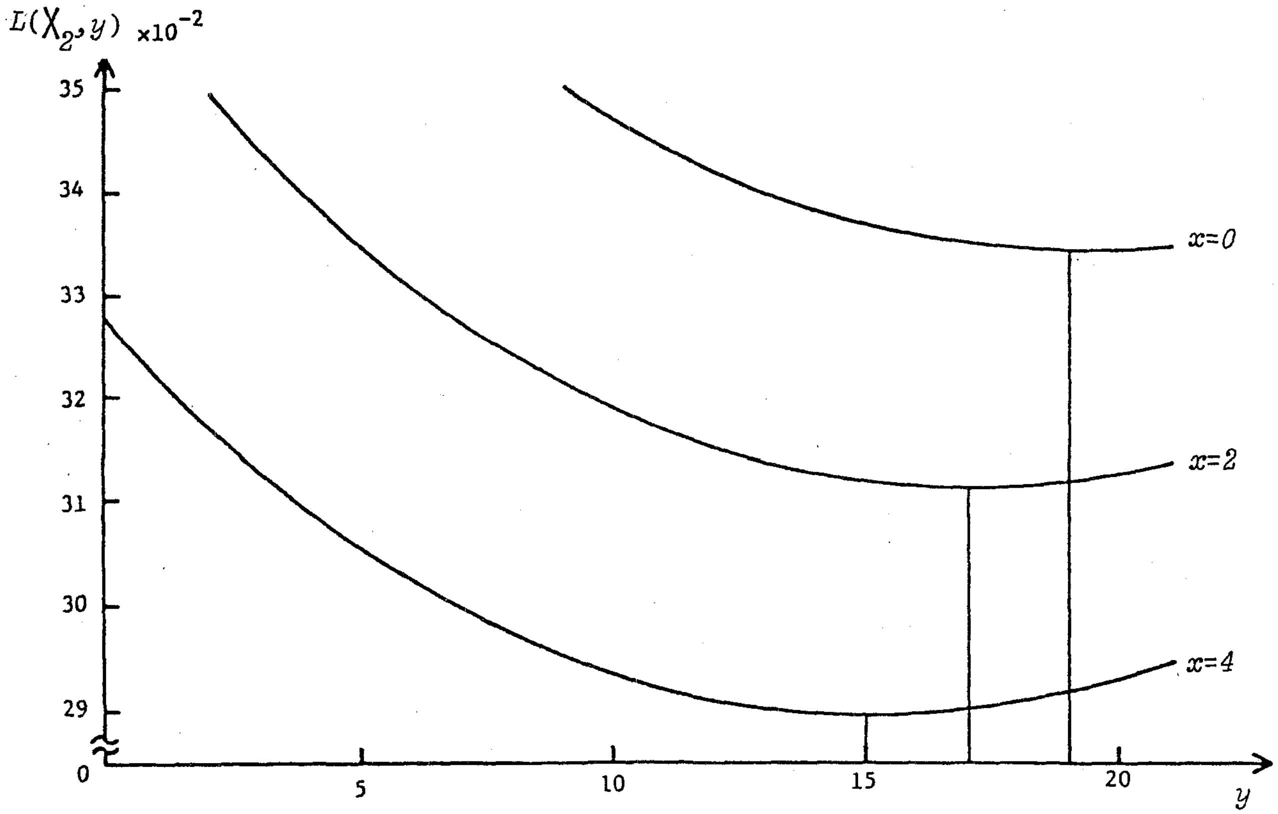

Figure2.4. Total expected cost when on-hand inventory $x=0,2,4$

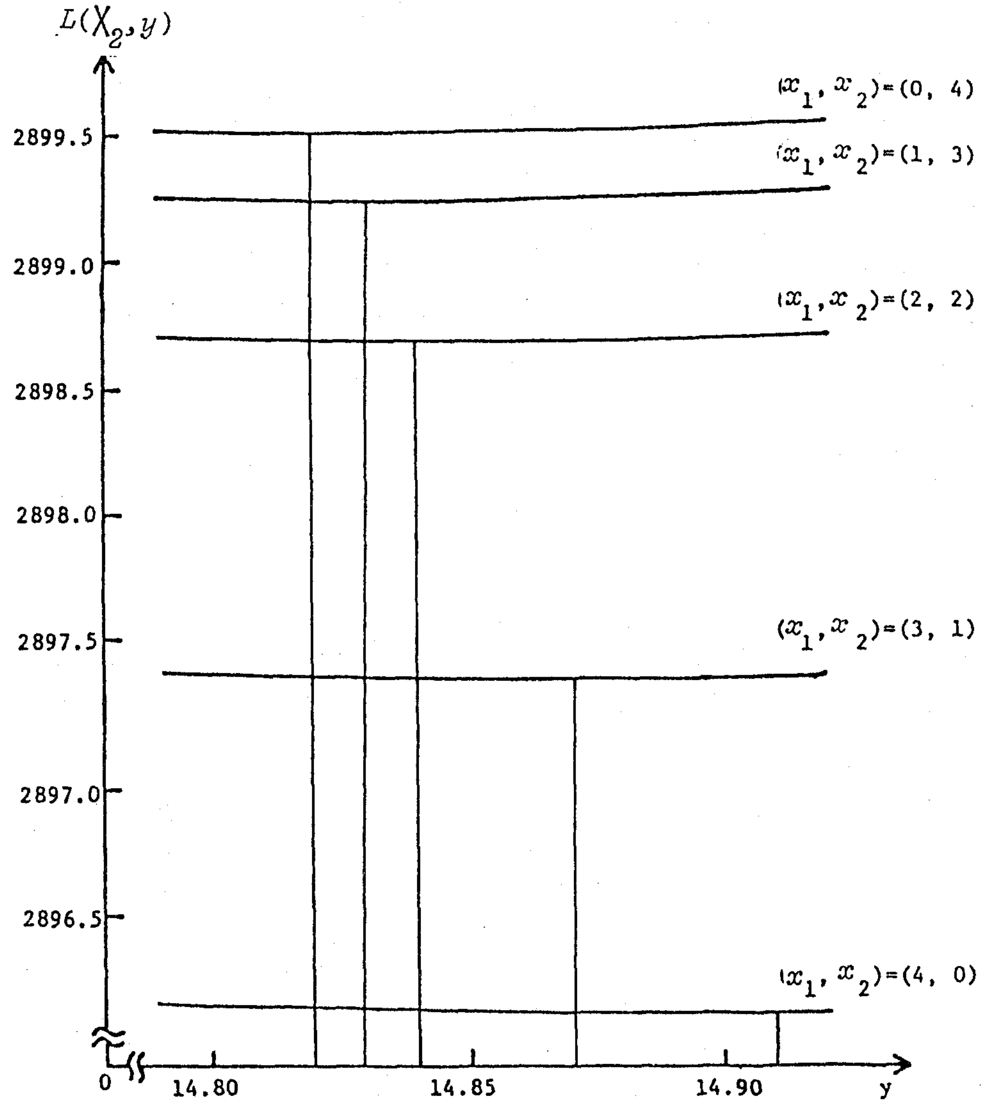

Figure 2.5. Total expected cost when status of inventory on hand
$(x_1, x_2)=(0,4)$, $(1,3)$, $(2,2)$, $(3,1)$, $(4,0)$

Table 2.4. The relation between the leadtime probability ℓ_1 and optimal ordering quantity $y(x_2)$

	ℓ_1	0.0	0.2	0.4	0.6
$y(X_2)$	$\lambda=5$ $\gamma=0.25$ ($\mu=20$ $\sigma=9$)	21.477	19.968	17.806	14.174
	$\lambda=1$ $\gamma=0.05$ ($\mu=20$ $\sigma=20$)	22.486	19.228	14.851	8.321

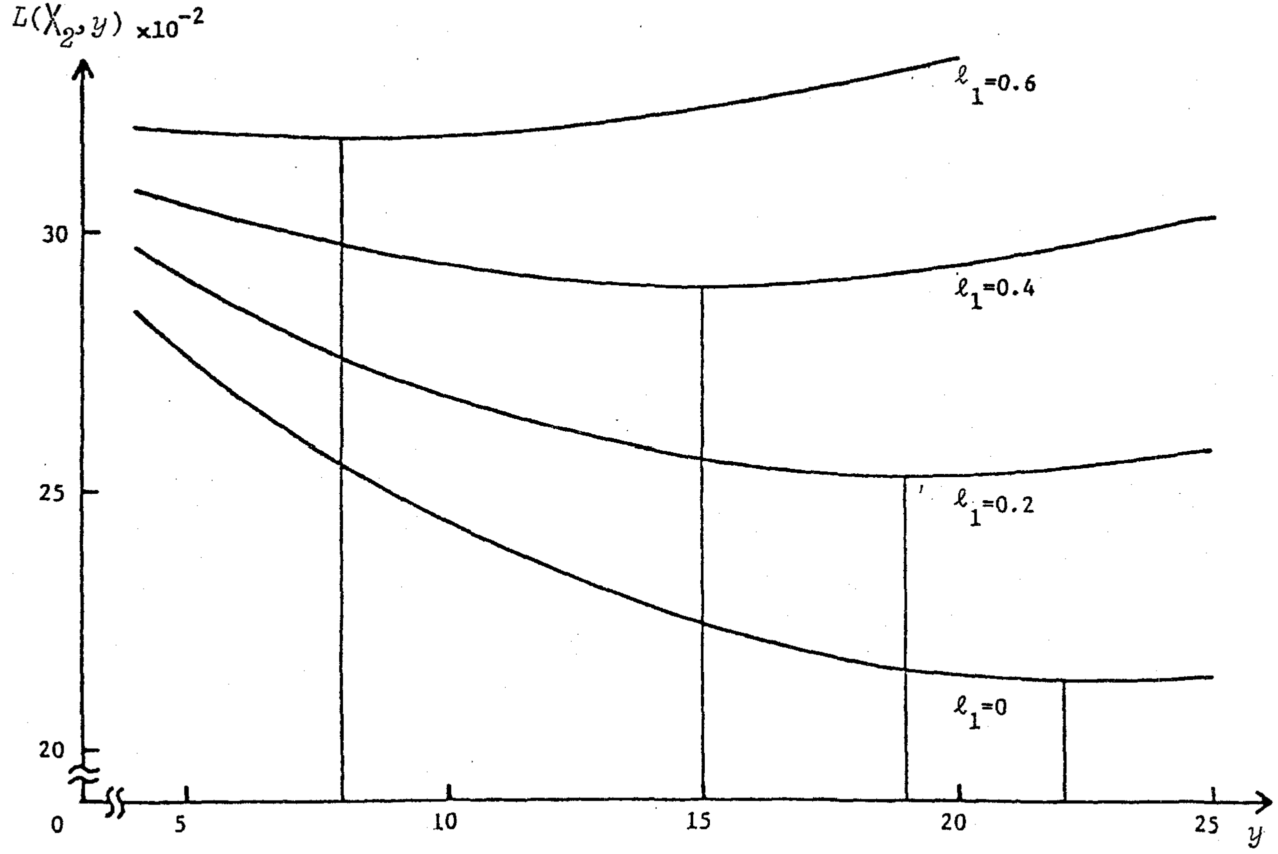

Figure 2.6.Total expected cost when the leadtime probability ℓ_1=0.0, 0.2, 0.4, 0.6

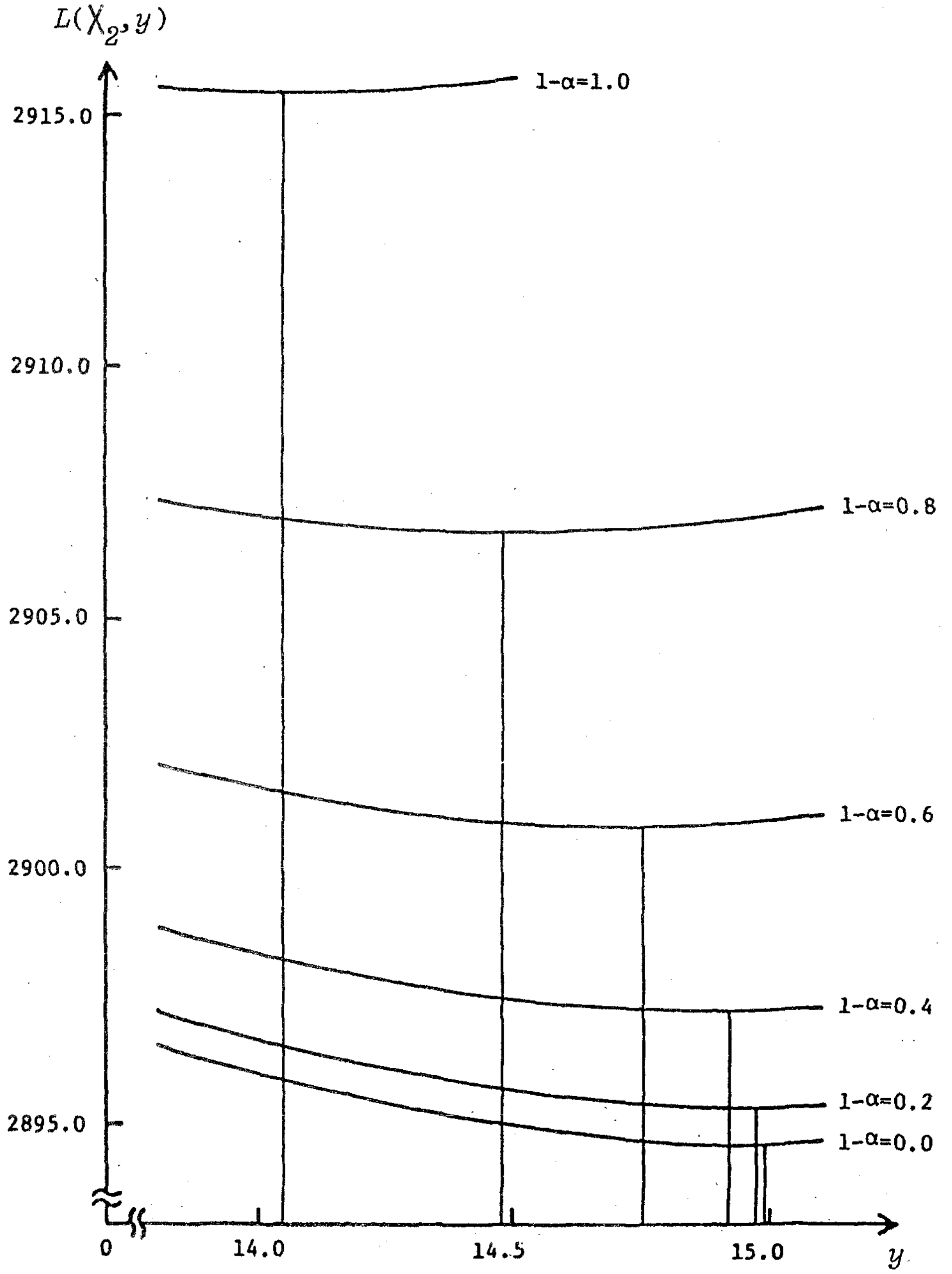

Figure 2.7. Total expected cost when the rate of excess perishability $1-\alpha = 1.0, 0.8, 0.6, 0.4, 0.2, 0.0$

Table 2.5. The relation between the rate of excess perishability occurring $(1-\alpha)$, and optimal ordering quantity $y(X_2)$ when leadtime 1.

	$1-\alpha$	1.0	0.8	0.6	0.4	0.2	0.0
$y(X_2)$	$\lambda=5$ $\gamma=0.25$ ($\mu=20$ $\sigma=9$)	17.644	17.754	17.802	17.806	17.806	17.806
	$\lambda=1$ $\gamma=0.05$ ($\mu=20$ $\sigma=20$)	14.056	14.477	14.765	14.908	14.963	14.993

2.5 Conclusion

In this chapter, we discussed the determination of the optimal ordering policies and their properties with respect to a perishable commodity with predetermined fixed lifetime subject to (0 or 1) procurement leadtime.

It was shown that the critical ordering point under the stochastic leadtime is less than that of the ordinary model without taking the stochastic leadtime into consideration. In addition, the probability level that the shortage does not occur is given and the optimal ordering policy was again derived. Further influence of the oldest inventory x_1 and β on the optimal ordering policy was clarified.

Sensitivity of some important factors such as the influences of leadtime probability, the rate of excess perishability and the status of inventory on hand upon the optimal ordering policies were analyzed. Next, set-up cost for placing an order was introduced and some characteristics were shown.

It is important to consider with respect to more general assumptions of procurement leadtime and excess perishability, though its generalization may make analysis more complicated and difficult.

Inventory depletion policy discussed in this chapter had been assumed to be FIFO issuing. But, in order to cope with the increase of customer serivice in need, customer-oriented inventory depletion policies, e.g., LIFO and etc., might be better than FIFO. From these point of view, we could develop our theme in using LIFO issuing policy for more general stochastic leadtime and stochastic excess perishability.

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CHAPTER III

INVENTORY CONTROL FOR PERISHABLE COMMODITIES WITH STOCHASTIC LEADTIME SUBJECT TO DIFFERENT SELLING PRICES

3.1 Introduction

In the recent situation of low economic development, competitions among companies are becoming severer. Under these circumstances, companies must consider not only the quantitative service but also the qualitative one when we discuss the problems of minimizing cost or maximizing profit. As for inventory problems, especially when taking account of fresh commodities or process foods, the quality of commodities has become an important factor as well as the amount of their inventory. Bulinskaya [1], Van Zyl [6], etc., treated the cases where lifetimes are one period or two, respectively. Nahmias [3] introduced perishing cost into the determination of optimal ordering policy for m periods lifetime model. Ishii et al. [2] and Nose et al. [5] generalized their model with respect to the relationship between the length of the procurement leadtime and the value of the commodities accepted. Generalizing the one period horizon models of Ishii et al. [2] and Nose et al. [5], this chapter discusses an inventory management for perishable commodities under two different selling prices and 0 or 1 leadtime in order to obtain more realistic optimal ordering policy and to derive its properties.

Section 3.2 states the assumptions and formulates our model. In Section 3.3, the optimal ordering policy is obtained. Section 3.4 discusses the influences of leadtime, rate of excess perishability and on-hand stock upon the optimal ordering policy. Section 3.5 gives a numerical example in order to illustrate the results of Sections 3.3 and 3.4. Finally, Section 3.6 concludes this chapter and discusses further research problems.

3.2 Problem formulation

The following assumptions are made throughout this chapter:

(1) A periodic review inventory model is considered for one planning period and single item. The period length is arbitrary but fixed.

(2) Ordering takes place at the start of a period and unit purchasing cost is charged.

(3) Maximum lifetime of the perishable commodity discussed in this chapter is m periods. If the commodity has not been depleted by demand until the period it reaches age m , then it perishes and must be discarded at a specified per-unit cost r .

(4) Stock is depleted by demand at the start of each period according to a FIFO policy and after commodities are placed into stock, deterioration proceeds by one stage at each period.

(5) Demand D_j in successive periods $j=1,2,\ldots$ is independent nonnegative random variables with known distribution function $F_j(\cdot)$ ($F_j(0)=0$) and density function $f_j(\cdot)$ that is continuous everywhere.

(6) When procurement leadtime ℓ is 0, the fresh stock with maximum lifetime m arrives and when $\ell=1$, the stock with lifetime $m-1$ or $m-2$ arrives. Leadtime 0 and 1 occur with probability ℓ_0 and ℓ_1, respectively, where $\ell_0+\ell_1=1$, $\ell_0>0$, $\ell_1\geq 0$, and when $\ell=1$, the stock with lifetime $m-1$ arrives at a constant rate $\alpha(0\leq\alpha\leq 1)$ and $m-2$ at $1-\alpha$. Here, $1-\alpha$ corresponds to the rate of excess perishability under the occurrence of leadtime 1.

(7) Shortage cost p and holding cost h are charged for unit short and unit carried over, respectively.

(8) The commodity whose remaining lifetime is only one period is sold at selling price R_1 and the other commodities are sold at R_2.
It may be reasonable to assume $R_2 \geqq R_1$.

In stocking perishable commodities, it is necessary to keep track of the amount of inventory on hand at each lifetime level. We list the following notations; x_i, X_p, B_p, $Q_n(u|X_{n-1})$ and $L(X_{m-1},y)$.

x_i ; the amount of commodity on hand with i periods of usable lifetime left,

X_p ; inventory level in stock, i.e., $X_p\equiv(x_1,x_2,\ldots,x_p)$, $p=1,2,\ldots,m-1$,

B_j ; after depleting all of the commodities $x_j,x_{j-1},\ldots,x_1$, the total unsatisfied demand until period j which should be satisfied by the inventory commodities whose remaining lifetimes are greater than j, i.e.,

$$B_j=[D_j+B_{j-1}-x_j]^+,\quad j=1,2,\ldots,m-1\ , \tag{3.1}$$

where $B_0=0$ and $[b]^+=max(b,0)$.

$Q_n(u|X_{n-1})$; the probability that the sum of D_n and B_{n-1} is less than a real number u, i.e.,

$$Q_n(u|X_{n-1})=Pr\{D_n+B_{n-1}\leq u\},\quad n=1,2,\ldots,m\ . \tag{3.2}$$

$L(X_{m-1}, y)$; the total expected cost function,

$U(X_{m-1}, y)$; the expected sales function.

The expected outdating quantities are obtained for the cases of $\ell=0$ and $\ell=1$ from the literature [2] as follows:

(I) The case of $\ell=0$: Using the equation (3.2), the expected number of units of y scheduled to perish after m periods is derived;

$$\int_0^\infty ud\{1-Q_m(y-u|X_{m-1})\}=\int_0^y Q_m(u|X_{m-1})du . \tag{3.3}$$

(II) The case of $\ell=1$: From the assumptions (6), remaining lifetime of αy is $m-1$ periods and that of $(1-\alpha)y$ is $m-2$ periods.

(i) The arrivals of $m-1$ periods lifetime commodities; The expected number of units, αy, scheduled to outdate after $m-1$ periods is derived as follows ;

$$\int_0^{\alpha y} Q_m(\alpha y-u|X_{m-1}+(1-\alpha)y\mathrm{e}_{m-1})du=\int_0^{\alpha y} Q_m(u|X_{m-1}+(1-\alpha)y\mathrm{e}_{m-1})du ,$$

$$\text{where, } \mathrm{e}_{m-1}=(0,0,\ldots,\overbrace{1}^{m-1}) . \tag{3.4}$$

(ii) The arrivals of $m-2$ periods lifetime commodities; The expected number of units of $(1-\alpha)y$ scheduled to outdate after $m-2$ periods is derived ;

$$\int_0^{(1-\alpha)y} Q_{m-1}((1-\alpha)y+x_{m-1}-u|X_{m-2})du=\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q_{m-1}(u|X_{m-2})du \tag{3.5}$$

Then the total expected cost function is expressed from the equations (3.3) to (3.5) together with the other costs as follows:

$$L(X_{m-1},y)=cy+\ell_0[h\int_0^{x+y}(x+y-u)f_1(u)du+p\int_{x+y}^{\infty}\{u-(x+y)\}f_1(u)du+r\int_0^{y}Q_m(u|X_{m-1})du]$$

$$+\ell_1[h\int_0^{x}(x-u)f_1(u)du+p\int_{x}^{\infty}(u-x)f_1(u)du+r\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}}Q_{m-1}(u|X_{m-2})du$$

$$+r\int_0^{\alpha y}Q_m(u|X_{m-1}+(1-\alpha)ye_{m-1})du]\ , \qquad (3.6)$$

where, $x\equiv\Sigma_{i=1}^{m-1}x_i$.

In order to obtain the profit function , we begin with constructing the sales function $U(X_{m-1},y)$. First of all, two cases of $\ell=0$ and $\ell=1$ are distinguished to constitute $U(X_{m-1},y)$ and $U_1(U_2)$ is defined as the sales quantity of y whose remaining lifetime is one period ; (greater than or equal to two periods).

(I) The case of $\ell=0$:

U_1 and U_2 of this case are denoted with U_1^0 and U_2^0 respectively.

$$U_2^0=\begin{cases}B_{m-1}=[D_{m-1}+B_{m-2}-x_{m-1}]^+\ , & B_{m-1}\leq y\\ y\ , & B_{m-1}>y\end{cases} \qquad (3.7)$$

The cumulative distribution function of U_2^0 is

$$Pr(U_2^0\leq u)=Q_{m-1}(u+x_{m-1}|X_{m-2})\ . \qquad (3.8)$$

Then, the expected sales quantity of U_2^0 is expressed as follows :

$$E[U_2^0]=y-\int_0^{y}Q_{m-1}(u+x_{m-1}|X_{m-2})du \qquad (3.9)$$

And the corresponding expected sales quantity of which the lifetime of commodity is one period, $E[U_1^0]$, is shown as follows :

$$E[U_1^0]=y-E[U_2^0]-\int_0^y Q_m(u|X_{m-1})du$$

$$=\int_0^y Q_{m-1}(u+x_{m-1}|X_{m-2})du-\int_0^y Q_m(u|X_{m-1})du \ . \qquad (3.10)$$

(II) The case of $\ell=1$:

To obtaining the expected outdating quantity in the case of $\ell=1$, two types, i.e., $m-1$ and $m-2$ periods lifetime commodities' arrivals are formulated each other.

(i) The arrivals of $m-1$ periods lifetime commodities ;

U_1 and U_2 of this case are denoted with U_1^1 and U_2^1 respectively.

Since

$$U_2^1 = \begin{cases} B_{m-1}=[D_{m-1}+B_{m-2}-x_{m-1}-(1-\alpha)y]^+, & B_{m-1}\leqq\alpha y \\ \alpha y & \end{cases} \qquad (3.11)$$

the cumulative distribution function of U_2^1 is

$$Pr(U_2^1 \leqq u)=Q_{m-1}((1-\alpha)+x_{m-1}+u|X_{m-2}) \ . \qquad (3.12)$$

Then, the expected sales quantity, $E[U_2^1]$, is expressed as follows:

$$E[U_2^1]=\alpha y-\int_0^{\alpha y} Q_{m-1}((1-\alpha)y+x_{m-1}+u|X_{m-2})du \qquad (3.13)$$

And the expected sales quantity, $E[U_1^1]$, is shown as follows:

$$E[U_1^1]=\alpha y-E[U_2^1]-\int_0^{\alpha y} Q_m(u|X_{m-1}+(1-\alpha)ye_{m-1})du$$

$$=\int_0^{\alpha y} Q_{m-1}((1-\alpha)y+x_{m-1}+u|X_{m-2})du-\int_0^{\alpha y} Q_m(u|X_{m-1}+(1-\alpha)ye_{m-1})du\ . \tag{3.14}$$

(ii) The arrivals of m-2 periods lifetime commodities;

In this case, U_1 and U_2 are denoted with U_1^2 and U_2^2, respectively.

$$U_2^{\,2}=\begin{cases}0 & , \quad B_{m-2}\leqq x_{m-1}\\ B_{m-2}-x_{m-1} & , \quad x_{m-1}<B_{m-2}\leqq x_{m-1}+(1-\alpha)y\\ (1-\alpha)y & , \quad B_{m-2}>x_{m-1}+(1-\alpha)y\end{cases} \tag{3.15}$$

The cumulative distribution function of U_2^2 is as follows;

$$Pr(U_2^2\leqq u)=Q_{m-2}(u+x_{m-2}|X_{m-3}) \tag{3.16}$$

Then, the expected sales quantity $E[U_2^2]$ is shown as follows;

$$E[U_2^{\,2}]=(1-\alpha)y-\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q_{m-2}(u+x_{m-2}|X_{m-3})du\ . \tag{3.17}$$

And the expected sales quantity of $E[U_1^2]$ is obtained as follows;

$$E[U_1^{\,2}]=(1-\alpha)y-E[U_2^{\,2}]-\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q_{m-1}(u|X_{m-2})du$$

$$=\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q_{m-2}(u+x_{m-2}|X_{m-3})du-\int_{x_{m-1}}^{(1-\alpha)y+x_{m-1}} Q_{m-1}(u|X_{m-2})du\ . \tag{3.18}$$

Taking account of the price assumption (8), from the equations (3.9), (3.10), (3.13), (3.14), (3.17) and (3.18), the expected sales function $U(X_{m-1},y)$ becomes as follows;

$$U(X_{m-1},y)= \ell_0\{R_2E[U_2^0]+R_1E[U_1^0]\}+\ell_1\{R_2(E[U_2^1]+E[U_2^2])$$

$$+R_1(E[U_1^1]+E[U_1^2])\}. \tag{3.19}$$

Thus, the expected profit function $J(X_{m-1},y)$ is written as follows;

$$J(X_{m-1},y)=U(X_{m-1},y)-L(X_{m-1},y), \tag{3.20}$$

where U and L are given in (3.6) and (3.9) respectively.

3.3 The optimal ordering policy

In order to prove the concavity of the profit function (3.20), i.e., $J(X_{m-1},y)$, the following Theorem 3.1, **Corollaries 3.1 and 3.2 are fully** utilized. And next proposition demonstrates the existence of an optimal ordering quantity $y(X_{m-1})$ under inventory level X_{m-1}.

Theorem 3.1 ([3])

$$Q_n(u|X_{n-1})=\int_0^u Q_{n-1}(v+x_{n-1}|X_{n-2})f_n(u-v)dv, \tag{3.21}$$

$$n=1,2,\ldots,m,$$

where $Q_0(u)=1$.

Corollary 3.1.

$$Q_m(u|X_{m-1}+(1-\alpha)y,e_{m-1})=\int_0^u Q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(u-v)dv. \tag{3.22}$$

Corollary 3.2

Assume that each demand distribution F_k possesses density f_k that is continuous everywhere. Then the functions $\partial Q_n(X_n)/\partial x_i$ are continuous over n **dimensional real space** R^n**. If** f_n **has a jump at 0,** then $\partial Q_n(X_n)/\partial x_i$ are all continuous over R^n for $i \leqq n-1$ and $\partial Q_n(X_n)/\partial x_n$ is continuous in all its arguments but possesses a jump at 0.

Proposition 3.1

When $\alpha=0$ and 1, the expected profit function, $J(X_{m-1},y)$, is a concave function of y. $y(X_{m-1})$ maximizing $J(X_{m-1},y)$ exists in $(-\infty,\infty)$, that is

$$J(X_{m-1},y(X_{m-1}))= max[J(X_{m-1},y)]. \tag{3.23}$$

Proof:

The convexity of the total expected cost function, $L(X_{m-1},y)$, has been already proved by Ishii et al. [2]. Thus, only the total expected sales function, $U(X_{m-1},y)$, needs to be considered.

$$\frac{\partial U(X_{m-1},y)}{\partial y} = \ell_0 [R_2\{1-Q_{m-1}(y+x_{m-1}|X_{m-2})\}$$

$$+R_1\{Q_{m-1}(y+x_{m-1}|X_{m-2})-Q_m(y|X_{m-1})\}]$$

$$+\ell_1 [R_2\{1-Q_{m-1}(y+x_{m-1}|X_{m-2})$$

$$+(1-\alpha)Q_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2})$$

$$-(1-\alpha)Q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})\}$$

$$+R_1\{Q_{m-1}(y+x_{m-1}|X_{m-2})-(1-\alpha)Q_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2})(2-F_m(\alpha y))$$

$$-\int_0^{\alpha y} Q_{m-1}(y+x_{m-1}-u|X_{m-2})f_m(u)du$$

$$+(1-\alpha)Q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})\}]. \qquad (3.24)$$

Note that the equation (3.24) is already arranged by the aid of Corollaries 3.1, 3.2 and the following transformation:

$$\int_0^{\alpha y}\int_0^u q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(u-v)dvdu$$

$$=\int_0^{\alpha y} f_m(u)Q_{m-1}(y-x_{m-1}-u|X_{m-2})du-Q_{m-1}(x_{m-1}+(1-\alpha)y|X_{m-2})F_m(\alpha y)$$

Then,

$$\frac{\partial^2 U(X_{m-1},y)}{\partial y^2} = \ell_0[-R_2 q_{m-1}(y+x_{m-1}|X_{m-2})$$

$$+R_1\{q_{m-1}(y+x_{m-1}|X_{m-2})-q_m(y|X_{m-1})\}]$$

$$+\ell_1[R_2\{-q_{m-1}(y+x_{m-1}|X_{m-2})$$

$$+(1-\alpha)^2 q_{m-1}(x_{m-1}+(1-\alpha)y|X_{m-2})$$

$$-(1-\alpha)^2 q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})\}$$

$$+R_1\{q_{m-1}(y+x_{m-1}|X_{m-2})$$

$$-(2-F_m(\alpha y))(1-\alpha)^2 q_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2})$$

$$-\int_0^{\alpha y} Q_{m-1}(y+x_{m-1}-u|X_{m-2})f_m(u)du$$

$$+(1-\alpha)Q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})$$

$$+(1-\alpha)^2 q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})\}] \qquad (3.25)$$

By the relation $R_1 \leq R_2$, the concavity of $U(X_{m-1},y)$ has been proved with respect to the case $\alpha=0$, 1.

After all, the concavity of $J(X_{m-1},y)$ has been proved from the knowledge of the convexity of $L(X_{m-1},y)$ [2]. Moreover, from the equation **(3.26)**,

$$\frac{\partial J(X_{m-1},y)}{\partial y} = \ell_0[R_2\{1-Q_{m-1}(y+x_{m-1}|X_{m-2})\}$$

$$+R_1\{Q_{m-1}(y+x_{m-1}|X_{m-2})-Q_m(y|X_{m-1})\}]$$

$$+\ell_1[R_2\{1-Q_{m-1}(y+x_{m-1}|X_{m-2})+(1-\alpha)Q_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2})$$

$$-(1-\alpha)Q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})\}$$

$$+R_1\{Q_{m-1}(y+x_{m-1}|X_{m-2})-(1-\alpha)Q_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2})(2-F_m(\alpha y))$$

$$-\int_0^{\alpha y} Q_{m-1}(y+x_{m-1}-u|X_{m-2})f_m(u)du$$

$$+(1-\alpha)Q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3})\}]$$

$$-c-\ell_0[h\int_0^{x+y} f_1(u)du-p\int_{x+y}^{\infty} f_1(u)du+rQ_m(y|X_{m-1})]$$

$$-\ell_1\ [(1-\alpha)rQ_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2})$$

$$+\alpha r\int_0^{\alpha y} Q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(\alpha y-v)dv$$

$$+(1-\alpha)r\int_0^{\alpha y}\int_0^{u} q_{m-1}(v+x_{m-1}+(1-\alpha)y|X_{m-2})f_m(u-v)dudv\ ]$$

$$=0, \tag{3.26}$$

the following inequalities (3.27) and (3.28) are obtained.

$$\lim_{y\to\infty} \frac{\partial J(X_{m-1},y)}{\partial y} < 0 . \tag{3.27}$$

$$\lim_{y\to-\infty} \frac{\partial J(X_{m-1},y)}{\partial y} > 0 . \tag{3.28}$$

Inequalities (3.27), (3.28) and the concavity of $J(X_{m-1},y)$ together prove Proposition 3.1.

Proposition 3.2

The configuration of the optimal ordering policy is described for the cases of $\alpha=1$ and $\alpha=0$.

(1) The case of $\alpha=1$:

When $x_{m-1}=0$, the following critical order policy is optimal;

$$\begin{cases} \text{order}, & \text{if} \quad c \leqq R_2-h\ell_0 \quad \text{or} \\ & \quad\; c>R_2-h\ell_0 \qquad x<x^*, \\ \text{not order}, & \text{otherwise}, \end{cases} \tag{3.29}$$

where the critical order-point, x^*, is obtained uniquely as follows:

$$x^*=\begin{cases} \hat{x}, & \text{if } F_1(\hat{x})=(R_2-c+p\ell_0)/\{\ell_0(h+p)\} \text{ and } c<R_2+p\ell_0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

When $x_{m-1}>0$,

$$\begin{cases} \text{order}, & \text{if} \quad c \leqq R_1-h\ell_0 \quad \text{or} \quad c>R_1-h\ell_0 \quad \text{and} \\ & \quad\; x_{m-1}<x^*_{m-1}(X_{m-2}) \\ \text{not order}, & \text{otherwise}, \end{cases} \tag{3.31}$$

is optimal where the critical order point $x^*_{m-1}(X_{m-2}) \geqq 0$ is derived as follows;

$$x^*_{m-1}(X_{m-2})=\begin{cases} \hat{x}_{m-1}(X_{m-2}), & \text{if } \hat{x}_{m-1}(X_{m-2}) \text{ satisfying} \\ & \left.\dfrac{\partial J(X_{m-1},y)}{\partial y}\right|_{y=0}=0 \text{ exists}, \\ 0 & , \text{otherwise}, \end{cases} \tag{3.32}$$

(2) The case of $\alpha=0$:

In this case ,

$$\begin{cases} \text{order, if } & c \leqq (R_1-h)\ell_0 - r\ell_1 \quad \text{or} \\ & c > (R_1-h)\ell_0 - r\ell_1 \quad \text{and,} \quad x_{m-1} < x^*_{m-1}(X_{m-2}), \\ \text{not order, otherwise,} & \end{cases} \tag{3.33}$$

where

$$x^*_{m-1}(X_{m-2}) = \begin{cases} \hat{x}_{m-1}(X_{m-2}), & \text{if } x_{m-1}(X_{m-2}) \text{ satisfying} \\ & \left.\dfrac{\partial J(X_{m-1},y)}{\partial y}\right|_{y=0} = 0, \text{ exists,} \\ 0 & \text{, otherwise,} \end{cases}$$

is the optimal ordering policy.

Proof:

(I) Case of $\alpha=1$

I-1) $x_{m-1}=0$:

As long as the relation $c<R_2-hl_0$ holds, the following inequality (3.34) is derived from the equation (3.26).

$$\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=1 \\ x_{m-1}=0 \\ y=0}} \geqq 0 . \tag{3.34}$$

From the inequality (3.34) and the concavity of $J(X_{m-1},y)$, the optimal ordering policy is to order $\hat{y}(X_{m-1})$ so as to satisfy $\partial J(X_{m-1},y)/\partial y=0$. On the other hand, as the relation $c>R_2-h\ell_0$ holds, $\hat{x}$ is defined so as to satisfy the relation,

$$\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=1\\x_{m-1}=0\\y=0}} = 0 \quad ,$$

that is,

$$F_1(\hat{x}) = \frac{R_2-c+p\ell_0}{\ell_0(h+p)} \,. \tag{3.35}$$

Considering the property of distribution function, $0 \leqq F_1(\hat{x}) \leqq 1$, the following condition is obtained.

$$R_2-h\ell_0<c<R_2+p\ell_0 \,. \tag{3.36}$$

As the case $R_2+p\ell_0>c$, there exists $\hat{x}$. Furthermore, when $x<\hat{x}$, the inequality (3.34) holds. This means that the optimal ordering policy is to order $y(X_{m-1})$. In turn, when $x \geqq \hat{x}$, the optimal ordering policy is not to order.

As the other case, $R_2+p\ell_0 \leqq c$, the value of the left side in inequality (3.34) is always to be negative. So there dose not exist $\hat{x}$. This means that the optimal ordering policy is always not to order.

I-2) $x_{m-1}>0$:

Setting $\alpha=1$ and $y=0$ in the equation (3.26), the following equation is obtained.

$$\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=1\\y=0}} = R_2+(R_1-R_2)Q_{m-1}(x_{m-1}|X_{m-2})$$
$$-[c-p\ell_0+\ell_0(h+p)F_1(x)]$$
$$=0. \tag{3.37}$$

And the partial derivative of the equation (3.37) with respect to x_{m-1} is as follows.

$$\frac{\partial}{\partial x_{m-1}}\left(\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=1\\y=0}}\right)<0 \tag{3.38}$$

As $x_{m-1}\to\infty$, the equation (3.37) becomes to be

$$\lim_{x_{m-1}\to\infty}\left(\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{y=0}\right) = R_1-\{c+h\ell_0\}. \tag{3.39}$$

From the inequality (3.38) and the equation (3.39), the optimal ordering policy is obtained as follows:

As long as the relation $c\leqq R_1-h\ell_0$ holds, the left hand side in the equation (3.39) is positive.

This means that the optimal ordering policy is to order $y(X_{m-1})$ so as to satisfy the equation (3.26) in $\alpha=1$.

On the other hand, as the relation $c>R_1-h\ell_0$ holds, $\hat{x}_{m-1}(X_{m-1})$ exists so as to satisfy the equation (3.37).

Furthermore, if $x_{m-1}<\hat{x}_{m-1}(X_{m-2})$, taking the inequality (3.38) into consideration, $y(X_{m-1})$ is ordered. And if $x_{m-1}\geqq\hat{x}_{m-1}(X_{m-1})$, the optimal ordering policy is not to order.

(II) Case of $\alpha=0$

Substituting $\alpha=0$ and $y=0$ in the equation (3.26), the following equation is obtained:

$$\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=0\\y=0}} = \ell_0[R_2\{1-Q_{m-1}(x_{m-1}|X_{m-2})\}+R_1Q_{m-1}(x_{m-1}|X_{m-2})]$$
$$+\ell_1[R_2\{1-Q_{m-2}(x_{m-1}+x_{m-2}|X_{m-3})\}$$

$$+R_1\{-Q_{m-1}(x_{m-1}|X_{m-2})+Q_{m-2}(x_{m-1}+x_{m-2}|X_{m-3})\}]$$

$$-[c+\ell_0\{(h+p)F_1(x)-p\}+\ell_1 rQ_{m-1}(x_{m-1}|X_{m-2})]$$

$$=0. \qquad (3.40)$$

And the partial derivative of the equation (3.40) with respect to x_{m-1} is as follows:

$$\frac{\partial}{\partial x_{m-1}}\left(\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=0\\ y=0}}\right)<0 \qquad (3.41)$$

As $x_{m-1}\to\infty$, the equation (3.40) becomes as follows :

$$\lim_{x_{m-1}\to\infty}\left(\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{\substack{\alpha=0\\ y=0}}\right)= R_1\ell_0-(c+h\ell_0+r\ell_1). \qquad (3.42)$$

From (3.41) and (3.42), the optimal ordering policy is summarized as follows :

As long as the relation $c\leqq(R_1-h)\ell_0-r\ell_1$ holds, the optimal ordering policy is to order $y(X_{m-1})$ so as to satisfy the equation (3.26) in $\alpha=0$.

On the other hand, as the relation $c>(R_1-h)\ell_0-r\ell_1$ holds, there exists $\hat{x}_{m-1}(X_{m-2})$ so as to satisfy the equation (3.40). Furthermore, if $x_{m-1}<\hat{x}_{m-1}(X_{m-2})$, taking the inequality (3.41) into consideration, $y(X_{m-1})$ is ordered. And if $x_{m-1}\geqq\hat{x}(X_{m-1})$, the optimal ordering policy is not to order.

3.4 Properties on the optimal ordering policy

First, $Q_m^{(i)}(y|X_{m-1})$ and $y^{(i)}(X_{m-1})$ are defined as follows;

$$Q_m^{(i)}(y|X_{m-1})=\begin{cases}\dfrac{\partial Q_m(y|X_{m-1})}{\partial y} & : \quad i=1\\[2ex] \dfrac{\partial Q_m(y|X_{m-1})}{\partial x_{m-i+1}} & : \quad i=2,\ \ldots,m\end{cases}$$

$$y^{(i)}(X_{m-1})=\frac{\partial y(X_{m-1})}{\partial x_{m-i}}\quad,\quad i=1,2,\ldots,m-1$$

Next inequality (3.43) which is presented by Nahmias [3] will be used in order to prove the following proposition 3.3 .

$$Q_m^{(i)}(y|X_{m-1})\geqq Q_m^{(i+1)}(y|X_{m-1}) \tag{3.43}$$

Proposition 3.3

When $\alpha=0$ or 1, then

$$-1\leqq y^{(i)}(X_{m-1})\leqq y^{(i+1)}(X_{m-1})\leqq 0\quad,\quad i=2,3,\ldots,m-2$$

(3.44)

and

$$y^{(1)}(X_{m-1})\leqq 0\ ,$$

hold.

Proof:

(I) Case of $\alpha=1$

Since $J(X_{m-1},y)$ achieves its maximum value at $y=y(X_{m-1})$, the following equation is obtained with respect to $i=2,3,\cdots,m-1$.

$$\frac{\partial}{\partial x_{m-i}}\left(\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{y=y(X_{m-1})}\right)=\ell_0[-R_2Q^{(i)}_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-R_2q_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})y^{(i)}(X_{m-1})$$

$$+R_1\{Q^{(i)}_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$+q_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})y^{(i)}(X_{m-1})$$

$$-Q^{(i+1)}_m(y(X_{m-1})|X_{m-1})-Q^{(1)}_m(y(X_{m-1})|X_{m-1})y^{(i)}(X_{m-1})\}]$$

$$+\ell_1[-R_2Q^{(i)}_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-R_2q_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})y^{(i)}(X_{m-1})$$

$$+R_1\{Q^{(i)}_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$+q_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})y^{(i)}(X_{m-1})$$

$$-Q_{m-1}(x_{m-1}|X_{m-2})f_m(y(X_{m-1}))y^{(i)}(X_{m-1})$$

$$-\int_0^{y(X_{m-1})}\{q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})y^{(i)}(X_{m-1})f_m(u)$$

$$+Q^{(i)}_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u)du\}]$$

$$-[\ell_0\{(h+p)f_1(x+y(X_{m-1}))(1+y^{(i)}(X_{m-1}))$$

$$+rQ^{(i+1)}_m(y(X_{m-1})|X_{m-1})+rQ^{(1)}_m(y(X_{m-1})|X_{m-1})y^{(i)}(X_{m-1})\}$$

$$+\ell_1 r\{Q_{m-1}(x_{m-1}|X_{m-2})f_m(y(X_{m-1}))y^{(i)}(X_{m-1})$$

$$+\int_0^{y(X_{m-1})}(q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})y^{(i)}(X_{m-1})f_m(u)$$

$$+Q_m^{(i)}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u))du\}], \qquad (3.45)$$

Solving the equation (3.45) for $y^{(i)}(X_{m-1})$, the following is gained :

$$y^{(i)}(X_{m-1}) = -\frac{E_1}{Z_1} \leqq 0, \qquad (3.46)$$

where,

$$Z_1 = \ell_0[(R_1-R_2)q_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-R_1 q_m(y(X_{m-1})|X_{m-1})-(h+p)f_1(x+y(X_{m-1}))$$

$$-rq_m(y(X_{m-1})|X_{m-1})]$$

$$+\ell_1[(R_1-R_2)q_{m-1}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-R_1 Q_{m-1}(x_{m-1}|X_{m-2})f_m(y(X_{m-1}))$$

$$-R_1\int_0^{y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u)du$$

$$-rQ_{m-1}(x_{m-1}|X_{m-2})f(y(X_{m-1}))$$

$$-r\int_0^{y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u)du]$$

$$< 0 \qquad (\because R_1 \leqq R_2), \tag{3.47}$$

$$E_1 = \ell_0[(R_1-R_2)Q_m^{(i)}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-R_1Q_m^{(i+1)}(y(X_{m-1})|X_{m-1})-(h+p)f_1(x+y(X_{m-1}))$$

$$-rQ_m^{(i+1)}(y(X_{m-1})|X_{m-1})]$$

$$+\ell_1[(R_1-R_2)Q_m^{(i)}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-(R_1+r)\int_0^{y(X_{m-1})} Q_m^{(i)}(y(X_{m-1})+x_{m-1}-u|X_{m-2})f_m(u)du]$$

$$\leqq 0 \qquad (\because R_1 \leqq R_2) \tag{3.48}$$

From the inequalities (3.46) and (3.48), considering the following relationship by Nahmias [3],

$$Q_m^{(i)}(y|X_{m-1}) \geqq Q_m^{(i+1)}(y|X_{m-1}) \;, \tag{3.49}$$

the inequality (3.50) is obtained.

$$y^{(i)}(X_{m-1}) \geqq -1. \tag{3.50}$$

Representing the numerator of $y^{(i)}(X_{m-1})$ and $y^{(i+1)}(X_{m-1})$ by $E_1^{(i)}$ and $E_1^{(i+1)}$ respectively and using inequality (3.49), the relation between $y^{(i)}(X_{m-1})$ and $y^{(i+1)}(X_{m-1})$ is obtained as follows :

$$y^{(i)}(X_{m-1})-y^{(i+1)}(X_{m-1})= - \frac{E_1^{(i)}-E_1^{(i+1)}}{Z_1} \leqq 0 \; . \tag{3.51}$$

Thus, together the inequalities (3.46),(3.50) and (3.51), the following relations are obtained :

$$-1 \leqq y^{(i)}(X_{m-1}) \leqq y^{(i+1)}(X_{m-1}) \leqq 0 \; , \quad i=2,3,\dots,m-2. \tag{3.52}$$

The relationship for the remained case $i=1$ is obtained as follows:

$$y^{(1)}(X_{m-1})= - \frac{E_1^0}{Z_1^0} \leqq 0 \; ,$$

where,

$$\begin{aligned} Z_1^0 = \; & \ell_0 [(R_1-R_2)Q_{m-1}^{(1)}(y(X_{m-1})+x_{m-1}|X_{m-2}) \\ & -R_1 Q_m^{(1)}(y(X_{m-1})|X_{m-1})-(h+p)f_1(x+y(X_{m-1})) \\ & -rQ_m^{(1)}(y(X_{m-1})|X_{m-1})] \\ & +\ell_1 [(R_1-R_2)Q_{m-2}^{(1)}(y(X_{m-1})+x_{m-1}+x_{m-2}|X_{m-3}) \\ & -R_1 Q_{m-1}^{(1)}(y(X_{m-1})+x_{m-1}|X_{m-2}) \\ & -rQ_{m-1}^{(1)}(y(X_{m-1})+x_{m-1}|X_{m-2})] < 0, \end{aligned}$$

$$\begin{aligned} E_1^0 = \; & \ell_0 [(R_1-R_2)Q_{m-1}^{(1)}(y(X_{m-1})+x_{m-1}|X_{m-2}) \\ & -R_1 Q_m^{(2)}(y(X_{m-1})|X_{m-1})-(h+p)f_1(x+y(X_{m-1})) \end{aligned}$$

$$-rQ_m^{(2)}(y(X_{m-1})|X_{m-1})]$$

$$+\ell_1[(R_1-R_2)Q_{m-2}^{(1)}(y(X_{m-1})+x_{m-1}+x_{m-2}|X_{m-3})$$

$$-R_1Q_{m-1}^{(1)}(y(X_{m-1})+x_{m-1}|X_{m-2})$$

$$-rQ_{m-1}^{(1)}(y(X_{m-1})+x_{m-1}|X_{m-2})]\leqq 0.$$

This proposition implies that the optimal ordering quantity is more sensitive to the increase of newer on-hand inventory than that of the older on-hand inventory, and also implies that the increase of on-hand inventory by one unit derives the decrease of optimal order quantity by less than one unit.

Proposition 3.4

Expected profit increases according to the increase of the ratio α .

Proof:

From the equation (3.22), the relation

$$Q_{m-1}((1-\alpha)y+x_{m-1}|X_{m-2}) \leqq Q_{m-2}((1-\alpha)y+x_{m-1}+x_{m-2}|X_{m-3}) , \tag{3.53}$$

is obtained.

From the equation (3.20) and the inequality (3.53),

$$\frac{\partial J(X_{m-1},y)}{\partial \alpha} \geqq 0. \tag{3.54}$$

is obtained. □

This proposition indicates that when $1-\alpha$, i.e., the ratio of excess perishability under the occurrence of leadtime 1, becomes closer to 1, the expected profit, $J(X_{m-1},y)$ decreases.

Given the fixed ordering cost K , and the optimal order quantity $y(X_{m-1})$, there exists $s(X_{m-1})$ $(\leq y(X_{m-1}))$ satisfying the following equation,

$$J(X_{m-1},s(X_{m-1}))=J(X_{m-1},y(X_{m-1}))-K \tag{3.55}$$

Here, if $s(X_{m-1})>0$, then $y(X_{m-1})$ should be ordered and if $s(X_{m-1})\leq 0$, then the optimal policy is not to order.

Differentiating both sides of the equation (3.55) with respect to x_{m-i} and noting

$$\left.\frac{\partial J(X_{m-1},y)}{\partial y}\right|_{y=y(X_{m-1})}=0$$

the following equation (3.56) is obtained.

$$s^{(i)}(X_{m-1})=\frac{J^{(i)}(X_{m-1},y(X_{m-1}))-J^{(i)}(X_{m-1},s(X_{m-1}))}{J^{(m)}(X_{m-1},s(X_{m-1}))} \tag{3.56}$$

where $s^{(i)}(X_{m-1})$ and $J^{(i)}(X_{m-1},y(X_{m-1}))$ are defined as follows:

$$s^{(i)}(X_{m-1})=\frac{\partial s(X_{m-1})}{\partial x_{m-i}} \quad , \quad i=1,2,\ldots,m-1 \ .$$

$$J^{(i)}(X_{m-1},s(X_{m-1}))=\frac{\partial J(X_{m-1},s(X_{m-1}))}{\partial x_{m-i}} \quad , \quad i=1,2,\ldots,m-1 \ .$$

Proposition 3.5

When $\alpha=0$ or 1, then

$$-1 \leqq s^{(i)}(X_{m-1}) \leqq s^{(i+1)}(X_{m-1}) \leqq 0 \quad , \quad i=2,3,\ldots,m-2$$

and

$$s^{(1)}(X_{m-1}) \leqq 0 \tag{3.57}$$

hold.

Proof:

(I) Case of $\alpha=1$:

Since the relation,

$$\begin{aligned} J^{(m,m)}(X_{m-1},y) \equiv & \frac{\partial J^{(m)}(X_{m-1},y)}{\partial y} \\ = & (R_1-R_2)Q_{m-1}^{(1)}(y+x_{m-1}|X_{m-2})-R_1Q_m^{(1)}(y|X_{m-1}) \\ & -\ell_0[(h+p)f_1(x+y)+rQ_m^{(1)}(y|X_{m-1})] \\ & -\ell_1 r\int_0^y Q_{m-1}^{(1)}(y+x_{m-1}-u|X_{m-2})f_m(u)du \leqq 0 , \end{aligned} \tag{3.58}$$

holds, $J^{(m)}(X_{m-1},y)$ is a non-increasing function with respect to y .

On the other hand, since the relation,

$$\begin{aligned} J^{(i,m)}(X_{m-1},y) = & (R_1-R_2)Q_{m-1}^{(i)}(y+x_{m-1}|X_{m-2})-R_1Q_m^{(i+1)}(y|X_{m-1}) \\ & -\ell_0[hf_1(x+y)+pf_1(x+y)+rQ_m^{(i+1)}(y|X_{m-1})] \\ & -\ell_1 rQ_m^{(i+1)}(y|X_{m-1}) \leqq 0 , \end{aligned} \tag{3.59}$$

is derived, $J^{(i)}(X_{m-1},y)$ is also a non-increasing function with respect to y.

From the equations (3.58) and (3.59), using the relationship $y(X_{m-1})\geqq s(X_{m-1})$ and the equation (3.56),

$$s^{(i)}(X_{m-1})\leqq 0 \quad , \qquad i=1,2,\ldots,m-1, \tag{3.60}$$

is derived.

Furthermore, denoting the numerator of $s^{(i)}(X_{m-1})-s^{(i-1)}(X_{m-1})$ with A_1, then

$$A_1 = R_2Q_{m-i}(X_{m-i})\{H_i(y(X_{m-1})|X^*_{m-i})-H_i(s(X_{m-1})|X^*_{m-i})\}$$

$$+rQ_{m-i}(X_{m-1})\{H_i(y(X_{m-1})|X_{m-1})-H_i(s(X_{m-1})|X_{m-1})\}>0, \tag{3.61}$$

is derived from the equation (3.56),

where $X^*_{m-i}=(x_{m-1},x_{m-2},\ldots,x_{m-i+1})$, and H_i is the same meanings in the Lemma 2.3.

From the equations (3.58) and (3.59),

$$s^{(i)}(X_{m-1})\geqq s^{(i-1)}(X_{m-1}) \quad , \qquad i=2,3,\ldots,m \ , \tag{3.62}$$

is obtained.

(II) Case of $\alpha=0$:

Similar to the case of $\alpha=1$, since the relation

$$J^{(m,m)}(X_{m-1},y)\leqq 0, \tag{3.63}$$

holds, $J^{(m)}(X_{m-1},y)$ is a non-increasing function with respect to y.

On the other hand, since the relation,

$$J^{(i,m)}(X_{m-1},y)\leqq 0, \tag{3.64}$$

holds similar to the case of $\alpha=1$, $J^{(i)}(X_{m-1},y)$ is also a non-increasing function with respect to y.

From the equations (3.63) and (3.64), using the relationship $y(X_{m-1})\geqq s(X_{m-1})$ and the equation (3.56),

$$s^{(i)}(X_{m-1})\leqq 0 \quad , \qquad i=1,2,\ldots,m-1 \quad , \tag{3.65}$$

is obtained.

Furthermore, denoting the numerator of $s^{(i)}(X_{m-1})-s^{(i-1)}(X_{m-1})$ with A_0, again

$$\begin{aligned}
A_0= \ & \ell_0[(R_2-R_1)Q_{m-i}(X_{m-i})\{H_{i-1}(y(X_{m-1})|X_{m-i}) \\
& -H_{i-1}(s(X_{m-1})|X_{m-1})\}+R_1Q_{m-i}(X_{m-i})\{H_i(y(X_{m-1})|X_{m-i}) \\
& -H_i(s(X_{m-1})|X_{m-i})\}] \\
& +\ell_1[(R_2-R_1)Q_{m-i}(X_{m-i})\{H_{i-2}(y(X_{m-1})+x_{m-1}|X_{m-i}) \\
& -H_{i-2}(s(X_{m-1})+x_{m-1}|X_{m-i})\}+R_1Q_{m-i}(X_{m-i})\{H_{i-1}(y(X_{m-1})+x_{m-1}|X_{m-i}) \\
& -H_{i-1}(s(X_{m-1})+x_{m-1}|X_{m-1})\}] \\
& +\ell_0rQ_{m-i}(X_{m-i})\{H_i(y(X_{m-1})|X^*_{m-i})-H_i(s(X_{m-1})|X^*_{m-1})\} \\
& +\ell_1rQ_{m-i}(X_{m-i})\{H_{i-1}(y(X_{m-1})+x_{m-1}|X^*_{m-i}) \\
& -H_{i-1}(s(X_{m-1})+x_{m-1}|X^*_{m-i})\}>0 \ ,
\end{aligned} \tag{3.66}$$

is derived from the equation (3.56) similar to the inequality (3.61).

From the equations (3.63) and (3.66),

$$s^{(i)}(X_{m-1})\geqq s^{(i-1)}(X_{m-1}), \quad i=2,3,\ldots,m, \tag{3.67}$$

is obtained.

3.5 Example

This section provides an example in order to illustrate the results of Sections 3.3 and 3.4. **The price and cost parameters are** (R_1,R_2,c,h,p,r) $=(300,350,100,10,100,20)$, $\ell_1=0.3$ and the maximum lifetime is $m=3$. The demand distribution function is assumed to be exponential with mean 20.

Figure 3.1 illustrates the total expected profit for on-hand inventory $x_1=15$ and $x_2=5$ when $\alpha=0.2$, 0.4, 0.6 and 0.8. The result shows the validity of Propositions 3.1 and 3.4. That is, the increase of α derives the increases of the expected profit and the optimal ordering quantity.

Figure 3.2 illustrates the validity of Proposition 3.2 on the restriction of $\alpha=1, x_2=0$ and $(R_1,R_2,h)=(200,250,250)$. In this case, the optimal ordering policy is determined by taking the inequalities (3.29) into consideration. Since the inequality, $c=100>R_2-h\ell_0=75$, holds, the order of quantity $y(X_{m-1})$ is placed according to the equality (3.30) iff $x<x^*(\fallingdotseq 50)$. The result shows that the optimal orders $y(X_2)=18$ and 11 are placed when $x_1=5$ and 15 respectively, and that no orders are placed when $x_1=50$ and 60.

Figures 3.3, 3.4 and 3.5 illustrate the relation between the age distribution of inventory on-hand x and optimal ordering quantity $y(X_2)$ for $x=20$,i.e., (x_2,x_1) $=(0,20),(5,15),(15,5),(20,0)$ with respect to $\alpha=0$, 0.8 and 1.0 respectively. These figures assert the increase of newer inventory on-hand makes more influences upon the decrease of the optimal ordering quantity than that of older one. The results coincide with Proposition 3.3.

Figure 3.6 illustrates the validity of Propositions 3.3 and 3.5. From the results of this figure, we can observe that the increase of older on-hand inventory makes more influences upon the increases of $y(X_2)$ and $s(X_2)$ than that of newer one. And the discrepancy between $y(X_2)$ and $s(X_2)$

inclines to decrease according to the increase of the proportion of the older on-hand inventory to the newer one. In addition, the solid line in Figure 3.6 shows the acceptable maximum fixed ordering cost for releasing order of quantity $y(X_2)$. This curve shows that the increase of newer on-hand inventory makes more difficult to raise fixed order cost than that of older one.

3.6 Conclusion

In this chapter, we discussed the determination of the optimal ordering policies for perishable commodity under different selling prices and stochastic leadtime. Some properties of this model such as the existence of the optimal ordering policy and its conditions, the influences of the rate of excess perishability, $1-\alpha$, and the status of on-hand inventory upon the optimal ordering policies were analyzed. Furthermore, a fixed charge cost for placing an order was introduced as a set-up cost and some additional characteristics were obtained. But unfortunately, except for the Proposition 3.4, any useful results could not be obtained with respect to $0<\alpha<1$.

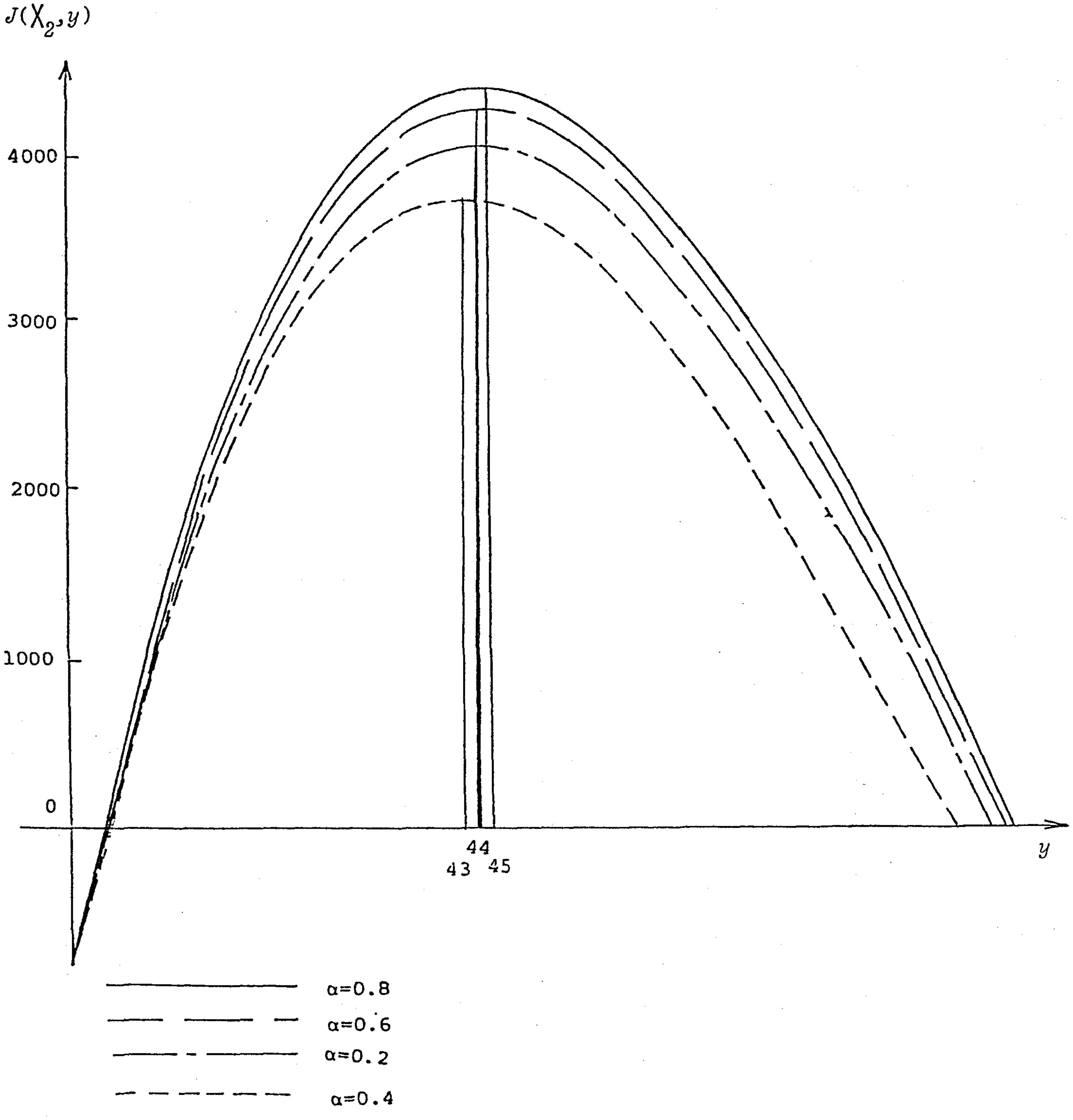

Figure 3.1.The influences of α upon the optimal ordering policy

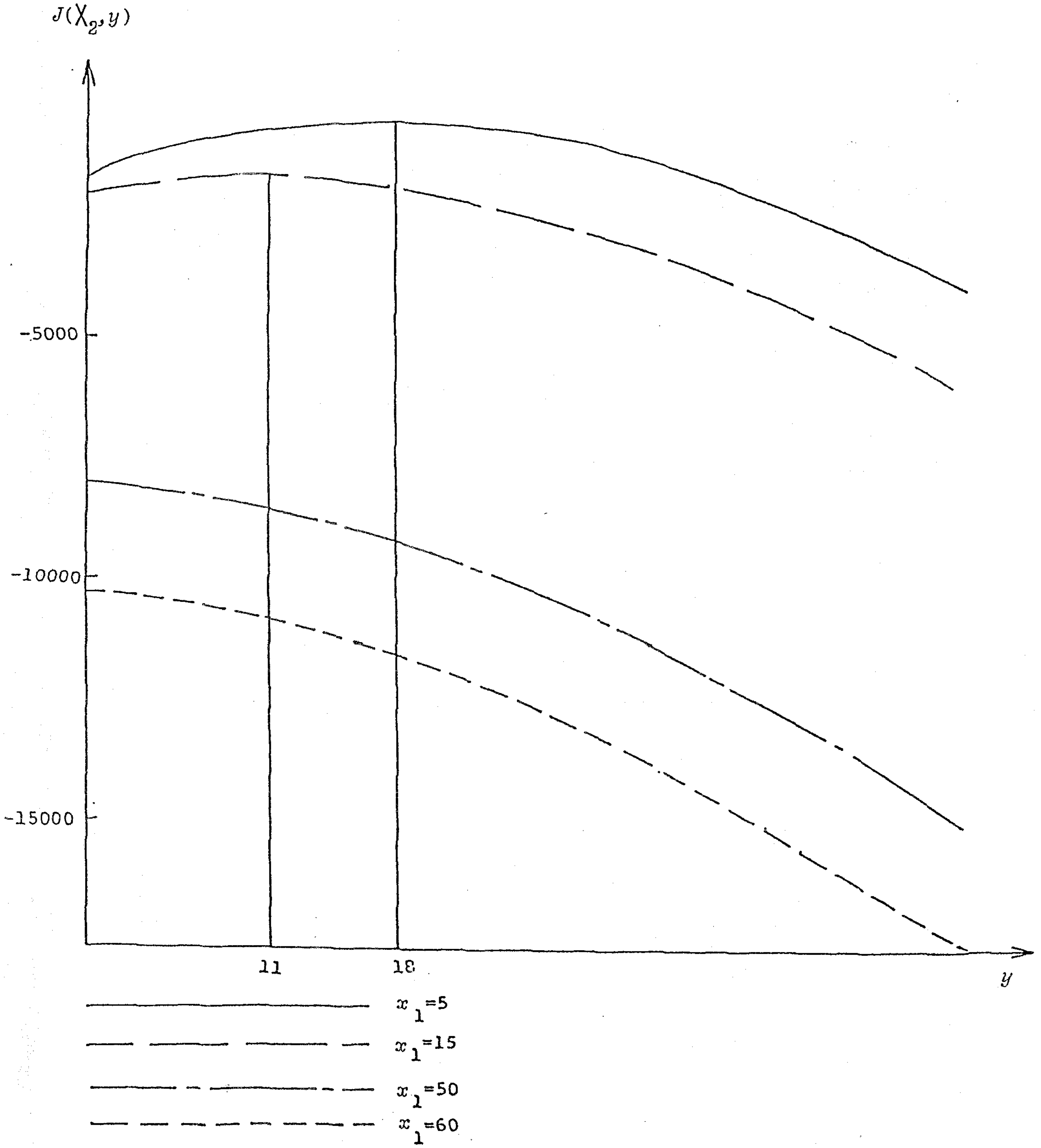

Figure3.2.The conditions for optimal ordering policy $y(X_2)$

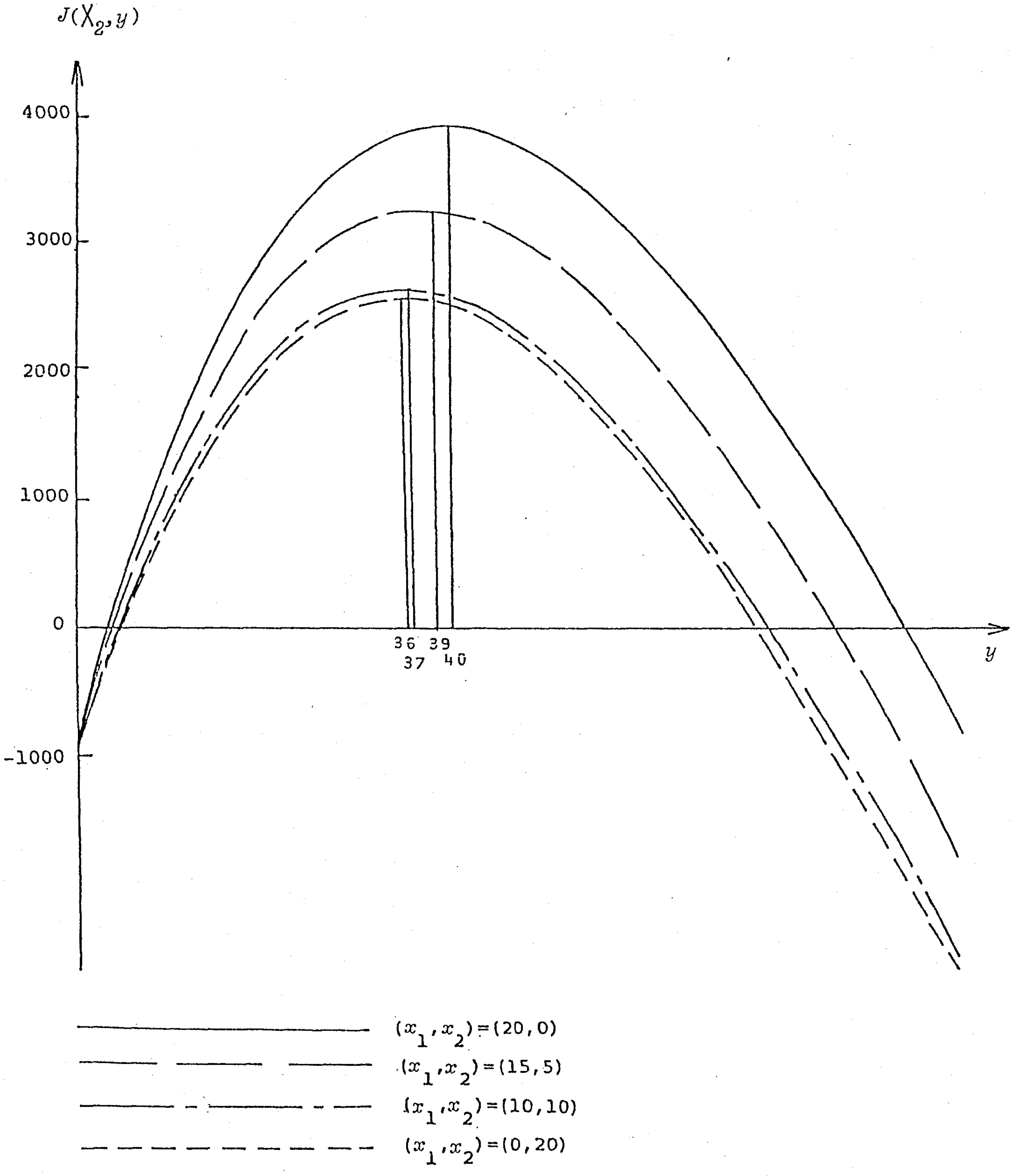

Figure 3.3 The influences of the age distribution of on-hand inventory upon the optimal ordering policy ($\alpha=0$)

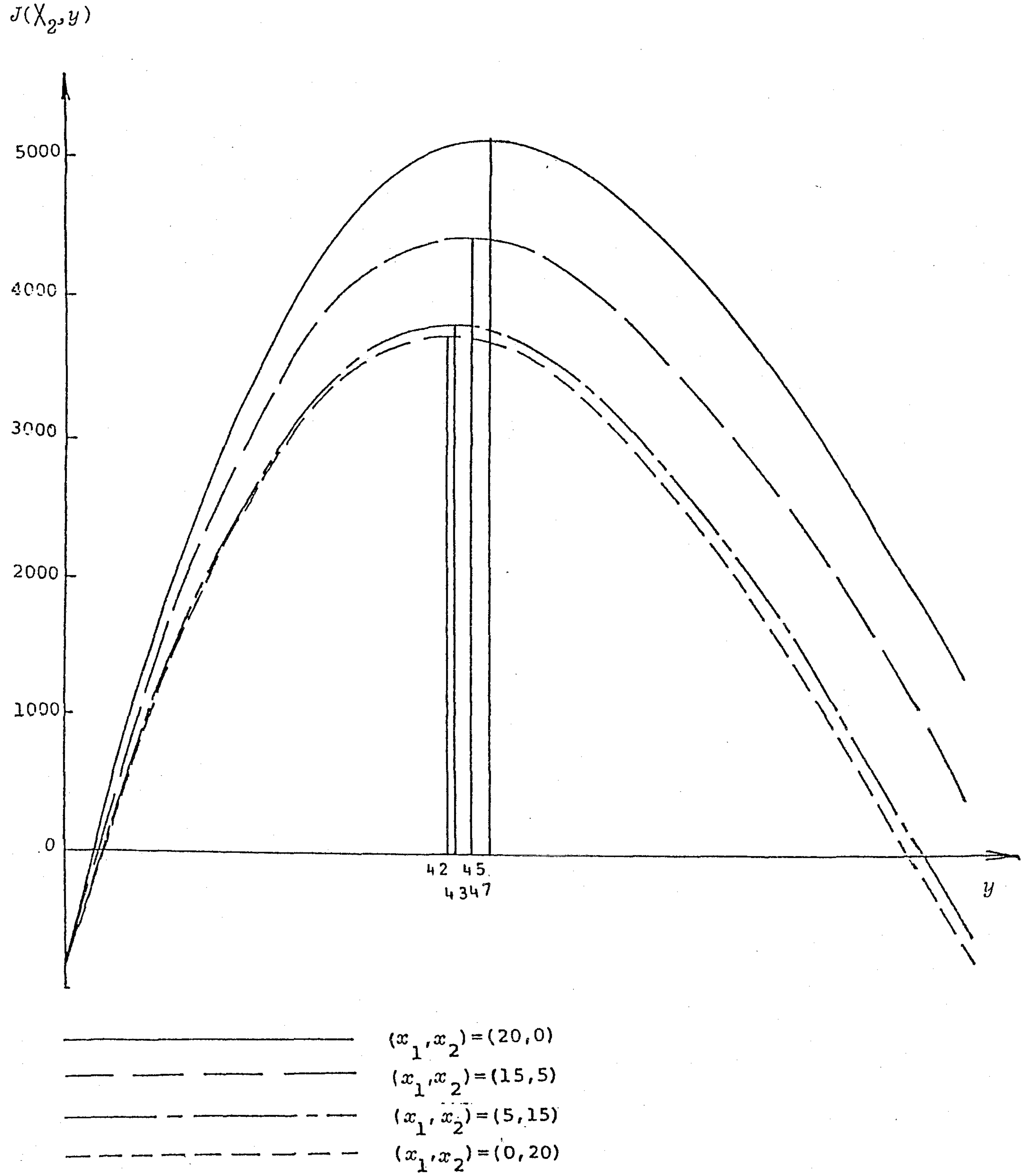

Figure 3.4 The influences of the age distribution of on-hand inventory upon the optimal ordering policy ($\alpha=0.8$)

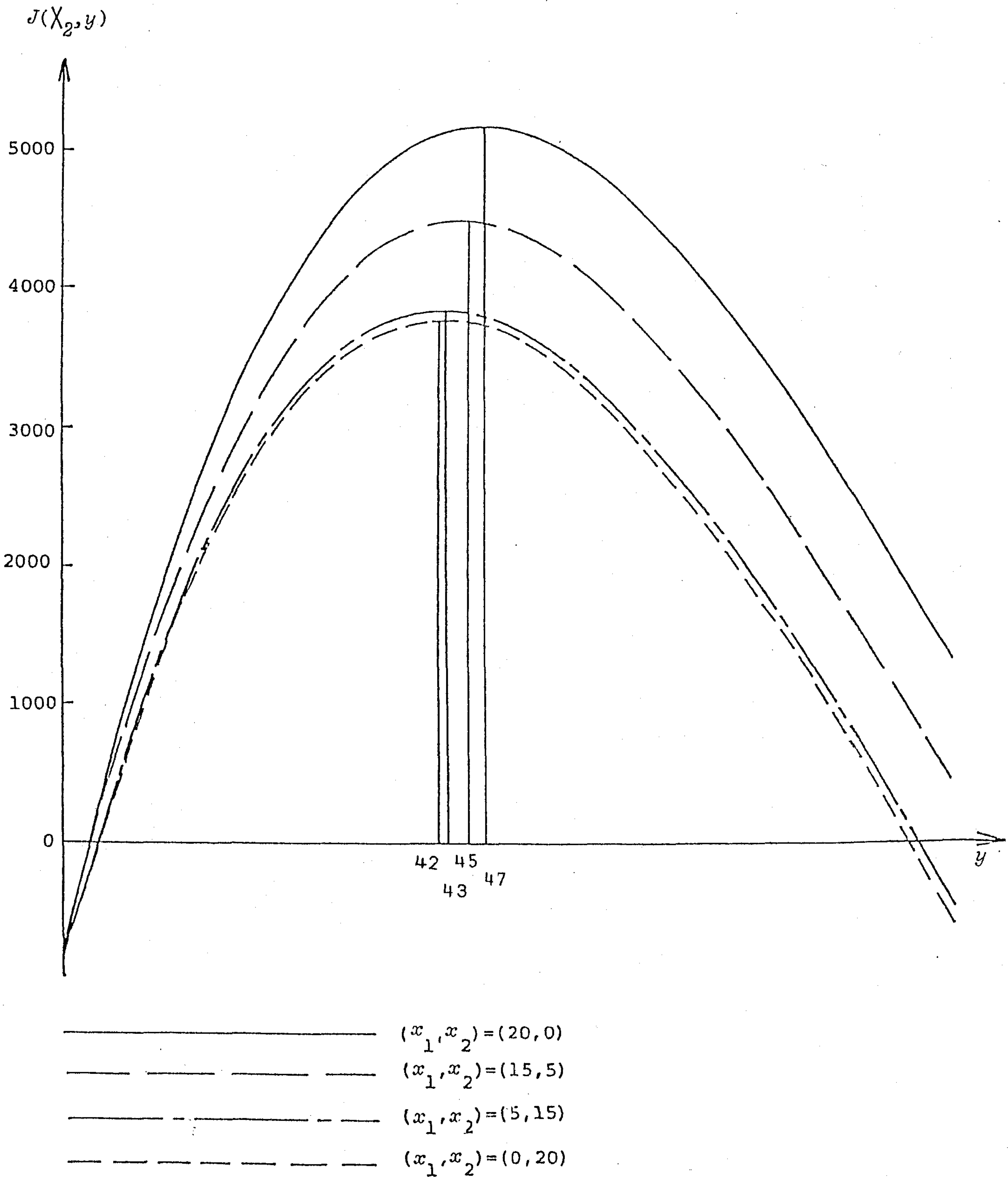

Figure 3.5. The influences of the age distribution of on-hand inventory upon the optimal ordering policy ($\alpha=1.0$)

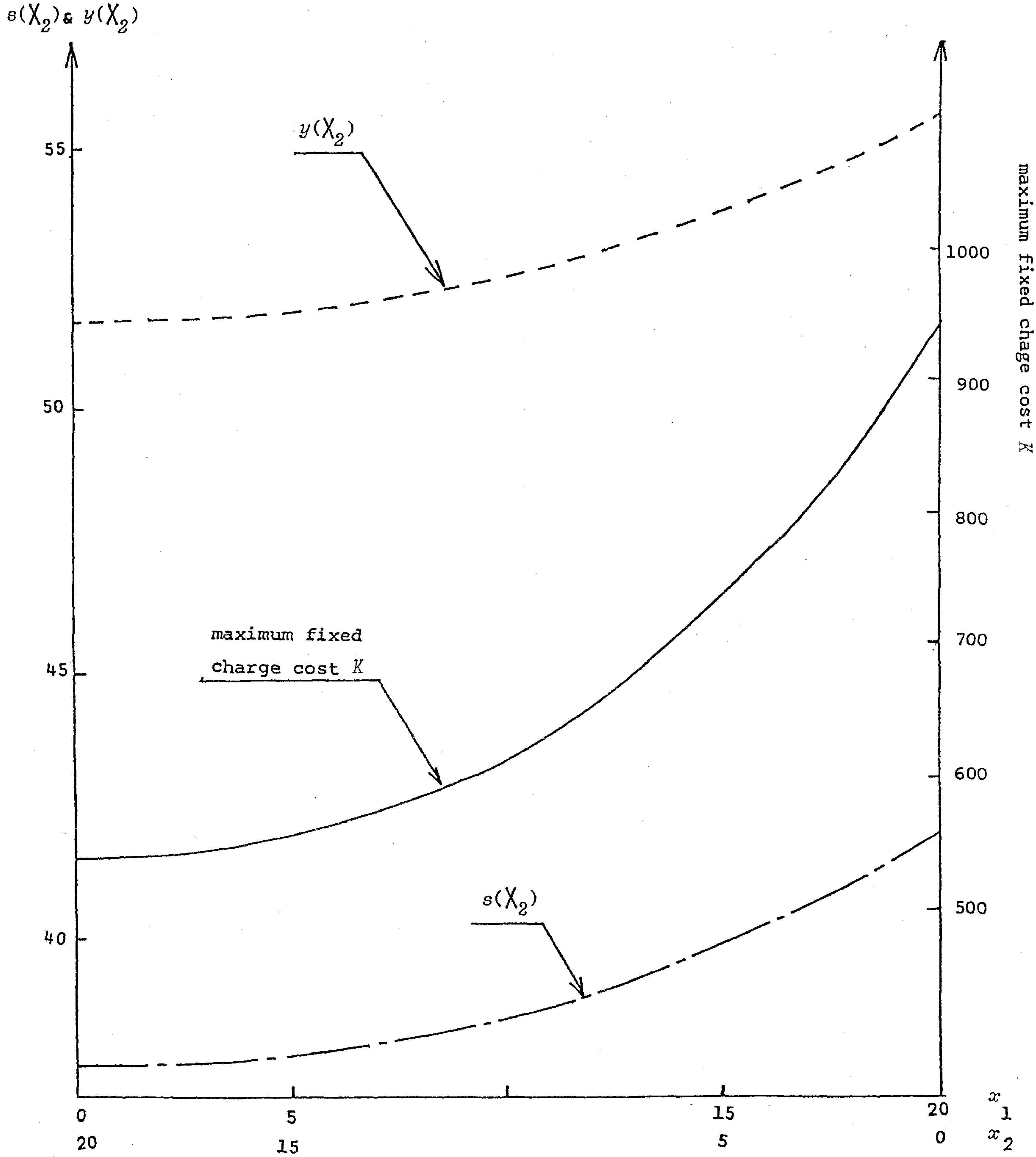

Figure 3.6,The influences of varying on-hand inventory status upon the optimal ordering policy when the fixed charge cost is considered

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CHAPTER IV

LIFO/FIFO ISSUING INVENTORY CONTROL FOR PERISHABLE COMMODITIES SUBJECT TO STOCHASTIC PROCUREMENT LEADTIME

4.1 Introduction

Problems associated with the perishable products arise in many areas of inventory and production management. Important works have been done for describing optimal ordering policies for item with predetermined maximum lifetime. Nahmias [2] has included explicitly the effect of the expected amount of perishable items among the ordered incoming commodity m periods ahead, by noting the above expectation depends upon the age distribution of inventory on band and the realizations of the demands over the next m preiods. Ishii et.al [1] and Nose et.al [3] described the model for perishable product with stochastic leadtime under the assumption of FIFO issuing policy. We have not discussed LIFO issuing policy for perishable inventory with stochastic leadtime.

In this chapter, inventory control policies for perishable commodities are considered with zero or one period stochastic leadtime on both LIFO and FIFO. Blood, photograhic film and fresh commodities are typical examples of our model. In Section 4.2, we describe some assumptions and notations which will be used throughout this chapter. In Section 4.3, we establish the LIFO model and clarify the existence of the unique optimal ordering policy.

In Section 4.4, we establish FIFO model and derive the similar properties with LIFO model. In Section 4.5, influences of fluctuation of the probability of leadtime ℓ_1 , inventory on hand, cost of unit holding, shortage and perishing of commodity in connection with the optimal ordering for elevating up to the desired level, are investigated on both LIFO and FIFO issuing policies.

4.2 Preliminaries

The following assumptions are made throughout this chapter.

(1) LIFO and FIFO issuing policies for a perishable commodity are considered subject to one time planning per cycle with respect to single kind of commodity, assuming periodic review of ordering.

(2) Unit purchasing cost c , unit holding cost h, unit shortage cost p and unit perishing cost r are introduced. Ordering is given at the start of each period, while other costs are incurred during each period.

(3) Maximum lifetime of the perishable commodity is assumed to be 2 periods in this chapter.

(4) Maximum inventory level is z and order is given so that the inventory level is kept to be z .

(5) Each demand in successive periods is independent and identically distributed nonnegative random variable with known distribution function $F(\cdot)$ with continuous probability density $f(\cdot)$.

(6) When leadtime ℓ is 0, new stock arrives with maximum lifetime 2, and when $\ell=1$, the stock arrives with remaining lifetime 1. Leadtime 0 occurs with probability ℓ_0 and leadtime 1 occurs with ℓ_1, where $\ell_0 + \ell_1 = 1$, $\ell_0 > 0$, $\ell_1 > 0$.

(7) The stock remained at the end of each period puts on one age monotonically.

(8) No backlogging is permitted.

We list the following notations;

x_j : the amount of commodity on hand which will perish at the end of j th period.

$z-x_j$: the amount of new commodity whose lifetime is two periods.

The following combinations are possible from the j th period to $j+1$ st, concerning the leadtime required, i.e., leadtime 0 or leadtime 1.

Table 4.1 Model of occuring leadtime

jth period	$j+1$st period	Case	Probability of occurence
ℓ_0	ℓ_0	Case 1	ℓ_0^2
	ℓ_1	Case 2	$\ell_0 \ell_1$
ℓ_1	ℓ_0	Case 3	$\ell_0 \ell_1$
	ℓ_1	Case 4	ℓ_1^2

$R_i^q(z,\ x_j)$: r.v. representing the amount of new commodity $z-x_j$ which will be perished if remain unsold at the end of $j+1$ st period, using policy q (LIFO or FIFO) for case i.

$$G_i^q(u\,|\,z, x_j) = P(\{R_i^q(z,\ x_j) \leqq u\}).$$

$ER_i^q(z,\ x_j)$: expected outdating of $R_i^q(z,\ x_j)$.

4.3 LIFO model

The LIFO assumption is equivalent to the freshest commodity first issuing doctrine. LIFO issuing policy gives the following transfer functions:

$$x_{j+1} = \begin{cases} [(z-x_j) - D_j]^+ & : \text{ leadtime } 0 \\ z-x_j & : \text{ leadtime } 1 \end{cases} \qquad (4.1)$$

where, $(a)^+ = \max(a, 0)$.

By the use of the equation (4.1), $R_i^{LIFO}(z, x_j)$, $G_i^{LIFO}(u|z,x_j)$ and $ER_i^{LIFO}(z,x_j)$ with respect to case $i\,(i=1,\ldots,4)$ are obtained as follows:

Case 1:

$$R_1^{LIFO}(z, x_j) = [x_{j+1} - \{D_{j+1} - (z-x_{j+1})\}^+]^+$$

$$= [(z-x_j-D_j)^+ - \{D_{j+1} - z + (z-x_j-D_j)^+\}^+]^+.$$

$$G_1^{LIFO}(u|z,x_j) = 1 - F(z-u)F(z-x_j-u).$$

$$ER_1^{LIFO}(z, x_j) = \int_0^{z-x_j} F(z-u)F(z-x_j-u)\,du. \qquad (4.2)$$

Similar to the Case 1, the other cases are gained as equalities (4.3) to (4.5).

Case 2:

$$ER_2^{LIFO}(z,\ x_j\ ) = \int_0^{z-x_j}\int_0^{z-x_j-u} F(z-x_j-u-d_j)f(d_j)dd_j du \quad . \qquad (4.3)$$

Case 3:

$$ER_3^{LIFO}(z,x_j) = \int_0^{z-x_j} F(z-u)\, du \quad . \qquad (4.4)$$

Case 4:

$$ER_4^{LIFO}(z,\ x_j) = \int_0^{z-x_j} F(z-x_j-u)du \quad . \qquad (4.5)$$

Proposition 4.1

Each $ER_i^{LIFO}(z,x_j)$, $i=1,\ldots, 4$, is an increasing convex function with respect to z for all $x_j \in [0,\infty)$.

Proof:

For the case $i = 1$, partially differentiating the equation (4.2) with respect to z once and twice result as follows:

$$\frac{\partial}{\partial z} ER_1^{LIFO}(z,\ x_j) = \frac{\partial}{\partial z}\int_0^{z-x_j} \{\ F(z-u)F(z-x_j-u)\}\, du$$

$$= F(z)F(z-x_j) \geqq 0. \qquad (4.6)$$

$$\frac{\partial^2}{\partial z^2} ER_1^{LIFO}(z,\ x_j) = f(z)F(z-x_j) + F(z)f(z-x_j) \geqq 0\ . \qquad (4.7)$$

For the case $i = 2$ partially differentiating the equation (4.3) with respect to z once and twice result as follows.

$$\frac{\partial}{\partial z} ER_2^{LIFO}(z, x_j) = \frac{\partial}{\partial z} \{\int_0^{z-x_j} \int_0^{z-x_j-u} F(z-x_j-u-d_j)f(d_j)dd_j du\}$$

$$= \int_0^{z-x_j} f(d_j)F(z-x_j-d_j)dd_j \geqq 0 \,. \qquad (4.8)$$

$$\frac{\partial^2}{\partial z^2} ER_2^{LIFO}(z, x_j) = \int_0^{z-x_j} f(z-x_j-d_j)f(d_j)dd_j + f(z-x_j)F(0) \geqq 0. \qquad (4.9)$$

For the case $i = 3$, partially differentiating the equation (4.4) with respect to z once and twice result as follows:

$$\frac{\partial}{\partial z} ER_3^{LIFO}(z, x_j) = \frac{\partial}{\partial z} \int_0^{z-x_j} F(z-u)du = F(z) \geqq 0 \,, \qquad (4.10)$$

$$\frac{\partial^2}{\partial z^2} ER_3^{LIFO}(z, x_j) = f(z) \geqq 0 \,. \qquad (4.11)$$

For the case $i = 4$, partially differentiating the equation (4.5) with respect to z once and twice result as follows:

$$\frac{\partial}{\partial z} ER_4^{LIFO}(z, x_j) = F(z-x_j) \geqq 0 \,. \qquad (4.12)$$

$$\frac{\partial^2}{\partial z^2} ER_4^{LIFO}(z, x_j) = f(z-x_j) \geqq 0 \,. \qquad (4.13)$$

Inequalities (4.6) to (4.13) complete the proof. □

Then the total expected cost function for the LIFO model becomes as follows from the equations (4.2) to (4.5) together with other costs.

$$L^{LIFO}(z, x_j) = c(z-x_j) + \ell_0[h\int_0^{z-x_j}(z-x_j-d_j)f(d_j)dd_j$$

$$+ p\int_z^{\infty}(d_j-z)f(d_j)dd_j]$$

$$+ \ell_1 p\int_{x_j}^{\infty}(d_j-x_j)f(d_j)dd_j$$

$$+ \ell_0^2\, r\int_0^{z-x_j}F(z-u)F(z-x_j-u)du$$

$$+ \ell_0\ell_1\, r\int_0^{z-x_j}\int_0^{z-x_j-u}F(z-x_j-u-d_j)f(d_j)dd_j du$$

$$+ \ell_0\ell_1\, r\int_0^{z-x_j}F(z-u)\,du + \ell_1^2\, r\int_0^{z-x_j}F(z-x_j-u)du \qquad (4.14)$$

In order to show the convexity of the cost function $L^{LIFO}(z,x_j)$, predescribed Proposition 4.1 is fully utilized.

Proposition 4.2

$L^{LIFO}(z, x_j)$ is a convex function of z under the assumptions of $x_j < F^{-1}((p\ell_0-c)/(p-r\ell_0^2+r\ell_0))$ and $1 \geq \ell_0 > c/p > 0$.

z^* minimizing $L^{LIFO}(z,x_j)$ exists in $[x_j, \infty)$, that is

$$L^{LIFO}(z^*,x_j) = \min_{z>x_j}[L^{LIFO}(z,x_j)].$$

Proof.

The partial derivative of the equation (4.14) with respect to z is as follows:

$$\frac{\partial L^{LIFO}(z, x_j)}{\partial z} = c + \ell_0[hF(z-x_j) - p(1-F(z))]$$

$$+ \ell_0^2 r\{ F(z)F(z-x_j) + 2\int_0^{z-x_j} F(z-u)f(z-x_j-u)du \}$$

$$+ \ell_0\ell_1 r\int_0^{z-x_j} f(d_j)F(z-x_j-d_j)dd_j$$

$$+ \ell_0\ell_1 r \{F(z) + \ell_1^2 r F(z-x_j)\}. \qquad (4.15)$$

Differentiating the equation (4.15) with respect to z once more, the following is obtained:

$$\frac{\partial^2 L^{LIFO}(z, x_j)}{\partial z^2} = \ell_0 [hf(z-x_j) + pf(z)]$$

$$+ \ell_0^2 r[f(z)F(z-x_j)+F(z)f(z-x_j)]$$

$$+ \ell_0\ell_1 r\int_0^{z-x_j} f(z-x_j-d_j)f(d_j)dd_j$$

$$+ \ell_0\ell_1 rf(z) + \ell_1^2 rf(z-x_j) \geqq 0 . \qquad (4.16)$$

This implies the convexity of $L^{LIFO}(z, x_j)$.

Moreover,

$$\lim_{z \to x_j} \frac{\partial L^{LIFO}(z, x_j)}{\partial z} = (-r\ell_0^2 + r\ell_0 + p)F(x_j) - p\ell_0 + c < 0 \qquad (4.17)$$

is obtained , when the following two inequalities are held.

$x_j < F^{-1}((p\ell_0 - c)/(p - r\ell_0^2 + r\ell_0))$ and $1 \geq \ell_0 > c/p > 0$.

$$\lim_{z \to \infty} \frac{\partial L^{LIFO}(z, x_j)}{\partial z} = c + \ell_0 h + r > 0 \,. \qquad (4.18)$$

Inequalities (4.16), (4.17) and (4.18) complete the proof. □

Proposition 4.3

As the occurrence probability of leadtime 0 , ℓ_0, increases, the optimal order-up-to-level, z., increases.

Proof.

From the equation (4.15), the partial derivative with respect to z of the partial derivative of $L^{LIFO}(z, x_j)$ with respect to ℓ_1 is obtained as follows:

$$\begin{aligned} \frac{\partial^2 L^{LIFO}(z, x_j)}{\partial z \cdot \partial \ell_0} &= hF(z - x_j) - p(1 - F(z)) + 2\ell_0 rF(z)F(z - x_j) \\ &+ (1 - 2\ell_0) r \int_0^{z - x_j} f(d_j) F(z - x_j - d_j)\, dd_j + (1 - 2\ell_0) rF(z) \\ &- 2(1 - \ell_0) rF(z - x_j) . \end{aligned} \qquad (4.19)$$

(Rewriting the above equation by the use of the relation $\partial L^{LIFO}(z,x_j)/\partial z=0$.)

$$= \ell_0 rF(z)F(z-x_j)-\ell_0 r\int_0^{z-x_j} f(d_j)F(z-x_j-d_j)\,dd_j$$
$$-2(1-\ell_0)rF(z-x_j)-\ell_0 rF(z)-\frac{c}{\ell_0}-\frac{(1-\ell_0)^2}{\ell_0}rF(z-x_j)$$
$$<\ell_0 rF(z)(F(z-x_j)-1) \leqq 0.$$

□

4.4 FIFO model

The FIFO assumption is equivalent to the oldest commodity first issuing doctrine. FIFO issuing policy gives the following transfer function:

$$x_{j+1} = \begin{cases} [z-x_j-(D_j-x_j)^+]^+ & : \text{leadtime } 0 \\ z-x_j & : \text{leadtime } 1 \end{cases} \qquad (4.20)$$

By the use of the equation (4.20), $R_i^{FIFO}(z,x_j)$, $G_i^{FIFO}(u|z,x_j)$ and $ER_i^{FIFO}(z, x_j)$ with respect to case i ($i=1, \ldots, 4$) are obtained as follows:

Case 1: Case 2:

$$R_1^{FIFO}(z, x_j) = R_2^{FIFO}(z, x_j)$$
$$= [x_{j+1} - D_{j+1}]^+$$
$$= [(z-x_j - (D_j-x_j)^+)^+ - D_{j+1}]^+.$$

$$G_1^{FIFO}(u|z,x_j) = G_2^{FIFO}(u|z,x_j)$$

$$= 1-F(x_j)F(z-x_j-u) - \int_0^{z-x_j-u} f(z-d_{j+1}-u)F(d_{j+1})dd_{j+1} .$$

$$ER_1^{FIFO}(z, x_j) = ER_2^{FIFO}(z, x_j)$$

$$= \int_0^{z-x_j} F(d_{j+1})F(z-d_{j+1})dd_{j+1} . \qquad (4.21)$$

Case 3: Case 4:

$$R_3^{FIFO}(z, x_j) = R_4^{FIFO}(z, x_j)$$

$$= [x_{j+1} - D_{j+1}]^+$$

$$= [z-x_j - D_{j+1}]^+ .$$

$$G_3^{FIFO}(u|z,x_j) = G_4^{FIFO}(u|z,x_j)$$

$$= 1 - F(z-x_j - u) .$$

$$ER_3^{FIFO}(z, x_j) = ER_4^{FIFO}(z, x_j)$$

$$= \int_0^{z-x_j} F(z-x_j - u)du . \qquad (4.22)$$

Proposition 4.4

Each $ER_i^{FIFO}(z, x_j)$, $i = 1, \ldots, 4$, is an increasing convex function with respect to z for all $x_j \in [0, \infty)$.

Proof.

For the case $i = 1$ and 2, partially differentiating the equation (4.21) with respect to z once and twice result as follows:

$$\frac{\partial}{\partial z} ER_1^{FIFO}(z, x_j) = \frac{\partial}{\partial z} ER_2^{FIFO}(z, x_j)$$

$$= F(z-x_j)F(x_j) + \int_0^{z-x_j} F(d_{j+1}) f(z-d_{j+1})\, dd_{j+1} \geq 0 . \quad (4.23)$$

$$\frac{\partial^2}{\partial z^2} ER_1^{FIFO}(z, x_j) = \frac{\partial^2}{\partial z^2} ER_2^{FIFO}(z, z_j)$$

$$= f(z-x_j)F(x_j) + \int_0^{z-x_j} f(d_{j+1}) f(z-d_{j+1})\, dd_{j+1} \geq 0 . \quad (4.24)$$

For the case $i = 3$ and 4, partially differentiating the equation (4.22) with respect to z once and twice result as follows:

$$\frac{\partial}{\partial z} ER_3^{FIFO}(z, x_j) = \frac{\partial}{\partial z} ER_4^{FIFO}(z, x_j) = F(z-x_j) \geq 0 . \quad (4.25)$$

$$\frac{\partial^2}{\partial z^2} ER_3^{FIFO}(z, x_j) = \frac{\partial^2}{\partial z^2} ER_4^{FIFO}(z, x_j) = f(z-x_j) \geq 0 . \quad (4.26)$$

□

Then the total expected cost function for the FIFO model is derived as follows from the equations (4.21) and (4.22) together with the other costs.

$$L^{FIFO}(z,\ x_j) = c\,(z-x_j)+ \ell_0[h\int_{x_j}^{z}(z-d_j)f\ (d_j)dd_j + h(z-x_j)\int_0^{x_j}f(d_j)dd_j$$

$$+\,p\int_z^{\infty}(d_j-z)f(d_j)dd_j + r\int_0^{z-x_j}F(d_{j+1})F(z-d_{j+1})dd_{j+1}]$$

$$+\ \ell_1[p\int_{x_j}^{\infty}(d_j-x_j)f(d_j)dd_j+ r\int_0^{z-x_j}F(z-x_j-u)du]\ . \qquad (4.27)$$

In order to show the convexity of the cost function $L^{FIFO}(z,\ x_j)$, predescribed Proposition 4.4 is fully utilized.

Proposition 4.5

Assuming $x_j< F^{-1}((p\ell_0-c)/((p+h)\ell_0))$ and $1 \geq \ell_0 > c/p > 0$, then $L^{FIFO}(z,x_j)$ is a convex function of z . z^* minimizing $L^{FIFO}(z,x_j)$ exists in $[x_j,\infty)$, that is $L^{FIFO}(z^*,x_j)= \min_{z>x_j}[L^{FIFO}(z,x_j)]$.

Proof.

The partial differentiation of the equation (4.27) with respect to z becomes as follows:

$$\frac{\partial L^{FIFO}(z,\ x_j)}{\partial z} = c + \ell_0[h\{F(z)-p\{1-F(z)\}$$

$$+r\int_0^{z-x_j}f(d_{j+1})F(z-d_{j+1})dd_{j+1}]+ \ell_1 rF(z-x_j). \qquad (4.28)$$

Again, differentiating the equation (4.28) with respect to z , the following is obtained:

$$\frac{\partial^2 L^{FIFO}(z, x_j)}{\partial z^2} = \ell_0 \, [(h+p)f(z) + r(f(z-x_j)F(x_j) + \int_0^{z-x_j} f(d_{j+1})f(z-d_{j+1})\,dd_{j+1})] + \ell_1 \, rf(z-x_j) \geqq 0 \, . \tag{4.29}$$

This implies the convexity of $L^{FIFO}(z, x_j)$.

Moreover,

$$\lim_{z \to x_j} \frac{\partial L^{FIFO}(z, x_j)}{\partial z} = ((p+h)F(x_j)-p)\ell_0+c < 0 \tag{4.30}$$

is obtained, when the following two inequalities hold,

$x_j < F^{-1}((p\ell_0-c)/(\ell_0(p+h)))$ and $1 \geqq \ell_0 > c/p > 0$.

$$\lim_{z \to \infty} \frac{\partial L^{FIFO}(z, x_j)}{\partial z} = c+\ell_0 h+r > 0. \tag{4.31}$$

Inequalities (4.29), (4.30) and (4.31) complete the proof. □

Proposition 4.6

As the occurrence probability of leadtime 1, ℓ_1 , increases, the optimal order-up-to-level z increases.

Proof.

From the equation (4.28), the partial derivative with respect to z of the partial derivative of $L^{FIFO}(z, x_j)$ with respect to ℓ_1 is obtained as follows:

$$\frac{\partial^2 L^{FIFO}(z, x_j)}{\partial z \cdot \partial \ell_0} = hF(z)-(1-F(z))p+r\int_0^{z-x_j} f(d_{j+1})F(z-d_{j+1})\,dd_{j+1} -rF(z-x_j). \tag{4.32}$$

(The above equation is rewritten by the use of the relation,

$\partial L^{FIFO}(z, x_j)/\partial z = 0.$)

$$= -rF(z-x_j)(1+(1-\ell_0)/\ell_0)-c/\ell_0 < 0 . \quad \square$$

4.5 Properties of LIFO/FIFO Models

In this section, the properties of the optimal order-up-to-levels on both LIFO and FIFO models are investigated with respect to on-hand inventory , x_j , and cost parameters.

Proposition 4.7

As the increase of on-hand inventory x_j the optimal order-up-to-levels on both LIFO and FIFO models increase.

Proof.

From the equation (4.15), the second partial derivative of $L^{LIFO}(z, x_j)$ with respect to z and x_j is obtained as follows:

$$\frac{\partial^2 L^{LIFO}(z, x_j)}{\partial z . \partial x_j} = -\ell_0 h f(z-x_j) - \ell_0^2 r F(z) f(z-x_j) - \ell_0 \ell_1 r \int_0^{z-x_j} f(d_j) f(z-x_j-d_j) dd_j$$

$$-\ell_1^2 r f(z-x_j) \leqq 0 . \tag{4.33}$$

From the equation (4.28), the second partial derivative of $L^{FIFO}(z, x_j)$ with respect to z and ℓ_1 is obtained as follows:

$$\frac{\partial^2 L^{FIFO}(z, x_j)}{\partial z . \partial x_j} = -\ell_0 r f(z-x_j) F(x_j) - \ell_1 r f(z-x_j) \leq 0 . \tag{4.34}$$

Inequalities (4.33) and (4.34) complete the proof. □

Next, the influences of the cost parameters, i.e., costs of holding, shortage and outdating are investigated upon the optimal order-up-to-level for both LIFO and FIFO issuing policies.

Proposition 4.8

As unit holding cost h increases, the optimal order-up-to-levels on both LIFO and FIFO models decrease.

Proof.

From the equations (4.15) and (4.28), the second partial derivative of $L^{LIFO}(z, x_j)$ and $L^{FIFO}(z, x_j)$ with respect to z and h are obtained as follows:

$$\frac{\partial^2 L^{LIFO}(z, x_j)}{\partial z \cdot \partial h} = \ell_0 F(z-x_j) \geqq 0. \tag{4.35}$$

$$\frac{\partial^2 L^{FIFO}(z, x_j)}{\partial z \cdot \partial h} = \ell_0 F(z) \geqq 0. \tag{4.36}$$

□

Proposition 4.9

As unit shortage cost p increases, the optimal order-up-to-levels on both LIFO and FIFO models increase.

Proof.

From the equations (4.15) and (4.28), the second partial derivative of $L^{LIFO}(z, x_j)$ and $L^{FIFO}(z, x_j)$ with respect to z and p are obtained as follows:

$$\frac{\partial^2 L^{LIFO}(z, x_j)}{\partial z \cdot \partial p} = - \ell_0 \{1-F(z)\} \leqq 0. \tag{4.37}$$

$$\frac{\partial^2 L^{FIFO}(z, x_j)}{\partial z \cdot \partial p} = - \ell_0 \{1-F(z)\} \leqq 0. \tag{4.38}$$

□

Proposition 4.10

As unit perishing cost r increases, optimal order-up-to-levels on both LIFO and FIFO models decrease.

Proof.

From the equations (4.15) and (4.28), the second partial derivative of $L^{LIFO}(z, x_j)$ and $L^{FIFO}(z, x_j)$ with respect to z and r are obtained as follows:

$$\frac{\partial^2 L^{LIFO}(z, x_j)}{\partial z \cdot \partial r} = \ell_0^2 F(z) F(z-x_j) + \ell_0 \ell_1 \int_0^{z-x_j} f(d_j) F(z-x_j-d_j)\, dd_j$$

$$+\ell_0 \ell_1 F(z) + \ell_1^2 F(z-x_j) \geqq 0. \tag{4.39}$$

$$\frac{\partial^2 L^{FIFO}(z, x_j)}{\partial z . \partial r} = \ell_0 \int_0^{z-x_j} f(d_{j+1}) F(z-d_{j+1})\, dd_{j+1} + \ell_1 F(z-x_j) \geqq 0. \tag{4.40}$$

□

4.6 Conclusion

In this chapter, an inventory control for perishable commodities subject to stochastic leadtime on both LIFO and FIFO issuing policies has discussed under zero or one unit leadtime. The existence of the optimal ordering policies on both LIFO and FIFO has been clarified and their uniqueness has been proved. Furthermore, the influences of the change of occurrence probability of leadime 1, on-hand inventory and costs of unit holding, shortage and outdating upon the optimal ordering policies have been considered on both LIFO and FIFO policies.

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CHAPTER V

INVENTORY CONTROL FOR DECAYING COMMODITIES SUBJECT TO STOCHASTIC LEADTIME AND STOCKOUT CONSTRAINT

5.1 Introduction

When we discuss the inventory problems for deteriorating items, coping with the stockout due to deteriorations becomes important. There are two types of deterioration, i.e., perishability and decay. The former type of commodities possesses a maximum usable lifetime and the latter does not. As for perishable commodities, optimal ordering policies under the chance constraint of shortages were studied by Ishii et al. [3]. But almost none of works had been concerned with decaying commodities subject to stockout constraint when procurement leadtime varies.

In this chapter, a (Q, r) inventory control system [2] with finite varying stochastic leadtime is considered for exponentially decaying commodities whose decaying ratio during the n-th period to the inventory level at the end of the $(n-1)$th period is equal to a constant.
Characteristics of optimal ordering decision, i.e., optimal ordering quantity and reorder point, are derived under given probability of stockout occurrence and their sensitivities of changes in decay are also clarified.

In addition, the optimal permitted probability of occurring stockout is investigated in order to obtain the optimal ordering decision.

5.2 Preliminaries

The decaying inventory model considered in this chapter is so-called (Q,r) stock control system with variable leadtime for finite range and with a chance constraint for shortages in which a quantity Q is ordered as soon as inventory on hand and on order reaches to a fixed reorder point r.

(1) Leadtime τ varies in the range $[\tau_0, \tau_1]$ and is subject to probability density $g(\tau)$ and distribution function $G(\tau)$.

(2) Probability of stockout occurrence is set not to be greater than $1-\alpha$. $(0 \leq \alpha \leq 1)$

$$Pr(\, I(\tau) \leq 0 \,) \leq 1-\alpha, \tag{5.1}$$

where, $I(\tau)$ is given as the inventory quantity at time τ and initial inventory $I(0)$ is specifed as r.

(3) Maximum inventory level is greater than reorder point.

$$Pr(\, Q + I(\tau) \geq r \,) = 1. \tag{5.2}$$

(4) Commodities considered in this chapter are exponentially decaying.

(5) Demand occurs with constant rate λ.

5.3 Constraints for Reorder Point and Ordering Quantity

When a time t is set and there does not exist any arrivals of orders during $[t, t+\tau)$, the inventory level at $t+\tau$, $I(t+\tau)$, is derived as follows ([1], [4]):

$$I(t+\tau) = e^{-\theta\tau}(I(t) + \lambda/\theta) - \lambda/\theta, \tag{5.3}$$

where θ is decaying ratio and λ is demand rate. Setting $t=0$ in (5.3),

$$I(\tau) = e^{-\theta\tau}(r + \lambda/\theta) - \lambda/\theta,$$

is obtained. Using assumption (2) the next inequality holds.

$$Pr\left(\tau \geq -\frac{1}{\theta}\ln\left(\frac{\lambda/\theta}{r+\lambda/\theta}\right) \right) \leq 1-\alpha. \tag{5.4}$$

The value y which satisfies the relation, $G(y)=\alpha$, is denoted with K_α. Then, the lower bound for reorder point r is derived from (5.4), i.e.,

$$r \geqq (e^{K_\alpha \theta} - 1)\lambda/\theta. \tag{5.5}$$

From assumption (3), the following inequality is obtained.

$$Q \geqq r - e^{-\theta\tau}(r+\lambda/\theta) + \lambda/\theta.$$

Together with (5.5), the lower bound for ordering quantity Q is obtained as follows :

$$Q \geqq e^{K_\alpha \theta}\lambda/\theta(1-e^{-\theta\tau}). \tag{5.6}$$

5.4 Formulation of Total Expected Cost Function

The planning horizon considered in this chapter is defined as duration between a released order point to the next one and its length is denoted with T . Then, the inventory level at time T is represented as follows :

$$\begin{aligned} I(T) &= e^{-\theta(T-\tau)}\{e^{-\theta\tau}(r+\lambda/\theta)+Q\}-\lambda/\theta \\ &= r. \end{aligned} \tag{5.7}$$

From the equation (5.7), T is expressed as the function of lead time :

$$T = \tau + \frac{1}{\theta}\ln\left(e^{-\theta\tau} + \frac{Q\theta}{r\theta+\lambda}\right). \tag{5.8}$$

In order to identify the total expected cost function, stock holding, decaying and shortage costs are constructed in the sequel.

Average inventory holding cost is expressed as follows :

$h(Q-\lambda T)/\theta$,

where h is stock holding cost for each commodity per unit time.

Average decaying cost is defined as follows :

$c(Q-\lambda T)$,

where c is decaying cost for each decaying commodity.

The cost for shortages is given by p when $\alpha=0$ and $(1-\alpha)p$ when α is in the range [0,1].

From the above descriptions, the total expected cost per unit time, $A(Q,r)$, can be defined as follows :

$$A(Q,r)=\int_{\tau_0}^{\tau_1}\left[\frac{K+Q(c+h/\theta)+(1-\alpha)p}{\tau+\frac{1}{\theta}\ln(e^{-\theta\tau}+Q\theta/(\lambda+r\theta))}-\lambda(c+h/\theta)\right]g(\tau)d\tau, \qquad (5.9)$$

where K is a fixed ordering cost.

Next, we define $\hat{Q}$ as follows :

$$\hat{Q}\triangleq\frac{Q\theta}{\lambda+r\theta}\geqq 1-e^{-\theta\tau_1}\geqq 0. \qquad (5.10)$$

Substituting (5.10) into (5.9), $A(\hat{Q}, r)$ is rewritten as follows :

$$A(\hat{Q}, r)=\int_{\tau_0}^{\tau_1}\left[\frac{K+\hat{Q}(r\theta+\lambda)(c+h/\theta)/\theta+(1-\alpha)p}{\tau+\frac{1}{\theta}\ln(e^{-\theta\tau}+\hat{Q})}-\lambda(c+h/\theta)\right]g(\tau)d\tau. \tag{5.11}$$

5.5 Properties on Reorder Point r and Ordering Quantity Q

Differentiating (5.11) with respect to r and using the inequality (5.10), the following proposition is obtained.

Proposition 5.1

$A(\hat{Q}, r)$ is a nondecreasing function with respect to r.

Proof: Differentiating $A(\hat{Q}, r)$ with respect to r, the following relation is obtained :

$$\frac{\partial A(\hat{Q}, r)}{\partial r}=\int_{\tau_0}^{\tau_1}\frac{\hat{Q}(c+h/\theta)}{\tau+\frac{1}{\theta}\ln(e^{-\theta\tau}+\hat{Q})}\cdot g(\tau)d\tau\geqq 0. \tag{5.12}$$

□

Next, differentiating (5.11) with respect to $\hat{Q}$, the equation (5.13) is derived.

$$\frac{\partial A(\hat{Q}, r)}{\partial \hat{Q}}=\int_{\tau_0}^{\tau_1}\left[\frac{(r\theta+\lambda)(c+h/\theta)\{\tau+1/\theta\cdot\ln(e^{-\theta\tau}+\hat{Q})\}}{\theta\left\{\tau+\frac{1}{\theta}\ln(e^{-\theta\tau}+\hat{Q})\right\}^2}-\frac{\dfrac{K+1/\theta\cdot\hat{Q}(r\theta+\lambda)(c+h/\theta)+(1-\alpha)p}{e^{-\theta\tau}+\hat{Q}}}{\theta\{\tau+1/\theta\cdot\ln(e^{-\theta\tau}+\hat{Q})\}^2}\right]g(\tau)d\tau. \tag{5.13}$$

The numerator of (5.13) is represented by $B(\hat{Q})$, that is,

$$B(\hat{Q}) = (r\theta+\lambda)(c+h/\theta)(\tau+1/\theta\cdot\ln(e^{-\theta\tau}+\hat{Q})) - \frac{K+1/\theta\cdot\hat{Q}(r\theta+\lambda)(c+h/\theta)+(1-\alpha)p}{e^{-\theta\tau}+\hat{Q}} \qquad (5.14)$$

Differentiating (5.14) with respect to $\hat{Q}$,

$$\frac{dB(\hat{Q})}{d\hat{Q}} = (r\theta+\lambda)(c+h/\theta)\frac{\hat{Q}}{\theta(e^{-\theta\tau}+\hat{Q})^2} + \frac{\hat{Q}}{(e^{-\theta\tau}+\hat{Q})^2} > 0, \qquad (5.15)$$

holds and the following proposition is obtained.

Proposition 5.2

$A(\hat{Q}, r)$ is either monotone increasing function with respect to $\hat{Q}$ or the function whose first partial derivative with respect to $\hat{Q}$ changes its sign once from minus to plus. That is, $A(\hat{Q}, r)$ is pseudo-convex with respect to $\hat{Q}$. □

Substituting (5.10) into (5.14),

$$B(1-e^{-\theta\tau_1}) \geqq (r\theta+\lambda)(c+h/\theta)\tau - \{K+(1-\alpha)p+1/\theta\cdot(1-e^{-\theta\tau_1})(r\theta+\lambda)(c+h/\theta)\},$$

and using (5.5), the lower bound for $B(\hat{Q})$ is obtained as follows :

$$B(\hat{Q}) \geqq e^{K_0\theta}(c+h/\theta)\{\tau - 1/\theta\cdot(1-e^{-\theta\tau_1})\} - \{K+(1-\alpha)p\}. \qquad (5.16)$$

In order to obtain the optimal ordering policy, we could consider two cases from (5.16).

(I) Case I: $e^{K_\alpha\theta}(c+h/\theta)\{\tau-1/\theta\cdot(1-e^{-\theta\tau_1})\}-\{K+(1-\alpha)p\}\geqq 0$, for any $\tau\in[\tau_0, \tau_1]$.

From (5.10), $\tilde{Q}$ is set to be the value of the lower bound, $1-e^{-\theta\tau_1}$, and

$$Q=1/\theta\cdot(1-e^{-\theta\tau_1})(\lambda+r\theta), \tag{5.17}$$

is obtained. Since the optimal reorder point, $r^*=\lambda/\theta\cdot(e^{K_\alpha\theta}-1)$, is gained from Proposition 1, the optimal ordering quantity, Q^*, could be derived as follows :

$$Q^*=\lambda/\theta\cdot(1-e^{-\theta\tau_1})e^{K_\alpha\theta}.$$

(II) Case II: $e^{K_\alpha\theta}(c+h/\theta)\{\tau-1/\theta\cdot(1-e^{-\theta\tau_1})\}-\{K+(1-\alpha)p\}<0$, for some $\tau\in[\tau_0, \tau_1]$.

Then also, there exists a unique solution for $\tilde{Q}$. That is, either $Q^*=Q$ so as to $\partial A(\tilde{Q},r)/\partial\tilde{Q}=0$ or Q^* which is again given by (5.17).

5.6 Properties on the Probability of Stockout Occurrence

In this section, the probability of stockout occurrence, $(1-\alpha)$, is considered with respect to the Case I. Substituting (5.17) into (5.11) and transforming by $q=G^{-1}(\alpha)$ for $\alpha\geq 1/2$, $A(\tilde{Q}, r)$ is rewritten as follows:

$$\hat{A}(q)=\int_{\tau_0}^{\tau_1}\frac{K+(c+h/\theta)(1-e^{-\theta\tau_1})e^{q\theta}/\theta+(1-G(q))p}{\tau+1/\theta\cdot\ln(e^{-\theta\tau}+1-e^{-\theta\tau_1})}g(\tau)\,d\tau-\lambda(c+h/\theta). \tag{5.18}$$

Then, first and second derivatives with respect to q are obtained as follows :

$$\frac{d\hat{A}(q)}{dq}=\int_{\tau_0}^{\tau_1}\frac{(c+h/\theta)(1-e^{-\theta\tau_1})e^{q\theta}-pg(q)}{\tau+1/\theta\cdot\ln(e^{-\theta\tau}+1-e^{-\theta\tau_1})}g(\tau)\,d\tau. \tag{5.19}$$

$$\frac{d^2\hat{A}(q)}{dq^2}=\int_{\tau_0}^{\tau_1}\frac{\theta(c+h/\theta)(1-e^{-\theta\tau_1})e^{q\theta}-pg'(q)}{\tau+1/\theta\cdot\ln(e^{-\theta\tau}+1-e^{-\theta\tau_1})}g(\tau)\,d\tau. \tag{5.20}$$

Combining the above equations (5.18) to (5.20), the following proposition is obtained.

Proposition 5.3

When lead time density $g(q)$ is unimodal and $g'(q) \leq 0$ is satisfied under the assumption $G(q) \geq 1/2$, $d^2\hat{A}(q)/dq^2 > 0$ holds. That is, $\hat{A}(q)$ is convex function with respect to q and the optimal value of q is obtained uniquely.

5.7 Properties of Decaying Ratio

As for Case I, the influences of decaying ratio θ, upon optimal reorder point r^* and optimal ordering quantity Q^* are considered.

From (5.5), optimal reorder point $r^*(\theta)$ is set as follows :

$$r^*(\theta) \triangleq (e^{K_a\theta} - 1)\lambda/\theta.$$

And the first derivative of $r^*(\theta)$ with respect to θ is obtained as follows:

$$\frac{dr^*(\theta)}{d\theta} = \frac{\lambda}{\theta^2}\{e^{K_a\theta}(K_a\theta - 1) + \lambda\}. \tag{5.21}$$

From (5.6), optimal ordering quantity $Q^*(\theta)$ is set as follows :

$$Q^*(\theta) \triangleq e^{K_a\theta}(1 - e^{-\theta\tau_1})\lambda/\theta.$$

Then the first derivative of $Q^*(\theta)$ with respect to θ is obtained as follows :

$$\frac{dQ^*(\theta)}{d\theta} = e^{K_a\theta}\frac{\lambda}{\theta}(1 - e^{-\theta\tau_1})\left(K_a - \frac{1}{\theta}\right) + \frac{\lambda}{\theta}\tau_1 e^{-\theta\tau_1} e^{K_a\theta}. \tag{5.22}$$

Combining (5.21) with (5.22), the following proposition is obtained.

Proposition 5.4

Optimal reorder point $r^*(\theta)$ and optimal order quantity $Q^*(\theta)$ have the following property with respect to decaying ratio θ .

(I) property of $r^*(\theta)$:

$\theta \geq 1/K_a$; increasing,

$\begin{cases} \theta < 1/K_a \\ \exp(K_a\theta)[K_a\theta - 1] + 1 > 0 \end{cases}$; increasing,

$\begin{cases} \theta < 1/K_a \\ \exp(K_a\theta)[K_a\theta - 1] + 1 \leq 0 \end{cases}$; decreasing.

(II) property of $Q^*(\theta)$: $\theta \geq 1/K_a$; increasing,

$\begin{cases} \theta < 1/K_a \\ [1 - \exp(-\theta\tau_1)](K_a - 1/\theta) + \tau_1 \exp(-\theta\tau_1) > 0 \end{cases}$; increasing,

$\begin{cases} \theta < 1/K_a \\ [1 - \exp(-\theta\tau_1)](K_a - 1/\theta) + \tau_1 \exp(-\theta\tau_1) \leq 0 \end{cases}$; decreasing.

5.8 Conclusion

In this chapter, a (Q, r) inventory control system with finite varying stochastic leadtime was discussed with respect to exponentially decaying commodities. Characteristics of optimal ordering decision, i.e., optimal ordering quantity and reorder point, were derived under given probability of stockout occurrence.

In addition, the optimal permitted probability of stockout occurrence was investigated in order to obtain the more effective optimal ordering decision.

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CHAPTER VI

ALLOCATION PROBLEM FOR PERISHABLE COMMODITIES

6.1 Introduction

We consider perishable commodities which become obsolete and cannot be used after a certain period of time ([1] [2] [3] [4]). Optimal allocation policies for perishable commodities were analyzed first by Prastacos [5], [6], [7]. These models were discussed under charging cost only for shortage and outdating. But, if we discuss the allocation problem, transportation cost should be also an important factor.

In this chapter, we discuss a single period allocation problem of perishable commodities for several fixed demand points based on an LIFO issuing and a rotation allocation policy considering shortage, outdating and transportation costs. Blood, photographic film and fresh commodities are typical examples of our model. Section 6.2 gives some assumptions and notations used throughout this chapter. Section 6.3 formulates the problem and clarifies the existence of the optimal allocation policy. Section 6.4 presents an algorithm for obtaining its optimal solution. In addition, we give an example in section 6.5 in order to examine the influences of shortage, outdating and transportation costs upon the optimal allocation policy.

6.2 Preliminaries

A periodic review inventory model is considered, for one planning horizon and single item. That is, ordering takes place at the start of a period and costs are incurred during a period. The period length is arbitrary but fixed. And the following assumptions are made to construct the model.

(1) Maximum lifetime of the perishable commodities is fixed and equal to m periods.

(2) Inventory is depleted by demand at the start of each period according to a LIFO issuing policy.

(3) There are n locations where demands occur.

(4) The stock remaining at the end of each period ages one period monotonously and is returned from each location to regional center. (rotation policy)

(5) If the commodities have not been depleted by demand until the period it reaches age m , then it perishes and must be discarded.

(6) Costs are charged at each location for every unit short, perish or transported.

(7) Demands at each location are independent random variables with known distribution $F_k(\cdot)$ and density $f_k(\cdot)$ $(k=1,2, \quad ,n)$.

(8) The following notations are used throughout this chapter.

s_k : shortage cost per unit of demand unfilled at location k

w_k : perishing cost per unit perished at location k

u_k : transportation cost per unit shipped from regional center to location k or returned from location k to regional center

N_k : allocated quantity not subject to outdating at location k (age 0 to age $m-2$)

B_k : allocated quantity subject to outdating at location k (age $m-1$)

$$\mathrm{N} = (N_1, N_2, \ldots, N_n)$$

$$\mathrm{B} = (B_1, B_2, \ldots, B_n)$$

$$B = \sum_{k=1}^{n} B_k$$

$$N = \sum_{k=1}^{n} N_k$$

6.3 LIFO Allocation Model

When newer units N and older units B are prepared, the total expected cost $C(\mathrm{N},\mathrm{B})$ is given by

$$C(\mathrm{N},\mathrm{B}) = \sum_{k=1}^{n} [s_k \int_{N_k+B_k}^{\infty} (x-N_k-B_k)\,dF_k(x) + w_k \int_{N_k}^{N_k+B_k} (N_k+B_k-x)\,dF_k(x)$$

$$+ w_k B_k F_k(N_k) + (N_k+B_k)u_k + u_k \int_0^{N_k} (N_k-x)\,dF_k(x)] \tag{6.1}$$

and we have

$$\frac{\partial C(\mathrm{N},\mathrm{B})}{\partial B_k} = (s_k+w_k)F_k(N_k+B_k)-s_k+u_k \,. \tag{6.2}$$

$$\frac{\partial^2 C(\mathrm{N},\mathrm{B})}{\partial B_k^2} = (s_k+w_k)f_k(N_k+B_k) \geqq 0 \tag{6.3}$$

$$\lim_{B_k \to \infty} \frac{\partial C(\mathrm{N},\mathrm{B})}{\partial B_k} = w_k+u_k > 0 \,.$$

$$\frac{\partial C(\mathrm{N},\mathrm{B})}{\partial N_k} = (s_k+w_k)F_k(N_k+B_k)-s_k+u_k+ (u_k-w_k)F_k(N_k) \,. \tag{6.4}$$

$$\frac{\partial^2 C(\mathrm{N},\mathrm{B})}{\partial N_k^2} = (s_k+w_k)f_k(N_k+B_k) + (u_k-w_k)f_k(N_k) \geqq 0 \tag{6.5}$$

where, $u_k > w_k$ is assumed throught this chapter for each k .

$$\lim_{N_k \to \infty} \frac{\partial C(N,B)}{\partial N_k} = 2u_k > 0$$

$$\begin{vmatrix} \frac{\partial^2 C(N,B)}{\partial N_k^2} & \frac{\partial^2 C(N,B)}{\partial N_k \partial B_k} \\ \frac{\partial^2 C(N,B)}{\partial B_k \partial N_k} & \frac{\partial^2 C(N,B)}{\partial B_k^2} \end{vmatrix} = (u_k - w_k)(s_k+w_k) f_k(N_k) f_k(N_k+B_k) \geq 0 \qquad (6.6)$$

Together the above inequalities (6.3), (6.5), (6.6) and the property of convexity, the following proposition is obtained.

Proposition 6.1

Total expected cost function $C(N,B)$ is convex with respect to N_k and B_k.

6.4 Algorithm for Optimal Allocation Policy

Introducing Lagrange multipliers λ and μ into equation (6.1), the following Lagrange function $\phi(N,B)$ is given.

$$\phi(N,B) = C(N,B) - \lambda \left(\sum_{k=1}^{n} B_k - B\right) - \mu \left(\sum_{k=1}^{n} N_k - N\right) . \qquad (6.7)$$

Then, we have

$$\frac{\partial \phi(N,B)}{\partial B_k} = (s_k+w_k) F_k(N_k+B_k) - s_k + u_k - \lambda . \qquad (6.8)$$

$$\frac{\partial \phi(N,B)}{\partial N_k} = (s_k+w_k) F_k(N_k+B_k) - s_k + u_k - \mu + (u_k - w_k) F_k(N_k) . \qquad (6.9)$$

An algorithm is developed to solve the following Kuhn-Tucker conditions with respect to N_k and B_k, $k=1, \ldots, n$.

$$\text{(I)} \quad \sum_{k=1}^{n} B_k = B, \quad \sum_{k=1}^{n} N_k = N$$

$$\text{(II)} \quad \frac{\partial\phi(N,B)}{\partial B_k} \geqq 0, \quad \frac{\partial\phi(N,B)}{\partial N_k} \geqq 0$$

$$\text{(III)} \quad B_k \frac{\partial\phi(N,B)}{\partial B_k} = 0, \quad N_k \frac{\partial\phi(N,B)}{\partial N_k} = 0$$

Algorithm

First, n locations are arranged and serially numbered as in the following ,i.e.,

$$u_1-s_1 \leqq u_2 - s_2 \leqq \ldots, \leqq u_n - s_n, \text{ and set,}$$

$$I_i = (u_i-s_i, u_{i+1}-s_{i+1}], \ i=1, \ldots, n$$

$$I_0 \triangleq (-\infty, u_1-s_1], \ u_{n+1}-s_{n+1} \triangleq +\infty$$

Then,our algorithms are given as follows by (i) $\sim$ (x).

(i) If $B=0$, execute (ii) and if $B\neq 0$, execute (v).

(ii) Set $i \leftarrow 0$. Then, arrange n locations and put serial numbers numbered as stated above .

(iii) Set $\mu \in I_i$. And the followings are obtained.

- $N_\ell=0$: $\ell=i+1, \ldots, n$.
- $N_\ell=F_\ell^{-1}\left(\frac{s_\ell-u_\ell+\mu}{s_\ell+u_\ell}\right)$: $\ell=1, \ldots, i$.

If $\Sigma_\ell F_\ell^{-1}(\frac{s_\ell - u_\ell + u_i - s_i}{s_\ell + u_\ell}) \leq N$ and $\Sigma_\ell F_\ell^{-1}(\frac{s_\ell - u_\ell + u_{i+1} - s_{i+1}}{s_\ell + u_\ell}) > N$, there exists μ_0 within I_i so that $\Sigma_{\ell=1}^{n} N_\ell = N$ and terminates the algorithm with $N_\ell (\ell=1,\ldots,n)$. Otherwise, execute (iv).

(iv) Set $i \leftarrow i+1$ and execute (iii).

(v) Set $i \leftarrow 1$.

(vi) Set $\lambda \in I_i$, $\mu \in I_i$. Then,

$B_\ell = 0$; $\ell = i+1, \ldots, n$ is obtained.

From the equation (6.8),

$$N_p + B_p = F_p^{-1}(\frac{s_p - u_p + \lambda_0}{s_p + w_p}) \; ; \; p=1, 2, \ldots, i$$

is obtained.

Introducing $B_p = 0$ into the equation (6.9),

$$N_p = F_p^{-1}(\frac{s_p - u_p + \mu_0}{s_p + u_p}) \; ; \; p=1, 2, \ldots, i$$

is gained

Two kinds of sets, J and K, are defined :

$$J \triangleq \{\text{location } p \mid p=1, \ldots, i, \; B_p \neq 0\}$$

$$K \triangleq \{\text{location } p \mid p=1, \ldots, i, \; B_p = 0\} \; .$$

Put each location number p $(=1, \ldots, i)$ into either type of sets K or J respectively and if both λ_0 and μ_0 exist within I_i so that

$$\sum_{p=1}^{i} (N_p + B_p) = \sum_{p \in J} F_p^{-1}(\frac{s_p - u_p + \lambda_0}{s_p + w_p}) + \sum_{p \in K} F_p^{-1}(\frac{s_p - u_p + \mu_0}{s_p + u_p}) = N + B$$

$$\sum_{p=1}^{i} N_p = \sum_{p\varepsilon J} F_p^{-1}\left(\frac{\mu_0-\lambda_0}{u_p-w_p}\right) + \sum_{p\varepsilon K} F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) = N,$$

then execute (ix).

Otherwise, set $q \leftarrow i+1$ and execute (vii).

(vii) Put each location $p = 1, \ldots, i$ into either type of sets K or J respectively and if λ_0 and μ_0 exist within I_i and I_q respectively so that

$$\sum_{p=1}^{q} N_p = \sum_{p\varepsilon J} F_p^{-1}\left(\frac{\mu_0-\lambda_0}{u_p-w_p}\right) + \sum_{p\varepsilon K}^{i} F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) + \sum_{p=i+1}^{q} F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) = N,$$

$$\sum_{p=1}^{i} (N_p+B_p) + \sum_{p=i+1}^{q} N_p = \sum_{p\varepsilon J} F_p^{-1}\left(\frac{s_p-u_p+\lambda_0}{s_p+w_p}\right) + \sum_{p\varepsilon K} F_p^{-1}\left(\frac{w_p-u_p+\mu_0}{s_p+u_p}\right)$$

$$+ \sum_{p=i+1}^{q} F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) = N+B,$$

and if $\dfrac{s_p-u_p+\lambda_0}{s_p+w_p} \geq \dfrac{\mu_0-\lambda_0}{u_p-w_p}$ $(p=1, 2, \ldots i)$,

then the optimal solution is obtained as follows :

$$B_p = F_p^{-1}\left(\frac{s_p-u_p+\lambda_0}{s_p+w_p}\right) - F_p^{-1}\left(\frac{\mu_0-\lambda_0}{u_p-w_p}\right) : p=1, \ldots, i \ (p\varepsilon J),$$

$$B_p = 0 : p=1, \ldots, i \ (p\varepsilon K),$$

$$N_p = F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) : p=i+1, \ldots, q,$$

$$N_p = F_p^{-1}\left(\frac{\mu_0-\lambda_0}{u_p-w_p}\right) : p=1, \ldots, i \ (p\varepsilon J) \quad N_p = F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) :$$

$p=1, \ldots, i$ $(p\varepsilon K)$.

Otherwise, execute (viii).

(viii) Set $q \leftarrow q+1$. If $q=n+1$, execute (x). Otherwise, execute (vii).

(ix) If

$$\frac{s_p-u_p+\lambda_0}{s_p+w_p} \geqq \frac{\mu_0-\lambda_0}{u_p-w_p} \quad ,$$

then the optimal solution is obtained as follows :

$$N_p=F_p^{-1}\left(\frac{\mu_0-\lambda_0}{u_p-w_p}\right) \quad : p=1, \ldots, i \ (p\varepsilon J), \ N_p=F_p^{-1}\left(\frac{s_p-u_p+\mu_0}{s_p+u_p}\right) : p=1,\cdots,i \ (p\varepsilon k),$$

$$B_p=F_p^{-1}\left(\frac{s_p-u_p+\lambda_0}{s_p+w_p}\right) - F_p^{-1}\left(\frac{\mu_0-\lambda_0}{u_p-w_p}\right) \quad : p=1, \ \ldots, \ i \ (p\varepsilon J),$$

$$B_p=0 \ : \ p=1, \ \ldots, \ i \ (p\varepsilon K),$$

$$N_p=0 \ : \ p=i+1, \ \ldots, \ n ,$$

$$B_p=0 \ : \ p=i+1, \ \ldots, \ n .$$

Otherwise, set $q \leftarrow i+1$ and execute (vii).

(x) If $i < n+1$, then set $i \leftarrow i+1$ and execute (vi).

Proposition 6.2

The increase of perishing cost w_k incurs the decrease of N_k+B_k.

(Proof)

From the equation (6.8),

$$N_k + B_k = F_k^{-1}\left(\frac{s_k-u_k+\lambda}{s_k+w_k}\right)$$

is obtained. With respect to the above equation, when w_k increases, $N_k + B_k$ decreases. If $N_k + B_k$ remains constant, we should increase λ. While λ increases, $N_\ell + B_\ell$ ($\ell \neq k$) increase at other locations. This causes $\sum_{i=1}^{n} (N_i + B_i) > N+B$. So, we must decrease $N_k + B_k$ instead of increasing λ.

Proposition 6.3

The increase of shortage cost s_k makes $N_k + B_k$ increase.

Proof:

From the equation (6.8), this proposition could be proved by the similar way to prove the proposition 6.2.

6.5 Numerical Example

This section provides an example in order to illustrate the results of Section 6.3 by the use of the proposed algorithm described in Section 6.4. The demand distribution functions at each location are the identical uniform distribution from 0 to 10. In this example, we consider n=3 locations and N=6 , B=2 .

The cost parameters(s_k, w_k, u_k)at location k are given in Table 6.1. Table 6.2 shows the optimal allocation policy and its total expected cost. Table 6.3 shows the influence of shortage cost s_ℓ upon the optimal allocation policy at location 1 in order to illustrate the proposition 6.3.

In this table, cost parameters are fixed except s_ℓ in Table 6.1. Table 6.4 shows the influence of transportaion cost u_ℓ upon the optimal allocation policy at location 1. In this table, cost parameters are fixed except u_ℓ in Table 6.1. Table 6.5 shows the influence of outdating cost w_ℓ upon the optimal allocation policy at location 1 in order to illustrate the proposition 6.2. In this table, cost parameters are fixed except w_ℓ in Table 6.1.

Table 6.1 Cost parameters

k \ Cost	s_k	u_k	w_k
1	5	10	5
2	10	15	5
3	15	20	5

Table 6.2 Optimal allocation policy and its total expected cost

location k	N	B	Cost
1	2.28169	2.0	
2	2.16901	0	210.549
3	1.54930	0	

Table 6.3 Influence of shortage cost upon the optimal allocation policy

s_ℓ	$N_\ell + B_\ell$
2	2.82132
3	3.08157
4	3.32361
5	4.28169
6	4.46866
7	4.64380
8	4.80818
9	4.96278

Table 6.4 Influence of transportation cost upon the optimal allocation policy

u_ℓ	$N_\ell + B_\ell$
6	6.20195
7	5.66771
8	5.17221
9	4.71137
10	4.28169
11	3.10627
12	2.69129
13	2.30179

Table 6.5 Influence of cost per unit perished item upon the optimal allocation policy

w_ℓ	$N_\ell + B_\ell$
2	4.48451
3	4.41690
4	4.34930
5	4.28169
6	4.21408
7	4.14648
8	4.07887
9	4.01162

6.6 Conclusion

In this chapter, the model for a single period allocation problem of perishable commodities based on an LIFO issuing policy has been formulated under the rotation allocation policy and the existence of the optimal allocation policy has been proved. An algorithm for obtaining its optimal allocation policy has been proposed. Furthermore, a numerical example is given for investigating the influences of shortage, transportation, and perishing cost, upon the optimal allocation policy.

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CHAPTER VII

CONCLUSION

We have discussed some structures and properties of mathematical models to describe the optimal ordering policies for perishable inventories. In this chapter, we summarize the main results of the dissertation.

Chapter II has been devoted specifically to examine inventory model for perishable commodities under the stochastic leadtime with mixed ages. Particularly the cases of zero and one period leadtime have been considered and the optimal ordering policies have been derived. Furthermore, the effect of certain important factors on the optimal ordering policies have been analyzed. Next, considering the fixed ordering cost, some other characteristics on the ordering policies have been obtained.

In Chapter III, we discussed the determination of the optimal ordering policies for perishable commodities with predetermined maximum lifetime under the different prices and stochastic leadtime. Some properties of this model such as the existence of the optimal ordering policies and conditions thereof, the influences of the rate of excess perishability and the status of inventory on hand upon the optimal ordering policies were analyzed. Furthermore, a fixed charge cost for placing an order was introduced as a set up cost and then some additional characteristics were obtained.

In Chapter IV, we discussed an inventory control for perishable commodities subject to stochastic leadtime on both LIFO and FIFO issuing policies under zero or one unit leadtime. The existence of the optimal ordering policies on both LIFO and FIFO has been confirmed and the uniqueness has been proved. Furthermore, the influence of the changes of occurrence probability of leadime 1, inventory on hand and inventory relevant costs upon the optimal ordering policies have been considered on both LIFO and FIFO policies.

In Chapter V, the inventory control system with finite varying stochastic leadtime was discussed in case of exponentially decaying commodities. Characteristics of the optimal ordering decision, i.e., optimal ordering quantity and reorder point were also derived under the given stockout probability.

In addition, the optimal admissible probability of stockout occurrence was investigated in order to obtain the more effective optimal ordering decision.

In Chapter VI, the model for a single period allocation problem of perishable commodities based on an LIFO issuing policy has been formulated under the rotatable allocation policy and the existence of the optimal allocation policy is proved. Furthermore, a numerical example was given and investigated the influences of shortage, transportation, and perishing costs were investigated upon the optimal allocation policy.

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