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# A LAW OF THE ITERATED LOGARITHM OF CHOVER TYPE FOR MULTIDIMENSIONAL LÉVY PROCESSES

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**Abstract**

An integral test for some multidimensional Lévy processes is given. A law of the iterated logarithm of Chover type is derived from it as a corollary. This law was pointed out by Chover in the case of one-dimensional symmetric stable processes with discrete time.

## 1. Introduction and result

Limit theorems have always been of central importance in probability theory. Among them we study the law of the iterated logarithm (abbreviated to LIL). The most fundamental result on the LIL for Lévy processes is the LIL for the Brownian motion $\{B_t : t \geq 0\}$ on $\mathbb{R}^1$. It is proved by Khintchine [6] in the following form:

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad \text{a.s.}$$

It still holds for $|B_t|$ in place of $B_t$, that is, it holds that

$$\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t\log\log t}} = 1 \quad \text{a.s.},$$

and this also holds for the multidimensional Brownian motion. However, even if we only consider the one-dimensional case, we cannot extend the LIL for the Brownian motion to that for strictly $\alpha$-stable processes $\{X_t : t \geq 0\}$. The reason is that we have

$$\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{2t\log\log t}} = \infty \quad \text{a.s.}$$

So Chover [3] paid attention to the classical LIL as follows: Let $Z_k$ $(k = 1, 2, 3, \ldots)$ be independent identically distributed on $\mathbb{R}^1$. Set $S_n = \sum_{k=1}^n Z_k$. If $E\exp(iuZ_1) = e^{-u^2}$ for all $u \in \mathbb{R}^1$, then it follows that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1 \quad \text{a.s.}$$

In this case the variables $S_n/\sqrt{n}$ satisfy $E\exp(iuS_n/\sqrt{n}) = e^{-u^2}$ again, so he thought that these variables must be cut down additionally by the factors $\sqrt{2\log\log n}^{-1}$ to achieve a finite limsup. In the case where the classical LIL does not hold, he expected the "cut down factors" $\sqrt{2\log\log n}^{-1}$ to appear otherwise, and proved the following: If $E\exp(iuZ_1) = e^{-|u|^\alpha}$ for some $\alpha$ with $0 < \alpha < 2$, then it follows that

$$\limsup_{n\to\infty}\left(\frac{|S_n|}{n^{1/\alpha}}\right)^{1/\log\log n} = e^{1/\alpha} \quad \text{a.s.}$$

Keeping this result in our mind, we consider the LIL of Chover type (for reference, see [12]). So we deduce an integral test to obtain the limsup behavior of multidimensional Lévy processes $\{X_t : t \geq 0\}$. And we derive the LIL of Chover type from its result. In the case of the multidimensional non-Gausian Lévy processes, we know a few result concerning $\limsup_{t\to\infty}|X_t|/h(t)$ for increasing positive functions $h(t)$. We only know Pruitt's result [9], which investigates $\limsup_{t\to\infty}|X_t|/t^{1/\eta}$ for $\eta > 0$ (see Remark 1.5). Here we get back to the LIL of Chover type. Its form is singular, but the classical LIL can be modified in a similar form, that is to say that its form is no rare. For example, in the case of the Brownian motion, we can get

$$\limsup_{t\to\infty}\left(\frac{|B_t|}{\sqrt{t}}\right)^{1/\log\log\log t} = e^{1/2} \quad \text{a.s.}$$

And, for example, in the case of strictly $\alpha$-stable subordinators $\{X_t : t \geq 0\}$ with $0 < \alpha < 1$, we know that

$$\liminf_{t\to\infty}\frac{X_t}{t^{1/\alpha}(\log\log t)^{-(1-\alpha)/\alpha}} = c \quad \text{a.s.,}$$

where $c$ is a finite positive constant (for example, see [2]). Hence, as we have $X_t \geq 0$ a.s., it follows that

$$\liminf_{t\to\infty}\left(\frac{X_t}{t^{1/\alpha}}\right)^{1/\log\log\log t} = e^{-(1-\alpha)/\alpha} \quad \text{a.s.}$$

In general, for non-trivial strictly $\alpha$-stable processes $\{X_t : t \geq 0\}$ on $\mathbb{R}^1$ with $0 < \alpha < 2$, we know an integral test, which shows that the LIL of Chover type holds (see the proof of Corollary 1.2 below). Its integral test was first given by Khintchine [7], and it implies that

$$\limsup_{t\to\infty}\frac{|X_t|}{t^{1/\alpha}h(t)} = 0 \quad \text{a.s.} \quad \text{or} \quad = \infty \quad \text{a.s.}$$

according as

$$\int_1^\infty \frac{dt}{th(t)^\alpha} < \infty \quad \text{or} \quad = \infty,$$

if $h(t)$ is increasing and $\lim_{t\to\infty} h(t) = \infty$. This fact can be shown by Yamamuro's result [15] for the case where $\rho((-\infty, 0)) > 0$ and $\rho((0,\infty)) > 0$, and a similar result is reported by Fristedt [4] for $\rho((0,\infty)) > 0$. Here $\rho$ is the Lévy measure of $\{X_t\}$. Our theorem is an extension of this fact.

We denote the inner product and the norm in $\mathbb{R}^d$ by $\langle x, y\rangle = \sum_{j=1}^d x_j y_j$ and $|x| = \sqrt{\langle x, x\rangle}$, respectively, for $x = (x_j)_{1\le j\le d}$ and $y = (y_j)_{1\le j\le d}$. By a Lévy process we mean a stochastically continuous process with stationary independent increments starting at the origin such that its sample functions are almost surely right-continuous with left limits. Throughout our paper we suppose that the process $\{X_t : t \ge 0\}$ is a Lévy process on $\mathbb{R}^d$. The characteristic function of $X_1$ is represented as

$$(1.1)\quad E[e^{i\langle z, X_1\rangle}] = \exp\left[-2^{-1}\langle z, Az\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|<1\}}(x)\right)\rho(dx) + i\langle z, \gamma\rangle\right],$$

where $A$ is a symmetric and nonnegative definite real $d\times d$ matrix, $\rho$ is a measure on $\mathbb{R}^d$ satisfying $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}^d}\min\{|x|^2, 1\}\rho(dx) < \infty$, and $\gamma \in \mathbb{R}^d$. Here $1_D$ is the indicator function of a set $D$. Define the function $\rho^*(u)$ for $u \in (0,\infty)$ by

$$\rho^*(u) = \rho\left(\{x \in \mathbb{R}^d : |x| > u\}\right).$$

We use the word "increase" in the wide sense in this paper. Our main theorem is the following.

**Theorem 1.1.** *Let $h(t)$ be an increasing positive function on $[c,\infty)$ with some $c > 0$, and let $\lim_{t\to\infty} h(t) = \infty$.*
(i) *If there exists a constant $\alpha$ with $0 < \alpha \le 1$ such that*

$$(1.2)\qquad 0 < \liminf_{u\to\infty} u^\alpha\rho^*(u) \le \limsup_{u\to\infty} u^\alpha\rho^*(u) < \infty$$

*and if*

$$(1.3)\qquad \int_c^\infty \frac{dt}{th(t)^\alpha} < \infty \quad (\textit{resp.} = \infty),$$

*in addition, and if we have $h(t)/\log t \to \infty$ as $t\to\infty$ for $\alpha = 1$, then we have*

$$(1.4)\qquad \limsup_{t\to\infty}\frac{|X_t|}{t^{1/\alpha}h(t)} = 0 \quad (\textit{resp.} = \infty) \quad \textit{a.s.}$$

(ii) *If* (1.2) *is satisfied for some $\alpha$ with $1 < \alpha < 2$ and if* (1.3) *is satisfied, then we have*

$$(1.5)\qquad \limsup_{t\to\infty}\frac{|X_t - tEX_1|}{t^{1/\alpha}h(t)} = 0 \quad (\textit{resp.} = \infty) \quad \textit{a.s.}$$

By using Theorem 1.1 we obtain the LIL of Chover type for multidimensional Lévy processes as follows.

**Corollary 1.2.** *If* (1.2) *is satisfied for some* $\alpha$ *with* $0 < \alpha \leq 1$, *or if* (1.2) *is satisfied for some* $\alpha$ *with* $1 < \alpha < 2$ *and* $EX_1 = 0$, *then we have*

$$\limsup_{t\to\infty}\left(\frac{|X_t|}{t^{1/\alpha}}\right)^{1/\log\log t} = e^{1/\alpha} \quad a.s. \tag{1.6}$$

EXAMPLE 1.1. Our theorem shows a very interesting property for some one-dimensional Lévy process: Let the characteristic function of $X_1$ be represented as

$$E[e^{izX_1}] = \exp\left[\int_0^\infty \left(e^{izx} - 1 - izx(1+x^2)^{-1}\right)x^{-2}\,dx\right].$$

Then we have $\liminf_{t\to\infty} X_t/(t\log t) = 1$ a.s. by Proposition 4 in [8]. And, as we have $EX_1 = \infty$, it follows that $\lim_{t\to\infty} X_t/t = \infty$ a.s. from Theorem 36.6 in [10]. Hence we have $X_t \geq 0$ for any sufficiently large $t$ a.s. We can obtain that $\liminf_{t\to\infty}(|X_t|/t)^{1/\log\log t} = e$ a.s. Therefore, from Corollary 1.2, we have

$$\lim_{t\to\infty}\left(\frac{|X_t|}{t}\right)^{1/\log\log t} = e \quad \text{a.s.} \tag{1.7}$$

REMARK 1.2. Let $\{X_t\}$ be non-trivial and strictly $\alpha$-stable with $0 < \alpha < 2$. Then (1.2) is satisfied for this $\alpha$. Here we note that $EX_1 = 0$ if $1 < \alpha < 2$. Hence our theorem and corollary apply.

REMARK 1.3. There are examples such that $\limsup_{t\to\infty} |X_t|/(t\log t)^{1/\alpha} = \infty$ a.s., though (1.6) holds. Such examples are one-dimensional strictly $\alpha$-stable subordinators with $0 < \alpha < 1$. Indeed, as we have $\int_2^\infty (t\log t)^{-1}\,dt = \infty$, it follows that $\limsup_{t\to\infty} |X_t|/(t\log t)^{1/\alpha} = \infty$ a.s. from Theorem 1.1.

REMARK 1.4. Suppose that (1.2) is satisfied for some $\alpha$ with $1 < \alpha < 2$. Then it automatically follows that $\int_{|x|\geq 1} |x|\rho(dx) < \infty$. Hence the characteristic function of $X_t - tEX_1$ is represented as

$$E[e^{i\langle z, X_t - tEX_1\rangle}] = \exp\left[t\left(-2^{-1}\langle z, Az\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle)\rho(dx)\right)\right]. \tag{1.8}$$

REMARK 1.5. Let the condition (1.2) be satisfied for some $\alpha$ with $0 < \alpha < 2$. Then we have

$$\limsup_{t\to\infty}\frac{|X_t|}{\sqrt{2t\log\log t}} = \infty \quad \text{a.s.} \tag{1.9}$$

Indeed, if the condition (1.2) be satisfied for some $\alpha$ with $0 < \alpha < 2$, we have

$$\alpha = \sup \left\{ \eta \in [0, 2] : \int_{|x|>1} |x|^\eta \rho(dx) < \infty \right\}.$$

Here we note that $\int_{|x|>1} |x|^\alpha \rho(dx) = \infty$. Now $X_t$ has the decomposition $X_t = Y_t + Z_t$, where $\{Y_t\}$ and $\{Z_t\}$ are two Lévy processes such that $Ee^{i\langle z, Y_1\rangle} = \exp[-2^{-1}\langle z, Az\rangle]$ and $Ee^{i\langle z, Z_1\rangle} = \exp\left[\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|<1\}}(x))\rho(dx) + i\langle z, \gamma\rangle\right]$. Then we have

$$\limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{2t \log\log t}} < \infty \quad \text{a.s.,} \tag{1.10}$$

while, by virtue of Pruitt's result [9], we have

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} |Z_s|}{t^{1/\eta}} = 0 \quad \text{a.s.} \quad (\text{resp.} = \infty \quad \text{a.s.}) \tag{1.11}$$

if $0 < \eta < \alpha$ (resp. $\eta > \alpha$). Now we have, for $\eta > 0$,

$$\limsup_{t\to\infty} t^{-1/\eta} \sup_{0\le s\le t} |Z_s| = \limsup_{t\to\infty} t^{-1/\eta}|Z_t|.$$

Hence, taking $\eta$ with $\alpha < \eta < 2$, we obtain (1.9) from (1.10) and (1.11).

## 2. Preliminaries

The lemma of the following type is pointed out by Toshiro Watanabe (see [13, Theorem 2]). There is another Watanabe's result [14] as paper related to [13]. Since we do not need his theorem in its full generality, we only state the fact in a simpler form without the proof.

**Lemma 2.1** (Toshiro Watanabe). *If* (1.2) *is satisfied for some* $\alpha$ *with* $0 < \alpha < 2$, *then there are two positive constants* $M_1$ *and* $M_2$ *such that, for any* $u > 1$,

$$M_1\rho^*(u) \le P(|X_1| > u) \le M_2\rho^*(u).$$

**Lemma 2.2.** *Let* $h(t)$ *be an increasing positive function on* $[c, \infty)$ *with some* $c > 0$, *and let* $\lim_{t\to\infty} h(t) = \infty$. *Then we have* $(t^{1/\alpha}h(t))^{-1}X_t \to 0$ *in probability as* $t \to \infty$ *if it satisfies one of the following two*:
(i) (1.2) *is satisfied for some* $\alpha$ *with* $0 < \alpha \le 1$, *and, in addition,* $\lim_{t\to\infty} h(t)/\log t = \infty$ *for* $\alpha = 1$;
(ii) (1.2) *is satisfied for some* $\alpha$ *with* $1 < \alpha < 2$ *and* $EX_1 = 0$.

Proof. Define the function $\Psi$ by $Ee^{i\langle z, X_1\rangle} = e^{-\Psi(z)}$. This is called the characteristic exponent (see [1]). Furthermore, we define two functions $\Psi_t^1$ and $\Psi_t^2$ by the real part and the imaginary part of $t\Psi((t^{1/\alpha}h(t))^{-1}z)$, respectively. Here we note that $Ee^{i\langle z,(t^{1/\alpha}h(t))^{-1}X_t\rangle} = e^{-(\Psi_t^1(z)+i\Psi_t^2(z))}$. If we have $\lim_{t\to\infty}\Psi_t^1(z) = 0$ and $\lim_{t\to\infty}\Psi_t^2(z) = 0$, it follows that $\lim_{t\to\infty} Ee^{i\langle z,(t^{1/\alpha}h(t))^{-1}X_t\rangle} = 1$. Hence, as we have $(t^{1/\alpha}h(t))^{-1}X_t$ converges in law to $\delta_0$ as $t \to \infty$, we can get $(t^{1/\alpha}h(t))^{-1}X_t \to 0$ in probability as $t \to \infty$. Here $\delta_0$ is a probability measure concentrated at 0. Therefore we shall show $\lim_{t\to\infty}\Psi_t^1(z) = 0$ and $\lim_{t\to\infty}\Psi_t^2(z) = 0$. In the first place we estimate $\Psi_t^1(z)$. We have

$$
\begin{aligned}
|\Psi_t^1(z)| &\le 2^{-1}t(t^{1/\alpha}h(t))^{-2}\langle z, Az\rangle + t\int_{\mathbb{R}^d}\big(1-\cos\langle z,(t^{1/\alpha}h(t))^{-1}x\rangle\big)\rho(dx)\\
&\le \frac{2^{-1}\langle z, Az\rangle}{t^{-1+2/\alpha}h(t)^2} + t\int_{|x|\le t^{1/\alpha}h(t)}\big(1-\cos\langle z,(t^{1/\alpha}h(t))^{-1}x\rangle\big)\rho(dx) + 2t\rho^*(t^{1/\alpha}h(t))\\
&= \frac{2^{-1}\langle z, Az\rangle}{t^{-1+2/\alpha}h(t)^2} + t\int_{|x|\le t^{1/\alpha}h(t)} 2\sin^2\frac{\langle z,(t^{1/\alpha}h(t))^{-1}x\rangle}{2}\rho(dx) + 2t\rho^*(t^{1/\alpha}h(t))\\
&\le \frac{2^{-1}\langle z, Az\rangle}{t^{-1+2/\alpha}h(t)^2} + \frac{2^{-1}|z|^2 t}{(t^{1/\alpha}h(t))^2}\int_{|x|\le t^{1/\alpha}h(t)}|x|^2\rho(dx) + 2t\rho^*(t^{1/\alpha}h(t))\\
&= I_1 \quad \text{(say)}.
\end{aligned}
$$

As we have

$$
\int_{|x|\le t^{1/\alpha}h(t)}|x|^2\rho(dx) = \int_0^{t^{1/\alpha}h(t)} 2u\rho\left(\{x : t^{1/\alpha}h(t)\ge |x| > u\}\right)du,
$$

then we can get, for all $t > c$ with $t^{1/\alpha}h(t) > 1$,

$$
\begin{aligned}
I_1 &\le \frac{2^{-1}\langle z, Az\rangle}{t^{-1+2/\alpha}h(t)^2} + \frac{2^{-1}|z|^2 t}{(t^{1/\alpha}h(t))^2}\int_0^{t^{1/\alpha}h(t)} 2u\rho^*(u)\,du + 2t\rho^*(t^{1/\alpha}h(t))\\
&= \frac{2^{-1}\langle z, Az\rangle}{t^{-1+2/\alpha}h(t)^2} + \frac{|z|^2 t}{(t^{1/\alpha}h(t))^2}\left(\int_0^1 + \int_1^{t^{1/\alpha}h(t)}\right)u\rho^*(u)\,du + 2t\rho^*(t^{1/\alpha}h(t))\\
&= I_2 \quad \text{(say)}.
\end{aligned}
$$

Furthermore, we shall use the following:

$$
\int_0^1 u\rho^*(u)\,du = 2^{-1}\int_{|x|\le 1}|x|^2\rho(dx) + 2^{-1}\rho(\{x : |x| > 1\}) < \infty.
$$

We put $C_1 = \int_0^1 u\rho^*(u)\,du$. As (1.2) is satisfied for some $\alpha$ with $0 < \alpha < 2$, we can take a positive constant $C_2$ such that

(2.1) $$\rho^*(u) \le C_2 u^{-\alpha}$$

for any $u > 1$. Hence we have, for any sufficiently large $t$,

$$\begin{aligned} I_2 &\le \frac{2^{-1}\langle z, Az\rangle}{t^{-1+2/\alpha}h(t)^2} + C_1\frac{|z|^2 t}{(t^{1/\alpha}h(t))^2} + C_2\frac{|z|^2 t}{(t^{1/\alpha}h(t))^2}\int_1^{t^{1/\alpha}h(t)} u^{1-\alpha}\,du + 2C_2 h(t)^{-\alpha} \\ &\le \frac{2^{-1}\langle z, Az\rangle + C_1|z|^2}{t^{-1+2/\alpha}h(t)^2} + C_2\left(\frac{|z|^2}{2-\alpha} + 2\right) \times h(t)^{-\alpha}. \end{aligned}$$

Thus, since we have $\lim_{t\to\infty} h(t) = \infty$, it follows that $\lim_{t\to\infty} \Psi_t^1(z) = 0$.

In the second place we estimate $\Psi_t^2$. We shall investigate the case of (i). Suppose that (1.2) is satisfied for some $\alpha$ with $0 < \alpha \le 1$, and, in addition, $\lim_{t\to\infty} h(t)/\log t = \infty$ for $\alpha = 1$. Then we have, for any sufficiently large $t$,

$$\begin{aligned} |\Psi_t^2(z)| &= \Bigg| t\int_{|x|<1}\big(\sin\langle (t^{1/\alpha}h(t))^{-1}z, x\rangle - \langle (t^{1/\alpha}h(t))^{-1}z, x\rangle\big)\rho(dx) \\ &\quad + t\int_{|x|\ge 1} \sin\langle (t^{1/\alpha}h(t))^{-1}z, x\rangle \rho(dx) + t\langle (t^{1/\alpha}h(t))^{-1}z, \gamma\rangle \Bigg| \\ &\le t\int_{|x|<1} |\langle (t^{1/\alpha}h(t))^{-1}z, x\rangle|^2\rho(dx) + t\int_{1\le|x|\le t^{1/\alpha}h(t)} |\sin\langle (t^{1/\alpha}h(t))^{-1}z, x\rangle|\rho(dx) \\ &\quad + t\int_{|x|>t^{1/\alpha}h(t)} |\sin\langle (t^{1/\alpha}h(t))^{-1}z, x\rangle|\rho(dx) + t|\langle (t^{1/\alpha}h(t))^{-1}z, \gamma\rangle| \\ &\le \frac{|z|^a 2}{t^{-1+2/\alpha}h(t)^2}\int_{|x|<1}|x|^2\rho(dx) + \frac{|z|}{t^{-1+1/\alpha}h(t)}\int_{1\le|x|\le t^{1/\alpha}h(t)}|x|\rho(dx) \\ &\quad + t\rho^*(t^{1/\alpha}h(t)) + \frac{|z||\gamma|}{t^{-1+1/\alpha}h(t)} = I_3 \quad \text{(say)}. \end{aligned}$$

Put $C_3 = \rho(\{x : |x| \ge 1\})$. Here, from (2.1), we have

$$\begin{aligned} &\int_{1\le|x|\le t^{1/\alpha}h(t)} |x|\rho(dx) \\ &= \int_0^1 \rho\big(\{x : 1 \le |x| \le t^{1/\alpha}h(t)\}\big)\,du + \int_1^{t^{1/\alpha}h(t)} \rho\big(\{x : u < |x| \le t^{1/\alpha}h(t)\}\big)\,du \\ &\le C_3 + C_2\int_1^{t^{1/\alpha}h(t)} u^{-\alpha}\,du \\ &= \begin{cases} C_3 + C_2(1-\alpha)^{-1}\big((t^{1/\alpha}h(t))^{1-\alpha} - 1\big) & \text{if } 0 < \alpha < 1, \\ C_3 + C_2\log(th(t)) & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

Hence, in the case where $0 < \alpha < 1$, we have

$$\begin{aligned} I_3 &\le \frac{|z|^2}{t^{-1+2/\alpha}h(t)^2}\int_{|x|<1}|x|^2\rho(dx) + \frac{C_3|z|}{t^{-1+1/\alpha}h(t)} \\ &\quad + \frac{|z|}{t^{-1+1/\alpha}h(t)}C_2(1-\alpha)^{-1}(t^{1/\alpha}h(t))^{1-\alpha} + \frac{C_2}{h(t)^\alpha} + \frac{|z||\gamma|}{t^{-1+1/\alpha}h(t)} \end{aligned}$$

$$= \frac{|z|^2}{t^{-1+2/\alpha}h(t)^2}\int_{|x|<1}|x|^2\rho(dx) + \frac{C_3|z|}{t^{-1+1/\alpha}h(t)} + C_2\frac{(1-\alpha)^{-1}|z|+1}{h(t)^\alpha} + \frac{|z||\gamma|}{t^{-1+1/\alpha}h(t)},$$

where we used (2.1) in the first inequality. And, in the case where $\alpha = 1$, we have

$$I_3 \le \frac{|z|^2}{th(t)^2}\int_{|x|<1}|x|^2\rho(dx) + \frac{C_3|z|}{h(t)} + C_2\frac{|z|(\log t + \log h(t)) + 1}{h(t)} + \frac{|z||\gamma|}{h(t)}.$$

Hence it follows that $\lim_{t\to\infty}\Psi_t^2(z) = 0$, because we have $\lim_{t\to\infty} h(t) = \infty$ for $\alpha \neq 1$ and $\lim_{t\to\infty} h(t)/\log t = \infty$ for $\alpha = 1$. Secondly, we shall investigate the case of (ii). Suppose that (1.2) is satisfied for some $\alpha$ with $1 < \alpha < 2$ and that $EX_1 = 0$. Then, since we have $EX_1 = 0$, the characteristic function of $X_1$ is represented as (1.8) in Remark 1.4. Hence we have

$$\begin{aligned}
|\Psi_t^2(z)| &= \left| t\int_{\mathbb{R}^d}\big(\sin\langle(t^{1/\alpha}h(t))^{-1}z, x\rangle - \langle(t^{1/\alpha}h(t))^{-1}z, x\rangle\big)\rho(dx)\right| \\
&\le t\int_{|x|\le t^{1/\alpha}h(t)} |\sin\langle(t^{1/\alpha}h(t))^{-1}z, x\rangle - \langle(t^{1/\alpha}h(t))^{-1}z, x\rangle|\rho(dx) \\
&\quad + t\int_{|x|>t^{1/\alpha}h(t)}\big(|\sin\langle(t^{1/\alpha}h(t))^{-1}z, x\rangle| + |\langle(t^{1/\alpha}h(t))^{-1}z, x\rangle|\big)\rho(dx) \\
&\le t\int_{|x|\le t^{1/\alpha}h(t)} |\langle(t^{1/\alpha}h(t))^{-1}z, x\rangle|^2\rho(dx) + 2t\int_{|x|>t^{1/\alpha}h(t)} |\langle(t^{1/\alpha}h(t))^{-1}z, x\rangle|\rho(dx) \\
&\le \frac{|z|^2}{t^{-1+2/\alpha}h(t)^2}\int_{|x|\le t^{1/\alpha}h(t)}|x|^2\rho(dx) + \frac{2|z|}{t^{-1+1/\alpha}h(t)}\int_{|x|>t^{1/\alpha}h(t)}|x|\rho(dx) \\
&= \frac{|z|^2}{t^{-1+2/\alpha}h(t)^2}J_1 + \frac{2|z|}{t^{-1+1/\alpha}h(t)}J_2 \quad \text{(say)}.
\end{aligned}$$

Now we investigate $J_1$ and $J_2$. Using (2.1), we have, for all $t > c$ with $t^{1/\alpha}h(t) > 1$,

$$\begin{aligned}
J_1 &= \int_0^{t^{1/\alpha}h(t)} 2u\rho(\{x : u < |x| \le t^{1/\alpha}h(t)\})\,du \\
&\le \left(\int_0^1 + \int_1^{t^{1/\alpha}h(t)}\right) 2u\rho^*(u)\,du \\
&\le 2C_1 + C_2\int_1^{t^{1/\alpha}h(t)} 2u^{1-\alpha}\,du \\
&\le 2C_1 + 2(2-\alpha)^{-1}C_2(t^{1/\alpha}h(t))^{2-\alpha}.
\end{aligned}$$

Using (2.1), we have, for all $t > c$ with $t^{1/\alpha}h(t) > 1$,

$$J_2 = \int_0^{t^{1/\alpha}h(t)} \rho(\{x : |x| > t^{1/\alpha}h(t)\})\,du + \int_{t^{1/\alpha}h(t)}^{\infty} \rho(\{x : |x| > u\})\,du$$

$$\leq C_2 \frac{\alpha}{\alpha - 1}(t^{1/\alpha}h(t))^{-\alpha+1}.$$

Thus we obtain that, for all $t > c$ with $t^{1/\alpha}h(t) > 1$,

$$\begin{aligned} |\Psi_t^2(z)| &\leq \frac{|z|^2}{t^{-1+2/\alpha}h(t)^2}\left\{2C_1 + 2(2-\alpha)^{-1}C_2(t^{1/\alpha}h(t))^{2-\alpha}\right\} \\ &\quad + \frac{2|z|}{t^{-1+1/\alpha}h(t)}C_2\frac{\alpha}{\alpha-1}(t^{1/\alpha}h(t))^{-\alpha+1} \\ &= C_1\frac{2|z|^2}{t^{-1+2/\alpha}h(t)^2} + C_2\frac{2(2-\alpha)^{-1}|z|^2 + 2\alpha(\alpha-1)^{-1}|z|}{h(t)^\alpha}. \end{aligned}$$

Hence we have $\lim_{t\to\infty}\Psi_t^2(z) = 0$ in both cases of (i) and (ii). We have completed the proof. □

**Lemma 2.3.** *Let $d = 1$. Let $n$ be an arbitrary positive integer. If $\{X_t\}$ is a symmetric Lévy process, then we have, for any $a > 0$,*

$$P\left(\sup_{0\leq t\leq n} X_t > a\right) \leq 2P(X_n > a). \tag{2.2}$$

Proof. As $X_t$ is right continuous in $t$ and $\{X_t\}$ has independent increments, we can obtain (2.2) by virtue of Lemma 1 in [11, p.372]. □

**Lemma 2.4.** *Let $n$ and $m$ be arbitrary positive integers with $n < m$. We have, for any $a > 0$,*

$$P\left(\sup_{n\leq t\leq m} |X_t| > 3a\right) \leq 3 \sup_{n\leq t\leq m} P(|X_t| > a). \tag{2.3}$$

Proof. Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables. Put $S_k = \sum_{i=1}^k Z_i$ for $k \geq 1$. Then it follows that

$$P\left(\max_{1\leq k\leq n} |S_k| > 3a\right) \leq 3 \max_{1\leq k\leq n} P(|S_k| > a).$$

Indeed, this is shown by using [10] Lemma 20.2. Hence, as $X_t$ is right continuous and $\{X_t\}$ has indepedent increments, we can get (2.3). □

A theorem similar to the following two lemmas is proved by C.C. Heyde in the case where $d = 1$ with certain conditions (see [5]). We cannot extend his theorem to the multidimensional case. But, if we suppose (1.2), then Heyde's proof is applicable to the multidimensional case without his conditions and we can get two lemmas.

**Lemma 2.5.** *Let $l(t)$ be an increasing positive function on $[c,\infty)$ with some $c>0$, and let $\lim_{t\to\infty} l(t)=\infty$. We suppose that $X_n/l(n)\to 0$ in probability as $n\to\infty$. If* (1.2) *is satisfied for some $\alpha$ with $0<\alpha<2$, then we have*

$$\limsup_{n\to\infty}\frac{P(|X_n|>l(n))}{n\rho^*(2^{-1}l(n))}<\infty. \tag{2.4}$$

*Here $n$ runs through positive integers.*

Proof. Set $S_n=X_n-X_n'$, where $\{X_n\}$ and $\{X_n'\}$ are independent, and $X_n$ and $X_n'$ are identical in law. As we have $X_n'/l(n)\to 0$ in probability as $n\to\infty$, then it follows that, for any sufficiently large $n$,

$$\begin{aligned}P(|S_n|>2^{-1}l(n)) &\ge P(|X_n'|<2^{-1}l(n))P(|X_n|>l(n))\\ &\ge 2^{-1}P(|X_n|>l(n)).\end{aligned}$$

Here we define, for $k=1,2,\ldots,n$,

$$Z_{k,n}=\begin{cases} S_k-S_{k-1} & \text{if } |S_k-S_{k-1}|\le l(n),\\ 0 & \text{otherwise.}\end{cases}$$

Set $S_{n,n}=\sum_{k=1}^n Z_{k,n}$. Hence we have, for any sifficiently large $n$,

$$\begin{aligned}P(|X_n|>l(n)) &\le 2P(|S_n|>2^{-1}l(n))\\ &= 2P(|S_n|>2^{-1}l(n), |S_k-S_{k-1}|>l(n)\quad\text{for some } k \text{ with } 1\le k\le n)\\ &\quad +2P(|S_n|>2^{-1}l(n), |S_k-S_{k-1}|\le l(n)\quad\text{for all } k \text{ with } 1\le k\le n)\\ &\le 2\sum_{k=1}^n P(|S_k-S_{k-1}|>l(n))+2P(|S_{n,n}|>2^{-1}l(n))\\ &= 2nP(|S_1-S_0|>l(n))+2P(|S_{n,n}|>2^{-1}l(n))\\ &\le 2nP(|X_1-X_1'|>l(n))+2\sum_{j=1}^d P\big(|S_{n,n}^j|>2^{-1}\sqrt{d}^{-1}l(n)\big)\\ &\le 4nP(|X_1|>2^{-1}l(n))+2\sum_{j=1}^d P\big(|S_{n,n}^j|>2^{-1}\sqrt{d}^{-1}l(n)\big),\end{aligned}$$

where $S_{n,n}=(S_{n,n}^1,S_{n,n}^2,\ldots,S_{n,n}^d)$. Furthermore, let $Z_{k,n}=(Z_{k,n}^1,Z_{k,n}^2,\ldots,Z_{k,n}^d)$. Here we shall use the following equality: For $k$ and $k'$ with $k\ne k'$,

$$EZ_{k,n}^jZ_{k',n}^j=EZ_{k,n}^jEZ_{k',n}^j=0.$$

Then we have

$$P\big(|S_{n,n}^j|>2^{-1}\sqrt{d}^{-1}l(n)\big)\le\frac{4d}{l(n)^2}E[|S_{n,n}^j|^2]$$

$$= \frac{4d}{l(n)^2} E\left[\sum_{k=1}^{n} (Z_{k,n}^j)^2\right] \le \frac{4dn}{l(n)^2} E[|Z_{1,n}|^2]$$
$$= \frac{4dn}{l(n)^2} \int_{|x|\le l(n)} |x|^2 P(S_1 \in dx).$$

Hence we obtain, for any sufficiently large $n$,

$$\begin{aligned}
\frac{P(|X_n| > l(n))}{nP(|X_1| > 2^{-1}l(n))} &\le 4 + 2\sum_{j=1}^{d} \frac{4dn}{l(n)^2} \cdot \frac{\int_{|x|\le l(n)} |x|^2 P(S_1 \in dx)}{nP(|X_1| > 2^{-1}l(n))} \\
&= 4 + \frac{8d^2}{l(n)^2} \cdot \frac{\int_0^{l(n)} 2uP(u < |S_1| \le l(n))\,du}{P(|X_1| > 2^{-1}l(n))} \\
&\le 4 + \frac{16d^2}{l(n)^2} \cdot \frac{\int_0^{l(n)} uP(|X_1 - X_1'| > u)\,du}{P(|X_1| > 2^{-1}l(n))} \\
&\le 4 + \frac{16d^2}{l(n)^2} \cdot \frac{\int_0^{l(n)} 2uP(|X_1| > 2^{-1}u)\,du}{P(|X_1| > 2^{-1}l(n))} = I \quad \text{(say).}
\end{aligned}$$

Fix an integer $m$ such that $l(m) > 2$. From Lemma 2.1 we have, for all $n$ with $n > m$,

$$\begin{aligned}
I &\le 4 + \frac{16d^2}{l(n)^2} \cdot \frac{\int_0^{l(m)} 2uP(|X_1| > 2^{-1}u)\,du}{P(|X_1| > 2^{-1}l(n))} + \frac{16d^2}{l(n)^2} \cdot \frac{\int_{l(m)}^{l(n)} 2uP(|X_1| > 2^{-1}u)\,du}{P(|X_1| > 2^{-1}l(n))} \\
&\le 4 + \frac{16d^2}{l(n)^2} \cdot \frac{l(m)^2}{M_1\rho^*(2^{-1}l(n))} + \frac{16d^2}{l(n)^2} \cdot \frac{M_2 \int_{l(m)}^{l(n)} 2u\rho^*(2^{-1}u)\,du}{M_1\rho^*(2^{-1}l(n))} \\
&\le 4 + \text{const.} \times \frac{1}{l(n)^{2-\alpha}} + \text{const.} \times \frac{\int_{l(m)}^{l(n)} u^{1-\alpha}\,du}{l(n)^{2-\alpha}} < \infty.
\end{aligned}$$

Here we used (1.2) in the last inequality. We have proved the lemma. □

**Lemma 2.6.** *Let $h(t)$ be an increasing positive function on $[c, \infty)$ with some $c > 0$, and let $\lim_{t\to\infty} h(t) = \infty$. If one of the two conditions* (i) *and* (ii) *in* Lemma 2.2 *is satisfied, then we have*

$$\liminf_{n\to\infty} \frac{P(|X_n| > n^{1/\alpha}h(n))}{n\rho^*(2n^{1/\alpha}h(n))} > 0. \tag{2.5}$$

*Here $n$ runs through positive integers.*

Proof. Put $l(n) = n^{1/\alpha}h(n)$. Set $Z_i = X_i - X_{i-1}$ for $i = 1, 2, \ldots, n$. Denote by $A_i$ and $B_i$ the events $\{|Z_i| > 2l(n)\}$ and $\{|\sum_{j=1, j\ne i}^{n} Z_j| < l(n)\}$, respectively, for $i = 1, 2, \ldots, n$. We denote the complement of the event $E$ by the symbol $\overline{E}$. Then we have

$$\begin{aligned}
P(|X_n| > l(n)) &\geq P\left(\bigcup_{i=1}^{n}(A_i \cap B_i)\right) = \sum_{i=1}^{n} P\left(\bigcap_{j=1}^{i-1}\overline{(A_j \cap B_j)} \cap (A_i \cap B_i)\right) \\
&\geq \sum_{i=1}^{n} P\left(\bigcap_{j=1}^{i-1}\overline{A_j} \cap (A_i \cap B_i)\right) \geq \sum_{i=1}^{n}\left\{P(A_i \cap B_i) - P\left(\bigcup_{j=1}^{i-1} A_j \cap A_i\right)\right\} \\
&\geq \sum_{i=1}^{n} P(A_i)\{P(B_i) - (i-1)P(A_1)\} \\
&\geq nP(A_1)\{P(B_1) - nP(A_1)\} = I \quad \text{(say)}.
\end{aligned}$$

From Lemma 2.1 and (1.2), we have, for any sufficiently large $n$,

$$\begin{aligned}
nP(A_1) &= nP(|X_1| > 2n^{1/\alpha}h(n)) \leq M_2 n\rho^*(2n^{1/\alpha}h(n)) \\
&\leq \text{const.} \times \frac{1}{h(n)^\alpha}.
\end{aligned}$$

Hence it follows that $\lim_{n\to\infty} nP(A_1) = 0$. And we have

$$P(\overline{B_1}) = P(|X_n - Z_1| \geq l(n)) \leq P(|X_n| \geq 2^{-1}l(n)) + P(|Z_1| \geq 2^{-1}l(n)).$$

As we have $X_n/l(n) \to 0$ in probability as $n \to \infty$ from Lemma 2.2, it follows that $\lim_{n\to\infty} P(B_1) = 1$. Take $\delta_0 > 0$ with $1 - 2\delta_0 > 0$. Then there is a positive integer $N$ such that $nP(A_1) < \delta_0$ and $P(B_1) > 1 - \delta_0$ for all $n \geq N$. Thus we obtain that, for all $n \geq N$,

$$I \geq nP(A_1) \times (1 - 2\delta_0).$$

Therefore, from Lemma 2.1, we have

$$P(|X_n| > l(n)) \geq (1 - 2\delta_0)nP(A_1) \geq M_1(1 - 2\delta_0)n\rho^*(2l(n)).$$

for all $n > N$, which implies (2.5). The lemma has been proved. □

**Lemma 2.7.** *Let $h(t)$ be an increasing positive function on $[c, \infty)$ with some $c > 0$, and let $\lim_{t\to\infty} h(t) = \infty$. If one of the two conditions* (i) *and* (ii) *in* Lemma 2.2 *is satisfied, then there is a positive constant $M$ such that, for any sufficiently large integer $r$,*

$$P\left(\sup_{2^r \leq t \leq 2^{r+1}} |X_t| > 2^{r/\alpha}h(2^r)\right) \leq M2^r\rho^*\big((16d)^{-1}2^{(r-1)/\alpha}h(2^{r-1})\big). \tag{2.6}$$

Proof. Set $l(t) = t^{1/\alpha}h(t)$. Let $X_t = (X_t^1, X_t^2, \ldots, X_t^d)$. From Lemma 2.4 we have

$$\begin{aligned}
P\left(\sup_{2^r \le t \le 2^{r+1}} |X_t| > l(2^r)\right) &\le 3 \sup_{2^r \le t \le 2^{r+1}} P\big(|X_t| > 3^{-1} l(2^r)\big) \\
&\le 3 \sup_{2^r \le t \le 2^{r+1}} P\big(|X_t| > 3^{-1} l(2^{-1}t)\big) \\
&\le 3 \sum_{j=1}^{d} \sup_{2^r \le t \le 2^{r+1}} P\big(|X_t^j| > (3d)^{-1} l(2^{-1}t)\big) \\
&= 3 \sum_{j=1}^{d} \sup_{2^r \le t \le 2^{r+1}} \big\{P\big(X_t^j > (3d)^{-1} l(2^{-1}t)\big) \\
&\qquad + P\big(X_t^j < -(3d)^{-1} l(2^{-1}t)\big)\big\}.
\end{aligned}$$

First, we shall estimate $P(X_t^j > (3d)^{-1} l(2^{-1}t))$. Denote by $W_t^j$ the symmetrized random variable of $X_t^j$. Here $W_t^j = X_t^j - \overline{X}_t^j$, where $X_t^j$ and $\overline{X}_t^j$ are independent and identically distributed. Letting $a(t)$ be a median of $X_t^j$, we obtain

$$\begin{aligned}
P\big(W_t^j > (4d)^{-1} l(2^{-1}t)\big) &= P\big(X_t^j - \overline{X}_t^j > (4d)^{-1} l(2^{-1}t)\big) \\
&\ge P\big(X_t^j - a(t) > (4d)^{-1} l(2^{-1}t)\big) P\big(\overline{X}_t^j - a(t) \le 0\big) \\
&\ge 2^{-1} P\big(X_t^j > (4d)^{-1} l(2^{-1}t) + a(t)\big) = I \quad \text{(say)}.
\end{aligned}$$

Now $a(t)/l(2^{-1}t)$ is a median of $X_t^j/l(2^{-1}t)$. Hence, since $X_t^j/l(2^{-1}t) \to 0$ in probability as $t \to \infty$ from Lemma 2.2, it is obvious that $\lim_{t\to\infty} a(t)/l(2^{-1}t) = 0$.

Consequently we have, for any sufficiently large $t$,

$$I \ge 2^{-1} P\big(X_t^j > (3d)^{-1} l(2^{-1}t)\big).$$

In the same way we have, for any sufficiently large $t$,

$$P\big(W_t^j < -(4d)^{-1} l(2^{-1}t)\big) \ge 2^{-1} P\big(X_t^j < -(3d)^{-1} l(2^{-1}t)\big).$$

Therefore, from Lemma 2.3, it follows that, for any sufficiently large integer $r$,

$$\begin{aligned}
P\left(\sup_{2^r \le t \le 2^{r+1}} |X_t| > l(2^r)\right) &\le 6 \sum_{j=1}^{d} \sup_{2^r \le t \le 2^{r+1}} \big\{P\big(W_t^j > (4d)^{-1} l(2^{-1}t)\big) \\
&\qquad + P\big(-W_t^j > (4d)^{-1} l(2^{-1}t)\big)\big\} \\
&\le 6 \sum_{j=1}^{d} \left\{P\left(\sup_{2^r \le t \le 2^{r+1}} W_t^j > (4d)^{-1} l(2^{r-1})\right)\right. \\
&\qquad \left. + P\left(\sup_{2^r \le t \le 2^{r+1}} (-W_t^j) > (4d)^{-1} l(2^{r-1})\right)\right\}
\end{aligned}$$

$$\le 12 \sum_{j=1}^{d} \Big\{ P\big(W^j_{2^{r+1}} > (4d)^{-1} l(2^{r-1})\big) \\ + P\left(-W^j_{2^{r+1}} > (4d)^{-1} l(2^{r-1})\right)\Big\} = J \quad \text{(say)}.$$

Now we have $X_n/(8d)^{-1}l(2^{-2}n) \to 0$ in probability as $n \to \infty$ from Lemma 2.2. Hence, by using Lemma 2.5, it follows that, for any sufficiently large $r$,

$$\begin{aligned} J = 12 \sum_{j=1}^{d} P\big(|W^j_{2^{r+1}}| > (4d)^{-1} l(2^{r-1})\big) &\le 24 \sum_{j=1}^{d} P\big(|X^j_{2^{r+1}}| > 2^{-1}(4d)^{-1} l(2^{r-1})\big) \\ &\le 24d \times P(|X_{2^{r+1}}| > (8d)^{-1} l(2^{r-1})) \\ &\le \text{const.} \times 2^{r+1} \rho^*((16d)^{-1} l(2^{r-1})). \end{aligned}$$

Therefore we got (2.6). We have completed the proof. □

## 3. Proof of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. From now on, in the case where $1 < \alpha < 2$, we suppose that $EX_1 = 0$ without loss of generality, because $X_t - tEX_1$ is a Lévy process with mean 0. And we suppose that (1.2) is satisfied for some $\alpha$ with $0 < \alpha < 2$, in addition, that $\lim_{t\to\infty} h(t)/\log t = \infty$ for $\alpha = 1$.

First, we suppose that $\int_c^\infty (th(t)^\alpha)^{-1}\, dt < \infty$. Let $n_r = 2^r$ for any positive integer $r$. Let $\delta > 0$. Put

$$B_r = \{|X_t| > \delta n_r^{1/\alpha} h(n_r) \quad \text{for some } t \text{ with } n_r \le t < n_{r+1}\}.$$

Set $l(t) = \delta t^{1/\alpha} h(t)$. By virtue of Lemma 2.7 we have, for any sufficiently large integer $r$,

$$P(B_{r+1}) = P\left(\sup_{n_{r+1} \le t < n_{r+2}} |X_t| > l(n_{r+1})\right) \le M n_{r+1} \rho^*((16d)^{-1} l(n_r)).$$

From (1.2) we obtain that, for any sufficiently large integer $m$,

$$\begin{aligned} \sum_{r=m}^{\infty} P(B_{r+1}) &\le 2M \sum_{r=m}^{\infty} \frac{(16d)^{-\alpha}\delta^\alpha 2^r h(2^r)^\alpha \rho^*((16d)^{-1}\delta 2^{r/\alpha} h(2^r))}{(16d)^{-\alpha}\delta^\alpha h(2^r)^\alpha} \\ &\le \text{const.} \times \sum_{r=m}^{\infty} \frac{1}{h(2^r)^\alpha} \le \text{const.} \times \int_{m-1}^{\infty} \frac{du}{h(2^u)^\alpha} = \text{const.} \times \int_{2^{m-1}}^{\infty} \frac{du}{uh(u)^\alpha} < \infty. \end{aligned}$$

Hence, by virtue of the first Borel-Cantelli lemma, we obtain that $P(\limsup_{r\to\infty} B_r) =$

0. Thus,

$$\begin{aligned} 1 &= P\left(\bigcup_{m=1}^{\infty}\bigcap_{r=m}^{\infty}\left\{|X_t| \le \delta n_r^{1/\alpha} h(n_r) \quad \text{for all } t \text{ with } n_r \le t < n_{r+1}\right\}\right) \\ &\le P\left(\bigcup_{m=1}^{\infty}\bigcap_{r=m}^{\infty}\left\{|X_t| \le \delta t^{1/\alpha} h(t) \quad \text{for all } t \text{ with } n_r \le t < n_{r+1}\right\}\right). \end{aligned}$$

Therefore we have

$$\limsup_{t\to\infty} \frac{|X_t|}{t^{1/\alpha} h(t)} \le \delta \quad \text{a.s.}$$

As $\delta \to 0$, we have

$$\limsup_{t\to\infty} \frac{|X_t|}{t^{1/\alpha} h(t)} = 0 \quad \text{a.s.}$$

Secondly, we suppose that $\int_c^{\infty}(th(t)^{\alpha})^{-1}\,dt = \infty$. Let $a > 0$. By virtue of Lemma 2.6 there is a positive constant $C_0$ such that, for any sufficiently large integer $n$,

$$P(|X_n| > an^{1/\alpha} h(2n)) \ge C_0 n \rho^*(2an^{1/\alpha} h(2n)).$$

Setting $D_r = X_{n_{r+1}} - X_{n_r}$, we obtain that, for any sufficiently large integer $r$,

$$P(|D_r| > an_r^{1/\alpha} h(n_{r+1})) = P(|X_{n_r}| > an_r^{1/\alpha} h(2n_r)) \ge C_0 n_r \rho^*(2an_r^{1/\alpha} h(2n_r)).$$

Hence we obtain that, for any sufficiently large integer $m$,

$$\sum_{r=m}^{\infty} P(|D_r| > an_r^{1/\alpha} h(n_{r+1})) \ge \text{const.} \times \sum_{r=m}^{\infty} \frac{1}{h(2n_r)^{\alpha}} \ge \text{const.} \times \int_{2^{m+1}}^{\infty} \frac{du}{uh(u)^{\alpha}} = \infty.$$

By virtue of the second Borel-Cantelli lemma, we almost surely have

$$|D_r| > an_r^{1/\alpha} h(n_{r+1})$$

for infinitely many $r$. Thus we almost surely have either

$$|X_{n_{r+1}}| > 2^{-1} an_r^{1/\alpha} h(n_{r+1})$$

for infinitely many $r$, or

$$|X_{n_r}| > 2^{-1} an_r^{1/\alpha} h(n_{r+1})$$

for infinitely many $r$. Therefore, since $h$ is increasing, we almost surely have

$$|X_{n_r}| > 2^{-1-1/\alpha} a n_r^{1/\alpha} h(n_r)$$

for infinitely many $r$. From the inequlity above, we obtain that

$$\limsup_{t\to\infty} \frac{|X_t|}{t^{1/\alpha} h(t)} \geq 2^{-1-1/\alpha} a \quad \text{a.s.}$$

As $a \to \infty$, we have

$$\limsup_{t\to\infty} \frac{|X_t|}{t^{1/\alpha} h(t)} = \infty \quad \text{a.s.}$$

The proof has been completed. □

Proof of Corollary 1.2. Let $\delta > 0$. Take $h(t) = (\log t)^{(1+\delta)/\alpha}$. As we have

$$\int_e^\infty \frac{dt}{t h(t)^\alpha} = \int_e^\infty \frac{dt}{t(\log t)^{1+\delta}} < \infty,$$

it follows that

$$\limsup_{t\to\infty} \frac{|X_t|}{t^{1/\alpha} (\log t)^{(1+\delta)/\alpha}} = 0 \quad \text{a.s.}$$

from Theorem 1.1. Then the limsup above implies that there is a positive constant $M$ such that

$$\frac{|X_t|}{t^{1/\alpha}} \leq M \exp\left[\frac{1+\delta}{\alpha} \log\log t\right]$$

for any sufficiently large $t$. Therefore we have

$$\limsup_{t\to\infty} \left(\frac{|X_t|}{t^{1/\alpha}}\right)^{1/\log\log t} \leq e^{(1+\delta)/\alpha} \quad \text{a.s.}$$

As $\delta \to 0$, we have

$$\limsup_{t\to\infty} \left(\frac{|X_t|}{t^{1/\alpha}}\right)^{1/\log\log t} \leq e^{1/\alpha} \quad \text{a.s.}$$

We prove the reverse inequality of the above one. Take $h(t) = (\log t)^{1/\alpha} (\log\log t)^{1/\alpha}$. As we have

$$\int_{e^2}^\infty \frac{dt}{t h(t)^\alpha} = \int_{e^2}^\infty \frac{dt}{t(\log t)(\log\log t)} = \infty,$$

it follows that

$$\limsup_{t\to\infty} \frac{|X_t|}{t^{1/\alpha}(\log t)^{1/\alpha}(\log\log t)^{1/\alpha}} = \infty \quad \text{a.s.}$$

from Theorem 1.1. Hence we have

$$\limsup_{t\to\infty} \left(\frac{|X_t|}{t^{1/\alpha}}\right)^{1/\log\log t} \geq e^{1/\alpha} \quad \text{a.s.}$$

We have completed the proof. □

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