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Osaka University
ON THE $\overline{\partial}$-COHOMOLOGY GROUPS OF STRONGLY $q$-CONCAVE MANIFOLDS

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0. Introduction

Let $X$ be a paracompact complex manifold of dimension $n$ and $\pi : E \rightarrow X$ be a holomorphic vector bundle. We denote by $\Omega^p(E)$ the germ of $E$-valued holomorphic $p$-forms, and by $H^q(X, \Omega^p(E))$ the sheaf cohomology group of $X$ of degree $q$ with coefficients in $\Omega^p(E)$. In 1955, Serre showed the following basic theorem with respect to complex analysis.

**Theorem** (Serre duality, cf: [15]). If $H^{q+i}(X, \Omega^p(E))$ ($i = 0, 1$) are Hausdorff, then $H^q(X, \Omega^p(E))$ is a Fréchet space, and its dual space and $H^{n-q}(X, \Omega^{n-p}(E^*))$ are isomorphic. Here, $E^*$ denotes the dual of $E$, and $H^k(X, \Omega'(E))$ denotes the compactly supported sheaf cohomology group of $X$ with coefficients in $\Omega'(E)$.

If $H^q(X, \Omega^p(E))$ is finite dimensional, then it is Hausdorff (cf: [15]). But, in general $H^q(X, \Omega^p(E))$ is not Hausdorff (cf: [8], [15]).

The cohomology groups of open manifolds were studied by Grauert [5] for solving Levi's problem, and his result played a fundamental role in the theory of singularities and hyperfunctions. As a natural extension of Grauert's work, it has been known that the finiteness of the cohomology groups results from on the convexity of manifolds:

$X$ is called strongly $q$-convex (resp. strongly $q$-concave) if there exists an exhaustion function $\Phi : X \rightarrow \mathbb{R}$ of class $C^\infty$ whose Levi form has at least $n-q+1$ positive (resp. $n-q+1$ negative) eigenvalues outside a compact subset $K$ of $X$. We call $K$ an exceptional set. In 1962, Andreotti and Grauert established finiteness theorems for cohomology groups which include the following theorem as a special case.
Theorem A-G (cf: Théorème 14 in [1]). Let $X$ be a strongly $q$-convex (resp. strongly $q$-concave) manifold of dimension $n$, and let $E$ be a holomorphic vector bundle over $X$. Then

$$\dim H^s(X, \Omega^r(E)) < \infty \text{ for } s \geq q \text{ (resp. } s < n-q).$$

They showed this theorem, using homological algebra and sheaf theory. Andreotti and Vesentini [3] showed this theorem for $q$-complete manifolds (i.e., $q$-convex manifolds with $K = \phi$), using so-called “Bochner-technique”. Moreover, they proved that when $X$ is strongly $q$-convex (resp. strongly $q$-concave), $H^k(X, \Omega^r(E))$ is finite dimensional for $s \leq n-q$ (resp. $s > q$) by using the method of [1]. At almost the same time, Hörmander [7] generalized the method for the $\overline{\partial}$-Neumann problem by J.J. Kohn, and proved Theorem A-G. Ohsawa [10] generalized the method of [3], [7] and gave an alternative proof of Theorem A-G. For further results, see [12], [13].

Andreotti and Vesentini [3] stated the following.

Theorem A-V. Let $X$ be a strongly $q$-concave manifold of dimension $n$, and let $\pi: E \to X$ be a holomorphic vector bundle over $X$. Then

$$H^{n-q}(X, \Omega^r(E)) \text{ is Hausdorff.}$$

This theorem has been extended by Andreotti and Kas [2], and Ramis [14] in the case where $X$ is a complex space and $E$ is a coherent analytic sheaf, by using homological algebra and sheaf theory. In 1988, Henkin and Leiterer [6] gave a proof of Theorem A-V in case $X$ is a $q$-concave domain of a compact complex manifold by integral formula.

In this paper, we use the method of $L^2$ estimate for $\overline{\partial}$ and give a straightforward proof of Theorem A-V. Moreover we show Hausdorffness of a certain cohomology group of a compact complex space by using the method. Particularly, we utilize not the basic estimate for differential forms satisfying $\overline{\partial}$-Neumann condition on a relatively compact $q$-concave domain with a smooth boundary, but one with respect to a complete hermitian metric on a strongly $q$-concave manifold. Application of such a method has not been well known since [10].

The $L^2$ method seems to have advantages since infinite dimensional cohomology groups seem to be better understood in the $L^2$ context. For instance, Takegoshi showed a harmonic representation theorem for some cohomology group by using an $L^2$ estimate for the $\overline{\partial}$-operator, and proved the torsion freeness theorem for higher direct image sheaves of semipositive vector bundle in [16].

The author expresses his hearty thanks to Professor T. Ohsawa who led him to this subject.
1. Preliminaries

(1) Hermitian Geometry

Let $X$ be a paracompact complex manifold of dimension $n$ and let $E$ be a holomorphic vector bundle over $X$ with a $C^\infty$ fiber metric $h$. Canonically, $h$ induces metrics along the fibers of $E^*$, $E$, $\wedge^n E$, $\otimes^n E$. We also denote by $\langle \cdot , \cdot \rangle$ (resp. $|\cdot |$) the pointwise inner product (resp. norm) with respect to the induced metrics. Let $ds^2$ be a hermitian metric on $X$ and let $\omega$ be the fundamental form associated to $ds^2$ and we denote the volume element by $dv = \frac{1}{n!} \omega \wedge \cdots \wedge \omega$. Let $A$ be the adjoint of the multiplication $L : u \rightarrow \omega \wedge u$ with respect to $ds^2$. We call $L$ the Lefschetz operator with respect to $ds^2$.

We denote by $C^{p,q}(X, E)$ the space of $E$-valued $(p, q)$-forms of class $C^\infty$ on $X$ and by $C^b_c(X, E)$ the space of the forms in $C^{p,q}(X, E)$ with compact supports. As usual we denote the exterior differentiation by $d$ and the $(1, 0)$ part of $d$ by $\partial$ and the $(0, 1)$ part of $d$ by $\bar{\partial}$. We set $D_E : = \partial + h^{-1} \partial h$, $D_{\bar{E}} : = h^{-1} \partial h = \partial + h^{-1}(\partial h)$, $\partial_E : = -\ast h^{-1} \partial h \ast$, $\bar{\partial} : = -\ast \partial \ast$.

Theorem 1.1 (cf : [10], [4]). We set $\tau=[\Lambda, d\omega]$, and denote its adjoint by $\tau^*$. Then

$$[D_E, \Lambda] = -\sqrt{-1}(\partial_E + \tau^*), \quad [\bar{\partial}, \Lambda] = \sqrt{-1}(\bar{\partial} + \tau^*),$$

$$\sqrt{-1}[D_E, \Lambda] = [\bar{\partial}, \partial_E] - [D_E, \bar{\partial}] + [\bar{\partial}, \tau^*] - [\tau^*, D_E].$$

We set $T_1 = \tau^*$ and $T_2 = \tau^*$. $T_i$ and the adjoints $T_i^*$ of $T_i$ ($i=1, 2$) are called the torsions of $ds^2$. $D^b_E$ is a multiplication of a $\text{Hom}(E, E)$-valued $(1, 1)$-form. We set $D^b_E = e(\Theta_h)$, $\Theta_h \in C^1(X, \text{Hom}(E, E))$. $\Theta_h$ is called the curvature form of $E$ with respect to $h$.

(2) Basic estimate

Let $H_1$ and $H_2$ be two Hilbert spaces and $T : H_1 \rightarrow H_2$ a closed linear operator with dense domain. We denote its domain, range and nullity by $D_T$, $R_T$, $N_T$, and the adjoint of $T$ by $T^*$. We set $(f, g) = \int_X \langle f, g \rangle dv$ for $f, g \in C^b_c(X, E)$. $C^b_c(X, E)$ is provided with the structure of a pre-Hilbert space with a norm $\|f\| = \sqrt{\langle f, f \rangle}$. $L^{p,q}(X, E, h, ds^2)$ denotes the space of integrable $E$-valued $(p, q)$-forms with respect to $ds^2$ and $h$ on $X$. We denote by $\bar{\partial} : L^{p,q}(X, E, h, ds^2) \rightarrow L^{p,q+1}(X, E, h, ds^2)$ the maximal closed extension of the original $\bar{\partial}$. Other operators are naturally extended to closed linear operators on $L^{p,q}(X, E, h, ds^2)$, we denote $D_2$ by $D^{b^q}_{2}$ and so on. In general, $D^{b^q}_{2} \subset D^{b^q}_{2}$. But it has been known due to Gaffney and Andreotti-Vesentini (cf : [3]) that if the hermitian metric $ds^2$ is complete, then $\bar{\partial}^* = \bar{\partial}_E$ and $D^{b^q}_{2} = \bar{\partial}$.

We say that the basic estimate holds at bi-degree $(p, q)$ if $ds^2$ is a complete
hermitian metric on $X$ and there exists a compact subset $K$ of $X$ and a constant $C_0$, satisfying

$$\|f\|^2 \leq C_0 \left( \|\bar{\partial}^* f\|^2 + \|\bar{\partial} f\|^2 + \int_K \langle f, f \rangle dv \right)$$

for all $f \in D^{q,l}_X \cap D^{q,s}_X$.

**Proposition 1.2** (cf: [7], [10]). Assume that the basic estimate holds at bi-degree $(p, q)$. Then $R^{p,q}_X$ and $R^{p,q+1}_X$ are closed and $\dim N^{p,q}_X/R^{p,q+1}_X < \infty$.

2. $L^2$ estimate on strongly $q$-concave manifolds

**Definition 2.1.** Let $X$ be a complex manifold of dimension $n$, and let $q$ be a positive integer. $X$ is said to be strongly $q$-concave if there is a real valued $C^\infty$ function $\Phi$ on $X$ satisfying

1) $X_c := \{ x \in X \mid \Phi(x) < c \} \subset X$ or $= X$ for any $c \in \mathbb{R}$,
2) the Levi form of $\Phi$ has at least $n - q + 1$ negative eigenvalues outside a compact subset $K$ of $X$.

We call $\Phi$ an exhaustion function and $K$ an exceptional set. A strongly $q$-concave manifold admits a bounded exhaustion function. In fact, if $\Phi$ is an unbounded satisfying 1) and 2), then $\overline{\Phi} = -\exp(-\Phi)$ is a bounded exhaustion function satisfying 1) and 2). From now on let $X$ be strongly $q$-concave, and $\Phi$ be an exhaustion function, and we assume $\sup_{x \in X} \Phi(x) = d < +\infty$, and $\inf_{x \in X} \Phi(x) = 0$. Moreover we assume that at least $n - q + 1$ negative eigenvalues of the Levi form of $\Phi$ are smaller than $-N$, and positive eigenvalues of the Levi form of $\Phi$ are smaller than $1/N$ for a positive integer $N \geq q + 3$ with respect to $ds^2$ (cf: [10]).

**Lemma 2.2.** Let $\mu$ be a $C^\infty$ function on $[0, d)$ with $\mu(0) = 0$, $\mu'(t) > 0$, and

$$\lim_{t \to d} \mu(t) = \infty, \int_0^d \mu(s) ds \geq C_1, \lim_{t \to d} \left\{ \int_0^t \mu(s) ds / \mu(t) \right\} = 0.$$ Then one can find a $C^\infty$ function $\mu_i$ on $[0, d)$ satisfying $\mu_i(t) \geq \mu(t)$ on $[0, d)$, $\mu_i(t) \geq \mu_i(t)$ on $[d_i, d)$, $\mu'(t) \leq \frac{1}{C_1} (\mu_i(t))^2$ on $[d_i, d)$ for $i = 1, 2$. Here, we can take $\mu_i(t) = \frac{1}{C_1} \int_0^t (\mu_i(s))^2 ds + L$, where $L$ is a constant.

**Proof.** See [9].

(1) **Basic estimate for $E$-valued $(0, p)$-form**

Given a $C^\infty$ function $\mu$ on $[0, d)$ satisfying the conditions of Lemma 2.2, we set
where $B$ is a positive number. We denote by $\omega_\delta$ the fundamental form associated to $ds_3^2$. As for the curvature form with respect to $h_3$, we have $\Theta_{h_3}=\Theta_h + B(\lambda(\Phi)\partial \overline{\partial} \Phi + 2\lambda(\Phi)^2 \partial \overline{\partial} \Phi \wedge \overline{\partial} \Phi)$.

**Lemma 2.3.** Let $\Gamma_1, \cdots, \Gamma_n$ be the eigenvalues of $\lambda(\Phi)\partial \overline{\partial} \Phi + 2\lambda(\Phi)^2 \partial \overline{\partial} \Phi$ at $x \in X$ with respect to $ds_3^2$. We assume $X_\delta, \supset K$, $C_1=2$ in Lemma 2.2. Then

1) $\sum_{\Gamma_1^k} \Gamma_i \leq -\lambda(\Phi(x))$ for $x \in X - X_{\delta_1}$.
2) $e(\Theta_h)$ and $e(d\omega_\delta)$ are locally bounded with respect to $h_\delta$ and $ds_3^2$.

**Proof.**
1) : See [10], Theorem 4.2.
2) : By Schwartz's inequality,

$$|e(\Theta_h)\varphi|_{h_\delta, \delta t} \leq |\Theta_h|_{h_\delta, \delta t} |\varphi|_{h_\delta, \delta t}$$

for $\varphi \in T_{\delta}^\infty \otimes E_x$.

As induced metric on $\text{Hom}(E, E) \simeq E \otimes E^*$ with respect to $ds_3^2$ is $h_\delta \otimes h_\delta^{-1}$, we have $|\Theta_h|_{h_\delta, \delta t} = |\Theta_h|_{h_\delta, \delta t} \leq |\Theta_h|_{h_\delta, ds_3^2}$. Therefore, we have

$$\sup_{x \in K} |e(\Theta_h)\varphi|_{h_\delta, \delta t} \leq \sup_{x \in K} |\Theta_h|_{h_\delta, ds_3^2} < \infty$$

for any compact set $K \subset X$.

On the other hand, we have for any $\varphi \in T_{\delta}^\infty \otimes E_x$,

$$|e(d\omega_\delta)\varphi|_{h_\delta, \delta t} = |[d\omega + i\lambda(\Phi) \overline{\partial} \Phi(\overline{\partial} \Phi - \partial \Phi)] \cdot \varphi|_{h_\delta, \delta t}$$

$$\leq |[d\omega]_{h_\delta, ds_3^2} + 2n\lambda(\Phi) |\overline{\partial} \Phi|_{h_\delta, ds_3^2} \cdot \varphi|_{h_\delta, \delta t}$$

Therefore, we have

$$\sup_{x \in K} |e(d\omega_\delta)\varphi|_{h_\delta, \delta t} \leq \sup_{x \in K} |[d\omega]_{h_\delta, ds_3^2} + 2n\lambda(\Phi) |\overline{\partial} \Phi|_{h_\delta, ds_3^2} < \infty$$

q.e.d.

The following proposition is basic for our purpose.

**Proposition 2.4.** Given a $C^\infty$ function $\mu$ on $[0, d)$ such that

1) $\mu$ satisfies the conditions of Lemma 2.3
2) $\sup_{x \in \Phi^{-1}(t)} |e(\Theta_h)\varphi|_{h_\delta, \delta t} < \mu(t)^2$ for any $t \in [d_1, d)$
3) $\sup_{x \in \Phi^{-1}(t)} |e(d\omega_\delta)\varphi|_{h_\delta, \delta t} < \mu(t)^2$ for any $t \in [d_1, d)$
4) \[ \int_0^d \sqrt{\mu(t)} \, dt = \infty \]

then the basic estimate holds at \((0, p) \) \((p \leq n - q - 1)\) with respect to \( \mu_1 \) and \( ds_1^2 \) for sufficiently large \( B \), where \( \lambda(t) = \mu_1(t) \) in Lemma 2.2.

Proof. See [10], Theorem 4.2.

(2) \( L^2 \) convergence for \( E \)-valued \((0, p)\)-forms

We denote by \( L^{p,q}_{loc}(X, E) \) the space of locally square integrable \( E \)-valued \((p, q)\)-forms on \( X \). \( L^{p,q}_{loc}(X, E) \) is a Fréchet space under the ordinary topology.

**Proposition 2.5.** Given \( f_j, f \in L^{p,q}_{loc}(X, E) \) with \( f_j \to f \) in \( L^{p,q}_{loc}(X, E) \), we can find a real valued \( C^\infty \) function \( \nu(t) \) on \([0, d)\) such that there exists a subsequence \( \{f_{i_\ell}\} \subset\{f_j\} \) with \( f_{i_\ell} \to f \) in \( L^{p,q}(X, E, ds^2, \overline{h}_\nu) \). Here \( \overline{h}_\nu = \exp(-\nu(\Phi)) \).

Proof. For any measurable set \( Y \subset X \), we denote by \( \|\cdot\|_{Y, \nu} \) the norm with respect to \( ds^2 \) and \( \exp(-\nu(\Phi)) \) on \( Y \). We fix any sequence \( \{d_i \in \mathbb{R} | i = 1, 2, \ldots \} \) with \( d_i \to d \).

We can find a real valued \( C^\infty \) function \( \nu_i \) on \([0, d)\) with \( \|f_j - f\|_{X, \nu_i} < \frac{1}{i} \).

Consider a real valued \( C^\infty \) function \( \nu \) on \([0, d)\) with \( \nu(x) \geq \max_{1 \leq i < l} \nu_j(x) \) on \([0, d_i]\) \((i = 1, 2, \ldots)\), and \( \|f\|_{X, \nu} < \infty \). We can select subsequences \( \{f_{i}\} \supset \{f_{i,i}\} \supset \cdots \{f_{k-1,i}\} \supset \{f_{k,i}\} \supset \cdots \) such that

\[ \|f_{k,i} - f\|_{X, \nu} < \frac{1}{i}, \quad \|f_{k,i} - f\|_{X \setminus X_{\nu}, \nu} < \frac{1}{k}. \]

Then we have

\[ \|f_{k,i} - f\|_{X, \nu} < \frac{2}{k}. \]

Therefore, \( f_{k,i} = f_{k,i} \to f \) in \( L^{p,q}(X, E, ds^2, \overline{h}_\nu) \).

q.e.d.

We set \( \overline{h}_\mu = \exp(-\mu(\Phi(x))) \), and \( ds^2_\mu = ds^2 + \mu(\Phi(x))\partial \Phi \otimes \overline{\partial} \Phi \). By the diagonalization for \( ds^2 \) and \( ds^2_\mu \), one can choose a basis \( \{\sigma_1, \ldots, \sigma_n\} \) of \( T^{1,0}_x \), which denotes the holomorphic cotangent space to \( X \) at \( x \), so that

\[ ds^2 = \sum_{i=1}^n \sigma_i \otimes \overline{\sigma}_i \quad \text{and} \quad ds^2_\mu = ds^2 + \mu(\Phi)\beta(x)\sigma_i \otimes \overline{\sigma}_i \]

at \( x \in X \)

where \( \{\sigma_1, \ldots, \sigma_n\} \) are the orthonormal basis of \( T^{1,0}_x \) with respect to \( ds^2 \), and \( \beta(x) \) is the non-negative \( C^\infty \) function on \( X \).
Proposition 2.6. Let $\mu(t), \mu_1(t), \mu_2(t)$ be as Lemma 2.2, satisfying $\mu(\Phi(x)) \geq \max\{1, \beta(x)\}$. Then we have $\|f\|^2_{\delta \mu^n, \delta \mu_2^n} \leq 2 \sup_{0 \leq t < d} \{\mu_1(t)^2 \cdot \exp(-\mu_1(t))\} \cdot \|f\|^2_{\delta \mu, \delta \mu_2} < \infty$. Here, $\|f\|^2_{\delta \mu^n, \delta \mu_2^n}$ denotes the norm with respect to $d\mu_2^n$ and $\overline{h}_{2\mu_2}$.

Proof. For $f \in L^{0, p}(X, E, d\mu_2^2, \overline{h}_{\mu_2})$ with $f = 0$ on $X$, \vspace{1cm}

$$
\|f\|^2_{\delta \mu^n, \delta \mu_2^n} = \int_X \langle f, f \rangle_{ds_2^2, h} \exp(-2\mu_2(\Phi(x)))dv_{\mu_1} \\
\leq \int_X \langle 1 + \mu_1(\Phi(x))\beta(x) \rangle \langle f, f \rangle_{ds_2^2, h} \exp(-2\mu_2(\Phi(x)))dv \\
\leq 2 \sup_{x \in X} \{\mu_1(\Phi(x))^2 \cdot \exp(-\mu_1(\Phi(x)))\} \cdot \int_X \langle f, f \rangle_{ds_2^2, h} \exp(-\mu_2(\Phi(x)))dv
$$

where $dv$ (resp. $dv_{\mu_1}$) is the volume element with respect to $d\mu_2^2$ (resp. $d\mu_1^2$).

In view of Lemma 2.2, $\mu_1(t) \geq \mu_2(t) \geq \mu(t)$ for $[d_1, d)$. Therefore,

$$
\|f\|^2_{\delta \mu^n, \delta \mu_2^n} \leq 2 \sup_{0 \leq t < d} \{\mu_1(t)^2 \cdot \exp(-\mu_1(t))\} \cdot \|f\|^2_{\delta \mu_1, \delta \mu_2} < \infty.
$$

q.e.d.

3. Proof of Theorem A-V

For $u \in L^{0, q}(X, E)$ and $v \in L^{0, q+1}(X, E)$, we denote $\delta u = v$ if the equation $(\forall, \delta \varphi) = (v, \varphi)$ holds for any $\varphi \in C_c^{0, q+1}(X, E)$. In view of Chapter 2, Proposition 3.1 in [10] and Proposition 4.5 in [17], we have only to show that for any $g \in L^{0, q}_c(X, E)$ such that there exists a sequence $\{f_i\} \subset L^{0, q-1}_c(X, E)$ with $\delta f_i \rightarrow g$ in $L^{0, q-1}_c(X, E)$, there exists $f \in L^{0, q-1}_c(X, E)$ such that $\delta f = g$.

In view of Proposition 2.5, there exists a real valued $C_\infty$ function $\nu$ on $[0, d)$ such that there exists a sequence $\{f_{\nu, i}\}$ with $\delta f_{\nu, i} \rightarrow g$ in $L^{0, q-1}(X, E, d\mu_2^2, \overline{h}_{\nu})$, where $\overline{h}_{\nu} = h \cdot \exp(-\nu(\Phi(x)))$.

Consider a real valued $C_\infty$ function $\mu$ on $[0, d)$ with

$$
\{\mu(t)\}^2 \geq \sup_{x \in \Phi^{-1}(t)} |\Theta_\delta|_{h, ds^2} \text{ for } t \in [d_1, d), \\
\{\mu(t)\}^2 \geq \sup_{x \in \Phi^{-1}(t)} (\|d\omega\|^2_{h, ds^2} + 2n\mu(t)|\delta \Phi |^2_{h, ds^2}) \text{ for } t \in [d_1, d), \\
\mu(t) \geq \max\{1, \beta(x)\} \text{ for } x \in X, \\
\mu(t) \geq \nu(t) \text{ for } t \in [0, d), \\
\int_{d_1}^{d} \mu(s)ds \geq 2, \int_{d}^{d} \sqrt{\mu(s)}ds = \infty, \lim_{t \rightarrow d} \left\{\int_{0}^{t} \mu(s)ds / \mu(t)\right\} = 0
$$

where $\beta(x)$ is the eigenvalue of $\delta \Phi \otimes \delta \Phi$ with respect to $d\mu_2^2$ at $x$.

We set $\lambda(t) := \mu(t)$, and $d\mu_3^2 := d\mu_2^2 + \lambda(t)(\delta \Phi) \otimes \delta \Phi$. $h_3 := h \exp(-B\mu_2(\Phi(x))) = \overline{h}_{B\mu_2}$, where $\mu_1(t)$ and $\mu_2(t)$ are in Lemma 2.2 and $B$ is a constant.
Then there exists a subsequence \( \{f_{n_k}\} \) with \( \partial f_{n_k} \to g \in L^{0,n-q}(X, E, d\bar{s}_I, h_\lambda) \) in view of Proposition 2.6, and the basic estimate holds at \( (0, n-q-1) \) with respect to \( ds_I^2 \) and \( h_\lambda \) for sufficiently large \( B \) in view of Proposition 2.4. Therefore, there exists \( f \in L^{0,n-q-1}(X, E, ds_I^2, h_\lambda) \) with \( \partial f = g \) by Proposition 1.2.

q.e.d.

4. Application

In this section, by using the basic estimate with respect to a complete hermitian metric we show Hausdorffness of a certain cohomology group.

Let \( X \) be a compact complex space of pure dimension \( \geq n \) whose singular points are isolated and \( X^* \) be the nonsingular part of \( X \). Let \( \pi : E \to X^* \) be a holomorphic vector bundle over \( X^* \) with a \( C^\infty \) fiber metric \( h \). We denote the canonical bundle of \( X^* \) by \( K_{X^*} \).

Suppose that \( \text{sing} \) consists of nonempty sets \( S_1 \) and \( S_2 \). Let \( \Phi \) be a family of closed subsets of \( X^* \) defined by \( \Phi = \{ C^X^* \text{ there exists a neighborhood } U \text{ of } S_1 \text{ such that } C/\pi \subset C \} \).

For any \( p_k \in S_1 (1 \leq k \leq l) \) we have a holomorphic embedding of the germ \( (X, p_k) \to (C^n, 0) \). We fix in the followings a holomorphic coordinate \( z(=z^{(p_k)})=(z_1, \ldots, z_n) \in C^n \) and the euclidian norm \( |z| \) of \( z \). We set \( X^*_c = X \cap \bigcup_{1 \leq k \leq l} \{ 0 < |z^{(p_k)}| < c \} \) for \( 0 < c \leq 1 \). We set \( F_c(z) = -\log((c-|z|^2) \cdot (\log(c/|z|))) \) and \( F(z) = F_1(z) \).

Then \( X^*_c \) is a complete Kähler manifold with respect to \( \partial \bar{\partial} F_c \).

We set \( L^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) = \{ f \in L^{p,q}_c(X^*, E) \text{ there exists a neighborhood } U \text{ of } S_1 \text{ such that } f|U \text{ is square integrable with respect to } \partial \bar{\partial} F \text{ and } h \} \). A sequence \( \{f_j\} \subset L^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) \) converges to \( f \in L^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) \) if \( f_j \to f \) in \( L^{p,q}_c(X^*, E) \) and there exists a neighborhood \( U \) of \( S_1 \) such that \( f_j \to f \) in \( L^{p,q}(U \setminus S_1, E, \partial \bar{\partial} F, h) \), and we write \( f_j \to f \) in \( L^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) \). We set \( H^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) = \text{Ker} \partial \bar{\partial} \cap L^{p,q}_c(X^*, E, \partial \bar{\partial} F, h)/\text{Im} \partial \bar{\partial} \cap L^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) \). \( H^{p,q}_c \) denotes the cohomology with supports in \( \Phi \). Then the sequence

\[ \rightarrow H^{0,q}_c(X^*, E) \to H^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) \to \lim H^{p,q}_c(U \setminus S_1, E, \partial \bar{\partial} F, h) \to \]

is exact. We set

\[ H^{p,q}_c(X^*, E, \partial \bar{\partial} F, h) = \text{Ker} \partial \bar{\partial} \cap L^{p,q}(X^*, E, \partial \bar{\partial} F, h)/\text{Im} \partial \bar{\partial} \cap L^{p,q}(X^*, E, \partial \bar{\partial} F, h) \].

**Proposition 4.1** (cf. [11]). Assume that \( K^\perp \otimes E \) is extendable to \( X \setminus S_2 \) as a holomorphic vector bundle. Then

\[ \lim H^{p,q}_c(U \setminus S_1, E, \partial \bar{\partial} F, h \exp(-mF)) = 0 \text{ for } q \geq 1 \]

and for sufficiently large \( m \).
Proof. We may assume $S_1 = \{p\}$. We set $c = 1$. As the curvature form of $K_{\tilde{\Theta}} \otimes I$ is, by the assumption, bounded with respect to the euclidian induced metric and $h$, there exists an integer $m_1$ such that the curvature form of $K_{\tilde{\Theta}} \otimes I$ is Nakano positive with respect to $h \exp(-m_1F)$. We set $m = m_1 + 1$. Then

$$(e(\Theta_{hm}) \Lambda u, u)_m \geq \|u\|_m^2$$

for $u \in C^0_{\lambda,q}(X_*^*, K_{\tilde{\Theta}} \otimes I)$. Here, $\Theta_{hm}$ denotes the curvature form of $K_{\tilde{\Theta}} \otimes I$ and $h_m : = h \exp(-mF)$ and $\cdot, \cdot)_m$, $\Lambda$ denote the norm, the inner product, the adjoint of Lefschetz operator with respect to $\partial \bar{\partial} F$ and $h_m$. Then we have

$$\|\partial u\|_m^2 + \|\partial h_m u\|_m^2 \geq \|u\|_m^2$$

for $u \in C^0_{\lambda,q}(X_*^*, K_{\tilde{\Theta}} \otimes I)$ by Kodaira-Nakano inequality. Therefore $H_{\bar{\partial}}^{\lambda,q}(X_*^*, E, \partial \bar{\partial} F, h \exp(-mF)) = 0$ for $q \geq 1$. Similarly we have $H_{\bar{\partial}}^{\lambda,q}(X_*^*, E, \partial \bar{\partial} F, h \exp(-m_cF_c)) = 0$ for $q \geq 1$ and any $0 < c < 1$ and sufficiently large integer $m_c$. Since $\partial \bar{\partial} F$ and $h \exp(-m_cF_c)$ is quasi-isometric near $S_1$ to $\partial \bar{\partial} F$ and $h_m$, we obtain $\lim_{m \to \infty} H_{\bar{\partial}}^{\lambda,q}(X_*^*, E, \partial \bar{\partial} F, h \exp(-mF)) = 0$ for $q \geq 1$.

q.e.d.

**Theorem 4.2.** Assume that $K_{\tilde{\Theta}} \otimes I$ is extendable to $X \setminus S_2$ as a holomorphic vector bundle. Then $H_{\bar{\partial}}^{0,n-1}(X^*, E)$ is Hausdorff.

Proof. By Proposition 4.1, we have only to show that for any $g \in L_{\bar{\partial}}^{0,n-1}(X^*, E, \partial \bar{\partial} F, h_m)$ such that there exists a sequence $(f_j) \subset L_{\bar{\partial}}^{0,n-2}(X^*, E, \partial \bar{\partial} F, h_m)$ with $\partial f_j \to g$ in $L_{\bar{\partial}}^{0,n-1}(X^*, E, \partial \bar{\partial} F, h_m)$, there exists $f \in L_{\bar{\partial}}^{0,n-2}(X^*, E, \partial \bar{\partial} F, h_m)$ such that $\partial f = g$.

Let $\rho$ be a $C^\infty$ function such that $\rho = 1$ on $X_*^\circ$ and $\rho = 0$ on $X^* \setminus X_*^\circ$. Then there exists a complete hermitian metric $d\bar{\partial}^2$ on $X^*$ and a fiber metric $\bar{h}$ of $E$ such that

1) $\partial f_j \to g$ in $L_{\bar{\partial}}^{0,n-1}(X^*, E, d\bar{\partial}^2, \bar{h})$.

2) there exists a compact subset $K$ of $X^*$ and a constant $C_0$, satisfying

$$\|(1 - \rho)u\|^2 \leq C_0(\|\bar{\partial} u\|_2^2 + \|\partial u\|_2^2)$$

for $u \in C^0_{\lambda,q}(X^*, E)$, $n - 2 \geq q$.

3) there exists a constant $C_1$, satisfying

$$\|\rho u\|^2 \leq C_1(\|u\|_2^2 + \|\bar{\partial} u\|_2^2 + \|\bar{\partial} u\|_2^2)$$

for $u \in C^0_{\lambda,q-2}(X^*, E)$

where $\|\cdot\|$ denotes the norm with respect to $d\bar{\partial}^2$ and $\bar{h}$.

Indeed, we set $d\bar{\partial}^2 = \partial \bar{\partial} F$ and $\bar{h} = h_m$ near $S_1$ in Proposition 4.1. Then we define $d\bar{\partial}^2$ and $\bar{h}$ near $S_2$ in the same way as in the construction of $d\bar{\partial}^2$ and $h_i$ in Section 3 because $X^*$ is strongly 1-concave (cf: [1]). By patching these metrics defined near $S_1$ and $S_2$ with any metric inbetween we obtain a complete hermitian
metric $d\bar{s}^2$ on $X^*$ and a fiber metric $\tilde{h}$ of $E$ enjoying the above mentioned properties. In fact, condition 1) and 2) are satisfied because $\text{supp}[(1-\rho)u] \subset X^* \setminus X_{\frac{1}{2}}^*$ and, by the assumption and Proposition 4.1 in [11], condition 3) is satisfied, too.

Therefore the basic estimate holds at bi-degree $(0, n-2)$.

q.e.d.

References


