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ON AN EXTENSION OF THE HILBERTIAN CENTRAL LIMIT THEOREM TO DIRICHLET FORMS

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Abstract

In a recent paper ([2]), Nicolas Bouleau provides a new tool, based on the language of Dirichlet forms, to study the propagation of errors and reinforce the historical approach of Gauss. In the same way that the practical use of the normal distribution in statistics may be explained by the central limit theorem, the aim of this paper is to underline the importance of a family of error structures by asymptotic arguments.

1. Introduction

The choice of a relevant mathematical language for speaking about errors and their propagations is an old topic. A new approach based on the theory of Dirichlet forms ([1], [9], [14]) has been recently suggested in [2], [3]. This method is a natural and powerful extension of the seminal works of Gauss ([2]) and it seems to be an appropriate framework to study the sensitivity to small changes of parameters in physical and financial models ([3], Chapter 7).

From now on, we shall call a term $(W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ an error structure, if $(W, \mathcal{W}, m)$ is a probability space, $\mathbb{D}$ is a dense sub-vector space of $L^2(W, \mathcal{W}, m)$ (also denoted by $L^2(m)$) and Γ is a positive symmetric bilinear map from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$ fulfilling:
1) the functional calculus of class $C^1 \cap \mathrm{Lip}$ meaning that if $U \in \mathbb{D}^n$, $V \in \mathbb{D}^p$, for $F \in C^1(\mathbb{R}^n, \mathbb{R}) \cap \mathrm{Lip} = \{C^1 \text{ and Lipschitz}\}$ and $G \in C^1(\mathbb{R}^p, \mathbb{R}) \cap \mathrm{Lip}$ one has $(F(U), G(V)) \in \mathbb{D}^2$ and

$$\Gamma[F(U), G(V)] = \sum_{i,j} \frac{\partial F}{\partial x_i}(U) \frac{\partial G}{\partial x_j}(V) \Gamma[U_i, V_j] \quad m\text{-a.e.}, \tag{1}$$

2) $1 \in \mathbb{D}$ (this implies $\Gamma[1, 1] = 0$),
3) the bilinear form $\mathcal{E}[F, G] = (1/2)\mathbb{E}_m[\Gamma[F, G]]$ defined on $\mathbb{D} \times \mathbb{D}$ is closed i.e. $\mathbb{D}$ is complete under the norm of the graph

$$\| \, . \, \|_{\mathbb{D}} = (\| \, . \, \|^2_{L^2(m)} + \mathcal{E}[\, . \,])^{1/2}.$$

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We always write $\Gamma[F]$ for $\Gamma[F, F]$ and $\mathcal{E}[F]$ for $\mathcal{E}[F, F]$.

From the hypotheses mentioned above, $\mathcal{E}$ is a local Dirichlet form and Γ its associated squared field operator. The property 1) is none other than the so-called Gauss's law of small errors propagation ([2]), thus, when $U \in \mathbb{D}$, the intuitive meaning of $\Gamma[U]$ is the conditional variance of the error on U given U. Moreover this first order calculus dealing with variances can naturally be reinforced by a calculus on biases involving the infinitesimal generator associated to $\mathcal{E}$ ([3], Chapter 3).

Thanks to property 3), the domain $\mathbb{D}$ is preserved by Lipschitz functions ([3], p.40): if $F : \mathbb{R}^n \to \mathbb{R}$ is a contraction in the following sense

$$|F(x) - F(y)| \leq \sum_{i=1}^{n} |x_i - y_i|$$

then for $U = (U_1, \ldots, U_n) \in \mathbb{D}^n$ one has $F(U) \in \mathbb{D}$ and

$$\Gamma[F(U_1, \ldots, U_n)]^{1/2} \leq \sum_{i=1}^{n} \Gamma[U_i]^{1/2}. \tag{2}$$

As mentioned in [4], one of the lacks of this new theory in practical cases is the need of *a priori* choices. In fact, for a rational treatment, error hypotheses should be obtained by statistical methods. In finite dimension, error structures are connected (through a robust identification) to statistical parametric methods thanks to Fisher information [4]. Moreover, this study can be reinforced by the refinement of the main limit theorems of the probability theory in our setting ([5], [6]).

In this way, Bouleau and Hirsch have introduced notions of independence and convergence for error structures that extend the independence and the convergence in distribution for random variables ([1], Chapter 5). By using these definitions, they prove a central limit theorem in finite dimension for erroneous random variables, the errors being modelised by error structures ([1], p.220). The main contribution of our paper is to propose an infinite dimensional extension of this result, at the very least, in the case of a separable Hilbert space. This finding, associated with the recent improvements of the Donsker theorem ([5], [6]), can explain the importance of the error structures of the Ornstein-Uhlenbeck type (structures where the measure is Gaussian and where Γ operates on cylinder functions as a first order differential operator with constant coefficients) in the applications.

From a technical point of view, the key stone of our study will be the notion of the vectorial domain of a Dirichlet form which was defined by Feyel and de La Pradelle ([8], p.900).

2. Preliminaries on error structures

2.1. Finite dimensional images and infinite products. Let us present the two fundamental algebraic operations on error structures that are compatible with the construction of probability spaces. We refer to [4] for their statistical interpretations.

DEFINITION 1. Let $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ be an error structure and U a random variable in $\mathbb{D}^d$. For $f \in C^1(\mathbb{R}^d, \mathbb{R}) \cap \mathrm{Lip}$, we put

$$\Gamma_U[f](x) = \mathbb{E}_m[\Gamma[f(U)] \mid U = x], \quad x \in \mathbb{R}^d.$$

Thus, $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_*m, C^1(\mathbb{R}^d, \mathbb{R}) \cap \mathrm{Lip}, \Gamma_U)$ is a closable error pre-structure in the sense of [3], p.44. Let U_*S be its smallest closed extension called the image structure of S by U.

DEFINITION 2. Let $S_n = (W_n, \mathcal{W}_n, m_n, \mathbb{D}_n, \Gamma_n)$, $n \geq 0$, be a family of error structures. The product structure $(W, \mathcal{W}, m, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} S_n$ is defined by $(W, \mathcal{W}, m) = \left(\prod_{n=0}^{\infty} W_n, \prod_{n=0}^{\infty} \mathcal{W}_n, \bigotimes_{n=0}^{\infty} m_n\right)$ with an explicit domain $\mathbb{D}$ ([1], p.203) and $\forall F \in \mathbb{D}$, $\Gamma[F] = \sum_{n=0}^{\infty} \Gamma_n[F]$ where the operator Γ_n acts on the n-th variable.

Thanks to the preceding definitions, it is easy to equip the fundamental spaces encountered in stochastic models (Wiener space, Monte Carlo space, Poisson space) with error structures, starting from elementary structures on $\mathbb{R}$ ([3], Chapter 6).

Now, in order to deal with Hilbert valued random variables, we first have to give sense to a coherent extension of the domain of an error structure.

2.2. Vectorial domain of an error structure.

DEFINITION 3. We say that an error structure S owns a gradient if both a separable Hilbert space $(\mathcal{H}, \| \ \|_{\mathcal{H}})$ and an operator ∇ from $\mathbb{D}$ into $L^2(m; \mathcal{H})$ (where $L^2(m; \mathcal{H})$ is the space of square integrable random variables with values in $\mathcal{H}$) called the gradient exist such that

$$\forall U \in \mathbb{D}, \quad \|\nabla U\|^2_{\mathcal{H}} = \Gamma[U].$$

Thus, according to (1), a gradient fulfills the classical chain rule.

From now on, we suppose that the error structure S satisfies the property mentioned in Definition 3. In practice, this assumption is not restrictive because Mokobodzki showed a gradient exists whenever $\mathbb{D}$ is separable ([1], p.242).

We introduce a slight variant of the gradient which is very useful when computing errors on Wiener space thanks to the Itô formula ([1], p.145 and [3], p.167). This notion has been introduced by Feyel and de La Pradelle in the Gaussian case and used

by Bouleau and Hirsch to prove important results concerning the regularity of solutions of stochastic differential equations with Lipschitz coefficients ([1], Chapter 4).

DEFINITION 4. Let $(\hat{W}, \hat{\mathcal{W}}, \hat{m})$ be a probability space which is a copy of $(W, \mathcal{W}, m)$ and J an isometry from $\mathcal{H}$ into $L^2(\hat{m})$. For $U \in \mathbb{D}$, let $U^{\#}$ be the derivative of U defined by

$$U^{\#} = J(\nabla U) \in L^2(m \times \hat{m}).$$

Of course we can suppose (what we shall do) that $\forall h \in \mathcal{H}$, $\mathbb{E}_{\hat{m}}[J(h)] = 0$. Thus, $\mathbb{E}_{\hat{m}}[U^{\#}] = 0$.

Let $(B, \| \ \|_B)$ be a separable Banach space and B' its topological dual space. Let $\langle \ , \ \rangle$ be the duality between B and B'. One of the main interests of the derivative is to allow a natural definition of a tensor product of $\mathbb{D}$ with B.

DEFINITION 5. Let $\mathbb{D}_B$ denote the vector space of random variables U in $L^2(m; B)$ such that there exists g in $L^2(m \times \hat{m}; B)$ such that

$$\forall \lambda \in B', \quad \langle \lambda, U \rangle \in \mathbb{D} \quad \text{and} \quad \langle \lambda, U \rangle^{\#} = \langle \lambda, g \rangle.$$

Then we put $g = U^{\#}$ and equip $\mathbb{D}_B$ with the norm

$$\|U\|_{\mathbb{D}_B} = \left(\|U\|^2_{L^2(m;B)} + \frac{1}{2} \|U^{\#}\|^2_{L^2(m \times \hat{m}; B)} \right)^{1/2}.$$

From the preceding definition we have obviously $\mathbb{D}_{\mathbb{R}^d} = \mathbb{D}^d$.

REMARK 1. Let B be a separable Hilbert space. If $U \in L^2(m; B)$ the following statements are equivalent:

i) $U \in \mathbb{D}_B$.

ii) There is an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of B such that $\forall i \in \mathbb{N}$, $\langle e_i, U \rangle \in \mathbb{D}$ and $\sum_{i=0}^{\infty} \mathcal{E}[\langle e_i, U \rangle] < \infty$.

iii) For all orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of B, $\forall i \in \mathbb{N}$, one has $\langle e_i, U \rangle \in \mathbb{D}$ and $\sum_{i=0}^{\infty} \mathcal{E}[\langle e_i, U \rangle] < \infty$.

Henceforth, we suppose that B satisfies the approximation hypothesis: There exists a sequence $(P_n)_{n \in \mathbb{N}}$ of continuous linear operators of finite rank from B into B such that $\forall x \in B$, $\lim_{n \to \infty} P_n(x) = x$ (this assumption holds when B owns a Schauder basis, particularly, when B is a separable Hilbert space or the Wiener space).

The following proposition extends the property (2) of stability of the domain $\mathbb{D}$ by contractions and the functional calculus (1) to $\mathbb{D}_B$.

Proposition 1. *Let F be a contraction from B into $\mathbb{R}$. If $U \in \mathbb{D}_B$ then $F(U) \in \mathbb{D}$ and $\Gamma[F(U)] \leq \mathbb{E}_{\hat{m}}[\|U^{\#}\|_B^2]$. Moreover, if we suppose that F is of class C^1, $F(U)^{\#} = \langle F'(U), U^{\#}\rangle$.*

Proof. Let us first suppose that $F \in C^1(B, \mathbb{R}) \cap \mathrm{Lip}$. For all $n \in \mathbb{N}$, we define the cylinder approximations of F by $F_n = F \circ P_n$. According to the Banach Steinhaus theorem we have $\sup_{N\in\mathbb{N}} \|P_n\| < \infty$ thus, by dominated convergence, $F_n(U) \xrightarrow[n\to\infty]{} F(U)$ in $L^2(m)$. Moreover, since F_n is a cylinder function, $F_n(U) \in \mathbb{D}$ and from the chain rule it follows that $F_n(U)^{\#} = \langle F'(P_n(U)), P_n(U^{\#})\rangle$. Hence, we easily obtain that $F_n(U)^{\#} \xrightarrow[n\to\infty]{} \langle F'(U), U^{\#}\rangle$ in $L^2(m \otimes \hat{m})$. Using the fact that the derivative is a closed operator, the conclusion holds.

When F is only a contraction, the result follows from an adaptation of the proof of Theorem 2.2.3 of [1], p.140. □

A direct consequence of the preceding proposition is to allow the construction of the image of S by an element of its vectorial domain by using the same idea as in Definition 1.

DEFINITION 6. For $U \in \mathbb{D}_B$, the term $(B, \mathcal{B}(B), U_*m, C^1(B, \mathbb{R}) \cap \mathrm{Lip}, \Gamma_U)$ where $\forall F \in C^1(B, \mathbb{R}) \cap \mathrm{Lip}$, $\Gamma_U[F] = \mathbb{E}_m[\Gamma[F(U)] \mid U]$, is a closable error pre-structure in the sense of [3], p.44. Let U_*S be its smallest closed extension, and $(\mathcal{E}_U, \mathbb{D}_U)$ the associated Dirichlet form. The structure U_*S is called the image of S by U or the Dirichlet law of U and $\forall F \in \mathbb{D}_U$, we have $\mathcal{E}_U[F] = \mathcal{E}[F(U)]$.

EXAMPLE 1. One of the simplest examples of error structures is the term

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), \gamma : u \mapsto u'^2)$$

where μ is the standard normal distribution on $\mathbb{R}$ and $H^1(\mu)$ the first Sobolev space associated to μ. We consider (Definition 2) the following product

$$S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), \gamma).$$

Let $(g_n)_{n\in\mathbb{N}}$ be the coordinate mappings of S and $(\phi_n)_{n\in\mathbb{N}}$ an orthonormal basis of $L^2([0, 1], dx)$. For $t \in [0, 1]$, we set

$$B_t = \sum_{n\in\mathbb{N}} \left(\int_0^t \phi_n(s)\, ds \right) g_n.$$

Thus, the continuous process $(B_t)_{t\in[0,1]}$ is a standard Brownian motion and we can easily see that it belongs to $\mathbb{D}_B$ where $B = C_0([0, 1], \mathbb{R})$ is the Wiener space. The image of

S by $(B_t)_{t\in[0,1]}$ is known as the Ornstein-Uhlenbeck error structure on B and let Γ_{OU} be its squared field operator. This structure possesses a gradient which is none other than the gradient in the Malliavin sense with an adjoint operator that extends the Itô integral ([15]). If we put $\mathcal{A} = \{f(\lambda_1, \ldots, \lambda_n); n \in \mathbb{N}, (\lambda_1, \ldots, \lambda_n) \in B' \text{ and } f \in C^1(\mathbb{R}^n, \mathbb{R}) \cap \mathrm{Lip}\}$, we can see that $\forall F = f(\lambda_1, \ldots, \lambda_n) \in \mathcal{A}$,

$$\Gamma_{OU}[F] = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(\lambda_1, \ldots, \lambda_n)a_{i,j} \tag{3}$$

where the coefficients $a_{i,j}$ only depend on (λ_i, λ_j). Thus, from now on, we will say that an error structure on a separable Banach space is of the Ornstein-Uhlenbeck type if the associated measure is Gaussian and if the operator Γ is of the form (3) on smooth cylinder functions.

Now, we extend the definition of the Dirichlet independence introduced by Bouleau and Hirsch in finite dimension ([1], p.217). Let $\otimes$ be the product of two error structures.

DEFINITION 7. For $(U, V) \in (\mathbb{D}_B)^2$, U and V are said to be Dirichlet independent if $U_*S \otimes V_*S = (U, V)_*S$. In other terms, the Dirichlet law of (U, V) is the product of the Dirichlet laws of U and V.

We can show that Theorem 4.1.4, p.218 of [1] remains valid in our framework, thus, we have the following characterization of the Dirichlet independence on $\mathbb{D}_B$.

Proposition 2. *For U and V in $\mathbb{D}_B$ to be Dirichlet independent, it is necessary and sufficient that the following four conditions are fulfilled*:
a) *U and V are independent on the probability space $(W, \mathcal{W}, m)$,*
b) $\forall(\lambda_1, \lambda_2) \in (B')^2$, $\mathbb{E}_m[\Gamma[\langle\lambda_1, U\rangle, \langle\lambda_2, V\rangle] \mid U, V] = 0$ *m-a.e.*,
c) $\forall\lambda \in B'$, $\mathbb{E}_m[\Gamma[\langle\lambda, U\rangle] \mid U, V] = \mathbb{E}_m[\Gamma[\langle\lambda, U\rangle] \mid U]$ *m-a.e.*,
d) $\forall\lambda \in B'$, $\mathbb{E}_m[\Gamma[\langle\lambda, V\rangle] \mid U, V] = \mathbb{E}_m[\Gamma[\langle\lambda, V\rangle] \mid V]$ *m-a.e.*

Finally, we introduce a notion of convergence on the vectorial domain that reinforces the convergence in distribution for random variables taking into account the underlying Dirichlet forms.

DEFINITION 8. We say that a sequence $(U_n)_{n\in\mathbb{N}}$ in $\mathbb{D}_B$ converges in Dirichlet law if there exists an error structure $\hat{S} = (B, \mathcal{B}(B), \nu, \hat{\mathbb{D}}, \hat{\Gamma})$ such that:
i) $(U_n)_*m \xrightarrow[n\to\infty]{} \nu$ weakly,
ii) $C^1(B, \mathbb{R}) \cap \mathrm{Lip} \subset \hat{\mathbb{D}}$ and $\forall F \in C^1(B, \mathbb{R}) \cap \mathrm{Lip}$, $\mathcal{E}[F(U_n)] \xrightarrow[n\to\infty]{} \hat{\mathcal{E}}[F]$.
For convenience, we shall say that $(U_n)_{n\in\mathbb{N}}$ converges in Dirichlet law towards $\hat{S}$.

In the next section, we prove an extension of the central limit theorem in Hilbert spaces (in the sense of the preceding definition).

3. Main result

We suppose that $S = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)$ owns a gradient $\nabla\colon \mathbb{D} \to L^2(m; \mathcal{H})$. Let $\#\colon \mathbb{D} \to L^2(m \times \hat{m})$ be a derivative operator. Although noncanonical, the choice of the isometry J is not specified because, according to Remark 1, such a choice leads to the same definition of $\mathbb{D}_H$ when H is a separable Hilbert space.

Theorem 1. *Let $(H, \langle\ \rangle)$ be a separable Hilbert space. Let $(U_n)_{n\in\mathbb{N}^*}$ be a sequence of centered random variables in $\mathbb{D}_H$, Dirichlet independent with the same Dirichlet law. If Σ is the covariance operator of U_1, then, $V_n = (U_1 + \cdots + U_n)/\sqrt{n}$ converges in Dirichlet law towards $\hat{S} = (H, \mathcal{B}(H), \nu, \hat{\mathbb{D}}, \hat{\Gamma})$ where*

i) *ν is a centered Gaussian measure on H with covariance operator Σ,*

ii) *$\forall F \in C^1(H, \mathbb{R}) \cap \mathrm{Lip},\ F \in \hat{\mathbb{D}}$ and*

$$\hat{\mathcal{E}}[F] = \frac{1}{2}\int_{H^2} \langle F'(x), y\rangle^2\, d\mu(x, y) \tag{4}$$

where μ is a centered Gaussian measure on H^2 with covariance operator K defined by

$$\langle Kx, y\rangle_{H^2} = \langle \Sigma x_1, y_1\rangle + 2\mathcal{E}[\langle U_1, x_2\rangle, \langle U_1, y_2\rangle]$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in H^2,

iii) *the form $(C^1(H,\mathbb{R})\cap \mathrm{Lip}, \hat{\mathcal{E}})$ is closable. Let $(\hat{\mathbb{D}}, \hat{\mathcal{E}})$ be its smallest closed extension that owns a squared field operator $\hat{\Gamma}$.*

Thus, V_n converges in Dirichlet law towards a structure of the Ornstein-Uhlen-beck type.

REMARK 2. The operator K is in the trace class because Σ is a covariance operator and $U_1 \in \mathbb{D}_H$ (Remark 1).

Before we turn to the proof, note the following. The hypothesis of Dirichlet independence allows us to consider that the variables U_i are of the form $U_i \circ g_i$ where the $(g_i)_{i\in\mathbb{N}^*}$ are the coordinate mappings of a product of error structures.

Indeed, define $s = (\Omega, \mathcal{A}, P, \mathbb{d}, \gamma) = (W, \mathcal{W}, m, \mathbb{D}, \Gamma)^{\mathbb{N}^*}$ and $e[\,.\,,\,.\,] = (1/2)\mathbb{E}_P[\gamma[\,.\,,\,.\,]]$. We set

$$\ell^2(\mathcal{H}) = \left\{(h_n)_{n\in\mathbb{N}^*};\ \forall i \in \mathbb{N}^*,\ h_i \in \mathcal{H} \text{ and } \sum_{i=1}^{\infty} \|h_i\|_{\mathcal{H}}^2 < \infty\right\}.$$

Classically, we can construct a gradient operator $\tilde{\nabla}$ for s setting

$$\tilde{\nabla}\colon F \in \mathbb{d} \mapsto (\ldots, \nabla_{[i]}[F], \ldots) \in L^2(P; \ell^2(\mathcal{H}))$$

where $\nabla_{[i]}$ means that the operator ∇ acts on the i-th variable of F. In the same way, if $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ is a copy of $(\Omega, \mathcal{A}, P)$, we obtain the following derivative operator $'$ for s:

$$\forall F \in \mathbb{d}, \quad F'(\omega, \hat{\omega}) = \sum_{i \in \mathbb{N}^*} J[\nabla_{[i]}[F](\omega)](\hat{\omega}_i) \in L^2(P \otimes \hat{P}) \tag{5}$$

where the series converge in $L^2(\hat{P})$. For all $i \in \mathbb{N}^*$, we put $X_i : \omega = (\omega_j)_{j \in \mathbb{N}^*} \in \Omega \mapsto U_i(\omega_i)$. Then, we can state the following easy lemma.

Lemma 1. a) *With the preceding notations,* $\forall i \in \mathbb{N}^*$, $X_i \in \mathbb{d}_H$ *and* $(X_i)'(\omega, \hat{\omega}) = U_i^{\#}(\omega_i, \hat{\omega}_i)$.

b) *The variables* $(X_i)_{i \in \mathbb{N}^*}$ *are Dirichlet independent with the same Dirichlet law.*

Let $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. The next statement will be used hereafter.

Lemma 2. $\forall F \in C^1(H, \mathbb{R}) \cap \mathrm{Lip}$, *we have* $\mathcal{E}[F(V_n)] = e[F(Z_n)]$.

Proof. We first suppose that F is a cylinder function. To lighten the notations we only consider the case $F = f(\langle x, \ . \rangle)$ where $x \in H$ and $f \in C^1(\mathbb{R}, \mathbb{R}) \cap \mathrm{Lip}$.

By the functional calculus we have

$$e[F(Z_n)] = \frac{1}{2n} \int_\Omega \sum_{i,j=1}^{n} f'(\langle Z_n, x \rangle)^2 \gamma[\langle x, X_i \rangle, \langle x, X_j \rangle] \, dP.$$

For every k, X_k is Dirichlet independent of $(X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n)$. Then, Proposition 2 entails

$$e[F(Z_n)] = \frac{1}{2n} \int_\Omega \sum_{i=1}^{n} f'(\langle Z_n, x \rangle)^2 \mathbb{E}_P[\gamma[\langle x, X_i \rangle] \mid X_i] \, dP.$$

Since $(X_1, \ldots, X_n)$ has the same law as $(U_1, \ldots, U_n)$ and $\gamma[\langle x, X_i \rangle](\omega) = \Gamma[\langle x, U_i \rangle](\omega_i)$, the conclusion holds.

For the general case, let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of H and P_n the projection on the vector space $\mathrm{vect}(e_0, \ldots, e_n)$ spanned by $(e_0, \ldots, e_n)$. Since $Z_n \in \mathbb{d}_H$ and $V_n \in \mathbb{D}_H$, using the same argument as in the proof of Proposition 1 we see that $\forall F \in C^1(H, \mathbb{R}) \cap \mathrm{Lip}$,

$$e[F \circ P_k(Z_n)] \xrightarrow[k \to \infty]{} e[F(Z_n)] \quad \text{and} \quad \mathcal{E}[F \circ P_k(V_n)] \xrightarrow[k \to \infty]{} \mathcal{E}[F(V_n)].$$

The result follows by uniqueness of the limit. □

Now we can come back to the proof of the theorem.

Proof of Theorem 1. The convergence in distribution of the sequence $(V_n)_* m$ towards ν is a consequence of the classical central limit theorem for Hilbert valued random variables ([13]).

Let us first show that $\forall F \in C^1(H, \mathbb{R}) \cap \mathrm{Lip}$, we have

$$\mathcal{E}[F(V_n)] \xrightarrow[n\to\infty]{} \hat{\mathcal{E}}[F].$$

According to Lemma 2, we study the asymptotic behavior of $e[F(Z_n)]$. Since $Z_n \in \mathbb{d}_H$, we obtain from Proposition 1 that

$$2e[F(Z_n)] = \int_\Omega \int_{\hat{\Omega}} \left\langle F'(Z_n),\, Z_n' \right\rangle^2 dP\, d\hat{P}.$$

Since the pairs (X_i, X_i') are i.i.d., the central limit theorem in Hilbert spaces ensures that (Z_n, Z_n') converges in distribution in H^2 towards a centered Gaussian measure μ with a covariance operator K fulfilling

$$\langle Kx, y\rangle_{H^2} = \mathbb{E}_P \mathbb{E}_{\hat{P}}[\langle (X_1, X_1'), x\rangle_{H^2} \langle (X_1, X_1'), y\rangle_{H^2}]$$

for all $x = (x_1, x_2)$, and $y = (y_1, y_2)$ in H^2.

Using the definition of X_1 and Lemma 1, it is easy to see that

$$\mathbb{E}_P[\langle X_1, x_1\rangle \langle X_1, y_1\rangle] = \mathbb{E}_m[\langle U_1, x_1\rangle \langle U_1, y_1\rangle] = \langle \Sigma x_1, y_1\rangle$$

and that

$$\mathbb{E}_P \mathbb{E}_{\hat{P}}[\langle X_1', x_2\rangle \langle X_1', y_2\rangle] = \mathbb{E}_m \mathbb{E}_{\hat{m}}[\langle U_1^{\#}, x_2\rangle \langle U_1^{\#}, y_2\rangle] = 2\mathcal{E}[\langle U_1, x_2\rangle, \langle U_1, y_2\rangle].$$

Now, as a consequence of Fubini's Theorem we obtain $\forall (z_1, z_2) \in \mathbb{R}^2$

$$\mathbb{E}_P \mathbb{E}_{\hat{P}}[\langle X_1, z_1\rangle \langle X_1', z_2\rangle] = \mathbb{E}_m \mathbb{E}_{\hat{m}}[\langle U_1, z_1\rangle \langle U_1^{\#}, z_2\rangle] = \mathbb{E}_m[\langle U_1, z_1\rangle \mathbb{E}_{\hat{m}}[\langle U_1^{\#}, z_2\rangle]].$$

Moreover from Definition 4 we have

$$\mathbb{E}_{\hat{m}}[\langle U_1^{\#}, z_2\rangle] = \mathbb{E}_{\hat{m}}[\langle U_1, z_2\rangle^{\#}] = 0.$$

Hence,

$$\langle Kx, y\rangle_{H^2} = \langle \Sigma x_1, y_1\rangle + 2\mathcal{E}[\langle U_1, x_2\rangle, \langle U_1, y_2\rangle].$$

Furthermore, using the independence of the (X_i, X_i')'s, it follows that

$$\mathbb{E}_P \mathbb{E}_{\hat{P}}[\|Z_n\|^2 + \|Z_n'\|^2] = \mathbb{E}_m[\|U_1\|^2] + \mathbb{E}_m \mathbb{E}_{\hat{m}}[\|U_1^{\#}\|^2] = \int_{H^2} \|x\|_{H^2}^2 \, d\mu(x),$$

thus, the $\|(Z_n, Z_n')\|_{H^2}^2$'s are uniformly integrable.

Since the function ϕ: $(x, y) \in H^2 \mapsto \langle F'(x), y\rangle^2$ is continuous and satisfies $\phi(x, y) \leq c\|(x, y)\|_{H^2}^2$, where c is a constant, we have

$$2\mathcal{E}[F(V_n)] \xrightarrow[n\to\infty]{} \int_{H^2} \langle F'(x), y\rangle^2 \, d\mu(x, y).$$

To conclude, it remains to show the closability of the form $\hat{\mathcal{E}}$ defined on $C^1(H, \mathbb{R}) \cap \mathrm{Lip}$ by (4). The proof is based on three lemmas whose proofs are left in Appendix.

Lemma 3. *Let β be an orthonormal basis of H, one has*

$$(C^1(H, \mathbb{R}) \cap \mathrm{Lip}, \hat{\mathcal{E}}) \text{ is closable} \iff (\mathcal{A}_\beta, \hat{\mathcal{E}}) \text{ is closable}$$

where $\mathcal{A}_\beta = \bigcup_{p\in\mathbb{N}^*} \{ f(\langle e_1, \ .\rangle, \ldots, \langle e_p, \ .\rangle);\ (e_1, \ldots, e_p) \in \beta^p,\ \ f \in C^1(\mathbb{R}^p, \mathbb{R}) \cap \mathrm{Lip}\}$.

Moreover, when one of the assertions is fulfilled their smallest closed extensions coincide.

Since Σ is a covariance operator, it is positive and belongs to the trace class. Thus ([12]), there exists an orthonormal basis $\beta_0 = (e_i)_{i\in\mathbb{N}}$ of H consisting of eigenvectors of Σ. According to the preceding lemma, we only have to prove that $(\mathcal{A}_{\beta_0}, \hat{\mathcal{E}})$ is closable. Let $(\sigma_i^2)_{i\in\mathbb{N}}$ be the corresponding eigenvalues, hence, the sequence $(\langle e_i, \ .\rangle)_{i\in\mathbb{N}}$, defined on $(H, \mathcal{B}(H), \nu)$, is a sequence of independent and centered Gaussian random variables on $\mathbb{R}$ with variances $(\sigma_i^2)_{i\in\mathbb{N}}$.

From $U_1 \in \mathbb{D}_H$, it follows from Proposition 1 that the bilinear operator

$$T : \begin{pmatrix} H^2 \to \mathbb{R} \\ (x, y) \mapsto \mathcal{E}[\langle x, U_1\rangle, \langle y, U_1\rangle] \end{pmatrix}$$

is continuous because

$$\mathcal{E}[\langle x, U_1\rangle, \langle y, U_1\rangle] \leq \|x\| \, \|y\| \mathbb{E}_{\hat{m}} \mathbb{E}_m[\|U_1^{\#}\|^2].$$

Therefore, there exists a bounded operator $C : H \to H$ such that $T(x, y) = \langle Cx, y\rangle$. The operator C is clearly self-adjoint and positive. Let us define

$$D_C = C^{1/2} D : \mathcal{A}_{\beta_0} \to L^2(\nu; H)$$

where D stands for the Fréchet derivative in H. Thus, we have the following equality: $\forall F \in \mathcal{A}_{\beta_0}$,

$$\hat{\mathcal{E}}[F] = \int_H \|D_C[F]\|^2 \, d\nu.$$

Therefore, the closability of the form $(\mathcal{A}_{\beta_0}, \hat{\mathcal{E}})$ is equivalent to the closability of D_C in $L^2(\nu; H)$. We have formulated our closability problem in terms of directional gradient in the sense of Goldys et al. in [10]. According to the following lemma we can impose that $\forall i \in \mathbb{N},\ \sigma_i^2 > 0$.

Lemma 4. *When we study the closability of* $(\mathcal{A}_{\beta_0}, \hat{\mathcal{E}})$ *we can suppose that* Σ *is injective.*

Thus, the operator $V = C^{1/2}\Sigma^{-1/2}$ is well defined on $\mathrm{dom}(V) = \Sigma^{1/2}(H)$ and we have the following result:

Lemma 5. $(\mathcal{A}_{\beta_0}, D_C)$ *is closable* $\Leftrightarrow$ $(\mathrm{dom}(V), V)$ *is closable.*

We show the closability of V: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\Sigma^{1/2}(H)$ fulfilling $x_n \xrightarrow[n\to\infty]{} 0$ and $Vx_n \xrightarrow[n\to\infty]{} u$. For all $n \in \mathbb{N}$, let h_n be the unique element in H such that $x_n = \Sigma^{1/2}(h_n)$. Then, it follows that $\int_W \langle h_n, U_1\rangle^2\, dm \xrightarrow[n\to\infty]{} 0$ and $\mathcal{E}[\langle h_n - h_m, U_1\rangle] \xrightarrow[n,m\to\infty]{} 0$. Since S is an error structure, the closedness property implies $\mathcal{E}(\langle h_n, U_1\rangle) = \|Vx_n\|^2 \xrightarrow[n\to\infty]{} 0$ and $u = 0$. Thus, the theorem is proved. □

Corollary 1. *When* B *is a separable Banach space isomorphic to* H*, the conclusion of* Theorem 1 *remains valid*: *Let* $(U_n)_{n\in\mathbb{N}^*}$ *be a sequence of centered random variables in* $\mathbb{D}_B$*, Dirichlet independent with the same Dirichlet law, then,* $(V_n)_{n\in\mathbb{N}^*}$ *converges in Dirichlet law towards an error structure of the Ornstein-Uhlenbeck type.*

Proof of Corollary 1. Let Φ be the isomorphism between B and H. According to [1], p.267, $(\Phi(U_n))_{n\in\mathbb{N}^*}$ is a sequence of centered random variables in $\mathbb{D}_H$ having the same Dirichlet law. From Proposition 2, we can show easily that these random variables are Dirichlet independent. Thus, applying Theorem 1, $(\Phi(U_1) + \cdots + \Phi(U_n))/\sqrt{n}$ converges in Dirichlet law towards an error structure of the Ornstein-Uhlenbeck type $\tilde{S}$. Finally, we immediately obtain that V_n converges towards ${\Phi^{-1}}_*\tilde{S}$ which is of the Ornstein-Ulhenbeck type because Φ^{-1} is linear. □

4. Concluding remarks

We use the notations of the preceding proof. Let us consider Λ the set of subsets of $\mathbb{N}$. If $u = \{i_1, \ldots, i_n\} \in \Lambda$, π_u is the canonical projection from H into $\mathrm{vect}(e_{i_1}, \ldots, e_{i_n})$ and ϕ_u the natural homeomorphism between $\mathbb{R}^n$ and $\mathrm{vect}(e_{i_1}, \ldots, e_{i_n})$. Let us define the following error structure

$$\hat{S}_u = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathcal{N}(0, \sigma_{i_1}) \otimes \cdots \otimes \mathcal{N}(0, \sigma_{i_n}), \hat{\mathbb{D}}_u, \hat{\Gamma}_u)$$

with $\forall f \in C^1(\mathbb{R}^n, \mathbb{R}) \cap \mathrm{Lip}$,

$$\hat{\Gamma}_u[f] = 2 \sum_{k,l=1}^{n} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_l} \mathcal{E}[\langle e_{i_k}, U_1\rangle, \langle e_{i_l}, U_1\rangle]$$

and where $\hat{\mathbb{D}}_u$ is the domain of the smallest closed extension of $(C^1(\mathbb{R}^n, \mathbb{R}) \cap \mathrm{Lip}, \hat{\mathcal{E}}_u)$.

We can see that $((\phi_u)_* \hat{S}_u)_{u \in \Lambda}$ is a projective system of error structures in the sense of [1], p.206. Since $(\pi_u)_* \hat{S} = (\phi_u)_* \hat{S}_u$, this projective system has a limit which is none other than $\hat{S}$. Thus, our result may be seen as the projective limit of the result of Bouleau and Hirsch in finite dimension ([1], p.220). Moreover, by the so-called Kwapien's theorem ([13], p.246) and by Corollary 1, it is easy to see that Theorem 1 extends to Banach spaces having type 2 and cotype 2. We can now wonder about the extension of such a result in more general settings. Unfortunately, the classical conditions for the central limit theorem to hold in Banach spaces having finite type and cotype ([13]) or in the Wiener space ([11]) seem to be, for the moment, insufficient to overcome the lack of orthogonality that is the keystone of our proof.

5. Appendix: Lemmas

5.1. Proof of Lemma 3. The implication "$(C^1(H, \mathbb{R}) \cap \mathrm{Lip}, \hat{\mathcal{E}})$ is closable $\Rightarrow$ $(\mathcal{A}_\beta, \hat{\mathcal{E}})$ is closable" is obvious. For the converse we suppose that $(\mathcal{A}_\beta, \hat{\mathcal{E}})$ is closable. Let $\hat{\mathbb{D}}_{\mathcal{A}_\beta}$ denote the domain of its smallest closed extension. We can see that $(\hat{\mathbb{D}}_{\mathcal{A}_\beta}, \hat{\mathcal{E}})$ possesses a gradient with values in $L^2(\nu; H)$ (which is the smallest closed extension of $(D_C, \mathcal{A}_\beta)$). Let J be the canonical isometry between H and a copy $\hat{H}$ of H and $(\hat{\mathbb{D}}_{\mathcal{A}_\beta})_H$ the associated vectorial domain. According to Remark 1, the identity mapping of H belongs to $(\hat{\mathbb{D}}_{\mathcal{A}_\beta})_H$. From Proposition 1, $C^1(H, \mathbb{R}) \cap \mathrm{Lip} \subset \hat{\mathbb{D}}_{\mathcal{A}_\beta}$ thus $(C^1(H, \mathbb{R}) \cap \mathrm{Lip}, \hat{\mathcal{E}})$ is closable and $\hat{\mathbb{D}} \subset \hat{\mathbb{D}}_{\mathcal{A}_\beta}$. The result follows.

5.2. Proof of Lemma 4. We set $N = \{e_i \in \beta_0 \mid \Sigma(e_i) = 0\}$. Suppose that $e_{i_1} \in N$. Using the locality of the form $\mathcal{E}$ ([1], p.28) we obtain $\mathcal{E}[\langle e_{i_1}, U_1\rangle] = 0$. Thus, if $F = f(\langle e_{i_1}, .\rangle, \ldots, \langle e_{i_p}, .\rangle) \in \mathcal{A}_{\beta_0}$ we have

$$\hat{\mathcal{E}}[F] = \sum_{k,l=2}^{p} \int_H f'_k f'_l(0, \langle e_{i_2}, .\rangle, \ldots, \langle e_{i_p}, .\rangle) \mathcal{E}[\langle e_{i_k}, U_1\rangle, \langle e_{i_l}, U_1\rangle] \, d\nu. \tag{6}$$

Putting

$$\mathcal{A}_{\beta_0 \setminus N} = \{F(\langle e_{i_1}, .\rangle, \ldots, \langle e_{i_n}, .\rangle); n \in \mathbb{N}, (e_{i_1}, \ldots, e_{i_n}) \in \beta_0 \setminus N, F \in C^1 \cap \mathrm{Lip}\}$$

we deduce from (6) that

$$(\mathcal{A}_{\beta_0 \setminus N}, \hat{\mathcal{E}}) \text{ is closable} \iff (\mathcal{A}_{\beta_0}, \hat{\mathcal{E}}) \text{ is closable.}$$

Since the restriction of Σ to $\mathrm{vect}(\beta_0 \setminus N)$ is injective, this entails the conclusion.

5.3. Proof of Lemma 5. The main ideas of the proof are taken from [10]. Let us define

$$\mathcal{P}(V^*) = \{Fk \mid F \in \mathcal{A}_{\beta_0},\ k \in \mathrm{dom}(V^*)\} \subset L^2(\nu; H).$$

In a natural way, we can extend D_C to $\mathcal{P}(V^*)$ putting

$$D_C[Fk] = D_C[F] \otimes k$$

where $\otimes$ is the tensor product on H. Using [10], p.4, we show that the operator

$$W[\psi](x) = -\mathrm{trace}(D_C[\psi](x)) + \langle \Sigma^{-1/2}x,\ V^*\psi(x)\rangle, \quad \mathrm{dom}(W) = \mathcal{P}(V^*)$$

is well defined and that $(W, \mathrm{dom}(W))$ and $(D_C, \mathcal{A}_{\beta_0})$ are adjoint to each other.

If we assume that V is closable, a classical result ([11], Theorem 5.28) gives that $\mathrm{dom}(V^*)$ is dense in H. Thus, $\mathrm{dom}(W)$ is dense in $L^2(\nu; H)$. Since W and D_C are adjoint, D_C is closable.

For the converse, let $(h_n)_{n\in\mathbb{N}}$ be a sequence in H such that

$$h_n \xrightarrow[n\to\infty]{} 0, \quad Vh_n \xrightarrow[n\to\infty]{} u.$$

Let f be in $C^1(\mathbb{R}, \mathbb{R}) \cap \mathrm{Lip}$ (with a Lipschitz constant equals to K) with $f(0) = 0$ and $f'(0) \neq 0$. We have

$$\int |f(\langle x,\ \Sigma^{-1/2}h_n\rangle)|^2\, d\nu(x) \leq K\|h_n\|^2,$$

hence, $f(\langle .,\ \Sigma^{-1/2}h_n\rangle) \xrightarrow[n\to\infty]{} 0$ in $L^2(\nu)$. Moreover

$$D_C[f(\langle .,\ \Sigma^{-1/2}h_n\rangle](x) - f'(0)u = [f'(\langle x,\ \Sigma^{-1/2}h_n\rangle) - f'(0)]Vh_n + [Vh_n - u]f'(0).$$

Since $\|Vh_n\|$ is bounded,

$$D_C[f(\langle .,\ \Sigma^{-1/2}h_n\rangle] \xrightarrow[n\to\infty]{} f'(0)u \quad \text{in} \quad L^2(\nu; H).$$

From Lemma 3, $(D_C, \mathcal{A}_{\beta_0})$ closable $\Rightarrow (D_C, C^1(H, \mathbb{R}) \cap \mathrm{Lip})$ closable. Thus, $u = 0$.

References

[1] N. Bouleau and F. Hirsch: Dirichlet Forms and Analysis on Wiener Space, de Gruyter, Berlin, 1991.

[2] N. Bouleau: *Calcul d'erreur complet lipschitzien et formes de Dirichlet*, J. Math. Pures Appl. (9) **80** (2001), 961–976.

[3] N. Bouleau: Error Calculus for Finance and Physics: the Language of Dirichlet Forms, de Gruyter, Berlin, 2003.

[4] N. Bouleau and C. Chorro: *Error structures and parameter estimation*, C.R. Math. Acad. Sci. Paris **338** (2004), 305–310.

[5] N. Bouleau: *Théorème de Donsker et formes de Dirichlet*, Bull. Sci. Math. **129** (2005), 369–380.

[6] C. Chorro: *Convergence in Dirichlet law of certain stochastic integrals*, Electron. J. Probab. **10** (2005), 1005–1025.

[7] R.M. Dudley: *Metric entropy and the central limit theorem in $C(S)$*, Ann. Inst. Fourier (Grenoble) **24** (1974), v, 49–60.

[8] D. Feyel and A. de La Pradelle: *Espaces de Sobolev Gaussiens*, Ann. Inst. Fourier (Grenoble) **39** (1989), 875–908.

[9] M. Fukushima: Dirichlet Forms and Symmetric Markov Processes, North-Holland, Amsterdam, 1980.

[10] B. Goldys, F. Gozzi and J.M.A.M. van Neerven: *On closability of directional gradients*, Potential Anal. **18** (2003), 289–310.

[11] T. Kato: Perturbation Theory for Linear Operators, Springer, Berlin, 1992.

[12] H.H. Kuo: Gaussian Measures in Banach Spaces, Lecture Notes in Math. **463**, Springer, Berlin, 1975.

[13] M. Ledoux and M. Talagrand: Probability in Banach Spaces, Springer, Berlin, 1991.

[14] Z.M. Ma and M. Röckner: Introduction to the Theory of (Nonsymmetric) Dirichlet Forms, Springer, Berlin, 1992.

[15] D. Nualart: The Malliavin Calculus and Related Topics, Springer, New York, 1995.

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