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ON THE GROWTH OF SOLUTIONS OF SEMI-LINEAR DIFFUSION EQUATION WITH DRIFT

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0. Introduction

This paper is devoted to the growing up problem of a semi-linear diffusion equation

$$(1) \qquad u_t = u_{xx} + k(x)u_x + F(u) \qquad t>0,\ x>0$$

with the initial-boundary conditions

$$(2) \qquad \lim_{x \downarrow 0} e^{B(x)} u_x(t, x) = 0 \qquad t>0,$$

$$(3) \qquad \lim_{t \downarrow 0} u(t, x) = f(x)^{(1)} \qquad x>0$$

where $0 \leqq f(x) \leqq 1$ and we write

$$B(x) = \int_1^x k(y)dy \qquad x>0;$$

i.e. the problem of finding criteria of whether a solution u of (1)-(3) grows up or fades away. Here and henceforth a solution u of (1)-(3) is said to *grow up* if

$$(4) \qquad \lim_{t \to \infty} u(t, x) = 1 \qquad \text{locally uniformly in } x>0\,,$$

and to *fade away* if

$$(5) \qquad \lim_{t \to \infty} u(t, x) = 0 \qquad \text{uniformly in } x>0\,.$$

In treating of this problem, we are mainly interested in the case that the initial function f is zero out-side a finite interval. We will call such f which is not identical to zero a *finite initial function* (abbreviated to f.i.f.).

Throughout this paper it will be assumed that $F(u)$ and $k(x)$ are real valued, continuous and continuously differentiable functions on $0 \leqq u \leqq 1$ and on $x>0$, respectively, and that they satisfy the following conditions

(1) The convergence in (3) is taken in the locally L_1 sense.

$$(6)\qquad F(0)=F(1)=0$$

$$(7)\qquad F(u)\begin{cases}\leqq 0 & 0<u<\mu\\ >0 & \mu<u<1\end{cases}\quad \text{for some } 0\leqq\mu<1$$

$$(8)\qquad \kappa\equiv\lim_{x\to\infty}k(x)\ \text{ exists and }\ \varliminf_{x\downarrow 0}k(x)>-\infty\,.$$

For a measurable function $f(x)$ with $0\leqq f(x)\leqq 1$, $x>0$ there exists one and only one classical solution $u(t, x)$ with values in $[0, 1]$ of the problem (1)-(3). We consider here only such a solution. The solution is positive for $t>0$ unless f is equal to zero almost everywhere. We will sometimes treat a positive solution of (1)-(2) without refering to the initial condition. The purpose of this paper is to offer sufficient conditions on F and k for one of the following statements to be true:

(i) all positive solutions grow up;
(ii) all solutions starting from f.i.f. fade away;
(iii) there is a soluton starting from f.i.f. which grows up;
(iv) there is a positive solution which fades away.

It can be proved that there exists the minimum of real numbers c for which the ordinary differential equation

$$(9)\qquad w''(x)+cw'(x)+F(w(x))=0$$

has a solution on R^1 (R^1 is the whole real line) such that $0<w(x)<1$ on R^1 and

$$(10)\qquad \lim_{x\to-\infty}w(x)=1 \quad\text{and}\quad \lim_{x\to+\infty}w(x)=0$$

(c.f., e.g., [7], [9], [11]). We denote it by c_0. When $\mu=0$, writing

$$\alpha=F'(0)\,,\quad c^*=2\alpha^{1/2}\,,$$

we have

$$c_0\geqq c^*\,.$$

(It may occur that $c_0>c^*$; this is always the case if $\mu=F'(0)=0$.) This constant c_0 serves as a critical value on the growing up problem as will be explained below.

To simplify the explanation let us assume $k(x)=\kappa$ for all $x>0$ in the following for the moment. Then Theorem 1 of the next section implies that: if $\kappa\geqq c_0$ (we must further restrict F slightly in case of $\mu=0$ with $c_0=c^*$), we have the statement (ii); if $\mu>0$ and $\kappa<c_0$ or if $\mu=0$ and $c^*<\kappa<c_0$, both (iii) and (iv) are true simultaneouly; if $\mu=0$ and $\kappa<c^*$, the statement (i) holds. When $k(x)$ is not constant, we have the similar statements under suitable restrictions on k.

In case of $\mu=0$ with $c^*<c_0$ the constant c^* serves as another critical value.

In the critical case $\kappa = c^*$ both of (i) and (iv) can occur depending on the behavior of $F(u)$ (near zero) and of $k(x)$. This will be seen in Theorems 2 and 3.

The following two examples will explain our motivation for the problem. As the first example put

$$k(x) = \frac{d-1}{x} \qquad (d \text{ is a positive integer}). \tag{11}$$

Then the solution $u(t, x)$ of (1) and (2) ((2) follows from (1), if $d \geqq 2$) gives a spherically symmetric solution $U(t, \underline{\underline{x}}) = u(t, |\underline{\underline{x}}|)$, $\underline{\underline{x}} = (x_1, \cdots, x_n) \in R^d$, $|\underline{\underline{x}}| = (\sum_{i=1}^n x_i^2)^{1/2}$, of d-dimensional semi-linear heat equation

$$U_t = \sum_{i=1}^d U_{x_i x_i} + F(U) \qquad t>0,\ \underline{\underline{x}} \in R^d\,. \tag{12}$$

The second example is an equation on the d-dimensional Riemannian manifold M with constant curvature -1;

$$U_t = \Delta U + F(U) \qquad t>0,\ \xi \in M \tag{13}$$

where $U = U(t, \xi)$ and Δ is the Laplace-Beltrami operator on M. Since the radial part of Δ is given by $\partial^2/\partial r^2 + (d-1)\coth r\, \partial/\partial r$ in a geodesic polar coordinate, where $r = r(\xi)$ is the distance from the origine of the coordinate system, if we put

$$k(x) = (d-1)\coth x \qquad x>0 \tag{14}$$

and if we set $U(t, \xi) = u(t, r(\xi))$ through a solution u of (1), then U becomes a symmetric solution of (13) whose space dependence comes only from the distance $r(\xi)$. In this example $\kappa = \lim k(x) = d-1$, $k(x) > \kappa$ for all $x>0$ and $k(x) - \kappa \sim 2e^{-2x}$ as $x \to +\infty$ (2).

The equation (12) was first introduced by R. A. Fisher [3] and by A. N. Kolmogorov *et al.* [12] in connection with population genetics. They all treated the case $\mu = 0$ and $F'(0) > 0$. The case $\mu > 0$ with $F(u) < 0$, $0 < u < \mu$, has also a population genetical interpretation (cf. [1]). When $\mu > 0$ with $F(u) = 0$, $0 < u < \mu$, the equation (12) with $d = 1$ serves as a mathematical model of a flame propagation in the chemical reactor theory (cf. [9], [10]). The conditions (6) and (7) are motivated by these applcations. The growing up problem for (12) has been studied by several authors ([2], [4], [6] and [11]). Some of their results or some weak versions of them will be obtained as special cases of Theorem 3 in this paper. N. Ikeda considered the same problem for (13) and obtained certain criteria for growing up or fading away (private communication). His result will be improved in Theorem 2.

(2) "$a(x) \sim b(x)$, as $x \to x_0$" means $\lim_{x \to x_0} a(x)/b(x) = 1$.

For a solution u of (12) with $d=1$ the papers [1], [5] and [12] examine its translation $u(t, x+\kappa t)$ on its growth as $t\to+\infty$ where κ is a real constant. This amounts to study the growing up problem for the equation

$$u_t = u_{xx}+\kappa u_x+F(u) \qquad t>0,\ x\in R^1. \tag{15}$$

The situation for the present problem (1)-(2) are essentially different from that for the equation (15) in the following two cases. The first one is the critical case $\kappa=c_0$ of $\mu>0$ or of $\mu=0$ with $c^*<c_0$, in which some solution of (15) starting from f.i.f. converges to a non-constant stationary solution (cf. [5], [14]). The second one is the case $c^*<\kappa<c_0$ of $\mu=0$, in which the situation does not differ from the case $|\kappa|<c_0$ on the growth of positive solutions of (15): they all grow up if $|\kappa|<c_0$ and $\mu=0$, so this time c^* does not serves as a critical value for the equation (15).

Finally let us give a simple but useful remark. Given a smooth positive function $a(y)$, $y>0$ with $\int_{0+} a^{-1/2}dy<+\infty$ and $\int^{+\infty} a^{-1/2}dy=+\infty$ a semi-linear diffusion equation on $y>0$

$$v_t = a(y)v_{yy}+b(y)v_y+F(v)$$

with

$$\lim_{y\downarrow 0} \exp\left(\int_1^y b(s)a(s)^{-1}ds\right)v_y(t, y) = 0$$

can be transformed to the equation (1) with (2) where k is given by

$$k(x) = a(y)^{-1/2}[b(y)-2^{-1}a'(y)],$$

through the change of variable $x=\int_0^y a^{-1/2}dy$.

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1. Results

In this section the results of this paper will be stated as theorems. We will use symbols c_0, c^* and α to denote constants defined in the introduction.

The first theorem asserts that c_0 (or c^* when $\mu=0$) is a critical value on the growth of solutions.

Theorem 1. (a) *If $\kappa<c_0$, then for any small positive constant a $(0<a<1-\mu)$ there exists a constant $L>0$ such that the solution of* (1)-(3) *with $f(x)=(\mu+a)I_{[0,L]}(x)$* (3) *grows up.*

(3) $I_{[0,L]}(x)=1$ for $0\leqq x\leqq L$ and $=0$ for $x>L$.

(b) *If* $\mu=0$ *and* $\kappa<c^*$, *then any positive solution of* (1)-(2) *grows up.*

(c) *Assume that* $\mu=0$ *and* $k(x)\geqq c^*+\delta$ *for* $x>0$ *with some* $\delta>0$, *or that* $\mu>0$ *and* $\int^{+\infty} e^{B(x)}dx=+\infty$. *Then there exists a continuous* f *such that* $f(x_0)>\mu$ *for some* $x_0>0$ *and the solution of* (1)-(3) *with this* f *fades away.*

(d) *Let* $k(x)\geqq c_0$ *for* $x>0$. *When* $\mu=0$ *and* $c_0=c^*$ *we further assume that*

$$\int_{0+} |F'(u)-\alpha|u^{-1}|\log u|du<+\infty. \tag{1-1}$$

Then any solution of (1)-(3) *starting from* f.i.f. *fades away.*

In the critical case $\kappa=c^*$ of $\mu=0$ an answer is given in the next two theorems.

Theorem 2. *Let* $\mu=0$, $\alpha>0$ *and* $\kappa=c^*$.

(a) *If* $k(x)\geqq c^*$ *for* $x>0$ *and*

$$\int_{0+} [F(u)-\alpha u]^+u^{-2}|\log u|du<+\infty,\text{[4]} \tag{1-2}$$

there is a positive solution of (1)-(2) *which fades away.*

(b) *If*

$$A\equiv\max\{\overline{\lim_{x\to\infty}}(k(x)-\kappa)x^2,\ -(2\kappa)^{-1}\} \tag{1-3}$$

and

$$\underline{\lim}_{u\downarrow 0}(F(u)-\alpha u)u^{-1}|\log u|>\alpha\cdot[(A\kappa/2+1/4)^{1/2}+1], \tag{1-4}$$

then any positive solution of (1)-(2) *grows up.*

Theorem 3. *Let* $\mu=0$ *and* $\alpha=\kappa=0$.

(a) *Assume there exists a pair of positive constants* a *and* M *such that*

$$F(uv)\leqq MuF(v) \qquad \text{for } 0<u<1 \text{ and } 0<v<a. \tag{1-5}$$

Provided either that for a constant $A>-1$

$$k(x)=\frac{A}{x}+O\left(\frac{1}{x^2}\right) \text{ as } x\to\infty \text{ and } \int_{0+}\frac{F(u)}{u^{2+2/(A+1)}}dy<+\infty \tag{1-6}$$

or that for a constant $0<\beta<1$

$$k(x)\asymp x^{-\beta} \text{ as } x\to\infty\text{[5]} \text{ and } \int_{0+}\frac{F(u)}{u^2}|\log u|^{2\beta/(1-\beta)}du<+\infty, \tag{1-7}$$

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(4) $A^+=\max\{A, 0\}$ for a real number A.

(5) "$a(x)\asymp b(x)$ as $x\to x_0$" means that $0<\underline{\lim}_{x\to x_0}a(x)/b(x)\leqq\overline{\lim}_{x\to x_0}a(x)/b(x)<+\infty$.

there is a positive solution u which fades away in such manner that $u(t, x) \asymp t^{-(A+1)/2}$ (*in case* (1–6)) *or* $-\log u(t, x) \asymp t^{(1-\beta)/(1+\beta)}$ (*in case* (1–7)) *as* $t \to \infty$ *locally uniformly in* $x > 0$.

(b) *Provided either that for a constant* $A > -1$

$$(1\text{–}8) \qquad k(x) = \frac{A}{x} + O\left(\frac{1}{x^2}\right) \text{ as } x \to \infty \text{ and } \lim_{u \downarrow 0} \frac{F(u)}{u^{1+2/(A+1)}} = +\infty$$

or that for a constant $0 < \beta < 1$

$$(1\text{–}9) \qquad k(x) \asymp x^{-\beta} \text{ as } x \to \infty \text{ and } \lim_{u \downarrow 0} F(u) u^{-1} |\log u|^{2\beta/(1-\beta)} = +\infty ,$$

any positive solution of (1)–(2) *grows up.*

REMARK 1. For the equation (13) with $d=2$ N. Ikeda obtained the condition $F(u) - 4^{-1}u \leqq u^{1+\delta}$ (for small u) with $\delta > 2/3$ in place of (1–2) and the condition $F(u) - 4^{-1}u \geqq 2u/|\log u|$ (for small u) in place of (1–4) to conclude the corresponding consequences similar to those of Theorem 2. (His results are implied by Theorem 2, since for k defined in (14) with $d=2$ we have $\kappa=1$, $\alpha=1/4$ under $\kappa=c^*$, and $A=0$ in (1–3).)

REMARK 2. The first part of Theorem 3 will be obtained as a corollary of a theorem in the section 5 which concerns a more general equation than (1)–(2). The latter is essentially due to Kobayashi *et al.* [11]. In Theorem 3 (b) the second part of (1–8) would be weakened to $\int_{0+} F(u) u^{-2-2/(A+1)} du = +\infty$. In fact Kobayashi *et al.* proved that this implies the conclusion of the theorem under additional restrictions on F when $k(x) = A/x$ and $A+1$ is a positive integer. (Unfortunately such a result can not be obtained by the method used in this paper.)

Given F and f, in order to guarantee that a solution grows up, we may impose a condition on $k(x)$ only for large values of x, as we have seen above. But when we ask whether a solution fades away or not, we can not disregard the behavior of $k(x)$ on finite intervals as the next theorem reveals.

Theorem 4. *Let* $F(u)/u$ *be a non-increasing function of* u. *Assume that* $\kappa < c^*$ *and that there exists a non-negative constant* $\delta < c^*$ *such that* $k(x) < \delta$ *on a interval whose length is larger than* $2\pi/(4\alpha - \delta^2)^{1/2}$. *Then there is the unique non-constant stationary solution of* (1)–(2) *which is bounded by a constant multiple of* $e^{-\kappa x/2}$. *And any positive solution of* (1)–(3) *with* $f(x) = 0(e^{-\kappa+\varepsilon)x/2})$ *for an* $\varepsilon > 0$ *converges to it uniformly on* $x > 0$.

REMARK 3. In Theorem 4 the monotonicity assumption on $F(u)/u$ is imposed to guarantee the uniqueness of the stationary solution satisfying the boundedness condition as mentioned there. The existance of such a solution is guaranteed for any F (satisfying (6) and (7)) if k is appropriately chosen on the analogy to the theorem. In that case some solution starting from f.i.f. neither grows up nor fades away.

2. Solutions of $w''+cw'+F(w)=0$

We give here some information about solutions of the ordinary differential equation (9), which combined with Lemma 5 in the next section, is very useful in solving the problem presented in the introduction.

We states here only necessary facts that are needed in the subsequent sections. Most of proofs for them are elementary and may be most easily obtained by observing trajectries drawn by $(w(x), w'(x))$ where $w(x)$ is a solution of (9) on a interval on the real line. (For the proofs readers may refer to [1], [7] or [14].)

If we say that w is a solution of (9) on (L, W), $L<M$, it is meant that w is defined and taking values in $[0, 1]$ on the interval (L, M) and solves (9) there. The followings are what we will need.

(i) If $\mu=0$, a decreasing solution of (9) on R^1 exists if and only if $c\geqq c_0$. Such a solution necessarily satisfies the boundary condition (10). If $F(u)/u\leqq\alpha$, then $c_0=c^*$.

(ii) If $\mu>0$, a decreasing solution of (9) on R^1 exists if and only if $c=c_0$.

(iii) Let $c<c_0$. Then there is a decreasing solution w of (9) on $(0, L)$, $0<L<+\infty$, with $w(L)=0$ and $w'(0)=0$. If w_1 is a solution of (9) on $(0, +\infty)$, then either $w_1(0)<w(0)$ or w_1 is identically equal to 1.

(iv) Let $\mu=0$ and $c>c^*$. There are two types of solutions of (9) on $(0, +\infty)$: one, say w_s, having a steep tail, i.e.

$$(2\text{–}1)\qquad \log w_s(x)\sim -b_+(c)x \quad \text{where } b_+(c)=2^{-1}[c+(c^2-4\alpha)^{1/2}]$$

and the other, say w_g, having a gentle tail, i.e.

$$(2\text{–}2)\qquad \log w_g(x)\sim -b_-(c)x \quad \text{where } b_-(c)=2^{-1}[c-(c^2-4\alpha)^{1/2}]\,.$$

A solution of the former type is unique up to translation, i.e., any two solutions defined up to $+\infty$ and having steep tails are obtained as a tralstion of each other. If $c>c_0$, there is steep one (defined on $[0, \infty)$) such that $w_s(0)=1$ (it is necessarily decreasing). If w_s is a solution having a steep tail and satisfying $w_s'(0)=0$, then necessarily $w_s(0)>w_g(0)$ for any solution w_g which has a gentle tail.

(v) Let $c=c_0$. Then there is a solution of (9) on R^1 satisfying (10) whose value at zero is 1/2. Such one is unique. We denote it by $w_{(0)}$. It holds that $w_{(0)}'(x)<0$ on R^1 and

$$\log w_{(0)}(x)\sim -b_+(c_0)x\,.$$

If $\mu>0$, every solution of (9) on $(0,+\infty)$ which vanishes at infinity is a translation of $w_{(0)}$.

(vi) Let $\mu=0$ and $c=c^*$. In this case whether there is a solution of (9) on $(0,+\infty)$ depends on the behavior of F near zero. If $F(u)-\alpha u\geqq 0$ for small u and $\int_{0+}(F(u)-\alpha u)u^{-2}du=+\infty$, there is no such solution. If F satisfies the condition (1–2), then there are solutions w_s and w_g of (9) on $(0,\infty)$ such that

(2–3) $$w_s(x)\sim C\cdot e^{-c^*x/2}$$

(2–4) $$w_g(x)\sim C\cdot xe^{-c^*x/2}\,,$$

where C denotes a positive constant. This time also the last statement of (iv) is true for w_s and w_g above. If in addition $c_0=c^*$, it holds either that w_s above is a translation of $w_{(0)}$ (which is defined in (v)), or that we can chose w_s so that $w_s(0)=1$. Under a somewhat stronger condition (1–1) w_s in the above is unique up to translation. (See Lemmas 2.2 and 2.3 in [14].)

Here is an example in which c_0 is given by an explicit formula. Let $G(u)$ be a continuous function on $0\leqq u\leqq 1$ which has the continuous first derivative there and the continuous second derivative on $0<u\leqq 1$ and satisies the conditions that $G(0)=G(1)=0$, $G'(0)=1$, $\lim_{u\downarrow 0}uG''(u)=0$ and $G(u)>0$ for $0<u<1$. For each pair of a real constant α and a positive constant β we set

$$F(u) = G(u)[\alpha+\beta^2(1-G'(u))]$$

Then $F'(0)=\alpha$ and F satisfies all requirements in the introduction execept (7). We assume that F defined above satisfies (7) (this is the case if G is concave and $\alpha+\beta^2(1-G'(1))>0$). A function $w(x)$, $x\in R^1$, given through the relation

$$x=\int_{1/2}^{w(x)}\frac{-1}{\beta G(u)}du$$

is a solution of (9) satisfying (10) with $c=\alpha/\beta+\beta$. It is not difficult to show that if $\alpha\leqq\beta^2$, then this w coincides with $w_{(0)}$ and $c_0=\alpha/\beta+\beta$ and that if $\alpha>\beta^2$ and if $G'(u)\leqq 1$ for $0<u<1$, then $\mu=0$ and $c_0=c^*$ (cf. [7] or [14]).

3. Preliminary lemmas

Before proceeding into the proof of Theorems 1 to 4 we prepare several lemmas. First we note that the latter condition(6) of (8) admits (2) as the boundary condition of the diffusion equation

(3–1) $$u_t = u_{xx}+k(x)u_x \qquad t>0,\ x>0\,.$$

(6) This can be replaced by $\int_0^1 e^{-B(y)}dy\int_0^y e^{B(x)}dx<+\infty$ (cf. [8]), under which almost all the statements of this paper are valid.

Let $p(t, x, y)$ denote the fundamental solution of the Cauchy problem for this equation with (2). The boundary condition (2) means that the system is conservative, i.e.

$$\int_0^{+\infty} p(t, x, y)dy = 1 .$$

We write, for g a bounded measurable function on $x>0$,

$$P_t\{g\}(x) = P_t g(x) = \int_0^{+\infty} p(t, x, y)g(y)dy .$$

Then the solution of (1)-(3) is obtained as the unique bounded solution of the integral equation

$$u(t, x) = e^{-at}P_t f(x)+\int_0^t e^{-as}P_s\{F^{(a)}(u(t-s, \cdot))\}(x)ds \tag{3–2}$$

where a is a real number and $F^{(a)}(u)=au+F(u)$. We will use the fact that the solution of (1)-(3) can be constructed through the following iteration procedure: putting

$$\begin{aligned} u_0(t, x) &= e^{-at}P_t f(x) \\ u_{n+1}(t, x) &= e^{-at}P_t f(x)+\int_0^t e^{-as}P_s\{F^{(a)}(u_n(t-s, \cdot))\}(x)ds \\ &\qquad \text{for } n = 0, 1, 2, \cdots, \end{aligned} \tag{3–3}$$

there exists $\lim_{n\to\infty} u_n(t, x)$, which is the solution of (1)-(3).

Lemma 1. *Let F_1 and F_2 be C^1-class functions on $0\leqq u\leqq 1$ with $F_i(0)=F_i(1)=0$ $(i=1, 2)$ and f_1 and f_2 be initial functions. Let u_i be corresponding solutions of (1)-(3) (where F and f are replaced by F_i and f_i, $i=1, 2$). If $F_1\leqq F_2$ and $f_1\leqq f_2$, then $u_1\leqq u_2$.*

Proof. Taking a in (3–2) so large that $F^{(a)}$ is increasing, we can esily see the lemma from the iteration procedure (3–3).

Setting $a=\gamma$ in (3–2), we see

$$e^{-\gamma t}P_t f(x)\leqq u(t, x)\leqq e^{\gamma t}P_t f(x)$$

where

$$\gamma = \sup_{0<u<1}|F'(u)| .$$

Similarly, by considering the equation satisfied by $1-u$,

$$u(t, x)\leqq 1-e^{-\gamma t}(1-P_t f(x)) .$$

It is noted that if f is not identical to zero, then

$$\inf_{0<x<N} P_t f(x) > 0 \qquad t>0\,,$$

for each positive number N. Thus we obtain

$$0< \inf_{0<x<N} u(t, x) \leqq \sup_{0<x<N} u(t, x) < 1 \qquad t>0$$

if f is neither identical to zero nor to unity.

Lemma 2. *Let f_n, $n=1, 2, \cdots$ and f be initial functions and u_n and u be corresponding solutions of* (1)-(3). *Suppose* $\int_0^N |f_n - f|\,dx \to 0$ *as* $n\to\infty$ *for each* $N>0$. *Then for each* $t>0$, $\lim_{n\to\infty} u_n(t, x) = u(t, x)$ *locally uniformly in* x.

Proof. It can be proved (by using the iteration procedure (3–3)) that $|u_n - u| \leqq e^{Kt} P_t\{|f_n - f|\}$ where $K = \sup_{0\leqq u<v\leqq 1}(F(u)-F(v))/(u-v)$. Then, noting that $\lim_{L\to\infty} P_t I_{(L,\infty)}(x)=0$ locally uniformly, the assertion of the lemma is immediate.

Lemma 3. *If an initial function f is non-increasing, then $u_x \leqq 0$ for $t>0$, $x>0$.*

Proof. Take $a=\sup_{0<u<1} F'(u)$ in (3–3). By the iteration procedure in constructing the solution and the monotonicity of $F^{(a)}$ it suffices to prove that if $g(x)$ is non-increasing, so is $P_t g(x)$ for each $t>0$. To see this we may assume that g' exists and is continuous, by virtue of the previous lemma. Put

$$Q(t, x, y) = \int_y^{+\infty} p(t, x, z)dz\,.$$

Then

$$P_t g(x) = g(0) + \int_0^{+\infty} g'(y) Q(t, x, y))dy\,. \tag{3–4}$$

Consider the difference $w(t, x) = Q(t, x_2, x) - Q(t, x_1, x)$ for $0<x_1<x_2$ and observe that it satisfies the equation $w_t = w_{xx} - k(x) w_x$ and boundary conditions $w(t, 0) = w(t, +\infty) = 0$. Since the initial value $w(0+, x) = I_{(x_1, x_2)}(x)$ is non-negative (a.s.), it follows that $w(t, x) \geqq 0$. Thus $Q(t, x, y)$ is non-decreasing in x and, by (3–4), $P_t g(x)$ is non-increasing if $g' \leqq 0$. This completes the proof.

Lemma 4. *Let k_i, $i=1, 2$, be C^1-class function on $x>0$, satisfying $k_i(x) = O(x)$ as $x\to\infty$ and $\underline{\lim}_{x\downarrow 0} k_i(x) > -\infty$ and u_i be solutions of* (1) *and* (3) *with k_i instead of k. Suppose that the common initial function f is non-increasing and that $\overline{\lim}_{x\downarrow 0} e^{B_1(x)}(u_1)_x(t, x) \leqq 0$ and $\underline{\lim}_{x\downarrow 0} e^{B_2(x)}(u_2)_x(t, x) \geqq 0$, where $B_i(x) = \exp\{\int_1^x k_i(y)dy\}$. If in addition $k_1 \leqq k_2$, then $u_1 \geqq u_2$.*

Proof. Let $\tilde{u}_1$ be the solution of (1)-(3) where k is replaced by k_1. Then by Lemma 3 $(\tilde{u}_1)_x \leqq 0$. Setting $w = \tilde{u}_1 - u_2$, we have $\overline{\lim}_{x\downarrow 0} e^{B_2(x)} w_x(t, x) \leqq 0$ and

$$w_t = w_{xx} + k_2 w_x + F'(\theta) w + (k_1 - k_2)(\tilde{u}_1)_x ,$$

where θ lies between $\tilde{u}_1$ and u_2. By the hypothesis of the lemma $(k_1 - k_2)(\tilde{u}_1)_x \geqq 0$. It then follows from Proposition of Appendix that $w \geqq 0$, i.e. $\tilde{u}_1 \geqq u_2$. From the same argument it follows that $\tilde{u}_1 \geqq u_1$. Thus the lemma has been proved.

The next lemma is a modification of Proposition 2.2 in [1], where it is applied to the growing up problem for the homogeneous equation (15).

Lemma 5. *Let an initial function f be continuous on $x \geqq 0$ and twice continuously differentiable on an interval (a, b), $0 \leqq a < b \leqq +\infty$. If it satisfies $f'' + kf' + F(f) \geqq 0$ on (a, b) and*

$$\text{(3–5)} \qquad f = 0 \text{ outside of } (a, b) \text{ and } \underline{\lim}_{x\downarrow 0} e^{B(x)} f'(x) \geqq 0$$

then the solution of (1)-(3) *is non-decreasing in t. If it satisfies $f'' + kf' + F(f) \leqq 0$ on (a, b) and*

$$\text{(3–6)} \qquad f = 1 \text{ outside of } (a, b) \text{ and } \overline{\lim}_{x\downarrow 0} e^{B(x)} f'(x) \leqq 0 ,$$

then the solution is non-increasing in t.

Proof. Let (3–5) be satisfied and consider the difference $w(t, x) = u(t, x) - f(x)$ on the half infinite strip $t > 0$, $a < x < b$. Then the standard comparison theorem concerning the parabolic equation (cf. Appendix) shows that $w \geqq 0$ in this strip. Hence we have

$$\text{(3–7)} \qquad u(t, x) \geqq f(x) \qquad t > 0,\ x > 0 .$$

Since $u^*(t, x) = u(t+s, x)$ is a solution of (1)-(2) starting from $u(s, \cdot)$, we conclude, by Lemma 1, that $u(t, x) \leqq u(t+s, x)$ $(s > 0)$, i.e. $u(t,x)$ is non-decreasing in t.

The other case is similarly treated. The proof is completed.

When (3–7) holds, we call the initial function f a *sub-steady state* for (1)-(2). A *super-steady state* is analogously defined. If f is a sub-(super-) steady state, then the corresponding solution of (1)–(3) is non-decreasing (non-increasing) in t. If initial functions f_1 and f_2 are both sub-(super-) steady states, then $f(x) = \max\{f_1(x), f_2(x)\}$ $(\min\{f_1(x), f_2(x)\})$ is also a sub-(super-)steady state.

Let $h(t, x) = P_t I_{[0,1]}(x)$. In the following lemma we give, without proof, estimates of $h(t, x)$ for large values of t. For a proof readers may refer to Titchmarsh [13].

Lemma 6. (1) *If $\kappa > 0$ and $k(x) = \kappa + Ax^{-2} + 0(x^{-3})$ as $x \to \infty$ with $A > -1/2$, then for each $x > 0$*

$$\varliminf_{t\to\infty} h(t, x)t^q \exp(\kappa^2 t/4) > 0$$

where $q=1+(A\kappa/2+1/4)^{1/2}$.

(ii) *If* $k(x)=Ax^{-1}+O(x^{-2})$ *as* $x\to\infty$ *with* $A>-1$, *then for each* $x>0$

$$h(t, x) \asymp t^{-(A+1)/2} \qquad \text{as } t\to\infty .$$

(iii) *If* $k(x)\asymp x^{-\beta}$ *with* $0<\beta<1$, *then for each* $x>0$

$$-\log h(t, x) \asymp t^{(1-\beta)/(1+\beta)} \qquad \text{as } t\to\infty .$$

REMARK 4. In (i) of the lemma if $k(x)>\kappa$ for all $x>0$ (necessarily $A\geqq 0$), we have also an upper estimate for $h(t,x)$ by the same estimator as in (i). This time the condition $k(x)>\kappa$ can not be removed.

4. Proof of Theorem 1 and Theorem 2 (a)

Proof of Theorem 1. (a) Let $\kappa<c_0$. Take $c=(\kappa+c_0)/2$ in (9). Then by (iii) of the section 2 there is a decreasing solution of (9) on $[0, L_1)$, $0<L_1<+\infty$, with $w(L_1)=0$ and $w'(0)=0$. Note that necessarily $w(0)<1$. Define a function g by

$$g(x) = w(x) \qquad \text{for } 0<x<L_1 \text{ and } =0 \text{ for } x>L_1 .$$

Let $v=v(t,x)$ be the solution of the following initial-boundary value problem

$$\text{(4–1)} \qquad \begin{cases} v_t = v_{xx}+cv_x+F(v) & t>0,\ x>0\,, \\ v(0, x) = g(x)\,, & x>0;\ v_x((t, 0)=0,\ t>0\,. \end{cases}$$

Then, by Lemma 5, $v(t,x)$ is a non-decreasing function of t for each x. Since $W(x)=\lim_{t\to\infty} v(t,x)$ is a stationary solution of (4–1), W is a solution of (9) on $x>0$. Clearly $W(0)\geqq w(0)$. Hence $W(x)\equiv 1$ follows from the latter half of (iii), and we conclude that $v(t,x)$ converges increasingly to unity.

There is a positive constant L_2 such that $k(x)<c$ for $x>L_2$. Let u be a solution of (1)-(3) starting from $f=(a+\mu)I_{[0,L]}$ $(0<a<1-\mu)$. Now we choose so large a constant L that $u(T,x)\geqq g(x-L_2)$ for $x>L_2$ with some T, which is possible since $g(0)=w(0)<1$ and $\lim_{T\to\infty}\lim_{L\to\infty} u(T,x)=1$ locally uniformly in x. Then from Lemma 3.4 it follows that $u(t+T,x)\geqq v(t, x-L_2)$, $x>L_2$, where v is defined in the first part of the proof. Consequently u grows up as desired.

(b) Let $\mu=0$ and $\kappa<c^*$. When $\alpha>0$, the assertion of (b) follows from Theorem 2 (b) (which will be proved in the section 6) through a comparison argument based on Lemmas 1 and 4 and the fact remarked just before Lemma 2. For example, letting $\kappa<\tilde\kappa<c^*$, we may choose $\tilde F\leqq F$ and $\tilde k\geqq k$ such that $\tilde k(x)=\tilde\kappa$ for large x and $\tilde F(u)=(\tilde\kappa^2/4)u+u|\log u|^{-1/2}$ for small u.

When $\alpha=0$, we have $\kappa<0$, and it follows that $P_t f(x)$ converges to a positive constant as $t\to\infty$ unless $f=0$ a.e., which combined with (a) deduces the conclusion of (b).

(c) By noting that the divergence condition $\int^{+\infty} e^{B(x)}dx=+\infty$ implies $\lim_{t\to\infty} P_t I_{[0,1]}\equiv 0$, the case of $\mu>0$ is almost trivial. Indeed if we put $f=(a+\mu)I_{[0,1]}$ in (3), we can choose a positive constant a so small that for the solution u of (1)-(3) $\sup_{x>0} u(1, x)<\mu$, and so $u(t+1, \cdot)\leqq P_t\{u(1, \cdot)\}\leqq e^{\gamma} P_{t+1} f$. Hence $\overline{\lim}\, u\leqq$const. $\lim P_t I_{[0,1]}=0$.

When $\mu=0$ and $k(x)\geqq c^*+\delta$, the result is derived from Theorem 2 (a) (which will be proved soon) combined with Lemmas 1 and 4.

(d) Let $k(x)\geqq c_0$. By Lemma 4 we can assume that $k(x)\equiv c_0$. First let $\mu=0$ with $c_0>c^*$ or $\mu>0$. Let u be a solution of (1)-(2) with $u(0, x)=O(e^{-bx})$ for some $b>b_+(c_0)$. Let us prove that u fades away. It is easily seen that $u(1, x)=O(e^{-bx})$. Then by (v) in the section 2 we can find a constant L so large that $u(1, x)\leqq w_{(0)}(x-L)$ for $x>0$, for $\sup_{x>0} u(1, x)<1$. Let u^* be the solution of (1)-(2) starting from $w_{(0)}(x-L)$ $(x>0)$. By Lemma 5 u^* is non-increasing in t. If $\mu>0$, by (v) there is no non-zero stationary solution of (1)-(2) that vanishes at infinity. If $\mu=0$ and $c_0>c^*$, any solution of (9) with $c=c_0$ having a steep tail as expressed in (2–1) is a translation of $w_{(0)}$ and can not be a stationary solution of (1)-(2) (whose derivative at zero must be zero). Hence the limit of u^* as t tend to infinity must be zero, for it is a stationary solution bounded above by $w_{(0)}(x-L)$. Since $u(t+1, x)\leqq u^*(t, x)$, u fades away.

When $\mu=0$ and $c_0=c^*$, we can get w_s in (vi), which is unique up to translation under the condition (1–1), to play the role of $w_{(0)}$ in the first part of this proof. Details are omitted.

Proof of Theorem 2 (a). The argument is analogous to those made in the above. We assume $k(x)\equiv c^*$. By the hypothesis (1–2) there exists solutions w_g and w_s of (9) with $c=c^*$ on $x>0$ as described in (2–4) and in (2–3), respectively. We can choose them so that $w_s(0)<w_g(0)$ and $w_s'(0)<0$. If $W(x)$ is a stationary solution of (1)-(2) with a steep tail, we have necessarily $W(0)>w_g(0)$ as being remarked in (vi) of the section 2. On the other hand the solution of (1)-(3) starting from w_s just chosen above decreases to a stationary solution, say $W_1(x)$. Since w_s be of steep tail, W_1 must be so or identically zero. But $W_1(0)<w_g(0)$ and we conclude $W_1\equiv 0$.

5. Proof of Theorem 3 (a)

In this section, taking up an equation more general than (1)-(3), we will give a sufficient condition for some positive solution of it to fade away. The result is essentially due to [11]. A slightly modified proof will be outlined. Theorem 3 (a) will be proved, by applying it, at the end of this section.

Let D be a domain of R^d and $\boldsymbol{B}$ the Banach space of all bounded measurable functions on D with the uniform norm: $||u||=\text{ess sup}_{\underline{x}\in D}|u(\underline{x})|$. Let T_t, $t\geqq 0$ be a contruction semi-group of bounded linear operators on $\boldsymbol{B}$ which are positive and measurable (i.e. $T_t u\geqq 0$ if $u\geqq 0$ and $T_t u(\underline{x})$ is measurable in $(t, \underline{x})\in[0, +\infty)\times D$ for each $u\in\boldsymbol{B}$). Clearly P_t introduced in the section 3 is such a semi-group. Given $u^0\in\boldsymbol{B}$ such that $0\leqq u^0\leqq 1$, there exists a unique solution $u(t)\in\boldsymbol{B}$, $t>0$, with $0\leqq u(t)\leqq 1$, of the integral equation

$$u(t) = T_t u^0+\int_0^t T_s F(u(t-s))ds. \tag{5-1}$$

The solution is obtained as the uniform limit of $u_n(t)$ which are inductively defined by

$$\begin{aligned} u_0(t) &= T_t u^0 \\ u_n(t) &= T_t u^0+\int_0^t T_s F(u_{n-1}(t-s))ds \qquad \text{for } n\geqq 1\,. \end{aligned} \tag{5-2}$$

Theorem 3 (a) follows from Lemma 6 and the following

Theorem 5. *Let $\mu=0$ in (7). Writing F as $F(u)=\alpha u+\xi(u)$, assume that there exists positive constants M and a $(a<1)$ such that*

$$0\leqq\xi(uv)\leqq Mu\xi(v) \qquad \textit{for } 0<u<1 \textit{ and } 0<v<a\,. \tag{5-3}$$

Let g be an element of $\boldsymbol{B}$ such that $0\leqq g(\underline{x})<a$, $\underline{x}\in D$ and assume there is a continuous function $h(t)$, $t\geqq 0$ such that

$$e^{\alpha t}||T_t g||<h(t)<a\,,\ t\geqq 0 \quad \textit{and} \quad K\equiv\int_0^{+\infty}\frac{\xi(h(s))}{h(s)}ds<+\infty\,,$$

Then the solution $u(t)$ of (5-1) with $u^0=\delta g$, where $\delta=2^{-1}\exp(-M^2K)$, satisfies the inequality

$$u(t, \underline{x})\leqq e^{\alpha t}T_t g(\underline{x})\,, \qquad t>0\,,\ \underline{x}\in D\,. \tag{5-4}$$

Proof. Let $y(t)$ be a solution of the initial value problem

$$dy(t)/dt = M\xi(h(t)y(t))/h(t)\,, \qquad y(0) = \delta \tag{5-5}$$

where δ is defined in the statement of the theorem. First we prove that $y(t)$ is defined on $t\geqq 0$ and satisfies

$$y(t)<1\,, \qquad t>0\,. \tag{5-6}$$

Let $t_0=\sup\{t;\, y(t)<1\}$. Then for $t<t_0$ we have, by (5-3), $dy/dt\leqq M^2y\xi(h)/h$ and so $\log y(t)-\log\delta\leqq M^2\int_0^t\xi(h(s))/h(s)ds<M^2K$ or

$$y(t)\leqq\delta\exp(M^2K) = 1/2<1 \qquad (t<t_0)\,.$$

Hence t_0 is infinite and we have (5–6).

Now for the proof of (5–4) it is enough to show (for $u^0=\delta g$)

$$u(t, \underline{x})\leqq y(t)e^{\alpha t}T_t g(\underline{x}), \qquad t>0, \underline{x}\in D.$$

Let u_n be the sequence defined by (5–2). It suffices to prove that for $n=0, 1, 2, \cdots$

$$u_n(t, \underline{x})\leqq y(t)e^{\alpha t}T_t g(\underline{x}), \qquad t>0, \underline{x}\in D. \tag{5–7}$$

This is trivial for $n=0$. Now we assume that (5–7) is true for $n-1$. Then, noticing $u_{n-1}(s)/y(s)h(s)<1$ and $y(s)h(s)<a$, we have

$$\begin{aligned}F(u_{n-1}(s, \underline{x}))&\leqq \alpha u_{n-1}(s, \underline{x})+M\cdot[u_{n-1}(s, \underline{x})/y(s)h(s)]\xi(y(s)h(s))\\ &\leqq e^{\alpha s}T_s g(\underline{x})[\alpha y(s)+M\xi(y(s)h(s))/h(s)]\end{aligned}$$

and by (5–2) and (5–5)

$$\begin{aligned}u_n(t, \underline{x})&\leqq T_t g(\underline{x})\,[\delta+\int_0^t e^{\alpha s}(\alpha y(s)+y'(s))ds]\\ &= T_t g(x)e^{\alpha t}y(t).\end{aligned}$$

Thus by induction we obtain (5–7) for all $n=0, 1, 2\cdots$ The proof is completed.

Proof of Theorem 3 (a). Calculate the indefinite integral $\int F(h(t))h(t)dt$ for $h(t)=Ct^{-(A+1)/2}$ or for $h(t)=C_1\exp\{-C_2t^{(1-\beta)/(1+\beta)}\}$ to see that it is transformed, through the change of variable $u=h(t)$, to a constant multiple of $\int F(u)u^{-2-2/(A+1)}du$ or of $\int F(u)u^{-2}|\log u|^{2\beta/(1-\beta)}du$, respectively. Then Theorem 3 (a) follows from Theorem 5 and Lemma 6, by taking $f(x)=\varepsilon I_{[0,1]}(x)$ with sufficiently small $\varepsilon>0$.

REMARK 5. The first part of Theorem 2 is not obtained from Theorem 5. For example if we let $k(x)\geqq\kappa=c^*$ and $k(x)=\kappa+Ax^{-2}+O(x^{-3})$ (as $x\to\infty$), we have $e^{-\alpha t}P_t I_{[0,1]}(x)\asymp t^{-q}$ as $t\to\infty$ where q is defined in Lemma 6, and so the sufficient condition for some positive solution to fade away, which is derived from Theorem 5, is $\int_{0+}\xi(u)u^{-2-1/q}du>+\infty$ (under (5–3)). This condition is much stronger than (1–2).

6. Proof of Theorem 2 (b) and Theorem 3 (b)

In the proof of "growing up" part of Theorem 2 and 3 we use two lemmas, in which we are concerned with a function denoted by $J(t; b)$ which is defined as the solution of

$$\frac{dJ}{dt} = F(J), \qquad t>0, J(0, b) = b. \tag{6–1}$$

The first lemma is taken from [11] (with a minor alternation). The result and its proof are valid for solutions of (5–1). The proof of [11] will be outlined.

Lemma 7. *If* $\mu=0$ *and* F *is convex on* $0<u<\eta$ *where* η *denotes a positive constant, then for any solution* u *of* (1)-(3) *we have a lower bound* $u(t, x)\geqq J(t; P_t f(x))$ *as long as* $\sup_{0<s<t, x>0} u(s, x)\leqq\eta$.

Proof. Set $F^*(u)=F(u)$ for $0\leqq u\leqq\eta$ and $=F'(\eta)u$ for $u>\eta$. Then F^* is a non-decreasing, convex and uniformly Lipschitz continuous function on $u\geqq 0$. Let u_n be the iterations defined in (3–3) in which we replace F by F^* and take $a=0$. Similarly we define $J_n(t; b)$ by

$$J_0(t; b) = b, \; J_n(t; b) = b+\int_0^t F^*(J_{n-1}(s; b))ds \qquad n\geqq 1.$$

By induction we see easily that $J_n(t; b)$ is convex in $b>0$. We now prove that for $i=0, 1, 2\cdots$

$$u_i(t, x)\geqq J_i(t; P_t f(x)) \qquad t>0, x>0. \tag{6–2}$$

If $i=0$, this is trivial. Assuming this holds for $i=n$, we estimate the integrand in the last term of (3–3) as follows:

$$\begin{aligned} P_{t-s}\{F^*(u_n(s, \cdot))\} &\geqq P_{t-s}\{F^*(J_n(s; P_s f))\} \\ &\geqq F^*(J_n(s; P_t f)) \end{aligned}$$

(in the last inequality we applied Jensen's inequality to the convex function $F^*(J_n(s, \cdot))$), and so we get (6–2) for $i=n+1$. Thus (6–2) has been proved. Let u^* and J^* be solutions of (1)-(3) and of (6–1), respectively, both with F^* in stead of F. By letting i tend to infinity in (6–2) we have $u^*(t, x)\geqq J^*(t; P_t f(x))$. This implies the desired inequality, because u or J is identical to u^* or J^*, respectively, at a time t as long as they do not reach η at all up to that time.

Lemma 8. *Let* $\mu=0$. *Let* $h(t)$ *be a differentiable function of* $t\geqq 0$ *with* $0<h<1$. *If* $\int_0^t [1+h'(s)/F(h(s))]ds$ *tends to infinity as* $t\to\infty$, *then* $\lim_{t\to\infty} J(t; h(t))=1$. (*Converse is also true.*)

Proof. Integrating the differential equation satisfied by J to obtain $\int_b^{J(t;b)} du/F(u)=t$, then differentiating the both sides of it with respect to b to find

$$\partial J(t; b)/\partial b = F(J(t; b))/F(b),$$

and setting $v(t)=J(t, h(t))$, we have $v'(t)=F(v(t))[1+h'(t)/F(h(t))]$ or equivalently

$$\int_{v(0)}^{v(t)} du/F(u) = \int_0^t [1+h'(s)/F(h(s))]ds.$$

The left-hand side tends to infinity if and only if $v(t)$ approaches to unity. This yields the conclusion of the lemma.

Proof of Theorem 2 (b). For simplicity we let $F(u)=\alpha u+a\cdot u|\log u|$ for $0<u<\eta$ where η and a are positive constants and η is taken so small that F is convex on this interval. This simplification will yield no loss of generality because of the strict inequality in (1–4). By (vi) in the section 2 there is no non-trivial solution of (9) with $c=c^*$ on $x>0$. Since $c_0\geqq c^*$ whenever $\mu=0$, we have $c_0>c^*$. Therefore by Theorem 1 (a) there is a positive constant L such that a solution of (1)-(3) with $f=\eta I_{[0,L]}$ grows up.

Let u be a positive solution of (1)-(3). By the fact remarked just before Lemma 2, we can assume that f is non-increasing and positive. We will prove

$$u(t_0, L)\geqq\eta \qquad \text{for some } t_0>0. \tag{6–3}$$

which implies that $u(t_0, \cdot)\geqq\eta I_{[0,L]}$ and so u grows up. Let u^* be a solution of

$$\begin{cases} u_t^* = u_{xx}^*+ku_x^*+F(u^*) & t>0,\ x>L \\ u^*(0, x) = f(x),\ x>L;\ u_u^*(t, L) = 0, & t>0 \end{cases} \tag{6–4}$$

and P_t^* the semi-group of operators associated with the linear part of (6–4). Then by Lemma 4 $u(t, x)\geqq u^*(t, x)$ $(x>L)$ and by Lemma 7 $u^*(t, x)\geqq J(t, P_t^*f^*(x))$ as long as $\sup_{0<s<t}u^*(s, L)\leqq\eta$, where f^* is a restriction of f on $x>L$. By the same reason why we could simplify the form of F near zero, we can suppose that $k(x)=\kappa+Ax^{-2}+O(x^{-3})$ as $x\to\infty$. Then from Lemma 6 it follows that $P_t^*f^*(L)>Kt^{-q}e^{-\alpha t}$ for large values of t, where $K>0$ and $q=1+[A\kappa/2+1/4]^{1/2}$. Let $h(t)$ be a smooth function of $t\geqq0$ such that $h(t)=Kt^{-q}e^{-\alpha t}$ for large t and $0<h(t)<P_t^*f^*(L)$ for all $t>0$. Then $u(t, L)\geqq u^*(t, L)\geqq J(t, h(t))$ as long as $u(s, L)<\eta$ for $s<t$. Since $a>\alpha q$ by the hypothesis (1–4) and since for large t

$$\frac{h'(t)}{F(h(t))} = \frac{-q/t-\alpha}{\alpha+a(1+o(1))/\alpha t} = -1+(a-\alpha q)(1+o(1))/\alpha^2 t,$$

we have $\lim_{t\to\infty}J(t, h(t))=1$ by Lemma 8. These prove (6–3). The proof is completed.

Proof of Theorem 3 (b) is carried out in a similar way, so is omitted.

7. Proof of Theorem 4

Let us prove Theorem 4. Let δ be a constant in the theorem. We write $K=2\pi(4\alpha-\delta^2)^{-1/2}$. By the hypothesis of the theorem there is a constant $L\geqq 0$ such that $k(x)\leqq\delta$ for $L<x<L+K$ and either $L=0$ or $k(L)=\delta$. Let $k^*(x)$ be a C^1-class function of $x\geqq 0$, satisfying $k^*(x)=\delta$ for $x>L$ and $k^*(x)\leqq k(x)$ for $x<L$. Given a constant a in (0, 1), let $w=w^{(a)}(x)$ be a solution of the equation

$$w''+k^*w'+F(w)=0$$

on $[L_1, L_2]$ where $0\leqq L_1\leqq L<L_2<+\infty$ such that $w(L)=a$, $w'(L)=0$, $w(L_2)=0$, $w'(x)\geqq 0$ for $L_1<x<L$, and either $L_1=0$ or $w(L_1)=0$. The existance of $w^{(a)}$ is readily proved if we note that $\delta<c^*$ and that if $w(x)$ satisfies the above equation in a neighborhood of x_0 and if $w'(x_0)=0$, $0<w(x_0)<1$, then w is strictly concave at x_0. For small a, $w^{(a)}$ restricted on the interval (L, L_2) is approximated by $z(x)=Ne^{-\delta x/2}\cdot\cos[2^{-1}(4\alpha-\delta^2)^{1/2}(x-M)]$ where constants N and M ($L<M<L+K/2$) are determined by $z'(L)=0$ and $z(L)=a$. This proves that L_2, which varies together with a, is bounded above by $L+K$ for sufficiently small a. For such a's let us define functions $f_{(a)}$ by $f_{(a)}=w^{(a)}$ on the interval (L_1, L_2) and $f_{(a)}=0$ outside of it. Since $w^{(a)\prime\prime}+kw^{(a)\prime}+F(w^{(a)})=(k-k^*)w^{(a)\prime}\geqq 0$ as far as $L_2<L+K$, by Lemma 5 $f_{(a)}$ is a sub-steady state, which is zero for $x>L+K$ and does not exceed a at all. Let $u_{(a)}$ be the solution of (1)-(2) starting from $f_{(a)}$. The monotone limit $W^*(x)=\lim_{t\to\infty}u_{(a)}(t, x)$ is a stationary solution of (1)-(2).

Next we construct super-steady states. Let c and x_0 be positive constants such that $\kappa>c>c^*$, $b_+(c)>\kappa/2$ (b_+ is defined in (2–1)) and $k(x)>c$ for $x>x_0$. Since $F(u)\leqq\alpha u$, $c^*=c_0$. Therefore there is a solution w_s of (9) on $x>M$ $(M>0)$ such that $w_s(M)=1$ and $\log w_s(x)\sim -b_+(c)x$ (and incidentally $w_s'<0$). (See (iv) in the section 2.) Take $M>x_0$. Then $w_s''+kw_s'+F(w_s)=(k-e)w_s'<0$, $x>M$, so a super-steady state can be made of w_s according to Lemma 5. A solution of (1)-(2) starting from it is denoted by $u^{(c,M)}$. The limit $W=\lim u^{(c,M)}$ is a stationary solution which is bounded by a constant multiple of $e^{-\lambda x/2}$ with $\kappa<\lambda<2b_+(c)$.

Let u be a positive solution of (1)-(3) with $f(x)=O(e^{-(\kappa+\delta)x/2})$. We can find constants a, c and M in the above so that

$$u_{(a)}(t, x)\leqq u(1+t, x)\leqq u^{(c,M)}(t, x)\,.$$

Hence if the uniqueness assertion on the stationary solution is proved, we can conclude that u converges to $W=W^*$.

Now we show the uniqueness. Let $V(x)$ be a stationary solution with $V(x)=O(e^{-\kappa x/2})$. We can assume $W\leqq V$, because, if not, we may replace W

by the limiting state of the solution starting from a super-steady state $f^*=\min\{V(x), W(x)\}$. Using the relation $g''+kg'=e^{-B}(d/dx)(e^{B}dg/dx)$ and the boundary condition satisfied by V and W at zero, we see

$$\int_0^N \left(\frac{F(W)}{W}-\frac{F(V)}{V}\right)WVe^{B}dx=\int_0^N[-V(W''+kW')+W(V''+kW')]e^{B}dx$$
$$=e^{B(N)}[-VW'+WV']_{x=N}.$$

The right-hand side of this equation can be made aribtrarily small for suitable large values of N, since $B(x)\sim\kappa x$ while $V(x)W(x)=O(e^{-(\lambda+\kappa)x/2})$ for some $\lambda>\kappa$. Hence, by the monotonicity of $F(u)/u$, we obtain the equality $W=V$. This complete the proof.

8. Appendix

We give here a proof of the following

Proposition. *Let $b(t, x)$ and $c(t, x)$ be continuous functions of $(t, x)\in(0, \infty)^2$ and $b^*(x)$ a continuous function of $x>0$ such that $b^*(x)\leqq b(t, x)$ for all t, $x>0$. Let $u(t, x)$ be a continuous function of $(t, x)\in[0, T)\times(0, L]$, $0<T\leqq\infty$, $0<L\leqq\infty$, and satisfy the differential inequality*

$$u_t\geqq u_{xx}+bu_x+cu \qquad \text{for } 0<t<T,\ 0<x<L.$$

When $L=\infty$, we further suppose that for a constant M

$$b(t, x)<Mx \qquad \text{for } t>0,\ x>1$$
$$c(t, x)<M\cdot(1+x^2) \quad \text{and} \quad |u(t, x)|<M\exp(Mx^2) \qquad \text{for } t,\ x>0,$$

and when $L<\infty$, we let $u(t, L)\geqq 0$. If $u(0, x)\geqq 0$, $0<x\leqq L$ and if

$$\overline{\lim_{x\downarrow 0}}\exp\left(\int_1^x b^*(y)dy\right)u_x(t, x)\leqq 0, \qquad 0<t<T,$$

then $u(t, x)\geqq 0$, $0<t<T$, $0<x\leqq L$.

Proof. The proof is given only for the case $L=\infty$, since the other case is analogously treated. Let $L=\infty$. Let $K(x)$ be a smooth function on $x\geqq 0$ with the following properties

$$K'(x)=0 \quad \text{for } 0<x\leqq 1; \qquad K'(x)\geqq 0 \quad \text{for } x>1$$
$$K(x)=2(M+1)x^2 \qquad \text{for } x>2.$$

For each pair of positive constants a and q we set

$$v(t, x)=\exp\left\{\frac{-K(x)}{1-at}-qt\right\}u(t, x) \qquad \text{for } (t, x)\in E_a$$

where $E_a=\{(t, x)\colon 0<t<1/2a,\ x>0\}$. It is seen that

$$\lim_{x\to\infty} v(t, x) = 0 \qquad \text{uniformly in } 0<t<1/2a$$

and

$$v_t \geqq v_{xx}+\hat{b}v_x+\hat{c}v\,,$$

where

$$\hat{b}(t, x) = b(t, x)+\frac{2K'(x)}{1-at}$$

$$\hat{c}(t, x) = \frac{K'(x)^2}{(1-at)^2}+\frac{K''(x)}{1-at}+\frac{K'(x)}{1-at}b(t, x)+c(t, x)-\frac{a}{(1-at)^2}K(x)-q\,.$$

We choose constants a and q so large that $\hat{c}<-1$ in E_a.

There exists a positive decreasing function $g(x)$ of $x>0$ which satisfies $g''+b^*g'-g=0$, $x>0$, $\lim_{x\to\infty} g=0$ and

$$\lim_{x\downarrow 0} \exp\left(\int_1^x b^*(y)dy\right)g'(x)<0\,.$$

(See Ito-McKean [8] §4.6.) Now let us prove that $w=v+\varepsilon g\geqq 0$ for each $\varepsilon>0$. Since $b^*\leqq b\leqq\hat{b}$ and $-\hat{c}-1>0$, we obtain

$$\text{(A-1)} \qquad \begin{aligned} w_t &\geqq w_{xx}+\hat{b}w_x+\hat{c}w+[(b^*-\hat{b})g'+(-\hat{c}-1)g] \\ &\geqq w_{xx}+\hat{b}w_x+\hat{c}w\,. \end{aligned}$$

We also have $\lim_{x\to\infty} w(t, x)=0$ uniformly in t and

$$\overline{\lim_{x\downarrow 0}} \exp\left(\int_1^x b^*(y)dy\right)w_x(t, x)<0\,.$$

Assuming the contrary let w become negative at some point of E_a. Then, by the boundary conditions satisfied by w, it must attain the negative minimum at a point in $0<t\leqq 1/2a$, $x>0$. But this is impossible by (A-1). Thus we have $v+\varepsilon g\geqq 0$ and, by letting $\varepsilon\downarrow 0$, $u\geqq 0$ in E_a. Step by step we can show $u\geqq 0$ in E_T.

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