<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Extendibility of G-maps to pseudo-equivalences to finite G-CW-complexes whose fundamental groups are finite</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Morimoto, Masaharu; Iizuka, Kunihiko</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 21(1) P.59-P.69</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1984</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/12172">https://doi.org/10.18910/12172</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/12172</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
0. Introduction

In this paper we let $G$ be a finite group. A. Assadi [2] and R. Oliver-T. Petrie [6] treated the following question. What is a necessary and sufficient condition, for given finite $G$-$CW$-complexes $X$ and $Y$ and a $G$-map $f: X \to Y$, to extend $f$ to a quasi-equivalence $f': X' \to Y$ (with some reservations)? Here a $G$-map is called a quasi-equivalence if it induces isomorphisms of fundamental groups and of integral homology groups. We apply the Oliver-Petrie theory to covering spaces to give a necessary and sufficient condition so that we may extend above $f$ to a pseudo-equivalence $f'': X'' \to Y$ (with some reservations), when $\pi_1(Y)$ is finite.

We take Oliver-Petrie [6] as our general reference and use their terms and notations.

Let $Y$ be a finite connected $G$-complex. Then $\tilde{G} = \pi_1(EG \times_G Y)$ acts on the universal covering space $\tilde{Y}$ of $Y$ as is shown in section 1 (compare the action with that of D. Anderson [1]). Assume $\pi_1(Y)$ is finite. Then $\tilde{G}$ is finite, so we have a $\tilde{G}$-poset $\tilde{\Pi} = \Pi(\tilde{Y})$ and a $G$-poset $\Pi = \Pi(Y)$. In section 3 we give a one to one correspondence $T$ from the set of $G$-families in $\Pi$ to the set of $\tilde{G}$-families in $\tilde{\Pi}$, and an isomorphism $\nu$ from $\Omega(\tilde{G}, \tilde{\Pi})$ to $\Omega(G, \Pi)$. A subgroup $\Delta_4(G, Y, \mathcal{F})$ of $\Delta(G, \mathcal{F})$ is defined by $\Delta_4(G, Y, \mathcal{F}) = \nu(\Delta(\tilde{G}, T(\mathcal{F})))$. Under certain conditions $\Delta_4(G, Y, \mathcal{F})$ agrees with the set

$$\{[M_f] \in \Omega(G, \Pi) | f: X \to Y \text{ is a pseudo-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}$$

(see Proposition 4.1), where $M_f$ is the mapping cone of $f$.

Our main results are:

**Theorem 1.** Let $X$ be a finite $G$-complex, $Y$ a finite connected $G$-complex with finite $\pi_1(Y)$, $f: X \to Y$ a skeletal $G$-map, and $\mathcal{F} \subset \Pi$ any connected $G$-
family containing $\mathcal{F}$. Let $\mathcal{F}' \subset \mathcal{F}$ be any subfamily containing $\mathcal{F}$. Assume $T(\mathcal{F})$ is simply generated. Then there exist a finite $G$-complex $X'$ and a pseudo-equivalence $f': X' \to Y$ extending $f$ with $X'/X$ an $\mathcal{F}'$-complex, if and only if

$$[M_f] \in \Delta_h(G, Y, \mathcal{F}) + \Omega(G, \mathcal{F}') \text{ in } \Omega(G, \Pi).$$

**Corollary 2.** Assume $G$ is not of prime power order. Let $Y$ be a finite connected $G$-complex with finite $\pi_1(Y)$, and $F_1$, $\ldots$, $F_k$ the connected components of $F = Y^G$. Then there is a subgroup $N_Y \subset \mathbb{Z}^k$ such that given any finite $G$-complex $F'$ and a map $\hat{f}: F' \to F$, there exist a finite $G$-complex $X$ with $X^G = F'$ and a pseudo-equivalence $f: X \to Y$ with $f^G = \hat{f}$ if and only if $$(\chi(F_1) - \chi(F'_1), \ldots, \chi(F_k) - \chi(F'_k)) \in N_Y,$$ $$(F'_i = \hat{f}^{-1}(F_i)).$$

Above $N_Y$ is the image of $\Delta_h(G, Y, \Pi)$ by the homomorphism $\psi: \Omega(G, \Pi) \to \mathbb{Z}^k$ defined in section 3 of [6]. Thus $N_Y$ is included in $n_Y$.

**Corollary 3.** Let $G$ and $Y$ be as above. Moreover we assume $F$ is connected and $G$ belongs to $\Omega^1$, i.e. $G/P$ is cyclic for some normal subgroup $P$ of $G$ of prime power order. Given any finite $G$-complex $F'$ and any map $\hat{f}: F' \to F$, there exist a finite $G$-complex $X$ with $X^G = F'$ and a pseudo-equivalence $f: X \to Y$ extending $\hat{f}$, if and only if $\chi(F) = \chi(F')$.

The proofs of Theorem 1 and Corollaries 2 and 3 are given in section 4.

In a subsequent paper we will calculate $N_Y$ in several cases.

In this paper we often omit the adjective skeletal from a skeletal $G$-map, however, a $G$-map should be understood to be a skeletal $G$-map when its mapping cone appears.

1. **A standard action of $\pi_1(EG \times_G Y)$ on the universal covering space of $Y$**

Let $Y$ be a connected $G$-complex, $p: \tilde{Y} \to Y$ the universal covering, $EG$ the universal principal $G$-bundle. Arbitrarily choose and fix base points $a_0$ of $Y$, $b_0$ of $\tilde{Y}$ with $p(b_0) = a_0$, and $c_0$ of $EG$. Let $q: EG \times Y \to EG \times_G Y$ be the canonical projection. We use $u_0 = (c_0, a_0)$ and $v_0 = q(u_0)$ as the base points of $EG \times Y$ and $EG \times_G Y$ respectively. We put $\pi = \pi_1(Y)$ and $\bar{G} = \pi_1(EG \times_G Y)$ in this section.

We define a map $k: Y \to EG \times_G Y$ by $k(y) = q(c_0, y)$ for $y \in Y$. The covering $q: EG \times Y \to EG \times_G Y$ induces the exact sequence

$$\{1\} \to \pi \to \bar{G} \to G \to \{1\}$$

where $j$ is the induced map by $k$, and $\sigma$ is the map obtained by identifying $\pi_0(G)$.
with $G$. We regard $\pi$ as a subgroup of $\tilde{G}$ through $j$.

In the following we illustrate a standard action of $G$ on $\bar{Y}$ such that

1. $p$ is $\sigma$-equivariant, i.e. for $g \in \tilde{G}$ and $b \in \bar{Y}$ $p(gb) = \sigma(g)p(b)$,

2. the induced CW-complex structure on $\bar{Y}$ by $p$ and the $G$-action make $\bar{Y}$ a $G$-complex,

3. the restriction of the $G$-action to $\pi$ agrees with the action given by M. Cohen [3; p. 12].

We denote by $r$ the projection from $EG \times Y$ to the second factor $Y$. Immediately $r(u_0) = a_0$ follows. We are going to give $gb$ for $g \in G$ and $b \in \bar{Y}$. An element $g$ of $\tilde{G}$ is represented by a path $\alpha: [0, 1] \to EG \times \bar{Y}$ with $\alpha(0) = \alpha(1) = v_0$.

There is a unique lift $L_\alpha(\alpha): [0, 1] \to EG \times Y$ of $\alpha$ (i.e. $q \circ L_\alpha(\alpha) = \alpha$) with $L_\alpha(\alpha)(0) = u_0$. The homomorphism $\sigma : \bar{G} \to G$ is given by the relation $\sigma(g)u_0 = L_\alpha(\alpha)(1)$. The path $\alpha$ gives two paths $\alpha' = r \circ L_\alpha(\alpha): [0, 1] \to Y$ and its lift $L_\alpha(\alpha') : [0, 1] \to \bar{Y}$ (i.e. $p \circ L_\alpha(\alpha') = \alpha$) with $L_\alpha(\alpha')(0) = b_0$. We have $\alpha'(0) = a_0$ and $\alpha'(1) = \sigma(g)a_0$.

For given $b \in \bar{Y}$, choose arbitrarily a path $\beta : [0, 1] \to \bar{Y}$ with $\beta(0) = b_0$ and $\beta(1) = b$. $\beta$ gives two paths $p \circ \beta : [0, 1] \to Y$ and $\beta' : [0, 1] \to Y$ defined by $\beta'(t) = \sigma(g)p(\beta(t))$ for $t \in [0, 1]$. We have $\beta'(0) = \sigma(g)a_0 = \alpha'(1)$ and

\[(1.1) \quad \beta'(1) = \sigma(g)p(b).\]

There is a unique lift $L_\beta(\beta') : [0, 1] \to \bar{Y}$ of $\beta'$ with $L_\beta(\beta')(0) = L_\beta(\alpha')(1)$. We
define $gb$ to be the point $L_p(\beta')(1)$.

By (1.1) we have $p(gb) = \sigma(g)p(b)$. That is, $p$ is $\sigma$-equivariant. The properties (2) and (3) follow immediately.

2. Remarks on $\mathcal{F}$-complexes

For a finite group $G$, a $G$-poset is axiomatically defined as follows. Let $\mathcal{S}(G)$ be the set of subgroups of $G$. By conjugation $G$ acts on $\mathcal{S}(G)$: $(g, H) \mapsto gHg^{-1}$ for $g \in G$, $H \in \mathcal{S}(G)$.

2.1. A partially ordered $G$-set $\Pi$ equipped with a $G$-map $\rho: \Pi \to \mathcal{S}(G)$ is called a $G$-poset if the following four conditions are satisfied: for $\alpha \in \Pi$, $\beta \in \Pi$ (i) $\rho(\alpha) \triangleleft G$, (ii) if $\alpha \leq \beta$ then $g\alpha \leq g\beta$ for $g \in G$, (iii) if $\alpha \equiv \beta$ then $\rho(\alpha) \equiv \rho(\beta)$, and (iv) for a subgroup $H$ of $\rho(\alpha)$ there exists a unique element $\gamma$ of $\Pi$ such that $\gamma \geq \alpha$ and $\rho(\gamma) = H$.

Typical examples of $G$-posets are $\Pi(X)$ for $G$-spaces $X$ (see [4] and [6]).

A $G$-subset of a $G$-poset $\Pi$ is called a $G$-family (in $\Pi$). A $\Pi$-complex $Z$ for a $G$-poset $\Pi$ is a finite $G$-complex with base point $\ast$ and subcomplexes $Z_{\alpha} \subset Z$, $(\ast \in Z_{\alpha})$, for all $\alpha \in \Pi$ such that $Z_{g\alpha} = gZ_{\alpha}$ for $g \in G$, $Z_{\alpha} \subset Z_{\beta}$ for $\alpha \leq \beta$, and $Z_{H} = \bigcup_{\rho(\gamma) = H} Z_{\gamma}$ for $H \triangleleft G$.

For a $G$-family $\mathcal{F}$ in $\Pi$ a $\Pi$-complex $Z$ is called an $\mathcal{F}$-complex if

$$Z_{\alpha} = \{\ast\} \cup \{Z_{\beta} | \beta \in \mathcal{F}, \beta \leq \alpha\}$$

for any $\alpha \in \Pi$.

2.2. Let $\Pi$ be a $G$-poset, $\mathcal{F}$ a $G$-family in $\Pi$, and $Z$ an $\mathcal{F}$-complex. For $\alpha \in \Pi$, $\beta \in \Pi$ and $x \in (Z_{\alpha} \cap Z_{\beta}) \setminus \{\ast\}$, there is a unique element $\gamma$ of $\mathcal{F}$ such that $\gamma \leq \alpha$, $\gamma \leq \beta$, $\rho(\gamma) = G_x$ and $x \in Z_{\gamma}$.

2.3. Let $\Pi$ be a $G$-poset, $Z$ a $\Pi$-complex. For each (non-equivariant) cell $c$ in $Z \setminus \{\ast\}$, there exists a unique element $\alpha(c) \in \Pi$ such that $\rho(\alpha(c)) = G_x$, $x \in c$, and $c \subset Z_{\alpha(c)}$. If $Z_{\beta} \supset c$ for $\beta \in \Pi$, then $\alpha(c) \leq \beta$. So we call $c$ of type $\alpha(c)$.

2.4. Let $\mathcal{F}$ be a $G$-family in $\Pi$. A $\Pi$-complex $Z$ is an $\mathcal{F}$-complex if and only if $\mathcal{F}$ contains $\alpha(c)$ for any cell $c$ in $Z \setminus \{\ast\}$.

Let $\Pi$ be a $G$-poset. For each $\alpha \in \Pi$, the $\Pi$-complexes $(\alpha)$ is the $G$-space $\{\ast\} \sqcup G/\rho(\alpha)$ with

$$(\alpha)_{\beta} = \{\ast\} \sqcup \{g \rho(\alpha) | g \in G, g \alpha \leq \beta\}$$

for $\beta \in \Pi$.

In the rest of this section we let $Y$ be a finite connected $G$-complex and
EXTENDIBILITY OF G-MAPS TO PSEUDO-EQUIVALENCES

Let $X$ be another finite $G$-complex, and $f$ a $G$-map from $X$ to $Y$. For $\alpha \in \Pi$, $X_\alpha \simeq X^\alpha \cap f^{-1}(|\alpha|)$. $X^+ = X \sqcup \{\ast\}$ (disjoint union) has a $\Pi$-complex structure given by $(X^+)_\alpha = X_\alpha \sqcup \{\ast\}$. We call this $\Pi$-complex structure the $\Pi$-complex structure induced by $f$.

2.5. Let $\alpha$ be an element of $\Pi$. For an arbitrary $G$-map $f$ from $X = G/\rho(\alpha)$ to $Y$ with $f(\rho(\alpha)) \subseteq |\alpha|$, the induced $\Pi$-complex $X^+$ by $f$ agrees with $(\alpha)$ as $\Pi$-complex.

2.6. Let $F$ be a finite CW-complex, and $\alpha$ an element of $\Pi$. For a $G$-map $f$ from $X = (G/\rho(\alpha)) \times F$ to $Y$ with $f(\rho(\alpha) \times F) \subseteq |\alpha|$, $[X^+] = \chi(F)[\alpha]$ in $\Omega(G, \Pi(Y))$.

**Proposition 2.7.** Let $\mathcal{F}$ be a $G$-family in $\Pi = \Pi(Y)$ containing $\mathcal{F}(Y)$. Then

$$\Omega(G, \mathcal{F}) = \{[M_f] \in \Omega(G, \Pi) \mid f: X \to Y \text{ is a } G\text{-map such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}.$$ 

Proof. Choose integers $z(\alpha), \alpha \in \mathcal{F}$, such that $[Y^+] = \sum \alpha z(\alpha)[\alpha], \alpha \text{ runs over } \mathcal{F}$. For any $\xi \in \Omega(G, \mathcal{F})$, there are integers $z'(\alpha), \alpha \in \mathcal{F}$, such that

$$\xi = \sum \alpha z(\alpha)[\alpha] - \sum \alpha z'(\alpha)[\alpha].$$

Take finite CW-complexes $F(\alpha)$ with $\chi(F(\alpha)) = z'(\alpha)$, and put $X = \sqcup \{(G/\rho(\alpha)) \times F(\alpha) | \alpha \in \mathcal{F}\}$. There is a $G$-map $f: X \to Y$ with $f(\rho(\alpha) \times F(\alpha)) \subseteq |\alpha|$. We have $[M_f] = [Y^+] - [X^+] = \xi$ by 2.6.

According to Proposition 1.6 of [6],

$$\Delta(G, \mathcal{F}) = \{[Z] \in \Omega(G, \Pi) \mid Z \text{ is a contractible } \mathcal{F}\text{-complex}\}.$$ 

Moreover we have the following.

**Proposition 2.8.** Let $\mathcal{F}$ be a connected $G$-family in $\Pi = \Pi(Y)$ such that $\mathcal{F}$ contains $\mathcal{F}(Y)$ and $\mathcal{F}$ is simply generated. Then

$$\Delta(G, \mathcal{F}) = \{[M_f] \in \Omega(G, \Pi) \mid f: X \to Y \text{ is a quasi-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}.$$ 

Proof. We prove that for given $\xi \in \Delta(G, \mathcal{F})$ there exist a finite $G$-complex $X$ and a (skeletal) $G$-map $f: X \to Y$ with $[M_f] = \xi$. For $\xi \in \Delta(G, \mathcal{F})$ there are a finite $G$-complex $X_0$ and a $G$-map $f_0: X_0 \to Y$ with $[M_{f_0}] = \xi$ by Proposition 2.7. By the same argument as Oliver-Petrie used at Steps 2 and 3 of the proof of [6; Proposition 2.9], we get a finite $G$-complex $X_1 \supset X_0$ and a $G$-map $f_1: X_1 \to Y$ extending $f_0$ such that $X_1/X_0$ is an $\mathcal{F}$-complex, $M_{f_1}$ is an $\mathcal{F}$-resolution.
and \( \{M_f\} = \xi \). Adding free cells to \( X \) appropriately if necessary, we may assume \( \dim X_i \geq 3 \). We use the same argument as was used in the proof (1) of [6; Theorem 3.2], and obtain a finite \( G \)-complex \( X \supset X_1 \) and a \( G \)-map \( f: X \to Y \) extending \( f_1 \) such that \( X/X_1 \) is a \( \hat{G} \)-complex and \( f \) is a quasi-equivalence. We have to check \( [M_f] = \xi \). Both \( [M_f] \) and \( [M_{f_2}] \) belong to \( J(G, \mathfrak{F}) \), and \( [X/X_1] = [M_{f_2}] - [M_f] \). We have \( \chi((X/X_0)_a) = 1 \) for \( \alpha \in \hat{G} \) by Proposition 2.6 of [6]. Since \( X/X_0 \) is an \( \hat{G} \)-complex, we have \( [X/X_0] = 0 \). That is \( [M_f] = [M_{f_2}] = \xi \).

3. Correspondences between the posets of a finite covering space and a base space

In this section we let \( G \) and \( \hat{G} \) be finite groups, \( \sigma: \hat{G} \to G \) a epimorphism, \( Y \) a finite connected \( G \)-complex, \( \check{Y} \) a finite connected \( \hat{G} \)-complex, and \( p: \check{Y} \to Y \) a \( \sigma \)-equivariant covering. We put \( \pi = \ker \sigma \). Moreover we assume that \( \pi \) acts freely and transitively on each fiber.

The \( \hat{G} \)-action on \( \check{Y} \) gives a \( \hat{G} \)-poset \( \check{\Pi} = \check{\Pi}(\check{Y}) \) and a \( \hat{G} \)-map \( p: \check{\Pi} \to \mathfrak{S}(\hat{G}) \). The set of \( G \)-families in \( \Pi \) is denoted by \( \mathfrak{F} \) and that of \( \hat{G} \)-families in \( \check{\Pi} \) is denoted by \( \check{\mathfrak{F}} \).

For arbitrarily given \( \alpha \in \check{\Pi} \), there is a unique element \( \beta \in \Pi \) such that \( \rho(\beta) = \sigma(\rho(\alpha)) \) and \( |\beta| \supseteq \rho(|\alpha|) \). This correspondence defines a map \( \mu: \check{\Pi} \to \Pi \).

For \( \alpha \in \Pi \), we denote the connected components of \( \rho^{-1}(|\alpha|) \) by \( A_1, \ldots, A_k \). We have \( \rho(A_i) = |\alpha| \) for any \( i = 1, \ldots, k \). Each \( A_i \) is fixed by a subgroup \( H_i \) of \( \hat{G} \) with \( \sigma(H_i) = \rho(\alpha) \), since \( \pi \) preserves each fiber. As \( \pi \) acts freely on each fiber, \( \sigma: H_i \to \rho(\alpha) \) is bijective. Each \( A_i \) is contained in a connected component \( B_i \) of the \( H_i \)-fixed point set of \( \check{Y} \). The projection \( \rho \) is \( \sigma \)-equivariant, so \( \rho(B_i) = |\alpha| \). We have \( A_i = B_i \). There is a unique element \( \beta_i \in \check{\Pi} \) such that \( \rho(\beta_i) = H_i \) and \( |\beta_i| = A_i \). We define a map \( \tau: \check{\Pi} \to \mathfrak{S}(\Pi) \) by \( \tau(\alpha) = \{\beta_1, \ldots, \beta_k\} \), where \( \mathfrak{S}(\check{\Pi}) \) denotes the set of subsets of \( \check{\Pi} \).

Immediately we have \( \mu(\tau(\alpha)) = \{\alpha\} \) for \( \alpha \in \Pi \). The above argument implies \( |\mu(\alpha)| = \rho(|\alpha|) \) for \( \alpha \in \check{\Pi} \). The following two diagrams are commutative:

\[
\begin{array}{ccc}
\check{\Pi} & \xrightarrow{\mu} & \Pi \\
\downarrow{p} & & \downarrow{\rho} \\
\mathfrak{S}(\hat{G}) & \xrightarrow{\sigma} & \mathfrak{S}(G) \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\check{\Pi} & \xrightarrow{\mu} & \Pi \\
\downarrow{\rho} & & \downarrow{\rho} \\
\mathfrak{S}(\hat{Y}) & \xrightarrow{p} & \mathfrak{S}(Y) \\
\end{array}
\]

where \( \mathfrak{S}(\hat{Y}) \) and \( \mathfrak{S}(Y) \) are the sets of subspaces of \( \hat{Y} \) and \( Y \) respectively. For \( \alpha \in \check{\Pi} \), \( \alpha \) is an element of \( \tau(\mu(\alpha)) \).

**Proposition and definition 3.1.** The following two equations define maps
$M: \tilde{F} \to F$ and $T: F \to \tilde{F}$,

$$M(\tilde{F}) = \{ \mu(\alpha) | \alpha \in \tilde{F} \} \text{ for } \tilde{F} \in \tilde{F},$$

$$T(F) = \cup \{ \tau(\alpha) | \alpha \in F \} \text{ for } F \in F.$$

We have $M \circ T = id_F$ and $T \circ M = id_F$.

We omit the proof.

**Lemma 3.2.** (i) If $\tilde{F} \in \tilde{F}$ is connected, then $M(\tilde{F})$ is connected, and $\tilde{M}(\tilde{F}) = M(\tilde{F})$.

(ii) If $F \in F$ is connected and contains $F(Y)$, then $T(F)$ is connected, and $\tilde{T}(\tilde{F}) = T(\tilde{F})$.

Proof. We prove (ii), and let (i) remain to be proved by the reader. We denote the maximal element of $\tilde{F}$ by $\tilde{m}$, so we have $\tau(\tilde{m}) = \{ \tilde{m} \}$. Since $m \in F$, $\tilde{m} \in T(\tilde{F})$. Assume $\alpha$ is an element of $\tilde{F}$ such that $\rho(\alpha)$ is of prime power order and $\{ \beta \in T(F) | \beta \leq \alpha \}$ is not empty. Since $M \circ T = id_F$ and $\mu$ preserves the order, $\{ \beta \in F | \beta \leq \mu(\alpha) \}$ is not empty. There is the unique maximal element $\gamma$ of $\{ \beta \in F | \beta \leq \mu(\alpha) \}$, $(\gamma = \mu(\alpha))$. Since $Y^*$ is a $\tilde{F}$-complex, we have $|\gamma| = |\mu(\alpha)|$ by Proposition 1.2 of [6]. There uniquely exists $\delta \in \tau(\gamma)$ with $\delta \leq \alpha$. For any $\beta \in T(F)$ with $\beta \leq \alpha$, we have $|\beta| \subset |\alpha| = |\delta|$ and $\mu(\beta) \leq \gamma$. Thus $\sigma(\rho(\beta)) = \rho(\mu(\beta)) \supset \rho(\gamma) = \sigma(\rho(\delta))$. Since $\pi = ker \sigma$ acts freely on each fiber, we see $\rho(\beta) \supset \rho(\delta)$. Therefore $\beta \leq \delta$, that is, $\delta$ is the unique maximal element of $\{ \beta \in T(F) | \beta \leq \alpha \}$. $T(F)$ is connected. This argument implies $T(\tilde{F}) = \tilde{T}(\tilde{F})$.

Let $X$ be another finite $G$-complex, and $f: X \to Y$ a skeletal $G$-map. Then $f$ induces the covering $f* p: \tilde{X} = f* \tilde{Y} \to X$,

$$\tilde{X} = \{(x, b) \in X \times \tilde{Y} | f(x) = p(b) \},$$

$(f* p)(x, b) = x$ for $(x, b) \in \tilde{X}$. $G$ acts on $\tilde{X}$ by $g(x, b) = (\sigma(g)x, gb)$ for $g \in G$, $(x, b) \in \tilde{X}$. $\tilde{X}$ has the CW-complex structure induced by $f* p$, and becomes a $G$-complex. A $G$-map $\tilde{f}: \tilde{X} \to \tilde{Y}$ is given by $\tilde{f}(x, b) = b$ for $(x, b) \in \tilde{X}$, and $\tilde{f}$ is skeletal.

**Lemma 3.3.** In the above situation, $\tilde{F}_f = T(\tilde{F}_f)$ and $\tilde{F}_f = M(\tilde{F}_f)$.

Proof. Firstly we show $M(\tilde{F}_f) \subseteq \tilde{F}_f$. For $\alpha \in \tilde{F}_f$, (i) $\rho(\alpha) \in Iso(|\alpha|)$ or (ii) $\rho(\alpha) \in Iso(\tilde{X}_\alpha)$. Assume the case (i). There exists a point $b \in |\alpha|$ with $G_b = \rho(\alpha)$. We have $G_{\rho(b)} = \sigma(G_b) = \sigma(\rho(\alpha)) = \rho(\mu(\alpha))$, and $\rho(\mu(\alpha)) \in Iso(|\mu(\alpha)|)$. Thus $\mu(\alpha) \in \tilde{F}_f$. Assume the case (ii). There exists a point $(x, b) \in \tilde{X}_\alpha$ with
By definition, $X_\alpha = \mathbb{R}^{(\alpha)} \cap f^{-1}(|\alpha|) = \{(x',b') \in X \times Y | f(x') = p(b')\}$, $x' \in X^{(\alpha)}$, $b' \in |\alpha|$. We have $G_{(x,b)} = \sigma^{-1}(G_x) \cap \tilde{G}_b$, and $\rho(\mu(\alpha)) = \sigma(p(\alpha)) = \sigma(G_{(x,b)}) = G_x$. Since $f^{-1}(|\mu(\alpha)|) = f^{-1}(p(|\alpha|))$, $x \in X^{(\alpha)} \cap f^{-1}(p(|\alpha|)) = X_{\mu(\alpha)}$. Thus $\mu(\alpha) \in \mathcal{F}_f$. We have $M(\mathcal{F}_f) \subset \mathcal{F}_f$.

Secondly we show $T(\mathcal{F}_f) \subset \mathcal{F}_f$. For $\alpha \in \mathcal{F}_f$, (iii) $\rho(\alpha) \in \text{Iso}(|\alpha|)$ or (iv) $\rho(\alpha) \in \text{Iso}(X_\alpha)$. Assume the case (iii). There exists a point $a \in |\alpha|$ with $\rho(a) = G_a$. Fix $\beta \in \tau(\alpha)$ and $b \in |\beta| \cap p^{-1}(a)$. $G_b$ contains $\rho(\beta)$. Since $\sigma: G_b \rightarrow G_a$ is bijective, $\rho(\beta) = \rho(a) = G_a$ implies $G_b = \rho(\beta)$. Thus $\rho(\beta) \in \text{Iso}(|\beta|)$, and $\beta \in \mathcal{F}_f$. We have $\tau(\alpha) \subset \mathcal{F}_f$. Assume the case (iv). There exists a point $x \in X_\alpha$ with $G_x = \rho(\alpha)$. Fix $\beta \in \tau(\alpha)$ and $b \in |\beta| \cap f^{-1}(a)$. Then $(x, b) \in X_\beta$. The isomorphism $\sigma: G_b \rightarrow G_{(x,b)}$ maps both $\rho(\beta)$ and $G_{(x,b)} = \sigma^{-1}(G_x) \cap \tilde{G}_b$ to $G_x$. We get $\rho(\beta) = G_{(x,b)}$, and $\beta \in \mathcal{F}_f$. Thus $\tau(\alpha) \subset \mathcal{F}_f$. We have $T(\mathcal{F}_f) \subset \mathcal{F}_f$.

By Proposition and definition 3.1, we have $\mathcal{F}_f = T(\mathcal{F}_f)$ and $\mathcal{F}_f = M(\mathcal{F}_f)$.

Let $\tilde{\mathcal{F}}$ be a $G$-family in $\tilde{I}$, $Z$ an $\tilde{\mathcal{F}}$-complex. The quotient space $Z = \tilde{Z}/\pi$ has a $\Pi$-complex structure given by

$$Z_\alpha = (\bigcup_{\beta \in \tau(\alpha)} Z_\beta)/\pi, \ \alpha \in \Pi.$$  

Moreover $Z$ becomes a $M(\tilde{\mathcal{F}})$-complex. For $\alpha \in \Pi$ we have

$$(3.4) \quad \chi(Z_\alpha) - 1 = (\chi(Z_\beta) - 1)/|\pi_\beta|,$$

where $\beta$ is an arbitrary element of $\tau(\alpha)$.

The correspondence $\tilde{Z} \rightarrow Z$ defines a homomorphism $\nu: \Omega(G, \tilde{\mathcal{F}}) \rightarrow \Omega(G, M(\tilde{\mathcal{F}}))$. By (3.4) $\nu$ is injective. If $\tilde{\mathcal{F}}' \subset \tilde{\mathcal{F}}$ then the following diagram is commutative:

$$\Omega(G, \mathcal{F}') \xrightarrow{\nu} \Omega(G, \tilde{\mathcal{F}})$$

$$\Omega(G, M(\mathcal{F}')) \xrightarrow{\nu} \Omega(G, M(\tilde{\mathcal{F}})),$$

where the horizontal arrows are the canonical maps.

**Proposition 3.5.** Let $\mathcal{F}$ be a $G$-family in $\Pi$, and put $\tilde{\mathcal{F}} = T(\mathcal{F})$. Then $\nu: \Omega(G, \tilde{\mathcal{F}}) \rightarrow \Omega(G, \tilde{\mathcal{F}})$ is an isomorphism.

**Proof.** It is sufficient to show that $\nu$ is surjective. Arbitrarily fix $\alpha \in \tilde{\mathcal{F}}$. Put $X = G/H$, $H = \rho(\alpha)$. There is a $G$-map $f: X \rightarrow Y$ with $f(1 \cdot H) \in Y_\alpha$. Let $\tilde{f}: \tilde{X} = \tilde{f}^* \tilde{Y} \rightarrow \tilde{Y}$ be the induced $G$-map. $(\tilde{X})^+$ has a $\tilde{I}$-complex structure induced by $f$. Take a point $(1 \cdot H, b) \in \tilde{X}$, so $f(1 \cdot H) = p(b)$, and put $\beta = \text{min.} \{ \gamma \in \tilde{I} | \tilde{X}_\gamma \ni (1 \cdot H, b) \}$, (that is, $(1 \cdot H, b)$ is a point of a cell of type $\beta$). Since $G$ acts transitively on $\tilde{X}, (\tilde{X})^+$ is a $\{ g\beta | g \in G \}$-complex. Since $\mu(\beta) = \alpha, (\tilde{X})^+$
is an \( \tilde{S} \)-complex. An easy calculation shows \( ((\tilde{X}^+)_\gamma \subset (X^+)_\gamma \) for any \( \gamma \in \Pi \).

Observing \( ((\tilde{X}^+)_\gamma \subset K \leq G \), we have \( ((\tilde{X}^+)_\gamma \subset (X^+)_\gamma \) for any \( \gamma \in \Pi \). Since \( X^+ = (\alpha) \) by 2.5, we have \( (\tilde{X}^+)_\gamma = (\alpha) \) as a \( \Pi \)-complex. Since \( \Omega(G, \mathcal{F}) \) is generated by \((\alpha)^\prime\)s, \( \nu \) is surjective.

**Proposition 3.6.** Let \( \mathcal{F} \) be a \( G \)-family in \( \Pi \), and \( \tilde{\mathcal{F}} = T(\mathcal{F}) \). Then we have \( \nu(\Delta(G, \mathcal{F})) \subset \Delta(G, \tilde{\mathcal{F}}) \).

Proof. Let \( Z \) be a contractible \( \mathcal{F} \)-complex. Then \( (Z, *) \) is a \( \pi \)-co-fibering pair and \( Z \setminus \{*\} \) is a numerable \( \pi \)-space. \( Z \) is \( \pi \)-contractible, and \( Z/\pi \) is contractible. By Proposition 1.6 of [6] we have \( \nu(\Delta(G, \mathcal{F})) \subset \Delta(G, \tilde{\mathcal{F}}) \).

4. **Proofs of the main results**

In this section we let \( Y \) be a finite connected \( G \)-complex with finite \( \pi_1(Y) \), \( p: \tilde{Y} \rightarrow Y \) the universal covering, and put \( G = \pi_1(EG \times_0 Y) \) and \( \pi = \pi_1(Y) \).

As was described in section 1, \( \tilde{Y} \) has the standard action of \( G \). We use the notations in section 3 for this situation.

For a \( G \)-family \( \mathcal{F} \) in \( \Pi = \Pi(Y) \), we define a subgroup \( \Delta_\pi(G, Y, \mathcal{F}) \) of \( \Omega(G, \mathcal{F}) \) by

\[
\Delta_\pi(G, Y, \mathcal{F}) = \{ \gamma \in \Omega(G, \mathcal{F}) \mid f: X \rightarrow Y \text{ is a pseudo-equivalence such that } X^+ \text{ is an } \mathcal{F}-\text{complex} \}.
\]

By Proposition 3.6 \( \Delta_\pi(G, Y, \mathcal{F}) \) is a subgroup of \( \Delta(G, \mathcal{F}) \).

**Proposition 4.1.** Let \( \mathcal{F} \) be a connected \( G \)-family in \( \Pi \) containing \( \mathcal{F}(Y) \). Assume \( T(\mathcal{F}) \) is simply generated, then

\[\Delta_\pi(G, Y, \mathcal{F}) = \{ \gamma \in \Omega(G, \mathcal{F}) \mid f: X \rightarrow Y \text{ is a pseudo-equivalence such that } X^+ \text{ is an } \mathcal{F}-\text{complex} \}.\]

Proof. By Lemma 3.3 we have \( \mathcal{T}(\tilde{Y}) = T(\mathcal{T}(Y)) \subset T(\mathcal{F}) \). By Lemma 3.2 (ii) and Proposition 2.8 we have

\[\Delta(\tilde{G}, T(\mathcal{F})) = \{ \gamma \in \Omega(\tilde{G}, \mathcal{T}) \mid f: \tilde{X} \rightarrow \tilde{Y} \text{ is a quasi-equivalence such that } (\tilde{X}^+) \text{ is a } T(\mathcal{F})-\text{complex} \}.
\]

Since \( \tilde{Y} \) is a numerable \( \pi \)-space, \( f \) is a \( \pi \)-homotopy equivalence. Thus the induced map \( f: X = \tilde{X}/\pi \rightarrow Y \) is a homotopy equivalence. On the other hand \( \nu([M_T]) = [M_T] \). Through the map \( \nu \) we have the consequence of the above proposition.

For a moment we assume Theorem 1 and prove the corollaries.

Proof of Corollary 2. We may assume \( F \) is not empty. In this case \( \tilde{G} \) is a semi-direct product of \( G \) by \( \pi \) as is well known. Let \( \alpha_1, \ldots, \alpha_\delta \) be the
elements of \( \Pi \) such that \(|\alpha_i|=F_i\) and \(\rho(\alpha_i)=G\), \(i=1, \ldots, k\). Oliver-Petrie defined a homomorphism \(\psi: \Omega(G, \Pi) \to \mathbb{Z}^k\) by
\[
\psi([Z]) = (\chi(Z_{\alpha_1})-1, \ldots, \chi(Z_{\alpha_k})-1)
\]
for a \( \Pi \)-complex \( Z \). The image of \( \Delta(G, \Pi) \) by \(\psi\) is denoted by \(N_\gamma\). We define \(N_\gamma\) as the image of \(\Delta_\alpha(G, Y, \Pi)\) by \(\psi\). Thus \(N_\gamma\) is a subgroup of \(n_\gamma\).

Oliver-Petrie defined a homomorphism \(\psi: \Omega(G, \Pi) \to Z^k\) by
\[
\psi \left( \left[ \alpha \right] \right) = \left( \chi(F_1) - \chi(F'), \ldots, \chi(F_1) - \chi(F') \right)
\]
for an \( \alpha \)-complex \( Z \). The image of \( J(G, \alpha) \) by \(\psi\) is denoted by \(\pi\). We define \(\pi\) as the image of \(\Delta_\gamma\) \( F' \), \( \Pi \) by \(\psi\).

Thus \(\pi\) is a subgroup of \(\pi\). Put \(\Xi_\gamma = \Pi\) and \(\mathcal{E}' = \{ \alpha \in \Pi | \rho(\alpha) \neq G \}\). Then for \(\alpha \in \mathcal{E}'\), \(\rho(\alpha)\) is of prime power order. For \(\alpha \in T(T(\mathcal{E}')) \rho(\alpha)\) is isomorphic to \(\rho(\mu(\alpha))\), so \(\rho(\alpha)\) is of prime power order. By Corollary 4.14 of [6], \(T(T(\mathcal{E}'))\) is simply generated. Put \(f' = \text{incl} \circ f: F' \to Y\). Since \(\ker \psi = \Omega(G, \mathcal{E}')\), we have \([M_\gamma] \in \Delta_\alpha(G, Y, \mathcal{E}') + \Omega(G, \mathcal{E}')\) if and only if \(\psi([M_\gamma]) \in N_\gamma\). On the other hand, \(\psi([M_\gamma]) = (\chi(F_1) - \chi(F'), \ldots, \chi(F_1) - \chi(F'))\). Thus we have the conclusion of Corollary 2.

**Proof of Corollary 3.** Since \( F \) is connected, \(n_\gamma = n_0 \mathbb{Z}\). By the assumption \(G \in \mathcal{Q}^1\), \(n_\gamma = \{0\}\) (see [5; p. 171]). We obtain \(N_\gamma = \{0\}\). Corollary 2 yields Corollary 3.

**Proof of Theorem 1.** Let \(q = f \ast p: X \to Y\) be the induced covering and \(f': X' \to X\) the induced map by \(f\). Since \(\mathcal{E} \supseteq \mathcal{E}_f \supseteq \mathcal{E}(Y)\), \(T(\mathcal{E})\) is connected.

Firstly we assume \( f' \) is extendible to \( f': X' \to Y\) as was mentioned in Theorem 1, (we may assume \( f' \) is skeletal without loss of generality). Let \(f': X' \to Y\) be the induced map by \(f'\). Since \( f' \) is a homotopy equivalence, \( f' \) is a \( \pi \)-homotopy equivalence. If we show \( X' \setminus X \) is a \( T(\mathcal{E}') \)-complex, we have \([M_\gamma] \in \Delta_\alpha(G, Y, \mathcal{E}') + \Omega(G, \mathcal{E}')\) by Theorem 3.2 of [6]. Through the map \(\nu\) we have \([M_\gamma] \in \Delta_\alpha(G, Y, \mathcal{E}') + \Omega(G, \mathcal{E}')\). So we prove \( X' \setminus X \) is a \( T(\mathcal{E}') \)-complex. For a cell \( c \) in \( X' \setminus X \), let \( \alpha \in \Pi \) be the type of \( c \). The isotropy group on \( c \) is \( \rho(\alpha) \) and that on \( q'(c) \) is \( \sigma(\rho(\alpha)) \), where \( q' = f' \ast p: X' \to X'\). Since \( f'(c) \subseteq \alpha, F', f'(c) \subseteq |\mu(\alpha)| \). The type of \( q'(c) \) is \( \mu(\alpha) \). By the assumption \( X' \setminus X \) is a \( \mathcal{E}' \)-complex, and so \( \mu(\alpha) \in \mathcal{E}'\). Thus \(\alpha \in T(\mathcal{E}')\). This means that \( X' \setminus X \) is a \( T(\mathcal{E}') \)-complex.

Secondly we assume \([M_\gamma] \in \Delta_\alpha(G, Y, \mathcal{E}') + \Omega(G, \mathcal{E}')\). Since \(\nu: \Omega(G, T(\mathcal{E}')) \to \Omega(G, \mathcal{E}')\) is injective and \(\nu(\Omega(G, T(\mathcal{E}'))) = \Omega(G, \mathcal{E}')\) by Proposition 3.5, we have \([M_\gamma] \in \Delta_\alpha(G, T(\mathcal{E}')) + \Omega(G, T(\mathcal{E}'))\). By Theorem 3.2 of [6] there exist a finite \( G \)-complex \( X' \setminus X \) and a (skeletal) pseudo-equivalence \( f': X' \to Y\) extending \( f \) such that \( X' \setminus X \) is a \( T(\mathcal{E}') \)-complex. Since \( Y \) is a numerable \( \pi \)-space, \( f' \) is a \( \pi \)-homotopy equivalence. Put \( X' = X' / \pi \). Then \( X' \supset X \) and the induced map \( f': X' \to Y \) by \( f' \) is a homotopy equivalence. Moreover \( X' \setminus X \) is an \( \mathcal{E}' \)-complex by the similar argument used in the first part. This completes the proof.
References


Department of Mathematics
Osaka University
Toyonaka, Osaka 560, Japan