Osaka University Knowledge Archive

Title	Extendibility of G-maps to pseudo-equivalences to finite G-CW-complexes whose fundamental groups are finite
Author(s)	Morimoto, Masaharu; Iizuka, Kunihiko
Citation	Osaka Journal of Mathematics. 1984, 21(1), p. 59-69
Version Type	VoR
URL	https://doi.org/10.18910/12172
rights	
Note	

Osaka University Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

Osaka University

Morimoto, M. and Iizuka, K.
Osaka J. Math.
21 (1984), 59–69

EXTENDIBILITY OF G-MAPS TO PSEUDO-EQUIVALENCES TO FINITE G-CW-COMPLEXES WHOSE FUNDAMENTAL GROUPS ARE FINITE

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

MASAHARU MORIMOTO AND KUNIHIKO IIZUKA

(Received August 4, 1982)

0. Introduction

In this paper we let G be a finite group. A. Assadi [2] and R. Oliver-T. Petrie [6] treated the following question. What is a necessary and sufficient condition, for given finite G-CW-complexes X and Y and a G-map $f: X \to Y$, to extend f to a quasi-equivalence $f': X' \to Y$ (with some reservations)? Here a G-map is called a quasi-equivalence if it induces isomorphisms of fundamental groups and of integral homology groups. We apply the Oliver-Petrie theory to covering spaces to give a necessary and sufficient condition so that we may extend above f to a pseudo-equivalence $f'': X'' \to Y$ (with some reservations), when $\pi_1(Y)$ is finite.

We take Oliver-Petrie [6] as our general reference and use their terms and notations.

Let Y be a finite connected G-complex. Then $\tilde{G}=\pi_1(EG\times_G Y)$ acts on the universal covering space $\tilde{Y}$ of Y as is shown in section 1 (compare the action with that of D. Anderson [1]). Assume $\pi_1(Y)$ is finite. Then $\tilde{G}$ is finite, so we have a $\tilde{G}$-poset $\tilde{\Pi}=\Pi(\tilde{Y})$ and a G-poset $\Pi=\Pi(Y)$. In section 3 we give a one to one correspondence T from the set of G-families in Π to the set of $\tilde{G}$-families in $\tilde{\Pi}$, and an isomorphism ν from $\Omega(\tilde{G}, \tilde{\Pi})$ to $\Omega(G, \Pi)$. A subgroup $\Delta_h(G, Y, \mathcal{F})$ of $\Delta(G, \mathcal{F})$ is defined by $\Delta_h(G, Y, \mathcal{F})=\nu(\Delta(\tilde{G}, T(\mathcal{F})))$. Under certain conditions $\Delta_h(G, Y, \mathcal{F})$ agrees with the set

$$\{[M_f]\in\Omega(G, \Pi)\,|\, f: X\to Y \text{ is a pseudo-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}$$

(see Proposition 4.1), where M_f is the mapping cone of f.

Our main results are:

Theorem 1. *Let X be a finite G-complex, Y a finite connected G-complex with finite $\pi_1(Y)$, $f: X \to Y$ a skeletal G-map, and $\mathcal{F}\subset\Pi$ any connected G-*

family containing $\mathcal{F}_f$. *Let* $\mathcal{F}'\subset\mathcal{F}$ *be any subfamily containing* $\hat{\mathcal{F}}$. *Assume* $T(\hat{\mathcal{F}})$ *is simply generated. Then there exist a finite G-complex* $X'\supset X$ *and a pseudo-equivalence* $f'\colon X'\to Y$ *extending* f *with* X'/X *an* $\mathcal{F}'$*-complex, if and only if*

$$[M_f]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}') \text{ in } \Omega(G, \Pi)\,.$$

Corollary 2. *Assume G is not of prime power order. Let Y be a finite connected G-complex with finite* $\pi_1(Y)$, *and* $F_1, \cdots, F_k$ *the connected components of* $F=Y^G$. *Then there is a subgroup* $N_Y\subset \boldsymbol{Z}^k$ *such that given any finite G-complex* F' *and a map* $\hat{f}\colon F'\to F$, *there exist a finite G-complex X with* $X^G=F'$ *and a pseudo-equivalence* $f\colon X\to Y$ *with* $f^G=\hat{f}$ *if and only if*

$$(\chi(F_1)-\chi(F_1'), \cdots, \chi(F_k)-\chi(F_k'))\in N_Y\,,$$
$$(F_i'=\hat{f}^{-1}(F_i))\,.$$

Above N_Y is the image of $\Delta_h(G, Y, \Pi)$ by the homomorphism $\psi\colon \Omega(G, \Pi)\to \boldsymbol{Z}^k$ defined in section 3 of [6]. Thus N_Y is included in n_Y.

Corollary 3. *Let G and Y be as above. Moreover we assume F is connected and G belongs to* $\mathcal{G}^1$, *i.e.* G/P *is cyclic for some normal subgroup P of G of prime power order. Given any finite G-complex* F' *and any map* $\hat{f}\colon F'\to F$, *there exist a finite G-complex X with* $X^G=F'$ *and a pseudo-equivalence* $f\colon X\to Y$ *extending* $\hat{f}$, *if and only if* $\chi(F)=\chi(F')$.

The proofs of Theorem 1 and Corollaries 2 and 3 are given in section 4.

In a subsequent paper we will calculate N_Y in several cases.

In this paper we often omit the adjective skeletal from a skeletal G-map, however, a G-map should be understood to be a skeletal G-map when its mapping cone appeares.

1. A standard action of $\pi_1(EG\times_G Y)$ on the universal covering space of Y

Let Y be a connected G-complex, $p\colon \tilde{Y}\to Y$ the universal covering, EG the universal principal G-bundle. Arbitrarily choose and fix base points a_0 of Y, b_0 of $\tilde{Y}$ with $p(b_0)=a_0$, and c_0 of EG. Let $q\colon EG\times Y\to EG\times_G Y$ be the canonical projection. We use $u_0=(c_0, a_0)$ and $v_0=q(u_0)$ as the base points of $EG\times Y$ and $EG\times_G Y$ respectively. We put $\pi=\pi_1(Y)$ and $\tilde{G}=\pi_1(EG\times_G Y)$ in this section.

We define a map $k\colon Y\to EG\times_G Y$ by $k(y)=q(c_0, y)$ for $y\in Y$. The covering $q\colon EG\times Y\to EG\times_G Y$ induces the exact sequence

$$\{1\}\to\pi\xrightarrow{j}\tilde{G}\xrightarrow{\sigma}G\to\{1\}$$

where j is the induced map by k, and σ is the map obtained by identifying $\pi_0(G)$

with G. We regard π as a subgroup of $\tilde{G}$ through j.

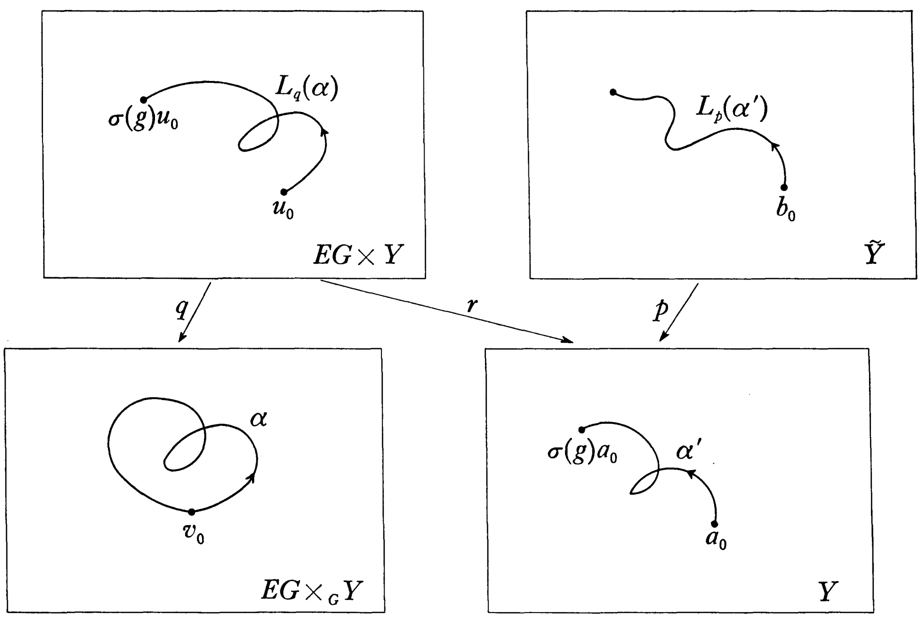
In the following we illustrate a standard action of $\tilde{G}$ on $\tilde{Y}$ such that

(1) p is σ-equivariant, i.e. for $g\in\tilde{G}$ and $b\in\tilde{Y}$ $p(gb)=\sigma(g)p(b)$,

(2) the induced CW-complex structure on $\tilde{Y}$ by p and the $\tilde{G}$-action make $\tilde{Y}$ a $\tilde{G}$-complex,

(3) the restriction of the $\tilde{G}$-action to π agrees with the action given by M. Cohen [3; p. 12].

We denote by r the projection from $EG\times Y$ to the second factor Y. Immediately $r(u_0)=a_0$ follows. We are going to give gb for $g\in\tilde{G}$ and $b\in\tilde{Y}$. An element g of $\tilde{G}$ is represented by a path α: $[0, 1]\to EG\times_G Y$ with $\alpha(0)=\alpha(1)=v_0$. There is a unique lift $L_q(\alpha)$: $[0, 1]\to EG\times Y$ of α (i.e. $q\circ L_q(\alpha)=\alpha$) with $L_q(\alpha)(0)=u_0$. The homomorphism σ: $\tilde{G}\to G$ is given by the relation $\sigma(g)u_0=L_g(\alpha)(1)$. The path α gives two paths $\alpha'=r\circ L_q(\alpha)$: $[0, 1]\to Y$ and its lift $L_p(\alpha')$: $[0, 1]\to\tilde{Y}$ (i.e. $p\circ L_p(\alpha')=\alpha$) with $L_p(\alpha')(0)=b_0$. We have $\alpha'(0)=a_0$ and $\alpha'(1)=\sigma(g)a_0$.



For given $b\in\tilde{Y}$, choose arbitrarily a path β: $[0, 1]\to\tilde{Y}$ with $\beta(0)=b_0$ and $\beta(1)=b$. β gives two paths $p\circ\beta$: $[0, 1]\to Y$ and β': $[0, 1]\to Y$ defined by $\beta'(t)=\sigma(g)p(\beta(t))$ for $t\in[0, 1]$. We have $\beta'(0)=\sigma(g)a_0=\alpha'(1)$ and

$$\beta'(1) = \sigma(g)p(b)\,. \tag{1.1}$$

There is a unique lift $L_p(\beta')$: $[0, 1]\to\tilde{Y}$ of β' with $L_p(\beta')(0)=L_p(\alpha')(1)$. We define gb to be the point $L_p(\beta')(1)$.

By (1.1) we have $p(gb)=\sigma(g)p(b)$. That is, p is σ-equivariant. The properties (2) and (3) follow immediately.

2. Remarks on $\mathcal{F}$-complexes

For a finite group G, a *G-poset* is axiomatically defined as follows. Let $\mathcal{S}(G)$ be the set of subgroups of G. By conjugation G acts on $\mathcal{S}(G)$: $(g, H)\mapsto gHg^{-1}$ for $g\in G$, $H\in\mathcal{S}(G)$.

2.1. A parcially ordered G-set Π equipped with a G-map ρ: $\Pi\to\mathcal{S}(G)$ is called a *G-poset* if the following four conditions are satisfied: for $\alpha\in\Pi$, $\beta\in\Pi$ (i) $\rho(\alpha)\subset G_\alpha$, (ii) if $\alpha\leqq\beta$ then $g\alpha\leqq g\beta$ for $g\in G$, (iii) if $\alpha\lneqq\beta$ then $\rho(\alpha)\supsetneqq\rho(\beta)$, and (iv) for a subgroup H of $\rho(\alpha)$ there exists a unique element γ of Π such that $\gamma\geqq\alpha$ and $\rho(\gamma)=H$.

Typical examples of G-posets are $\Pi(X)$ for G-spaces X (see [4] and [6]). A G-subset of a G-poset Π is called a *G-family* (in Π). A *Π-complex* Z for a G-poset Π is a finite G-complex with base point $*$ and subcomplexes $Z_\alpha\subset Z$, $(*\in Z_\alpha)$, for all $\alpha\in\Pi$ such that $Z_{g\alpha}=gZ_\alpha$ for $g\in G$, $Z_\alpha\subset Z_\beta$ for $\alpha\leqq\beta$, and
$$Z^H = \bigvee_{\rho(\alpha)=H} Z_\alpha \quad \text{for } H\leqq G\,.$$
For a G-family $\mathcal{F}$ in Π a Π-complex Z is called an *$\mathcal{F}$-complex* if
$$Z_\alpha = \{*\}\cup\bigcup\{Z_\beta\,|\,\beta\in\mathcal{F},\ \beta\leqq\alpha\}$$
for any $\alpha\in\Pi$.

2.2. Let Π be a G-poset, $\mathcal{F}$ a G-family in Π, and Z an $\mathcal{F}$-complex. For $\alpha\in\Pi$, $\beta\in\Pi$ and $x\in(Z_\alpha\cap Z_\beta)\setminus\{*\}$, there is a unique element γ of $\mathcal{F}$ such that $\gamma\leqq\alpha$, $\gamma\leqq\beta$, $\rho(\gamma)=G_x$ and $x\in Z_\gamma$.

2.3. Let Π be a G-poset, Z a Π-complex. For each (non-equivariant) cell c in $Z\setminus\{*\}$, there exists a unique element $\alpha(c)\in\Pi$ such that $\rho(\alpha(c))=G_x$, $x\in c$, and $c\subset Z_{\alpha(c)}$. If $Z_\beta\supset c$ for $\beta\in\Pi$, then $\alpha(c)\leqq\beta$. So we call c *of type* $\alpha(c)$.

2.4. Let $\mathcal{F}$ be a G-family in Π. A Π-complex Z is an $\mathcal{F}$-complex if and only if $\mathcal{F}$ contains $\alpha(c)$ for any cell c in $Z\setminus\{*\}$.

Let Π be a G-poset. For each $\alpha\in\Pi$, the Π-complexes (α) is the G-space $\{*\}\amalg G/\rho(\alpha)$ with
$$(\alpha)_\beta = \{*\}\amalg\bigcup\{g\rho(\alpha)\,|\,g\in G,\ g\alpha\leqq\beta\}$$
for $\beta\in\Pi$.

In the rest of this section we let Y be a finite connected G-complex and

$\Pi=\Pi(Y)$.

Let X be another finite G-complex, and f a G-map from X to Y. For $\alpha\in\Pi$ $X_\alpha=X^{\rho(\alpha)}\cap f^{-1}(|\alpha|)$. $X^+=X\coprod\{*\}$ (disjoint union) has a Π-complex structure given by $(X^+)_\alpha=X_\alpha\coprod\{*\}$. We call this Π-complex structure the Π-complex structure *induced by* f.

2.5. Let α be an element of Π. For an arbitrary G-map f from $X=G/\rho(\alpha)$ to Y with $f(\rho(\alpha))\in|\alpha|$, the induced Π-complex X^+ by f agrees with (α) as Π-complex.

2.6. Let F be a finite CW-complex, and α an element of Π. For a G-map f from $X=(G/\rho(\alpha))\times F$ to Y with $f(\rho(\alpha)\times F)\subset|\alpha|$, $[X^+]=\chi(F)[\alpha]$ in $\Omega(G,\Pi(Y))$.

Proposition 2.7. *Let $\mathcal{F}$ be a G-family in $\Pi=\Pi(Y)$ containing $\mathcal{F}(Y)$. Then*

$$\Omega(G,\mathcal{F})=\{[M_f]\in\Omega(G,\Pi)\mid f\colon X\to Y \text{ is a } G\text{-map such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}.$$

Proof. Choose integers $z(\alpha)$, $\alpha\in\mathcal{F}$, such that $[Y^+]=\sum z(\alpha)[\alpha]$, where α runs over $\mathcal{F}$. For any $\xi\in\Omega(G,\mathcal{F})$, there are integers $z'(\alpha)$, $\alpha\in\mathcal{F}$, such that

$$\xi=\sum_{\alpha\in\mathcal{F}}z(\alpha)[\alpha]-\sum_{\alpha\in\mathcal{F}}z'(\alpha)[\alpha].$$

Take finite CW-complexes $F(\alpha)$ with $\chi(F(\alpha))=z'(\alpha)$, and put $X=\coprod\{(G/\rho(\alpha))\times F(\alpha)\mid\alpha\in\mathcal{F}\}$. There is a G-map $f\colon X\to Y$ with $f(\rho(\alpha)\times F(\alpha))\subset|\alpha|$. We have $[M_f]=[Y^+]-[X^+]=\xi$ by 2.6.

According to Proposition 1.6 of [6],

$$\Delta(G,\mathcal{F})=\{[Z]\in\Omega(G,\Pi)\mid Z \text{ is a contractible } \mathcal{F}\text{-complex}\}.$$

Moreover we have the following.

Proposition 2.8. *Let $\mathcal{F}$ be a connected G-family in $\Pi=\Pi(Y)$ such that $\mathcal{F}$ contains $\mathcal{F}(Y)$ and $\hat{\mathcal{F}}$ is simply generated. Then*

$$\Delta(G,\mathcal{F})=\{[M_f]\in\Omega(G,\Pi)\mid f\colon X\to Y \text{ is a quasi-equivalence such that } X^+ \text{ is an } \mathcal{F}\text{-complex}\}.$$

Proof. We prove that for given $\xi\in\Delta(G,\mathcal{F})$ there exist a finite G-complex X and a (skeletal) G-map $f\colon X\to Y$ with $[M_f]=\xi$. For $\xi\in\Delta(G,\mathcal{F})$ there are a finite G-complex X_0 and a G-map $f_0\colon X_0\to Y$ with $[M_{f_0}]=\xi$ by Proposition 2.7. By the same argument as Oliver-Petrie used at Steps 2 and 3 of the proof of [6; Proposition 2.9], we get a finite G-complex $X_1\supset X_0$ and a G-map $f_1\colon X_1\to Y$ extending f_0 such that X_1/X_0 is an $\hat{\mathcal{F}}$-complex, M_{f_1} is an $\mathcal{F}$-resolution and $\{M_{f_1}\}=\xi$. Adding free cells to X appropriately if necessary, we may assume $\dim X_1 \geqq 3$. We use the same argument as was used in the proof (1) of [6; Theorem 3.2], and obtain a finite G-complex $X \supset X_1$ and a G-map f: $X \to Y$ extending f_1 such that X/X_1 is a $\hat{\mathcal{F}}$-complex and f is a quasi-equivalence. We have to check $[M_f]=\xi$. Both $[M_f]$ and $[M_{f_0}]$ belong to $\Delta(G, \mathcal{F})$, and $[X/X_0]=[M_{f_0}]-[M_f]$. We have $\chi((X/X_0)_\alpha)=1$ for $\alpha \in \hat{\mathcal{F}}$ by Proposition 2.6 of [6]. Since X/X_0 is an $\hat{\mathcal{F}}$-complex, we have $[X/X_0]=0$. That is $[M_f] =[M_{f_0}]=\xi$

3. Correspondences between the posets of a finite covering space and a base space

In this section we let G and $\tilde{G}$ be finite groups, σ: $\tilde{G} \to G$ a epimorphism, Y a finite connected G-complex, $\tilde{Y}$ a finite connected $\tilde{G}$-complex, and p: $\tilde{Y} \to Y$ a σ-equivariant covering. We put $\pi = \ker \sigma$. Moreover we assume that π acts freely and transitively on each fiber.

The $\tilde{G}$-action on $\tilde{Y}$ gives a $\tilde{G}$-poset $\tilde{\Pi}=\Pi(\tilde{Y})$ and a $\tilde{G}$-map $\tilde{\rho}$: $\tilde{\Pi} \to \mathcal{S}(\tilde{G})$. The set of G-families in Π is denoted by $\boldsymbol{F}$ and that of $\tilde{G}$-families in $\tilde{\Pi}$ is denoted by $\tilde{\boldsymbol{F}}$.

For arbitrarily given $\alpha \in \tilde{\Pi}$, there is a unique element $\beta \in \Pi$ such that $\rho(\beta)=\sigma(\tilde{\rho}(\alpha))$ and $|\beta| \supset p(|\alpha|)$. This correspondence defines a map μ: $\tilde{\Pi} \to \Pi$.

For $\alpha \in \Pi$, we denote the connected components of $p^{-1}(|\alpha|)$ by $A_1, \cdots, A_k$. We have $p(A_i)=|\alpha|$ for any $i=1, \cdots, k$. Each A_i is fixed by a subgroup H_i of $\tilde{G}$ with $\sigma(H_i)=\rho(\alpha)$, since π preserves each fiber. As π acts freely on each fiber, σ: $H_i \to \rho(\alpha)$ is bijective. Each A_i is contained in a connected component B_i of the H_i-fixed point set of $\tilde{Y}$. The projection p is σ-equivariant, so $p(B_i)=|\alpha|$. We have $A_i=B_i$. There is a unique element $\beta_i \in \tilde{\Pi}$ such that $\tilde{\rho}(\beta_i)=H_i$ and $|\beta_i|=A_i$. We define a map τ: $\Pi \to \mathcal{S}(\Pi)$ by $\tau(\alpha)=\{\beta_1, \cdots, \beta_k\}$, where $\mathcal{S}(\tilde{\Pi})$ denotes the set of subsets of $\tilde{\Pi}$.

Immediately we have $\mu(\tau(\alpha))=\{\alpha\}$ for $\alpha \in \Pi$. The above argument implies $|\mu(\alpha)|=p(|\alpha|)$ for $\alpha \in \tilde{\Pi}$. The following two diagrams are commutative:

$$\begin{array}{ccc} \tilde{\Pi} & \xrightarrow{\mu} & \Pi \\ \downarrow{\scriptstyle\tilde{\rho}} & & \downarrow{\scriptstyle\rho} \\ \mathcal{S}(\tilde{G}) & \xrightarrow{\sigma} & \mathcal{S}(G) \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{\Pi} & \xrightarrow{\mu} & \Pi \\ \downarrow{\scriptstyle|\,|} & & \downarrow{\scriptstyle|\,|} \\ \mathcal{S}(\tilde{Y}) & \xrightarrow{p} & \mathcal{S}(Y), \end{array}$$

where $\mathcal{S}(\tilde{Y})$ and $\mathcal{S}(Y)$ are the sets of subspaces of $\tilde{Y}$ and Y respectively. For $\alpha \in \tilde{\Pi}$, α is an element of $\tau(\mu(\alpha))$.

Proposition and definition 3.1. *The following two equations define maps*

M: $\tilde{\boldsymbol{F}}\to\boldsymbol{F}$ and T: $\boldsymbol{F}\to\tilde{\boldsymbol{F}}$,

$$M(\tilde{\mathcal{F}}) = \{\mu(\alpha)\,|\,\alpha\in\tilde{\mathcal{F}}\} \quad \textit{for } \tilde{\mathcal{F}}\in\tilde{\boldsymbol{F}},$$
$$T(\mathcal{F}) = \cup\,\{\tau(\alpha)\,|\,\alpha\in\mathcal{F}\} \quad \textit{for } \mathcal{F}\in\boldsymbol{F}.$$

We have $M\circ T=id_{\boldsymbol{F}}$ and $T\circ M=id_{\tilde{\boldsymbol{F}}}$.

We omit the proof.

Lemma 3.2. (i) *If $\tilde{\mathcal{F}}\in\tilde{\boldsymbol{F}}$ is connected, then $M(\tilde{\mathcal{F}})$ is connected, and $\widehat{M(\tilde{\mathcal{F}})}=M(\hat{\tilde{\mathcal{F}}})$.*

(ii) *If $\mathcal{F}\in\boldsymbol{F}$ is connected and contains $\mathcal{F}(Y)$, then $T(\mathcal{F})$ is connected, and $\widehat{T(\mathcal{F})}=T(\hat{\mathcal{F}})$.*

Proof. We prove (ii), and let (i) remain to be proved by the reader. We denote the maximal element of $\tilde{\Pi}$ by $\tilde{m}$, so we have $\tau(m)=\{\tilde{m}\}$. Since $m\in\mathcal{F}$, $\tilde{m}\in T(\mathcal{F})$. Assume α is an element of $\tilde{\Pi}$ such that $\tilde{\rho}(\alpha)$ is of prime power order and $\{\beta\in T(\mathcal{F})\,|\,\beta\leqq\alpha\}$ is not empty. Since $M\circ T=\text{id}$ and μ preserves the order, $\{\beta\in\mathcal{F}\,|\,\beta\leqq\mu(\alpha)\}$ is not empty. There is the unique maximal element γ of $\{\beta\in\mathcal{F}\,|\,\beta\leqq\mu(\alpha)\}$, $(\gamma=\widehat{\mu(\alpha)})$. Since Y^+ is a $\mathcal{F}$-complex, we have $|\gamma|=|\mu(\alpha)|$ by Proposition 1.2 of [6]. There uniquely exists $\delta\in\tau(\gamma)$ with $\delta\leqq\alpha$. For any $\beta\in T(\mathcal{F})$ with $\beta\leqq\alpha$, we have $|\beta|\subset|\alpha|=|\delta|$ and $\mu(\beta)\leqq\gamma$. Thus $\sigma(\tilde{\rho}(\beta))=\rho(\mu(\beta))\supset\rho(\gamma)=\sigma(\tilde{\rho}(\delta))$. Since $\pi=\ker\sigma$ acts freely on each fiber, we see $\tilde{\rho}(\beta)\supset\tilde{\rho}(\delta)$. Therefore $\beta\leqq\delta$, that is, δ is the unique maximal element of $\{\beta\in T(\mathcal{F})\,|\,\beta\leqq\alpha\}$. $T(\mathcal{F})$ is connected. This argument implies $T(\hat{\mathcal{F}})=\widehat{T(\mathcal{F})}$.

Let X be another finite G-complex, and f: $X\to Y$ a skeletal G-map. Then f induces the covering f^*p: $\tilde{X}=f^*\tilde{Y}\to X$,

$$\tilde{X}=\{(x, b)\in X\times\tilde{Y}\,|\,f(x)=p(b)\},$$

$(f^*p)(x, b)=x$ for $(x, b)\in\tilde{X}$. $\tilde{G}$ acts on $\tilde{X}$ by $g(x, b)=(\sigma(g)x, gb)$ for $g\in\tilde{G}$, $(x, b)\in\tilde{X}$. $\tilde{X}$ has the CW-complex structure induced by f^*p, and becomes a $\tilde{G}$-complex. A $\tilde{G}$-map $\tilde{f}$: $\tilde{X}\to\tilde{Y}$ is given by $\tilde{f}(x, b)=b$ for $(x, b)\in\tilde{X}$, and $\tilde{f}$ is skeletal.

Lemma 3.3. *In the above situation, $\mathcal{F}_{\tilde{f}}=T(\mathcal{F}_f)$ and $\mathcal{F}_f=M(\mathcal{F}_{\tilde{f}})$.*

Proof. Firstly we show $M(\mathcal{F}_{\tilde{f}})\subset\mathcal{F}_f$. For $\alpha\in\mathcal{F}_{\tilde{f}}$, (i) $\tilde{\rho}(\alpha)\in\text{Iso}(|\alpha|)$ or (ii) $\tilde{\rho}(\alpha)\in\text{Iso}(\tilde{X}_\alpha)$. Assume the case (i). There exists a point $b\in|\alpha|$ with $\tilde{G}_b=\tilde{\rho}(\alpha)$. We have $G_{p(b)}=\sigma(\tilde{G}_b)=\sigma(\tilde{\rho}(\alpha))=\rho(\mu(\alpha))$, and $\rho(\mu(\alpha))\in\text{Iso}(|\mu(\alpha)|)$. Thus $\mu(\alpha)\in\mathcal{F}_f$. Assume the case (ii). There exists a point $(x, b)\in\tilde{X}_\alpha$ with

$\tilde{G}_{(x,b)}=\tilde{\rho}(\alpha)$. By definition $\tilde{X}_{\alpha}=\tilde{X}^{\tilde{\rho}(\alpha)}\cap\tilde{f}^{-1}(|\alpha|)=\{(x',b')\in X\times\tilde{Y}|f(x')=p(b'), x'\in X^{\sigma(\tilde{\rho}(\alpha))}, b'\in|\alpha|\}$. We have $\tilde{G}_{(x,b)}=\sigma^{-1}(G_x)\cap\tilde{G}_b$, and $\rho(\mu(\alpha))=\sigma(\tilde{\rho}(\alpha))=\sigma(\tilde{G}_{(x,b)})=G_x$. Since $f^{-1}(|\mu(\alpha)|)=f^{-1}(p(|\alpha|))$, $x\in X^{\rho(\mu(\alpha))}\cap f^{-1}(p(|\alpha|))=X_{\mu(\alpha)}$. Thus $\mu(\alpha)\in\mathcal{F}_f$. We have $M(\mathcal{F}_{\tilde{f}})\subset\mathcal{F}_f$.

Secondly we show $T(\mathcal{F}_f)\subset\mathcal{F}_{\tilde{f}}$. For $\alpha\in\mathcal{F}_f$, (iii) $\rho(\alpha)\in\mathrm{Iso}(|\alpha|)$ or (iv) $\rho(\alpha)\in\mathrm{Iso}(X_{\alpha})$. Assume the case (iii). There exists a point $a\in|\alpha|$ with $\rho(\alpha)=G_a$. Fix $\beta\in\tau(\alpha)$ and $b\in|\beta|\cap p^{-1}(a)$. $\tilde{G}_b$ contains $\tilde{\rho}(\beta)$. Since $\sigma\colon\tilde{G}_b\to G_a$ is bijective, $\sigma(\tilde{\rho}(\beta))=\rho(\alpha)=G_a$ implies $\tilde{G}_b=\rho(\beta)$. Thus $\tilde{\rho}(\beta)\in\mathrm{Iso}(|\beta|)$, and $\beta\in\mathcal{F}_{\tilde{f}}$. We have $\tau(\alpha)\subset\mathcal{F}_{\tilde{f}}$. Assume the case (iv). There exists a point $x\in X_{\alpha}$ with $G_x=\rho(\alpha)$. Fix $\beta\in\tau(\alpha)$ and $b\in|\beta|$ with $f(x)=p(b)$. Then $(x,b)\in\tilde{X}_{\beta}$. The isomorphism $\sigma\colon\tilde{G}_b\to G_{p(b)}$ maps both $\tilde{\rho}(\beta)$ and $\tilde{G}_{(x,b)}=\sigma^{-1}(G_x)\cap\tilde{G}_b$ to G_x. We get $\tilde{\rho}(\beta)=\tilde{G}_{(x,b)}$, and $\beta\in\mathcal{F}_{\tilde{f}}$. Thus $\tau(\alpha)\subset\mathcal{F}_{\tilde{f}}$. We have $T(\mathcal{F}_f)\subset\mathcal{F}_{\tilde{f}}$.

By Proposition and definition 3.1, we have $\mathcal{F}_{\tilde{f}}=T(\mathcal{F}_f)$ and $\mathcal{F}_f=M(\mathcal{F}_{\tilde{f}})$.

Let $\tilde{\mathcal{F}}$ be a $\tilde{G}$-family in $\tilde{\Pi}$, $\tilde{Z}$ an $\tilde{\mathcal{F}}$-complex. The quotient space $Z=\tilde{Z}/\pi$ has a Π-complex structure given by

$$Z_{\alpha}=(\bigcup_{\beta\in\tau(\alpha)}\tilde{Z}_{\alpha})/\pi,\ \alpha\in\Pi\,.$$

Moreover Z becomes a $M(\tilde{\mathcal{F}})$-complex. For $\alpha\in\Pi$ we have

$$\chi(Z_{\alpha})-1=(\chi(\tilde{Z}_{\beta})-1)/|\pi_{\beta}|\,, \tag{3.4}$$

where β is an arbitrary element of $\tau(\alpha)$.

The correspondence $\tilde{Z}\to Z$ defines a homomorphism $\nu\colon\Omega(\tilde{G},\tilde{\mathcal{F}})\to\Omega(G, M(\tilde{\mathcal{F}}))$. By (3.4) ν is injective. If $\tilde{\mathcal{F}}'\subset\tilde{\mathcal{F}}$ then the following diagram is commutative:

$$\begin{array}{ccc}\Omega(\tilde{G},\tilde{\mathcal{F}}') & \longrightarrow & \Omega(\tilde{G},\tilde{\mathcal{F}})\\ \downarrow\nu & & \downarrow\nu\\ \Omega(G,M(\tilde{\mathcal{F}}')) & \to & \Omega(G,M(\tilde{\mathcal{F}}))\,,\end{array}$$

where the horizontal arrows are the canonical maps.

Proposition 3.5. *Let $\mathcal{F}$ be a G-family in Π, and put $\tilde{\mathcal{F}}=T(\mathcal{F})$. Then $\nu\colon\Omega(\tilde{G},\tilde{\mathcal{F}})\to\Omega(G,\mathcal{F})$ is an isomorphism.*

Proof. It is sufficient to show that ν is surjective. Arbitrarily fix $\alpha\in\mathcal{F}$. Put $X=G/H$, $H=\rho(\alpha)$. There is a G-map $f\colon X\to Y$ with $f(1\cdot H)\in Y_{\alpha}$. Let $\tilde{f}\colon\tilde{X}=f^{*}\tilde{Y}\to\tilde{Y}$ be the induced $\tilde{G}$-map. $(\tilde{X})^{+}$ has a $\tilde{\Pi}$-complex structure induced by f. Take a point $(1\cdot H,b)\in\tilde{X}$, so $f(1\cdot H)=p(b)$, and put $\beta=\min.\{\gamma\in\tilde{\Pi}|\tilde{X}_{\gamma}\ni(1\cdot H,b)\}$, (that is, $(1\cdot H,b)$ is a point of a cell of type β). Since $\tilde{G}$ acts transitively on $\tilde{X}$, $(\tilde{X})^{+}$ is a $\{g\beta|g\in\tilde{G}\}$-complex. Since $\mu(\beta)=\alpha$, $(\tilde{X})^{+}$

is an $\tilde{\mathcal{F}}$-complex. An easy calculation shows $((\tilde{X})^+/\pi)_\gamma \subset (X^+)_\gamma$ for any $\gamma \in \Pi$. Observing $((\tilde{X})^+/\pi)^K$ for $K \leqq G$, we have $((\tilde{X})^+/\pi)_\gamma = (X^+)_\gamma$ for any $\gamma \in \Pi$. Since $X^+ = (\alpha)$ by 2.5, we have $(\tilde{X})^+/\pi = (\alpha)$ as a Π-complex. Since $\Omega(G, \mathcal{F})$ is generated by (α)'s, ν is surjective.

Proposition 3.6. *Let $\mathcal{F}$ be a G-family in Π, and $\tilde{\mathcal{F}} = T(\mathcal{F})$. Then we have $\nu(\Delta(\tilde{G}, \tilde{\mathcal{F}})) \subset \Delta(G, \mathcal{F})$.*

Proof. Let $\tilde{Z}$ be a contractible $\tilde{\mathcal{F}}$-complex. Then $(\tilde{Z}, *)$ is a π-cofibering pair and $\tilde{Z} \setminus \{*\}$ is a numerable π-space. $\tilde{Z}$ is π-contractible, and $\tilde{Z}/\pi$ is contractible. By Proposition 1.6 of [6] we have $\nu(\Delta(\tilde{G}, \tilde{\mathcal{F}})) \subset \Delta(G, \mathcal{F})$.

4. Proofs of the main results

In this section we let Y be a finite connected G-complex with finite $\pi_1(Y)$, $p: \tilde{Y} \to Y$ the universal covering, and put $\tilde{G} = \pi_1(EG \times_G Y)$ and $\pi = \pi_1(Y)$. As was described in section 1, $\tilde{Y}$ has the standard action of $\tilde{G}$. We use the notations in section 3 for this situation.

For a G-family $\mathcal{F}$ in $\Pi = \Pi(Y)$, we define a subgroup $\Delta_h(G, Y, \mathcal{F})$ of $\Omega(G, \mathcal{F})$ by

$$\Delta_h(G, Y, \mathcal{F}) = \nu(\Delta(\tilde{G}, T(\mathcal{F}))) .$$

By Proposition 3.6 $\Delta_h(G, Y, \mathcal{F})$ is a subgroup of $\Delta(G, \mathcal{F})$.

Proposition 4.1. *Let $\mathcal{F}$ be a connected G-family in Π containing $\mathcal{F}(Y)$. Assume $T(\hat{\mathcal{F}})$ is simply generated, then*

$$\Delta_h(G, Y, \mathcal{F}) = \{[M_f] \in \Omega(G, \Pi) \mid f: X \to Y \textit{ is a pseudo-equivalence such that } X^+ \textit{ is an } \mathcal{F}\textit{-complex}\}.$$

Proof. By Lemma 3.3 we have $\mathcal{F}(\tilde{Y}) = T(\mathcal{F}(Y)) \subset T(\mathcal{F})$. By Lemma 3.2 (ii) and Proposition 2.8 we have

$$\Delta(\tilde{G}, T(\mathcal{F})) = \{[M_{\tilde{f}}] \in \Omega(\tilde{G}, \tilde{\Pi}) \mid \tilde{f}: \tilde{X} \to \tilde{Y} \text{ is a quasi-equivalence such that } (\tilde{X})^+ \text{ is a } T(\mathcal{F})\text{-complex}\}.$$

Since $\tilde{Y}$ is a numerable π-space, $\tilde{f}$ is a π-homotopy equivalence. Thus the induced map $f: X = \tilde{X}/\pi \to Y$ is a homotopy equivalence. On the other hand $\nu([M_{\tilde{f}}]) = [M_f]$. Through the map ν we have the consequence of the above proposition.

For a moment we assume Theorem 1 and prove the corollaries.

Proof of Corollary 2. We may assume F is not empty. In this case $\tilde{G}$ is a semi-direct product of G by π as is well known. Let $\alpha_1, \cdots, \alpha_k$ be the elements of Π such that $|\alpha_i|=F_i$ and $\rho(\alpha_i)=G$, $i=1, \cdots, k$. Oliver-Petrie defined a homomorphism ψ: $\Omega(G, \Pi)\to \boldsymbol{Z}^k$ by

$$\psi([Z]) = (\chi(Z_{\alpha_1})-1, \cdots, \chi(Z_{\alpha_k})-1)$$

for a Π-complex Z. The image of $\Delta(G, \Pi)$ by ψ is denoted by n_Y. We define N_Y as the image of $\Delta_h(G, Y, \Pi)$ by ψ. Thus N_Y is a subgroup of n_Y. Put $\mathcal{F}=\Pi$ and $\mathcal{F}'=\{\alpha\in\Pi\,|\,\rho(\alpha)\neq G\}$. Then for $\alpha\in\hat{\mathcal{F}}$ $\rho(\alpha)$ is of prime power order. For $\alpha\in T(\hat{\mathcal{F}})$ $\tilde{\rho}(\alpha)$ is isomorphic to $\rho(\mu(\alpha))$, so $\tilde{\rho}(\alpha)$ is of prime power order. By Corollary 4.14 of [6] $T(\hat{\mathcal{F}})$ is simply generated. Put $f'=\mathrm{incl}\circ\hat{f}$: $F'\to Y$. Since $\ker\psi$ is $\Omega(G, \mathcal{F}')$, we have $[M_{f'}]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}')$ if and only if $\psi([M_{f'}])\in N_Y$. On the other hand $\psi([M_{f'}])=(\chi(F_1)-\chi(F_1'), \cdots, \chi(F_k)-\chi(F_k'))$. Thus we have the conclusion of Corollary 2.

Proof of Corollary 3. Since F is connected, $n_Y=n_G\boldsymbol{Z}$. By the assumption $G\in\mathcal{G}^1$, $n_Y=\{0\}$ (see [5; p. 171]). We obtain $N_Y=\{0\}$. Corollary 2 yields Corollary 3.

Proof of Theorem 1. Let $q=f*p$: $\tilde{X}=f*\tilde{Y}\to X$ be the induced covering and $\tilde{f}$: $\tilde{X}\to\tilde{Y}$ the induced map by f. Since $\mathcal{F}\supset\mathcal{F}_f\supset\mathcal{F}(Y)$, $T(\mathcal{F})$ is connected.

Firstly we assume f is extendible to f': $X'\to Y$ as was mentioned in Theorem 1, (we may assume f' is skeletal without loss of generality). Let $\tilde{f}'$: $\tilde{X}'=f'\tilde{Y}\to\tilde{Y}$ be the induced map by f'. Since f' is a homotopy equivalence, $\tilde{f}'$ is a π-homotopy equivalence. If we show $\tilde{X}'/\tilde{X}$ is a $T(\mathcal{F}')$-complex, we have $[M_{\tilde{f}}]\in\Delta(\tilde{G}, T(\mathcal{F}))+\Omega(\tilde{G}, T(\mathcal{F}'))$ by Theorem 3.2 of [6]. Through the map ν we have $[M_f]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}')$. So we prove $\tilde{X}'/\tilde{X}$ is a $T(\mathcal{F}')$-complex. For a cell c in $\tilde{X}'\backslash\tilde{X}$, let $\alpha\in\tilde{\Pi}$ be the type of c. The isotropy group on c is $\tilde{\rho}(\alpha)$ and that on $q'(c)$ is $\sigma(\tilde{\rho}(\alpha))$, where $q'=f'*p$: $\tilde{X}'\to X'$. Since $\tilde{f}'(c)\subset|\alpha|, f'(q'(c))\subset|\mu(\alpha)|$. The type of $q'(c)$ is $\mu(\alpha)$. By the assumption X'/X is a $\mathcal{F}'$-complex, and so $\mu(\alpha)\in\mathcal{F}'$. Thus $\alpha\in T(\mathcal{F}')$. This means that $\tilde{X}'\backslash\tilde{X}$ is a $T(\mathcal{F}')$-complex.

Secondly we assume $[M_f]\in\Delta_h(G, Y, \mathcal{F})+\Omega(G, \mathcal{F}')$. Since ν: $\Omega(\tilde{G}, T(\mathcal{F}))\to\Omega(G, \mathcal{F})$ is injective and $\nu(\Omega(\tilde{G}, T(\mathcal{F}')))=\Omega(G, \mathcal{F}')$ by Proposition 3.5, we have $[M_{\tilde{f}}]\in\Delta(\tilde{G}, T(\mathcal{F}))+\Omega(\tilde{G}, T(\mathcal{F}'))$. By Theorem 3.2 of [6] there exist a finite $\tilde{G}$-complex $\tilde{X}'\supset\tilde{X}$ and a (skeletal) pseudo-equivalence $\tilde{f}'$: $\tilde{X}\to\tilde{Y}$ extending $\tilde{f}$ such that $\tilde{X}'/\tilde{X}$ is a $T(\mathcal{F}')$-complex. Since $\tilde{Y}$ is a numerable π-space, $\tilde{f}'$ is a π-homotopy equivalence. Put $X'=\tilde{X}'/\pi$. Then $X'\supset X$ and the induced map f': $X'\to Y$ by $\tilde{f}'$ is a homotopy equivalence. Moreover X'/X is an $\mathcal{F}'$-complex by the similar argument used in the first part. This completes the proof.

References

[1] D.R. Anderson: *Torsion invariants and actions of finite groups*, Michigan Math. J. **29** (1982), 27–42.

[2] A.H. Assadi: *Finite group actions on simply-connected manifolds and CW complexes*, Mem. Amer. Math. Soc. **257** (1982).

[3] M.M. Cohen: A course in simple-homotopy theory, Graduate Texts in Math. 10, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[4] K.H. Dovermann and T. Petrie: *G surgery* II, Mem. Amer. Math. Soc. **260** (1982).

[5] R. Oliver: *Fixed-point set of group actions on finite acyclic complexes,* Comment. Math. Helv. **50** (1975), 155–177.

[6] R. Oliver and T. Petrie: *G-CW-surgery and* $K_0(\mathbf{Z}G)$, Math. Z. **179** (1982), 11–42.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560, Japan