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## ON THE NUMBER OF THE LATTICE POINTS IN THE AREA

$$0 < x < n, 0 < y \leq ax^k/n.$$

ISAO MIYAWAKI

(Received October 18, 1974)

### 1. Introduction

Let  $S_a^{(k)}(n)$  be the number of the lattice points in the area  $0 < x < n$ ,  $0 < y \leq ax^k/n$ , where  $k$  and  $n$  are positive integers and  $a$  is a positive integer which is prime to  $n$ . Then we have

$$S_a^{(k)}(n) = \sum_{x=1}^{n-1} [ax^k/n],$$

where  $[ ]$  denotes the Gauss symbol. Let

$$ax^k/n = [ax^k/n] + \overline{\{ax^k/n\}},$$

where  $\overline{\{ax^k/n\}}$  denotes the fractional part of  $ax^k/n$ . Then we have

$$\sum_{x=1}^{n-1} ax^k/n = S_a^{(k)}(n) + \sum_{x=1}^{n-1} \overline{\{ax^k/n\}}$$

or

$$S_a^{(k)}(n) = \sum_{x=1}^{n-1} ax^k/n - \sum_{x=1}^{n-1} \overline{\{ax^k/n\}}.$$

We put

$$S_a^{(k)}(n) = \sum_{x=1}^{n-1} ax^k/n - \frac{n-1}{2} + c_a^{(k)}(n),$$

$$c_a^{(k)}(n) = \frac{n-1}{2} - \sum_{x=1}^{n-1} \overline{\{ax^k/n\}}.$$

If we suppose that  $S_a^{(k)}(n)$  behaves approximately as  $\sum_{x=1}^{n-1} ax^k/n - \frac{n-1}{2}$  then  $c_a^{(k)}(n)$  can be regarded as error term. T. Honda has conjectured the followings.

**Conjecture 1.** For a fixed  $k$  and any positive real number  $\varepsilon$  we have

$$c_a^{(k)}(n) = O(n^{(k-1)/k+\varepsilon}),$$

for  $a=1$ .

**Conjecture 2.**  $c_1^{(2)}(n) \geq 0$  and  $c_1^{(2)}(n) = 0$  if and only if  $n$  is an integer of the following type

$$n = p_1 \cdots p_j,$$

where  $p_1, \dots, p_j$  are distinct primes and each  $p_i$  is equal to 2 or congruent to 1 modulo 4.

In this paper we shall give the complete proof of the above conjectures. Conjecture 1 is true not only in the case  $a=1$  but also in the case  $a$  is any positive integer which is prime to  $n$ . In the case  $k$  is odd,  $c_a^{(k)}(n)$  is a very simple quantity. On the other hand in the case  $k$  is even,  $c_a^{(k)}(n)$  is an interesting quantity which is rather difficult to handle. For example,  $c_1^{(2)}(n)$  can be expressed in terms of the class numbers of imaginary quadratic fields whose discriminants are divisors of  $n$ . For the even  $k > 2$ ,  $c_a^{(k)}(n)$  is also related to some class numbers of some subfields of the cyclotomic field  $\mathbb{Q}(\zeta)$  where  $\zeta$  is a primitive  $n$ -th root of unity.

I would like to express my deep gratitude to Professor T. Honda for his presenting this problem to me.

## 2. Preliminaries

For positive integers  $k$ ,  $n$  and an integer  $x$ , we denote by  $N^{(k)}(x, n)$  the number of the elements of the set

$$\{y \in \mathbb{Z} \mid y^k \equiv x \pmod{n}, \quad 0 \leq y < n\}.$$

**Lemma 1.** Let  $n = \prod_{i=1}^j p_i^{e_i}$  be the prime decomposition of  $n$ . Then we have

$$N^{(k)}(x, n) = \prod_{i=1}^j N^{(k)}(x, p_i^{e_i}).$$

**Proof.** Consider the following map

$$f: \mathbb{Z}/n\mathbb{Z} \rightarrow \prod_{i=1}^j \mathbb{Z}/p_i^{e_i}\mathbb{Z}, \quad (f(a \pmod{n}) = \prod_{i=1}^j a \pmod{p_i^{e_i}}).$$

We can easily see that this  $f$  is a ring isomorphism. From this we can immediately obtain the lemma.

Let  $n$  be a positive integer which is not equal to 1. We denote by  $(\mathbb{Z}/n\mathbb{Z})^\times$  the unit group of the residue ring  $\mathbb{Z}/n\mathbb{Z}$ . We put

$$\Gamma(n) = \{\chi \mid \chi: (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow U, \text{ homomorphism}\},$$

where  $U = \{z \in \mathbb{C} \mid |z| = 1\}$ . Then  $\Gamma(n)$  is an abelian group isomorphic to  $(\mathbb{Z}/n\mathbb{Z})^\times$ . An element  $\chi$  of  $\Gamma(n)$  is extended on  $\mathbb{Z}$  by setting

$$\chi(a) = \begin{cases} 0 & \text{if } (a, n) \neq 1 \\ \chi(a \bmod n) & \text{otherwise.} \end{cases}$$

This function is denoted by  $\chi$ , and is called a character modulo  $n$ . If  $\chi$  has always the value 1 for any  $a$  such that  $(a, n)=1$ , then  $\chi$  is called the trivial character modulo  $n$ , and denoted by 1. If  $\chi$  is a non-trivial character modulo  $n$  and there is no character  $\chi'$  of  $(\mathbb{Z}/n'\mathbb{Z})^\times$  with a proper divisor  $n'$  of  $n$  satisfying  $\chi'(a)=\chi(a)$  for any  $(a, n)=1$ , then  $\chi$  is called a primitive character modulo  $n$ . Any non-trivial character  $\chi$  modulo  $n$  can be uniquely decomposed to the following form

$$\chi = \chi_0 \chi',$$

where  $\chi_0$  is the trivial character modulo  $n$  and  $\chi'$  is a primitive character modulo  $n'$  with some divisor  $n'$  of  $n$ . We call this  $n'$  the conductor of  $\chi$  and denote it by  $f_\chi$ . If  $\chi$  is a primitive character modulo some  $n$ , then we call  $\chi$  simply primitive. In this case the conductor  $f_\chi$  is equal to  $n$ . Let  $n = \prod_{i=1}^j p_i^{e_i}$  be the prime decomposition of  $n$ . Then we have  $(\mathbb{Z}/n\mathbb{Z})^\times = \prod_{i=1}^j (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^\times$ . Therefore if  $\chi$  is a character modulo  $n$ , then  $\chi$  has the following unique decomposition

$$(1) \quad \chi = \prod_{i=1}^j \chi_i,$$

where each  $\chi_i$  is a character modulo  $p_i^{e_i}$ . It is clear that  $\chi$  is primitive, if and only if each  $\chi_i$  is primitive. Let  $\chi$  be a character modulo  $n$ . Then we put  $H_\chi = -\frac{1}{n} \sum_{a=1}^n \chi(a)a$ .

**Lemma 2.** *Let  $\chi$  be a non-trivial character modulo  $n$ . If  $\chi(-1)=1$  then we have  $H_\chi=0$ .*

**Proof.** First we should note  $\chi(n)=0$ . Then we have

$$\begin{aligned} H_\chi &= \frac{-1}{2n} \left( \sum_{a=1}^{n-1} \chi(a)a + \sum_{a=1}^{n-1} \chi(-a+n)(-a+n) \right) \\ &= \frac{-1}{2n} \left( \sum_{a=1}^{n-1} \chi(a)a + \sum_{a=1}^{n-1} \chi(-a)(-a+n) \right) \\ &= \frac{-1}{2n} \sum_{a=1}^{n-1} \chi(a)(a+(-a+n)) \\ &= -\frac{1}{2} \sum_{a=1}^{n-1} \chi(a) = 0. \end{aligned}$$

We put

$$\Gamma^k(n) = \{\chi \in \Gamma(n) \mid \chi^k = 1\}.$$

**Lemma 3.** *Let  $p$  be a prime number. Then we have*

$$(i) \quad N^{(k)}(b, p^e) = \sum_{\substack{\chi \in \Gamma^{(k)}(p^e) \\ f_\chi = p}} \chi(b) = 1 + \sum_{\substack{\chi \text{ primitive} \\ f_\chi \neq p \\ \chi^k = 1}} \chi(b)$$

if  $(b, p) = 1$ ,

$$(ii) \quad N^{(k)}(b, p) = 1 + \sum_{\substack{f_\chi = p \\ \chi^k = 1}} \chi(b).$$

Proof. If we note that  $\Gamma^{(k)}(p^e)$  is the character group of the factor group  $(\mathbf{Z}/p^e\mathbf{Z})^\times/(\mathbf{Z}/p^e\mathbf{Z})^{\times k}$  and  $\chi(b)$  is zero for any  $(b, p^e) \neq 1$ , then we can easily obtain the lemma.

**Lemma 4.** *We denote by  $\#\Gamma^{(k)}(n)$  the number of the elements of the set  $\Gamma^{(k)}(n)$ . Let  $p$  be a prime. Then we have*

$$(i) \quad \#\Gamma^{(k)}(p^e) = (p-1, k) \quad \text{if } (p, k) = 1,$$

$$(ii) \quad \#\Gamma^{(k)}(p^e) = \begin{cases} p^{e-1}(p-1, k) & \text{if } e_0 + 1 \geq e, \\ (p+2) & \text{if } e_0 + 1 < e, \end{cases}$$

where we define  $e_0$  by

$$p^{e_0} \mid k, \quad e_0 > 0,$$

$$(iii) \quad \#\Gamma^{(k)}(2^e) = \begin{cases} 2^{e-1} & \text{if } e \leq e_0 + 2 \\ 2^{e_0+1} & \text{if } e \geq e_0 + 3, \end{cases}$$

where we define  $e_0$  by

$$2^{e_0} \mid k, \quad e_0 > 0.$$

Especially for a fixed  $k$ , there is a constant  $c_0$  such that

$$\#\Gamma^{(k)}(p^e) \leq c_0$$

for any  $p$  and  $e$ .

Proof. If we note the following facts

$$(\mathbf{Z}/p^e\mathbf{Z})^\times \cong \mathbf{Z}/(p-1)p^{e-1}\mathbf{Z} \quad \text{if } p \neq 2,$$

$$(\mathbf{Z}/2^e\mathbf{Z})^\times \cong \mathbf{Z}/2\mathbf{Z} + \mathbf{Z}/2^{e-2}\mathbf{Z} \quad \text{if } e \geq 2,$$

$$(\mathbf{Z}/p^e\mathbf{Z})^\times/(\mathbf{Z}/p^e\mathbf{Z})^{\times k} \cong \Gamma^{(k)}(p^e),$$

then we have immediately the lemma 4.

### 3. Main theorem and its proof

Let  $n \geq 2$  be a positive integer and  $n = \prod_{i=1}^j p_i^{e_i}$  be the prime decomposition of  $n$ . We define index sets  $A(n)$  and  $B(n)$  as follows

$$\begin{aligned} A(n) &= \{1, 2, \dots, j\} \\ B(n) &= \{i \in A(n) \mid e_i \geq 2\}. \end{aligned}$$

For a subset  $\alpha = \{\alpha_1, \dots, \alpha_l\}$  of the set  $A(n)$  we denote by  $d_\alpha$  the integer

$$\begin{aligned} d_\alpha &= \prod_{i=1}^l p_{\alpha_i}, \quad \text{if } \alpha \neq \phi \\ d_\phi &= 1. \end{aligned}$$

For a fixed positive integer  $k$ , we put

$$e_i = ks_i + r_i, \quad s_i \geq 0, \quad 1 \leq r_i \leq k,$$

and

$$n_0 = n_0^{(k)} = \prod_{i=1}^j p_i^{(k-1)s_i + r_i - 1}.$$

Let  $d$  be a positive divisor of  $n$ . Then we put

$$\begin{aligned} n(d) &= n^{(k)}(d) = n/(d^k, n), \\ d^*(n) &= d^*(d)^{(k)} = d^k/(d^k, n). \end{aligned}$$

Under the above notation we have the following proposition.

**Proposition 1.**

$$\begin{aligned} c_\alpha^{(k)}(n) &= \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid n, \chi^k = 1}} \overline{\chi(a)} H_\chi - \left[ \sum_{\substack{\alpha \subset B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left\{ \frac{(d_\alpha^k, n)/d_\alpha - 1}{2} + \frac{(d_\alpha^k, n)}{d_\alpha} \right. \right. \\ &\quad \left. \left. \cdot c_{\alpha d_\alpha^{(k)}(n)}^{(k)}(n(d_\alpha)) - \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid n, \chi^k = 1 \\ (f_\chi, d_\alpha) = 1}} \chi(d_\alpha) \overline{\chi(a)} H_\chi \right\} \right], \end{aligned}$$

where we denote by  $\mu(\cdot)$  the Möbius function.

Proof. By the definition of  $c_\alpha^{(k)}(n)$  we have

$$c_\alpha^{(k)}(n) = \frac{n-1}{2} - \frac{1}{2} \sum_{x=1}^{n-1} N^{(k)}(a^{-1}x, n),$$

where we consider  $a^{-1}x$  in  $(\mathbb{Z}/n\mathbb{Z})^\times$ . If  $(x, d_{B(n)}) = 1$  then by Lemma 1 and Lemma 2 we have

$$N^{(k)}(a^{-1}x, n) = \prod_{i=1}^j (1 + \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid p_i^{e_i}, \chi^k = 1}} \chi(a^{-1}x)).$$

Therefore we get

$$\begin{aligned}
c_a^{(k)}(n) &= \frac{n-1}{2} - \left[ \frac{1}{n} \sum_{x=1}^{n-1} \prod_{i=1}^j (1 + \sum_{\substack{\chi \text{ primitive} \\ f_x \mid p_i^e, \chi^k=1}} \chi(a^{-1}x))x \right. \\
&\quad + \sum_{\substack{\alpha \in B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left\{ \frac{1}{n} \sum_{x=1}^{(n/d_\alpha)-1} \prod_{i \in \alpha} (1 + \sum_{\substack{\chi \text{ primitive} \\ f_x \mid p_i^e, \chi^k=1}} \chi(a^{-1}d_\alpha x))d_\alpha x \right. \\
&\quad \left. \left. - \sum_{x=1}^{(n/d_\alpha)-1} \left\{ \overline{\frac{a(d_\alpha x)^k}{n}} \right\} \right\} \right] \\
&= \frac{n-1}{2} - \frac{n(n-1)}{2n} - \frac{1}{n} \sum_{\substack{\chi \text{ primitive} \\ f_x \mid n, \chi^k=1}} \sum_{x=1}^{n-1} \chi(a^{-1}x)x - \sum_{\substack{\alpha \in B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \\
&\quad \cdot \left[ \frac{d_\alpha \cdot (n/d_\alpha)((n/d_\alpha)-1)}{2} - \frac{d_\alpha}{n} \sum_{\substack{\chi \text{ primitive} \\ f_x \mid n, \chi^k=1 \\ (f_x, d_\alpha)=1}} \sum_{x=1}^{(n/d_\alpha)-1} \chi(a^{-1}d_\alpha x)x \right. \\
&\quad \left. - \sum_{x=1}^{(n/d_\alpha)-1} \left\{ \overline{\frac{ad_\alpha^*(n)x^k}{n(d_\alpha)}} \right\} \right],
\end{aligned}$$

where we should note that

$$\frac{1}{n} \sum_{x=1}^{n-1} \chi(x)x = \frac{1}{n} \sum_{x=1}^{f_x-1} \sum_{i=0}^{(n/f_x)-1} \chi(x)(x+if_x) = \frac{1}{n} \frac{n}{f_x} \sum_{x=1}^{f_x-1} \chi(x)x = -H_x.$$

Then we have

$$\begin{aligned}
c_a^{(k)}(n) &= \sum_{\substack{\chi \text{ primitive} \\ f_x \mid n, \chi^k=1}} \overline{\chi(a)} H_x - \left[ \sum_{\substack{\alpha \in B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left\{ \frac{(n/d_\alpha)-1}{2} - \sum_{\substack{\chi \text{ primitive} \\ f_x \mid n, \chi^k=1 \\ (f_x, d_\alpha)=1}} \overline{\chi(a)} \chi(d_\alpha) H_x \right. \right. \\
&\quad \left. \left. - \frac{n}{d_\alpha n(d_\alpha)} \sum_{x=1}^{(n/d_\alpha)-1} \left\{ \overline{\frac{ad_\alpha^*(n)x^k}{n(d_\alpha)}} \right\} \right\} \right].
\end{aligned}$$

On the other hand we see that

$$-\sum_{x=1}^{(n/d_\alpha)-1} \left\{ \overline{\frac{ad_\alpha^*(n)x^k}{n(d_\alpha)}} \right\} = c_{\alpha \alpha^*(n)}^{(k)}(n(d_\alpha)) - \frac{n(d_\alpha)-1}{2}.$$

Therefore we have

$$\begin{aligned}
c_a^{(k)}(n) &= \sum_{\substack{\chi \text{ primitive} \\ f_x \mid n, \chi^k=1}} \overline{\chi(a)} H_x - \sum_{\substack{\alpha \in B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left[ \frac{(n/d_\alpha)-1}{2} - \frac{n}{d_\alpha n(d_\alpha)} \cdot \frac{n(d_\alpha)-1}{2} \right. \\
&\quad \left. - \sum_{\substack{\chi \text{ primitive} \\ f_x \mid n, \chi^k=1 \\ (f_x, d_\alpha)=1}} \overline{\chi(a)} \chi(d_\alpha) H_x + \frac{n}{d_\alpha n(d_\alpha)} c_{\alpha \alpha^*(n)}^{(k)}(n(d_\alpha)) \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid n, \chi^k=1}} \overline{\chi(a)} H_\chi - \sum_{\substack{\alpha \in B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left[ \frac{(n/d_\alpha n(d_\alpha)) - 1}{2} \right. \\
&\quad \left. + \frac{n}{d_\alpha(n(d_\alpha))} c_{\alpha \alpha^*(n)}^{(k)}(n(d_\alpha)) - \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid n, \chi^k=1 \\ (f_\chi, d_\alpha)=1}} \overline{\chi(a)} \chi(d_\alpha) H_\chi \right].
\end{aligned}$$

But by the definition of  $n(d)$  we have

$$\frac{n}{n(d_\alpha)} = \frac{n}{\frac{n}{(d_\alpha^k, n)}} = (d_\alpha^k, n).$$

Therefore we get

$$\begin{aligned}
c_\alpha^{(k)}(n) &= \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid n, \chi^k=1}} \overline{\chi(a)} H_\chi - \sum_{\substack{\alpha \in B(n) \\ \alpha \neq \phi}} \mu(d) \left[ \frac{((d_\alpha^k, n)/d_\alpha) - 1}{2} \right. \\
&\quad \left. + \frac{(d_\alpha^k, n)}{d_\alpha} c_{\alpha \alpha^*(n)}^{(k)}(n(d_\alpha)) - \sum_{\substack{\chi: \text{ primitive} \\ f_\chi \mid n, \chi^k=1 \\ (f_\chi, d_\alpha)=1}} \overline{\chi(a)} \chi(d_\alpha) H_\chi \right].
\end{aligned}$$

Thus Proposition 1 is proved

Let  $\chi$  be a non-trivial character modulo  $n$  such that  $\chi^k=1$ . Then we define the integer  $n(\chi)=n^{(k)}(\chi)$  as follows,

$$\begin{aligned}
n(\chi) &= \prod_{p: \text{ prime}} p^{[v_p(n/f_\chi)/k] + \varepsilon_{p,n}} \\
\varepsilon_{p,n} = \varepsilon_{p,n}^{(k)} &= \begin{cases} 0 & \text{if } p \mid f_\chi \text{ or } v_p\left(\frac{n}{f_\chi}\right) - k \left[ v_p\left(\frac{n}{f_\chi}\right) \frac{1}{k} \right] \leq 1, \\ 1 & \text{otherwise,} \end{cases}
\end{aligned}$$

where we denote by  $v_p(\cdot)$  the normalized  $p$ -adic exponential valuation of the field of the rational numbers  $\mathbb{Q}$ . Then we can easily obtain the following two remarks.

REMARK 1. For a prime  $p$  if  $p$  divides  $n(\chi)$ , then  $p^2$  divides  $n/f_\chi$ .

REMARK 2. If  $n(\chi)$  is divisible by  $d$ , then  $n/(d^k, n) \equiv 0 \pmod{f_\chi}$ .

**Lemma 5.** *Let  $n$  be a positive integer. For distinct primes  $p_1, \dots, p_j$  such that  $p_i^2 \mid n$  ( $i=1, \dots, j$ ), we put  $d_0 = p_1 \cdots p_j$  and  $n(d_0) = n/(d_0^k, n)$ . Let  $\chi$  be a character modulo  $n(d_0)$ . Then  $\chi$  induces the character modulo  $n$  through the homomorphism  $(\mathbb{Z}/n\mathbb{Z})^\times \rightarrow (\mathbb{Z}/n(d_0)\mathbb{Z})^\times$ . Denoting this also  $\chi$  we have that if  $d$  divides  $n(d_0)(\chi)$  then  $dd_0$  divides  $n(\chi)$ .*

Proof. We shall show that  $v_p(dd_0) \leq v_p(n(\chi))$  for every prime  $p$ . We consider the two cases.

The case I.  $p \neq p_i$  ( $i = 1, \dots, j$ ).

By the definition of  $n(d_0)$  we have

$$v_p(n) = v_p(n(d_0))$$

and

$$v_p(n/f_x) = v_p(n(d_0)/f_x).$$

It follows from this

$$\varepsilon_{p,n} = \varepsilon_{p,n(d_0)}.$$

From this and by the definition of  $d$  we have

$$\begin{aligned} v_p(dd_0) &= v_p(d) \leq [v_p(n(d_0)/f_x)/k] + \varepsilon_{p,n(d_0)} \\ &= [v_p(n/f_x)/k] + \varepsilon_{p,n} \\ &= v_p(n(X)). \end{aligned}$$

Thus Lemma 5 is proved in our case.

The case II.  $p = p_i$  (for some  $i$ )

By the definition of  $n(d_0)$  we have

$$v_p(n(d_0)/f_x) = \begin{cases} v_p(n/f_x) - k & \text{if } p^k \mid n, \\ 0 & \text{if } p^k \nmid n. \end{cases}$$

Therefore we shall consider the two cases.

(i) The case  $v_p(n(d_0)/f_x) = v_p(n/f_x) - k$ .

In this case we have

$$\begin{aligned} v_p(n/f_x) - k \left[ v_p(n/f_x) \frac{1}{k} \right] &= v_p(n(d_0)/f_x) + k - k[v_p(n(d_0)/f_x)/k + 1] \\ &= v_p(n(d_0)/f_x) - k[v_p(n(d_0)/f_x)/k]. \end{aligned}$$

This shows that  $\varepsilon_{p,n} = \varepsilon_{p,n(d_0)}$ . Noting this we have

$$\begin{aligned} v_p(dd_0) &= 1 + v_p(d) \leq 1 + [v_p(n(d_0)/f_x)/k] + \varepsilon_{p,n(d_0)} \\ &= 1 + [v_p(n/f_x)/k - 1] + \varepsilon_{p,n} \\ &= [v_p(n/f_x)/k] + \varepsilon_{p,n} \\ &= v_p(n(X)). \end{aligned}$$

This also completes the proof of Lemma 5 in our case.

(ii) The case  $v_p(n(d_0)/f_x) = 0$

In this case we should note that  $v_p(f_x) = 0$ . Then we have

$$v_p(n(d_0)/f_x) - k[v_p(n(d_0)/f_x)/k] = 0.$$

It follows

$$\varepsilon_{p,n(d_0)} = 0.$$

This shows  $v_p(d)=0$ . On the other hand we have

$$v_p(n) \geq 2 + v_p(n(d_0)).$$

This shows that

$$[v_p(n/f_\chi)/k] > 0.$$

or

$$v_p(n/f_\chi) - k \left[ v_p(n/f_\chi) \frac{1}{k} \right] > 1, \quad (\text{i.e., } \varepsilon_{p,n} = 1).$$

Therefore  $[v_p(n/f_\chi)/k] + \varepsilon_{p,n}$  is positive in both cases. Then we have

$$\begin{aligned} v_p(dd_0) &= v_p(d_0) = 1 \leq [v_p(n/f_\chi)/k] + \varepsilon_{p,n} \\ &= v_p(n(\chi)). \end{aligned}$$

Thus Lemma 5 is completely proved.

The following lemma is a converse of Lemma 5 in a sense.

**Lemma 6.** *Let  $\chi$  be a character modulo  $n$  and  $d$  be a positive divisor of  $n(\chi)$ . Let  $p_1, \dots, p_j$  be distinct primes each of which is a divisor of  $d$ . If we put  $d_0 = p_1 \cdots p_j$ , and  $d = d_0 d'$  with a positive integer  $d'$ , then  $\chi$  is a character modulo  $n(d_0)$  and  $d'$  is a divisor of  $n(d_0)(\chi)$ .*

Proof. The former assertion is obvious by Remark 2. So we shall show the latter half in the same manner as in Lemma 5. Let  $p$  be a prime.

(I) The case  $p \neq p_i$  ( $i=1, \dots, j$ )

In this case we can show that  $v_p(n(\chi)) = v_p(n(d_0)(\chi))$  by the same method as in the case (I) of Lemma 5. Then we have

$$v_p(d') = v_p(d) \leq v_p(n)(\chi) = v_p(n(d_0)(\chi))$$

(II) The case  $p = p_i$  (for some  $i$ ).

In this case we have

$$v_p(d) \leq v_p(n(\chi)).$$

This shows that

$$[v_p(n/f_\chi)/k] > 0$$

or

$$[v_p(n/f_\chi)/k] = 0 \quad \text{and} \quad \varepsilon_{p,n} = 1.$$

Therefore we shall consider the two cases.

(i) The case  $[v_p(n/f_\chi)/k] > 0$ .

In this case we can easily see that

$$\begin{aligned} v_p(n/f_x)/k &= v_p\left(\frac{1}{f_x} \frac{n}{(p^k, n)}\right) \frac{1}{k} + 1 \\ &= v_p\left(\frac{1}{f_x} \frac{n}{(d_0^k, n)}\right) \frac{1}{k} + 1. \end{aligned}$$

Therefore we have

$$\begin{aligned} v_p(d') &= v_p(d) - 1 \leq [v_p(n/f_x)/k] + \varepsilon_{p,n} - 1 \\ &= [v_p(n(d_0)/f_x)/k] + 1 + \varepsilon_{p,n} - 1. \end{aligned}$$

But we can show by the same method as in the case (II)-(i) of Lemma 5 that  $\varepsilon_{p,n} = \varepsilon_{p,n(d_0)}$ . Therefore it follows

$$v_p(d') \leq v_p(n(d_0)(\chi)).$$

(ii) The case  $[v_p(n/f_x)/k] = 0$  and  $\varepsilon_{p,n} = 1$ .

In this case we have

$$v_p(d') = v_p(d) - 1 \leq \varepsilon_{p,n} - 1 = 0.$$

This shows that

$$v_p(d') = 0.$$

Therefore we have

$$v_p(d') \leq v_p(n(d_0)(\chi)).$$

These complete the proof of Lemma 6.

Now we are in a position to state our main Theorem.

**Theorem 1.** *Notation being as above. Then*

$$\begin{aligned} c_a^{(k)}(n) &= \frac{n_0 - 1}{2} + \sum_{\substack{\chi \text{ primitive} \\ \chi^k = 1 \\ f_x \mid n}} \chi^{-1}(a) H_x \left\{ \sum_{d \mid n(\chi)} \frac{(d^k, n)}{d} \chi^{-1}\left(\frac{d^k}{(d^k, n)}\right) \right. \\ &\quad \left. \cdot \left( \sum_{\substack{d_\alpha \mid n(d)(\chi) \\ (d_\alpha, f_x) = 1 \\ \alpha \subset B(n)}} \mu(d_\alpha) \chi(d_\alpha) \right) \right\}. \end{aligned}$$

Proof. Let  $n = \prod_{i=1}^j p_i^{e_i}$  be the prime decomposition of  $n$ . Then we put  $s(n) = \sum_{i=1}^j (e_i - 1)$ . We shall prove our theorem by the induction with respect to  $s(n)$ . If  $s(n) = 0$ , i.e.,  $n$  is a square-free integer, then by taking  $B(n) = \phi$  in Proposition 1 we get

$$c_a^{(k)}(n) = \sum_{\substack{\chi \text{ primitive} \\ \chi^k=1 \\ f_\chi \mid n}} \chi^{-1}(a) H_\chi .$$

On the other hand, in this case we have  $n_0=1$ ,  $n(\chi)=1$  and  $B(n)=\phi$ . This shows that our theorem is true in our case. If  $s(n)>0$ , then we assume that the theorem is valid for any  $m$  such that  $s(m)< s(n)$ . Now we can easily see that  $s(n(d_\alpha)) < s(n)$  with respect to  $n(d_\alpha)$  of Proposition 1. Therefore by the assumption we have

$$(2) \quad c_{d_\alpha^*(n)\alpha}^{(k)}(n(d_\alpha)) = \frac{n(d_\alpha)_0 - 1}{2} + \sum_{\substack{\chi \text{ primitive} \\ \chi^k=1 \\ f_\chi \mid n(d_\alpha)}} \chi^{-1}(d_\alpha^*(n)\alpha) H_\chi$$

$$\cdot \left\{ \sum_{d \mid n(d_\alpha)(\chi)} \frac{(d^k, n(d_\alpha))}{d} \chi^{-1}\left(\frac{d^k}{(d^k, n(d_\alpha))}\right) \right. \\ \left. \cdot \left( \sum_{\substack{d_\beta \mid (n(d_\alpha))(d)(\chi) \\ (d_\beta, f_\chi)=1 \\ \beta \subset B(n(d_\alpha))}} \mu(d_\beta) \chi(d_\beta) \right) \right\} .$$

Hereafter we shall only consider primitive characters which take values  $k$ -th roots of unity or zero, though we shall not mention it explicitly. From (2) and Proposition 1 we get

$$c_\alpha^{(k)}(n) = \sum_{f_\chi \mid n} \chi^{-1}(a) H_\chi - \sum_{\substack{\alpha \subset B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left[ \left( \frac{(d_\alpha^k, n)}{2} - 1 \right) + \frac{(d_\alpha^k, n)}{d} \right. \\ \left. \cdot \left\{ \frac{n(d_\alpha)_0 - 1}{2} + \sum_{f_\chi \mid n(d_\alpha)} \chi^{-1}(d_\alpha^*(n)\alpha) H_\chi \sum_{d \mid n(d_\alpha)(\chi)} \right. \right. \\ \left. \left. \cdot \frac{(d^k, n(d_\alpha))}{d} \chi^{-1}\left(\frac{d^k}{(d^k, n(d_\alpha))}\right) \right. \right. \\ \left. \left. \cdot \sum_{\substack{\beta \subset B(n(d_\alpha)) \\ d_\beta \mid n(d_\alpha)(d)(\chi) \\ (d_\beta, f_\chi)=1}} \mu(d_\beta) \chi(d_\beta) \right\} - \sum_{\substack{f_\chi \mid n \\ (f_\chi, d_\alpha)=1}} \chi(d_\alpha) \chi^{-1}(a) H_\chi \right] .$$

Therefore if we prove the following two facts (I) and (II), then the proof of Theorem 1 is completed.

$$(I) \quad - \sum_{\substack{\alpha \subset B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left\{ \frac{(d_\alpha^k, n)}{2} - 1 + \frac{(d_\alpha^k, n)(n(d_\alpha)_0 - 1)}{2d_\alpha} \right\} = \frac{n_0 - 1}{2} .$$

$$\begin{aligned}
\text{(II)} \quad & \sum_{f_x \mid n} \chi^{-1}(a) H_x - \sum_{\substack{\alpha \subset B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) \left[ \left\{ \frac{(d_\alpha^k, n)}{d_\alpha} \sum_{f_x \mid n(d_\alpha)} \chi^{-1}(d_\alpha^*(n)a) H_x \right. \right. \\
& \left. \left. - \sum_{\substack{d \mid n(d_\alpha)(\chi) \\ (d_\alpha, f_x) = 1}} \frac{(d^k, n(d_\alpha))}{d} \chi^{-1} \left( \frac{d^k}{(d^k, n(d_\alpha))} \right) \sum_{\substack{\beta \subset B(n(d_\alpha)) \\ d_\beta \mid n(d_\alpha)(d)(\chi) \\ (d_\beta, f_x) = 1}} \mu(d_\beta) \chi(d_\beta) \right\} \right. \\
& \left. - \sum_{\substack{f_x \mid n \\ (f_x, d_\alpha) = 1}} \chi(d_\alpha) \chi^{-1}(a) H_x \right] \\
& = \sum_{f_x \mid n} \chi^{-1}(a) H_x \sum_{d \mid n(\chi)} \frac{(d^k, n)}{d} \chi^{-1} \left( \frac{d^k}{(d^k, n)} \right) \sum_{\substack{\alpha \subset B(n) \\ d_\alpha \mid n(d)(\chi) \\ (d_\alpha, f_x) = 1}} \mu(d_\alpha) \chi(d_\alpha) .
\end{aligned}$$

First we shall prove (I). By the definition of  $n(d_\alpha)$  we get

$$n(d_\alpha)_0 = \left( \frac{n}{(d_\alpha^k, n)} \right)_0$$

and

$$n(d_\alpha)_0 \frac{(d_\alpha^k, n)}{d_\alpha} = \left( \frac{n}{(d_\alpha^k, n)} \right)_0 \frac{(d_\alpha^k, n)}{d_\alpha} .$$

By examining  $p$ -adic valuation of  $(n/(d_\alpha^k, n))_0 \cdot ((d_\alpha^k, n)/d_\alpha)$  for each  $p$  such that  $p \mid n$ , we can easily see that

$$n(d_\alpha)_0 \frac{(d_\alpha^k, n)}{d_\alpha} = n_0 .$$

On the other hand we have

$$- \sum_{\substack{\alpha \subset B(n) \\ \alpha \neq \phi}} \mu(d_\alpha) = - \sum_{\substack{d \mid d_{B(n)} \\ d \neq 1}} \mu(d) = - ((\sum_{d \mid d_{B(n)}} \mu(d)) - 1) = 1 .$$

It follows (I).

Next we shall prove (II). We can rewrite the left hand side of (II) to the following formula

$$\begin{aligned}
\text{(3)} \quad & \sum_{f_x \mid n} \chi^{-1}(a) H_x \left[ \left\{ \sum_{\substack{\alpha \subset B(n) \\ (d_\alpha, f_x) = 1}} \mu(d_\alpha) \chi(d_\alpha) \right\} \right. \\
& - \left\{ \sum_{\substack{\alpha \subset B(n) \\ \alpha \neq \phi \\ f_x \mid n(d_\alpha)}} \sum_{d \mid n(d_\alpha)(\chi)} \sum_{\substack{\beta \subset B(n(d_\alpha)) \\ d_\beta \mid n(d_\alpha)(d)(\chi) \\ (d_\beta, f_x) = 1}} \mu(d_\alpha) \cdot \frac{(d_\alpha^k, n)}{d_\alpha} \cdot \frac{(d_\alpha^k, n(d_\alpha))}{d} \right. \\
& \left. \left. \cdot \chi^{-1} \left( \frac{d_\alpha^*(n)d^k}{(d^k, n(d_\alpha))} \right) \mu(d_\beta) \chi(d_\beta) \right\} \right] .
\end{aligned}$$

Here we note that

$$\frac{(d_\alpha^k, n)}{d_\alpha} \cdot \frac{(d^k, n(d_\alpha))}{d} = \frac{(d_\alpha^k, n) \left( d^k, \frac{n}{(d_\alpha^k, n)} \right)}{dd_\alpha} = \frac{((dd_\alpha)^k, n)}{dd_\alpha}$$

and

$$\frac{d_\alpha^*(n)d^k}{(d^k, n(d_\alpha))} = \frac{d_\alpha^k}{(d_\alpha^k, n)} \cdot \frac{d^k}{\left( d^k, \frac{n}{(d_\alpha^k, n)} \right)} = \frac{(dd_\alpha)^k}{((dd_\alpha)^k, n)}.$$

And by Lemma 5 we note that

$$dd_\alpha | n(\chi).$$

By the definition of  $n(d)$  we can easily see that

$$(n(d_\alpha))(d) = n(dd_\alpha).$$

Then we can rewrite the inside of the bracket of (3) as follows

$$(4) \quad \left\{ \sum_{\substack{\alpha \subset B(n) \\ (d_\alpha, f_\chi) = 1}} \mu(d_\alpha) \chi(d_\alpha) \right\} - \left\{ \sum_{\substack{d \mid n(\chi) \\ d \neq 1}} \frac{(d^k, n)}{d} \chi^{-1} \left( \frac{d^k}{(d^k, n)} \right) \right. \\ \left. \cdot \sum_{\substack{d = d'd_\alpha \\ d' \mid n(d_\alpha)(\chi) \\ \alpha \subset B(n) \\ \alpha \neq \emptyset \\ f_\chi \mid n(d_\alpha)}} \mu(d_\alpha) \sum_{\substack{\beta \subset B(n(d_\alpha)) \\ d_\beta \mid n(d)(\chi) \\ (d_\beta, f_\chi) = 1}} \mu(d_\beta) \chi(d_\beta) \right\}.$$

Here we can easily see that if  $\beta \subset B(n)$  and  $d_\beta \mid n(d)(\chi)$  then  $\beta \subset B(n(d_\alpha))$ . This shows that we may change  $B(n(d_\alpha))$  of the last term of (4) for  $B(n)$ . Moreover by Lemma 6 we see that  $d_\alpha \mid d$  implies that  $f_\chi \mid n(d_\alpha)$  and  $d' \mid n(d_\alpha)(\chi)$ . Therefore we may exclude these conditions of (4). Then we have

$$(4) = \left\{ \sum_{\substack{\alpha \subset B(n) \\ (d_\alpha, f_\chi) = 1}} \mu(d_\alpha) \chi(d_\alpha) \right\} - \left\{ \sum_{\substack{d \mid n(\chi) \\ d \neq 1}} \frac{(d^k, n)}{d} \chi^{-1} \left( \frac{d^k}{(d^k, n)} \right) \right. \\ \left. \cdot \sum_{\substack{\beta \subset B(n) \\ d_\beta \mid n(d)(\chi) \\ (d_\beta, f_\chi) = 1}} \mu(d_\beta) \chi(d_\beta) \sum_{\substack{d = d'd_\alpha \\ \alpha \subset B(n) \\ \alpha \neq \emptyset}} \mu(d_\alpha) \right\} \\ = \left\{ \sum_{\substack{\alpha \subset B(n) \\ (d_\alpha, f_\chi) = 1}} \mu(d_\alpha) \chi(d_\alpha) \right\} + \left\{ \sum_{\substack{d \mid n(\chi) \\ d \neq 1}} \frac{(d^k, n)}{d} \chi^{-1} \left( \frac{d^k}{(d^k, n)} \right) \sum_{\substack{\beta \subset B(n) \\ d_\beta \mid n(d)(\chi) \\ (d_\beta, f_\chi) = 1}} \mu(d_\beta) \chi(d_\beta) \right\} \\ = \sum_{d \mid n(\chi)} \frac{(d^k, n)}{d} \chi^{-1} \left( \frac{d^k}{(d^k, n)} \right) \sum_{\substack{\alpha \subset B(n) \\ (d_\alpha, f_\chi) = 1 \\ d_\alpha \mid n(d)(\chi)}} \mu(d_\alpha) \chi(d_\alpha).$$

which implies (II). Thus the proof of Theorem 1 is completed.

Let  $\mathbf{Q}(\sqrt{D})=K$  be a quadratic extension field of  $\mathbf{Q}$  with discriminant  $D$ . We denote by  $\left(\frac{D}{n}\right)$  or  $\chi_D(n)$  the Kronecker's symbol of  $K$ . Then  $\left(\frac{D}{\cdot}\right)$  is a primitive character modulo  $|D|$ .

REMARK 3. Conversely it is well-known that every primitive character of degree 2 is of such type.

Let  $h(D)$  be the class number of  $K=\mathbf{Q}(\sqrt{D})$  and  $2w_D$  be the number of the roots of unity in  $K$ . Then the following Lemma 7 is well-known.

**Lemma 7.** *Notation being as above. Then we have*

$$H_{\chi_D} = \begin{cases} 0 & \text{if } D > 0, \\ \frac{h(D)}{w_D} & \text{if } D < 0. \end{cases}$$

REMARK 4. It is also well-known that if  $\left(\frac{D}{-1}\right)=1$  then  $D > 0$  and if  $\left(\frac{D}{-1}\right)=-1$  then  $D < 0$ .

**Corollary 1.** *In the case  $k=2$  we have*

$$c_a^{(2)}(n) = \frac{n_0 - 1}{2} + \sum_{\substack{|D| \mid n \\ D < 0}} \left(\frac{D}{a}\right)^{-1} \frac{h(D)}{w_D} \sum_{d \mid n(\chi_D)} d \prod_{\substack{p \mid n(d)(\chi_D) \\ (p, D) = 1}} \left\{1 - \left(\frac{D}{p}\right)\right\},$$

where  $D$  runs over all the discriminants of the imaginary quadratic fields dividing  $n$ .

Proof. By the definition of  $n(\chi_D)$  we can easily see that if  $d$  divides  $n(\chi_D)$  then  $d^2$  divides  $n$ . It follows

$$\frac{(d^2, n)}{d} = d \quad \text{and} \quad \frac{(d^2, n)}{d^2} = 1.$$

Therefore by Remark 3, Remark 4, Lemma 2, Lemma 7 and the above facts, Theorem 1 implies our Corollary.

Our Corollary in the case  $a=1$  and  $n=\text{prime}$  is obtained by T. Honda in [2].

**Corollary 2.** *If  $k=2$  then  $c_1^{(2)}(n) \geq 0$ . Moreover  $c_1^{(2)}(n)=0$ , if and only if  $n$  is of the following type*

$$n = p_1 \cdots p_j \quad \text{or} \quad 2p_1 \cdots p_j,$$

where  $p_1, \dots, p_j$  are distinct primes each of which is congruent to 1 modulo 4.

Proof. The first assertion is obvious from Corollary 1. We shall prove the second assertion. If  $c_1^{(2)}(n)=0$  then  $n$  must be square-free, because if  $n$  is not square-free then  $n_0 > 1$ , which implies  $c_1^{(2)}(n) > 0$ . Consequently we have by Corollary 1

$$c_1^{(2)}(n) = \sum_{|D| \mid n} \frac{h(D)}{w_D}.$$

If there exists some  $p$  such that  $p \mid n$  and  $p \equiv 3 \pmod{4}$ , then  $-p$  is the discriminant of  $\mathbf{Q}(\sqrt{-p})$ . This shows

$$c_1^{(2)}(n) \geq \frac{h(-p)}{w_{-p}} > 0.$$

Thus  $n$  must be an integer of such type as in our Corollary. The converse is clear.

**Corollary 3.** *If  $k$  is an odd integer, then we have*

$$c_a^{(k)}(n) = \frac{n_0 - 1}{2},$$

therefore  $|c_a^{(k)}(n)| < n^{(k-1)/k}$ .

Proof. Let  $\chi$  be any character modulo  $n$  of degree  $k$ . Then we have

$$\chi(-1)^2 = \chi((-1)^2) = 1$$

and

$$\chi(-1)^k = 1.$$

This shows  $\chi(-1) = 1$ . Therefore by Lemma 2 we have  $H_\chi = 0$ . This shows the first assertion of our Corollary by Theorem 1. We can immediately obtain the second assertion by a simple calculation.

**REMARK 5.**  $c_1^{(k)}(n)$  is not always non-negative for even  $k > 2$ . For example  $c_1^{(4)}(29) = -2$ . (See the table of at the end of the section 5.)

#### 4. Proof of Conjecture 1

Let  $\chi$  be a primitive character modulo  $f_\chi$ . Then we define the Dirichlet's  $L$ -function by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

We denote by  $G(\chi)$  the Gauss's sum with respect to  $\chi$ , i.e.,

$$G(\chi) = \sum_{a=1}^{f_\chi} \chi(a) \zeta^a,$$

where  $\zeta = \exp(2\pi i/f_\chi)$ . Then the following two lemmas are well-known. (See Hasse [1] and Prachar [3]).

**Lemma 8.**

$$|L(1, \chi)| < 3 \log f_\chi.$$

**Lemma 9.**

$$L(1, \chi) = \frac{\pi i G(\chi)}{f_\chi^2} \sum_{a=1}^{f_\chi} \bar{\chi}(a) a.$$

Moreover

$$G(\chi)G(\bar{\chi}) = \chi(-1)f_\chi,$$

in particular

$$|G(\chi)| = \sqrt{f_\chi}.$$

**Lemma 10.**

$$|H_\chi| < \sqrt{f_\chi} \log f_\chi.$$

Proof. By Lemma 8 and Lemma 9 we have

$$\begin{aligned} |H_\chi| &= \left| \frac{1}{f_\chi} \cdot \frac{L(1, \bar{\chi}) f_\chi^2}{\pi i G(\bar{\chi})} \right| \\ &< \frac{f_\chi}{|G(\bar{\chi})|} \log f_\chi = \sqrt{f_\chi} \log f_\chi. \end{aligned}$$

It is obvious that  $f_{\bar{\chi}}$  is equal to  $f_\chi$ . This completes the proof.

We denote by  $\delta(n)$  the number of prime divisors of  $n$ .

**Lemma 11.** *For any positive number  $\varepsilon$  and a given positive constant  $A$  we have*

$$A^{\delta(n)} = O(n^\varepsilon),$$

where  $O$  denotes the Landau's large  $O$ -symbol.

Proof. We may suppose  $A > 1$ . Let  $p_0$  be a sufficiently large prime number such that

$$\frac{\log A}{\log p_0} < \varepsilon.$$

We denote by  $\delta_0$  the number of primes which are less than  $p_0$  and by  $\delta'(n)$  the number of prime divisors of  $n$  each of which is not smaller than  $p_0$ . Then we can easily see that

$$\delta(n) \leq \delta'(n) + \delta_0.$$

By the definition of  $\delta'(n)$  we have

$$p_0^{\delta'(n)} \leq n.$$

Therefore we have

$$\delta'(n) \leq \frac{\log n}{\log p_0}.$$

From this we get

$$\begin{aligned} A^{\delta(n)} &\leq A^{\delta'(n)+\delta_0} = A^{\delta_0} A^{\delta'(n)} \\ &= A^{\delta_0} n^{\log n} A^{\delta'(n)} = A^{\delta_0} n^{\delta'(n) \log A / \log n} \\ &\leq A^{\delta_0} n^{(\log n / \log p_0) * (\log A / \log n)} \leq A^{\delta_0} n^{\varepsilon}. \end{aligned}$$

This completes the proof.

**Lemma 12.** *For any positive number  $\varepsilon$  we have*

$$\sum_{d|n} 1 = O(n^\varepsilon).$$

Proof. See Prachar [3]-I-Satz 5.2

Now we shall prove Conjecture 1.

**Theorem 2.** *For any positive number  $\varepsilon$  and a fixed positive integer  $k$  we have*

$$c_a^{(k)}(n) = O(n^{(k-1)/k + \varepsilon}).$$

Proof. By Theorem 1 we have

$$|c_a^{(k)}(n)| \leq \frac{n_0 - 1}{2} + \sum_{f_\chi|n} |H_\chi| \sum_{d|n(\chi)} \frac{(d^k, n)}{d} \prod_{\substack{p|n(d)(\chi) \\ (p, f_\chi)=1}} |1 - \chi(p)|.$$

We have already known that

$$n_0 \leq n^{(k-1)/k}.$$

Therefore we shall show that

$$\sum_{f_\chi|n} |H_\chi| \sum_{d|n(\chi)} \frac{(d^k, n)}{d} \prod_{\substack{p|n(d)(\chi) \\ (p, f_\chi)=1}} |1 - \chi(p)| = O(n^{(k-1)/k + \varepsilon}).$$

First we get by Lemma 11

$$\prod_{\substack{p|n(d)(\chi) \\ (p, f_\chi)=1}} |1 - \chi(p)| \leq \prod_{p|n} 2 = 2^{\delta(n)} = O(n^\varepsilon).$$

Next we get

$$\sum_{f_\chi|n} 1 \leq \prod_{p|n} \left( \sum_{\substack{\chi \\ f_\chi=p^\alpha}} 1 \right).$$

But by Lemma 3 we know that

$$\sum_{\substack{\chi \\ f_\chi = p^\alpha}} 1 < A, \quad \text{for some positive constant } A.$$

Hence by Lemma 11 we also get

$$\sum_{\substack{\chi \\ f_\chi \mid n}} 1 < A^{\delta(n)} = O(n^\varepsilon).$$

Lastly we shall show that

$$\left( |H_x| \sum_{d \mid n(\chi)} \frac{(d^k, n)}{d} \right) / n^{(k-1)/k} = O(n^\varepsilon)$$

We transform this into

$$\left( |H_x| \sum_{d \mid n(\chi)} \frac{(d^k, n)}{d} \right) / n^{(k-1)/k} = \frac{|H_x|}{f_\chi^{(k-1)/k}} \cdot \sum_{d \mid n(\chi)} \left( \frac{(d^k, n)}{d} \right) / \left( \frac{n}{f_\chi} \right)^{(k-1)/k}.$$

Then we have by Lemma 10

$$|H_x| / f_\chi^{(k-1)/k} \leq (f_\chi^{1/2} / f_\chi^{(k-1)/k}) \log f_\chi \leq \log f_\chi.$$

Moreover by Remark 2 we can easily see that

$$\left( \frac{(d^k, n)}{d} \right) / \left( \frac{n}{f_\chi} \right)^{(k-1)/k} < 1.$$

From these and by Lemma 12 we have

$$\begin{aligned} \left( |H_x| \sum_{d \mid n(\chi)} \frac{(d^k, n)}{d} \right) / n^{(k-1)/k} &\leq \sum_{d \mid n(\chi)} \log n \\ &< \log n \sum_{d \mid n} 1 \\ &= O(n^\varepsilon). \end{aligned}$$

This completes the proof of our Theorem.

## 5. Number theoretic properties of some $c_a^{(k)}(n)$ .

**Lemma 13.** *Let  $k$  be a positive integer and  $p$  be a prime number which is prime to  $k$ . We denote by  $k_0$  the greatest common divisor of  $k$  and  $p-1$ . Then we have*

$$N^{(k)}(x, p) = N^{(k_0)}(x, p).$$

Proof. If  $x \equiv 0 \pmod{p}$  then the lemma is trivial. Hence we assume  $x \not\equiv 0 \pmod{p}$ . Consider the following sequence of groups and homomorphisms

$$\{1\} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^{\times(p-1)/k_0} \xrightarrow{g_1} (\mathbb{Z}/p\mathbb{Z})^{\times} \xrightarrow{g_2} (\mathbb{Z}/p\mathbb{Z})^{\times k_0} \xrightarrow{g_3} (\mathbb{Z}/p\mathbb{Z})^{\times k_0} \longrightarrow \{1\},$$

where we define the homomorphisms  $g_1$ ,  $g_2$  and  $g_3$  as follows

$$\begin{aligned} g_1(a) &= a^{\frac{p-1}{k_0}} \in (\mathbb{Z}/p\mathbb{Z})^{\times(p-1)/k_0}, \\ g_2(a) &= a^{k_0} \in (\mathbb{Z}/p\mathbb{Z})^{\times k_0}, \\ g_3(a) &= a^{k_0} \in (\mathbb{Z}/p\mathbb{Z})^{\times k_0}. \end{aligned}$$

By the definition of  $k_0$ , we see that  $k/k_0$  is prime to  $(p-1)/k_0$ . This shows that  $g_3$  is an isomorphism and the above sequence is exact. By the definition of  $N^{(k_0)}(x, p)$  and  $N^{(k)}(x, p)$  we see that  $N^{(k_0)}(x, p)$  is not zero if and only if  $x \in \text{Im}(g_2) = (\mathbb{Z}/p\mathbb{Z})^{\times k_0}$  and  $N^{(k)}(x, p)$  is not zero if and only if  $x \in \text{Im}(g_3 \circ g_2) = (\mathbb{Z}/p\mathbb{Z})^{\times k_0}$ . Therefore  $N^{(k_0)}(x, p)$  is not zero if and only if so is  $N^{(k)}(x, p)$ . If  $x \in (\mathbb{Z}/p\mathbb{Z})^{\times k_0}$  then  $N^{(k_0)}(x, p) = \#\text{Ker}(g_2) = \#\text{Ker}(g_3 \circ g_2) = N^{(k)}(x, p)$ . Thus Lemma 13 is proved.

**Proposition 2.** *Let  $p_1, \dots, p_j$  be distinct primes each of which is prime to  $k$  and  $k_i$  be the greatest common divisor of  $k$  and  $p_i - 1$ . If we denote by  $k_0$  the least common multiple of  $k_1, \dots, k_j$ , then*

$$c_a^{(k)}(p_1 \cdots p_j) = c_a^{(k_0)}(p_1 \cdots p_j).$$

**Proof.** By Lemma 13 it is obvious that

$$N^{(k)}(x, p) = N^{(k_0)}(x, p) = N^{(k_0)}(x, p).$$

Then by Lemma 1 we have

$$N^{(k)}(x, p_1 \cdots p_j) = N^{(k_0)}(x, p_1 \cdots p_j).$$

On the other hand we have already shown in the proof of Proposition 1 that

$$c_a^{(k)}(n) = \frac{n-1}{2} - \frac{1}{n} \sum_{i=1}^{n-1} N^{(k)}(a^{-1}x, n),$$

where we consider  $a^{-1}x$  in  $(\mathbb{Z}/n\mathbb{Z})^{\times}$ . Therefore we can immediately obtain the lemma.

**Lemma 14.** *Let  $p$  be a prime such that*

$$p-1 \equiv 0 \pmod{2k}$$

*and  $\chi$  be a character of modulo  $p$  of degree  $k$ , then*

$$\chi(-1) = 1.$$

**Proof.** If we put  $p-1=2mk$  with a positive integer  $m$ , then the order of  $-1$  in  $(\mathbb{Z}/p\mathbb{Z})^{\times}$  is  $mk$ . Therefore there exists some  $x_0 \in (\mathbb{Z}/p\mathbb{Z})$  such that

$$x_0^{m^k} \equiv -1 \pmod{p},$$

which implies  $\chi(-1) = \chi(x_0^m)^k = 1$ .

**Proposition 3.** *Let  $p_1, \dots, p_j$  be distinct primes each of which is prime to  $k$  and congruent to 1 modulo  $2k$ , then*

$$c_a^{(k)}(p_1 \cdots p_j) = 0.$$

Proof. We put  $n = p_1 \cdots p_j$ . Let  $\chi$  be any character of conductor  $f_\chi | n$ , then by the decomposition (1) in §2 of  $\chi$  and Lemma 14 we see that  $\chi(-1) = 1$ . Therefore by Lemma 2 and Theorem 1 we can immediately obtain our Proposition.

In the case  $k=2$ , we have obtained the very beautiful formula for  $c_a^{(2)}(n)$  in corollary 2. But when  $k$  is an even integer  $> 2$ ,  $c_a^{(k)}(n)$  is more complicated. From now on till the end of the this section we shall only consider the case  $k=4$  and  $n=p$ , where  $p$  is a prime. If  $p=2$ , then  $c_a^{(4)}(2)=0$  and there is nothing to say. If  $p \equiv 3 \pmod{4}$ , then  $c_a^{(4)}(p) = c_a^{(2)}(p)$  by Proposition 2. Further if  $p \equiv 1 \pmod{8}$ , then  $c_a^{(4)}(p)=0$  by Proposition 3. Therefore we may confine ourselves to the cases  $p \equiv 5 \pmod{8}$ .

Let  $p$  be a prime which is congruent to 5 modulo 8. Then the unit group  $(\mathbb{Z}/p\mathbb{Z})^\times$  of the residue ring  $\mathbb{Z}/p\mathbb{Z}$  is a cyclic group of order  $p-1$  which is divisible by 4. We denote by  $H$  (respectively  $H_0$ ) the unique subgroup of  $(\mathbb{Z}/p\mathbb{Z})^\times$  of index 4 (respectively 2). Let  $K$  be the  $p$ -th cyclotomic field i.e.,  $K = \mathbb{Q}(\zeta)$ , where  $\zeta = \exp\left(\frac{2\pi i}{p}\right)$ . Then there exists the subfield  $L$  (respectively  $L_0$ ) corresponding to the group  $H$  (respectively  $H_0$ ). As the order of  $-1$  is 2,  $H$  does not contain  $-1$  but  $H_0$  contains it. This shows that  $L$  is a totally imaginary field and  $L_0$  is the maximal totally real subfield of  $L$ . Hence we obtain the following diagram

$$\begin{array}{ccc} K = \mathbb{Q}(\zeta) & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ L & \longrightarrow & H(\nexists -1) \\ \downarrow & & \downarrow \\ L_0 & \longrightarrow & H_0(\exists -1) \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^\times. \end{array}$$

Hereafter till the end of the this section we shall use the following notations.

$$\zeta = \exp\left(\frac{2\pi i}{p}\right)$$

$h$  = the class number of  $L$

$h_0$  = the class number of  $L_0$

$$h^* = h/h_0$$

$E$  = the unit group of  $L$

$E_0$  = the unit group of  $L_0$

$w$  = the number of the roots of unity of  $L$

By the condition on  $p$  we can easily see that the element 2 is not a quadratic residue of modulo  $p$ . This shows that the group  $(\mathbb{Z}/p\mathbb{Z})^\times/H$  is generated by the class represented by 2. We shall denote by  $\chi^{(j)}$  ( $j=0, 1, 2, 3$ ) the character of  $(\mathbb{Z}/p\mathbb{Z})^\times/H$  which takes value  $\sqrt{-1}^j$  at the class 2 mod  $H$ . From these characters we obtain the characters modulo  $p$  in the sense of section 2 and we also denote them by  $\chi^{(j)}$  ( $j=0, 1, 2, 3$ ). We can easily see that these characters except  $\chi^{(0)}$  have the conductor  $p$ . Then the group of characters  $\{\chi^{(j)} | j=0, 1, 2, 3\}$  corresponds to  $L$  and  $\{\chi^{(0)}, \chi^{(2)}\}$  corresponds to  $L_0$ . Now we quote the following formula for  $h^*$  from Hasse [1].

**Lemma 15.** *Let  $E'$  be the group generated by  $E_0$  and the roots of unity contained in  $L$ . Then we have*

$$h^* = Qw \prod_{j=1,3} \frac{1}{2p} \left( \sum_{x=1}^{p-1} -\chi^{(j)}(x)x \right),$$

where  $Q$  is defined by  $Q=[E; E']$ . In our case we can easily see  $Q=1$ .

Proof. See Hasse [1] III-(\*).

**Theorem 3.** *If we use the above notation, then we have*

$$h^* = \frac{w}{4} \left\{ \left( \frac{c_1^{(4)}(p)}{2} \right)^2 + \left( \frac{c_2^{(4)}(p)}{2} \right)^2 \right\}.$$

Proof. We put

$$\frac{1}{p} \sum_{x=1}^{p-1} \chi^{(1)}(x)x = a + bi \quad a, b \in \mathbb{Q}.$$

Then we have

$$\frac{1}{p} \sum_{x=1}^{p-1} \chi^{(3)}(x)x = a - bi.$$

We shall prove that

$$(7) \quad a = -\frac{c_1^{(4)}(p)}{2},$$

$$(8) \quad b = -\frac{c_2^{(4)}(p)}{2}.$$

By the definition of  $a$  we get

$$\begin{aligned}
a &= \frac{1}{p} \left\{ \sum_{\substack{x=1 \\ x \equiv y^4 \pmod{p}}}^{p-1} x - \sum_{\substack{x=1 \\ x \equiv y_1^2 \pmod{p} \\ x \not\equiv y_2^2 \pmod{p}}}^{p-1} x \right\} \\
&= \frac{1}{p} \left\{ 2 \sum_{\substack{x=1 \\ x \equiv y^4 \pmod{p}}}^{p-1} x - \sum_{\substack{x=1 \\ x \equiv y^2 \pmod{p}}}^{p-1} x \right\}.
\end{aligned}$$

As  $p \equiv 1 \pmod{4}$ , if  $x \equiv y^2 \pmod{p}$  then  $-x \equiv y'^2 \pmod{p}$  for some  $y' \in \mathbb{Z}/p\mathbb{Z}$ . From this we get

$$(9) \quad \sum_{\substack{x=1 \\ x \equiv y^2 \pmod{p}}}^{p-1} x = \frac{p(p-1)}{4}.$$

On the other hand we have by the definition of  $c_1^{(4)}(p)$

$$(10) \quad c_1^{(4)}(p) = \frac{p-1}{2} - \frac{4}{p} \sum_{\substack{x=1 \\ x \equiv y^4 \pmod{p}}}^{p-1} x.$$

By (9) and (10) we have

$$\begin{aligned}
a &= \left( \frac{p-1}{4} - \frac{c_1^{(4)}(p)}{2} \right) - \frac{p-1}{4} \\
&= -\frac{c_1^{(4)}(p)}{2}.
\end{aligned}$$

Thus we obtain the formula (7). Next we shall prove (8). By the definition of  $b$  we have

$$\begin{aligned}
b &= \left\{ \frac{1}{p} \sum_{\substack{x=1 \\ x \equiv 2y^4 \pmod{p}}}^{p-1} x - \sum_{\substack{x=1 \\ x \equiv 2y_1^2 \\ x \not\equiv 2y_2^2 \pmod{p}}}^{p-1} x \right\} \\
&= \frac{1}{p} \left\{ 2 \sum_{\substack{x=1 \\ x \equiv 2y^4 \pmod{p}}}^{p-1} x - \sum_{\substack{x=1 \\ y \equiv 2y^2 \pmod{p}}}^{p-1} x \right\} \\
&= \frac{1}{p} \left\{ 2 \sum_{\substack{x=1 \\ x \equiv 2y^4 \pmod{p}}}^{p-1} x - \frac{p(p-1)}{4} \right\}.
\end{aligned}$$

On the other hand by the definition of  $c_2^{(4)}(p)$  we have also

$$c_2^{(4)}(p) = \frac{p-1}{2} - \frac{4}{p} \sum_{\substack{x=1 \\ x \equiv 2y^4 \pmod{p}}}^{p-1} x.$$

Therefore we obtain

$$\begin{aligned}
 b &= \left( \frac{p-1}{4} - \frac{c_2^{(4)}(p)}{2} \right) - \frac{p-1}{4} \\
 &= -\frac{c_2^{(2)}(p)}{2}.
 \end{aligned}$$

Thus we have completed the proof of our Theorem.

REMARK 6. We can easily see that

$$\begin{aligned}
 w &= 10 && \text{if } p = 5, \\
 w &= 2 && \text{otherwise.}
 \end{aligned}$$

For the even  $k > 2$  it can be considered that  $c_a^{(k)}(p)$ 's have similar relations to some relative class numbers. But for the composite  $n$ 's such relations are more complicated. We shall give the table of  $h^*$ ,  $c_1^{(4)}(p)$  and  $c_2^{(4)}(p)$ .

Table ( $p \equiv 5(8)$ ,  $p < 500$ )

| $p$ | $c_1^{(4)}(p)$ | $c_2^{(4)}(p)$ | $h^*$ |
|-----|----------------|----------------|-------|
| 5   | 6/5            | 2/5            | 1     |
| 13  | 2              | 2              | 1     |
| 29  | -2             | 2              | 1     |
| 37  | 2              | -2             | 1     |
| 53  | -2             | -2             | 1     |
| 61  | 2              | -2             | 1     |
| 101 | -6             | 2              | 5     |
| 109 | 10             | 6              | 17    |
| 149 | 6              | 6              | 9     |
| 157 | 2              | 6              | 5     |
| 173 | -6             | -2             | 5     |
| 181 | 14             | 2              | 25    |
| 197 | -2             | -6             | 5     |
| 229 | 6              | 10             | 17    |
| 269 | 10             | -2             | 13    |
| 277 | -6             | 10             | 17    |
| 293 | 6              | -6             | 9     |
| 317 | 2              | 10             | 13    |
| 349 | -6             | -2             | 5     |
| 389 | 18             | 2              | 41    |
| 397 | 2              | -10            | 13    |
| 421 | 2              | 14             | 25    |
| 461 | -2             | -14            | 25    |

## 6. An afterthought

We shall give an another elementary proof of Corollary 3.

**Proposition 4.** *If the following congruence equation has a solution*

$$(10) \quad x^k \equiv -1 \pmod{n},$$

then

$$c_a^{(k)}(n) = \frac{n_0^{(k)} - 1}{2}.$$

**Proof.** If (10) has a solution, then it is clear that

$$N^{(k)}(x, n) = N^{(k)}(-x, n) = N^{(k)}(n-x, n).$$

Hence by the defintion of  $c_a^{(k)}(n)$  we have

$$\begin{aligned} c_a^{(k)}(n) &= \frac{n-1}{2} - \frac{1}{n} \sum_{x=1}^{n-1} N^{(k)}(a^{-1}x, n)x \\ &= \frac{n-1}{2} - \frac{1}{2n} \sum_{x=1}^{n-1} \{N^{(k)}(a^{-1}x, n)x + N^{(k)}(a^{-1}(n-x), n)(n-x)\} \\ &= \frac{n-1}{2} - \frac{1}{2n} \sum_{x=1}^{n-1} nN^{(k)}(a^{-1}x, n), \end{aligned}$$

where we consider  $a^{-1}x$  in  $\mathbb{Z}/n\mathbb{Z}$ . But we can easily see that

$$\sum_{x=0}^{n-1} N^{(k)}(a^{-1}x, n) = n.$$

From this it follows that

$$\begin{aligned} c_a^{(k)}(n) &= \frac{n-1}{2} - \frac{1}{2}(n - N^{(k)}(0, n)) \\ &= \frac{N^{(k)}(0, n) - 1}{2}. \end{aligned}$$

But by a simple computation we get

$$N^{(k)}(0, n) = n_0^{(k)}.$$

Thus we obtain Proposition 4.

Considering the definition of  $c_a^{(k)}(n)$ , if  $ax^k \equiv 0 \pmod{n}$  then  $\left[ \frac{ax^k}{n} \right] = \frac{ax^k}{n}$ , but we suppose that  $\left[ \frac{ax^k}{n} \right]$  is approximately  $\frac{ax^k}{n} - \frac{1}{2}$ . Therefore  $\frac{n_0^{(k)} - 1}{2}$  can be considered the known error term. From this point of view we had better to

consider that  $d_a^{(k)}(n) = c_a^{(k)}(n) - \frac{n_0^{(k)} - 1}{2}$  is the essential error term. The proof of Theorem 2 shows that the order of  $d_a^{(k)}(n)$  is less than  $n^{((k-1)/k) + \varepsilon}$  for any  $\varepsilon > 0$ . The Corollary 2 is true with slight modification of  $d_a^{(k)}(n)$ .

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### References

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