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ON THE BOUNDARY BEHAVIOUR OF HARMONIC MAPS

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Introduction

In the classical theory of functions, one can find many theorems on the boundary behaviour of meromorphic functions defined on the unit disc. Above all, the theorem of Plessner is well-known. A meromorphic function on a domain D of a complex plane is an analytic map of D into a Riemann sphere, and it is natural to attempt to generalize the theorem of Plessner to an analytic map between Riemann surfaces. An immediate generalization is rejected since the angular limit is meaningless on Riemann surfaces. The first successful contribution to this problem is due to Constantinescu-Cornea [4]. Their argument based essentially on the notion of fine filters converging to minimal boundary points of the Martin compactification, which is defined by L. Naïm [11] originally. In their book [5], it is given in a course of systematic development of compactifications. Among various compactifications, the Martin's and the Wiener's are of great importance. The boundary behaviour of an analytic map φ from a hyperbolic Riemann surface into another R' is described simply in the Wiener compactification, that is, at each Wiener boundary point $\tilde{x}$ either φ has a limit or the image of every neighbourhood of $\tilde{x}$ is dense in R'.

In this paper, we shall give a new proof of the theorem of Plessner in accordance with the following idea: *making use of the relations between the Wiener and the Martin boundaries, we may transmit the results obtained in the Wiener compactification to the Martin's*. At the same time, we may proceed with above program in an axiomatic setting. In fact, we can consider a harmonic map between harmonic spaces satisfying the axioms of Brelot and obtain a theorem of Plessner type.

In §1, we recall the definitions, give the notations which will be used later and list up the hypotheses assumed in this paper. §2 is devoted to lemmas used in the following. The study of boundary behaviour of a harmonic map leads us into the investigation of some cluster sets. The properties of cluster sets are investigated in §3, and Lemma A and Lemma B are fundamental tools for our study. The theorem of Plessner type mentioned above is stated in §4 with the theorem of Fatou type. In Corollary 2 to Theorem 5, a relation between a fine cluster set and a different sort of cluster set are considered. In the classical case of the unit disc, the latter is a certain tangential one. And it gives a result analogous to a result of Bagemihl [1] concerning an angular cluster set and a horocyclic cluster set. The author wishes to express his hearty thanks to Prof. F-Y. Maeda for his valuable remarks and advices.

1. Preliminaries

Let X be a harmonic space in the sense of Brelot[1], that is, X is a locally compact connected Hausdorff space satisfying the axioms 1, 2 and 3 of Brelot. We assume that X is *non-compact* and *has a countable base of open sets.*

The family of harmonic spaces possessing positive potentials (resp. positive harmonic functions) will be denoted by $\mathcal{P}$ (resp. $\mathcal{H}$). Thus, for example, $X\in\mathcal{P}\cup\mathcal{H}$ means that there exists a positive superharmonic function on X.

We assume $X\in\mathcal{P}$ and the *axiom of proportionality*: for every $a\in X$, potentials with a single point support $\{a\}$ are all proportional.

As in [3] and [10], we define the Martin compactification X^M of X, the Martin boundary $\Delta=X^M-X$ and the minimal boundary Δ_1 of Δ.

Let us denote by $\mathcal{W}(X)$ the family of all Wiener functions on X[2]. We assume that *the constant functions are Wiener functions.*

The Wiener compactification X^W of X is a compact Hausdorff space containing X as a dense subset and all Wiener functions are extended continuously on X^W and separate points of X^W. We know that the Wiener compactification and the Martin compactification are resolutive. Consequently, we may consider harmonic measures on each boundary which are denoted by ω^W and ω^M respectively.

The harmonic boundary of X^W will play an important role in our investigation. It is denoted by Γ^W.

As in the classical case, where X is a hyperbolic Riemann surface[3], X^M is considered as a quotient space of X^W, that is, there exists a continuous map π of X^W onto X^M mapping each point of X onto itself.

We define

$$\Delta_1^* = \{x\in\Delta_1;\ \pi^{-1}(x)\cap\Gamma^W\neq\emptyset\}\ .$$

Then, $\Delta_1^*=\pi(\Gamma^W)\cap\Delta_1$ and $\omega^M(\Delta-\Delta_1^*)=0$.

Next, let X' be another harmonic space. We assume that X' *has a countable basis of open sets,* $X'\in\mathcal{P}\cup\mathcal{H}$ and *constant functions are Wiener functions on* X'.

Here, we shall give a remark on resolutivity of compactification. In order to include a compact case, we define a compactification X'^* of X' to be resolutive if for every bounded continuous function f on X'^* the restriction of f to X' is a

1) Cf. [2]. We note that X may be compact.
2) Cf. [6].
3) Cf. [5].

Wiener function. When X' is compact, X' itself is considered as a resolutive compactification, since all bounded continuous functions on X' are Wiener functions. When X' is non-compact and $X' \in \mathcal{P}$, the present definition is equivalent to the original one, that is, every bounded continuous function on $X'^* - X'$ is resolutive with respect to the Dirichlet problem[4].

Let φ be a harmonic map[4] of X into X'. We define some cluster sets at a minimal Martin boundary point: for $x \in \Delta_1^*$

$\tilde{\varphi}(x) = \cap \{\overline{\varphi(\tilde{U} \cap X)};\ \tilde{U}$ is an open neighbourhood of $\pi^{-1}(x) \cap \Gamma^W$ in $X^W\}$
and for $x \in \Delta_1$

$$\hat{\varphi}(x) = \cap \{\overline{\varphi(E \cap X)};\ X - E \text{ is thin at } x^{5)}\} ,$$

where closures are taken in a compactification X'^*.

Summing up the hypotheses, we assume

X: axioms of Brelot 1, 2 and 3; axiom of proportionality; countable basis of open sets; non-compactness; $X \in \mathcal{P}$; $1 \in \mathcal{W}(X)$.

X': axioms of Brelot 1, 2 and 3; countable basis of open sets; $X' \in \mathcal{P} \cup \mathcal{H}$; $1 \in \mathcal{W}(X')$.

2. Auxiliary lemmas

Let φ be a harmonic map of X into X', X^W be the Wiener compactification of X and X'^* be an arbitrary resolutive compactification of X'. For $\tilde{x} \in \Delta^W = X^W - X$, we define

$\varphi^*(\tilde{x}) = \cap \{\overline{\varphi(\tilde{U} \cap X)};\ \tilde{U}$ is an open neighbourhood of $\tilde{x}$ in $X^W\}$,
where the closure is taken in X'^*, and

$$\Delta_P = \{\tilde{x} \in \Delta^W;\ \varphi^*(\tilde{x}) = X'^*\} ,$$

and

$$\Delta_F = \{\tilde{x} \in \Delta^W;\ \varphi^*(\tilde{x}) \text{ consists of a single point}\} .$$

We remark that φ is extended continuously on Δ_F.

Lemma 1. *Using above notations we have*

a) $\Delta^W = \Delta_P \cup \Delta_F$,

b) Δ_P *is an open and closed subset of* Γ^W.

c) *if* $\tilde{U}$ *is an open subset of* X^W *and* $\tilde{U} \cap \Delta_P \neq \emptyset$, *then there exists a component of* $\tilde{U} \cap X$ *on which* φ *is not a Fatou map.*

The proof is carried out quite in the same way as [6] (p. 54).

In [9], the author investigated the relations between the Wiener and the

4) Cf. [6].
5) Cf. [8].

Martin boundary of a hyperbolic Riemann surface. Most parts of the results obtained there are readily extended to the case of a harmonic space. The following lemma will be of use in our present study.

Lemma 2. *For $f\in\mathcal{W}(X)$ we can find a set $\tilde{N}\subset\Delta^W$ of $d\omega^W$-harmonic measure zero such that*

$$\lim_{a\to\tilde{x}} f(a) = \text{fine}\lim_{a\to\pi(\tilde{x})} f(a)^{6)} \qquad \textit{for every } \tilde{x}\in\Delta^W-\tilde{N}\,.$$

For, $f=h_f^X+q^{7)}$, where q is a Wiener potential[7]. We may assume $f\geq 0$. Putting $u_0=h_1^X$, we have $h_f^X=v_B+v_S$, where v_B is u_0-quasi-bounded and v_S is u_0-singular, i.e., $v_B=\lim_{n\to\infty}(h_f^X\wedge nu_0)$ and $v_S\wedge u_0=0$[8], the latter is equivalent to the fact that inf (v_S, u_0) is a potential. *A* u_0-quasi-bounded harmonic function v_B has a fine limit $g(x)$ at $d\omega^M$-almost every boundary point $x\in\Delta$ and the limit $\tilde{g}(\tilde{x})$ of v_B at $\tilde{x}\in\Delta^W$ coincides with $g[\pi(\tilde{x})]$ $d\omega^W$-almost everywhere. We know that a Wiener potential has the limit zero at every point $\tilde{x}\in\Gamma^W$ and has the fine limit zero at $d\omega^M$-almost every point of Δ. Combining these results we can derive the lemma.

We shall remark that the limits in Lemma 2 may be considered on a subset of Γ^W since Γ^W is the carrier of the harmonic measure ω^W.

Lemma 3. *Let X'^* be a metrizable compactification of X' and G' be an open set of X'^*. Then the following sets are Borel sets*:

$$\begin{aligned} A &= \{x\in\Delta_1;\ \phi(x)\subset G'\}\ , \\ B &= \{x\in\Delta_1;\ \phi(x)\cap G'=\emptyset\}\ , \\ C &= \{x\in\Delta_1;\ \phi(x)\cap G'\neq\emptyset\}\ , \\ D &= \{x\in\Delta_1;\ \phi(x) \textit{ contains at least two points}\}\ . \end{aligned}$$

Proof. Without loss of generality, we may assume $G'\neq X'^*$. Set

$$F'_n=\{x'\in X'^*;\ \rho(x', X'^*-G')\geq 1/n\}\ ,$$

where ρ denotes the distance in the metric of X'^*. F'_n is compact and

$$G'=\bigcup_{n=1}^{\infty}F'_n\,.$$

It is readily seen that

$$\{x\in\Delta_1;\ \phi(x)\subset G'\}=\bigcup_{n=1}^{\infty}\{x\in\Delta_1;\ X-\varphi^{-1}(F'_n\cap X') \text{ is thin at } x\}\ .$$

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6) For the notion of fine limit, see [8].

7) For the notation h_f^X and the definition of a Wiener potential, see [6].

8) $u\wedge v$ denotes the greatest harmonic minorant of u and v.

Since $X-\varphi^{-1}(F'_n \cap X')$ is open in X, the set

$$\{x\in\Delta_1;\ X-\varphi^{-1}(F'_n\cap X') \text{ is thin at } x\}$$

is a Borel subset of Δ_1[9]. We know that Δ_1 is a G_δ-set. Thus A is a Borel set.

If F' is a closed set of X'^*, then there exists a decreasing sequence $\{G'_n\}$ of open sets such that $F'=\bigcap_{n=1}^{\infty} G'_n$. Then, we have

$$\{x\in\Delta_1;\ \phi(x)\subset F'\} = \bigcap_{n=1}^{\infty}\{x\in\Delta_1;\ \phi(x)\subset G'_n\}$$

and we conclude that

$$\{x\in\Delta_1;\ \phi(x)\subset F'\}$$

is a Borel set. Thus,

$$B=\{x\in\Delta_1;\ \phi(x)\subset X'^*-G'\} \quad\text{and}\quad C=\Delta_1-B$$

are Borel sets.

Finally, let $\{U'_n\}$ be a countable basis of open sets for X'^*. We consider pairs of indices (m, n) such that $\overline{U'_m}\subset U'_n$. Then,

$$D=\bigcup_{(m,n)}\{x\in\Delta_1;\ \phi(x)\cap U'_m\neq\emptyset,\ \phi(x)\cap[X'^*-\overline{U'_n}]\neq\emptyset\}$$

is a Borel set.

3. Some results on cluster sets

In the following, let X'^* be a metrizable and resolutive compactification of X'. We define

$$\hat{P}=\{x\in\Delta_1;\ \phi(x)=X'^*\}\ ,$$
$$\hat{F}=\{x\in\Delta_1;\ \phi(x) \text{ consists of a single point}\}$$

and

$$P^*=\{x\in\Delta_1^*;\ \tilde{\phi}(x)=X'^*\}\ .$$

It is readily seen that $\hat{P}$ and $\hat{F}$ are Borel sets and $\pi(\Delta_P)\subset P^*$.

Lemma A. *Let A be a $d\,\omega^M$-measurable subset of Δ, G' be an open set of X'^* and f' be finite continuous function on X'^* whose carrier is contained in G'. If $\overline{G'}\cap\overline{\phi(A)}=\emptyset$, where $\phi(A)=\bigcup_{x\in A}\phi(x)$, then*

$$\lim_{a\to\tilde{x}} f'[\varphi(a)]=0 \qquad d\,\omega^W\text{-a.e. on } \pi^{-1}(A)\cap\Gamma^W \text{[10]}.$$

9) Cf. [8], Lemma 2.
10) "$d\,\omega^W$ (resp. $d\,\omega^M$)-*a.e.*" means "*except a set of $d\,\omega^W$ (resp. $d\,\omega^M$)-measure zero*".

Proof. Put $u_A=H^M_{\chi_A}$, *i.e.*, the Dirichlet solution for the characteristic function χ_A of A on Δ, and $f=f'\circ\varphi$. Then, it is derived that fu_A is a Wiener function on X. For, if we choose an open set G'_1 containing the carrier of f' and is contained with its closure in G', then $\varphi^{-1}(G'_1)$ is thin at every point of A, thus $\hat{R}^{\varphi^{-1}(G'_1)}_{u_A}$ is a potential[11]. Hence fu_A is a Wiener function on X[12]. By lemma 2,

$$\lim_{a\to\tilde{x}} f(a)u_A(a) = \text{fine}\lim_{a\to\pi(\tilde{x})} f(a)u_A(a) \qquad d\,\omega^W\text{-}a.e. \text{ on } \Delta^W .$$

Since fine $\lim fu_A=0$ $d\,\omega^M$-*a.e.* on Δ,

$$\lim fu_A = 0 \qquad \text{on } \Gamma^W . \tag{3.1}$$

On the other hand, since

$$\lim_{a\to\tilde{x}} u_A(a) = \text{fine}\lim_{a\to\pi(\tilde{x})} u_A(a) \qquad d\,\omega^W\text{-}a.e. \text{ on } \Delta^W$$

and

$$\text{fine}\lim_{a\to x} u_A(a) = 1 \qquad d\,\omega^M\text{-}a.e. \text{ on } A,$$

we have

$$\lim u_A = 1 \qquad d\,\omega^W\text{-}a.e. \text{ on } \pi^{-1}(A) . \tag{3.2}$$

Lemma A is derived immediately from (3.1) and (3.2), q.e.d..

Lemma B. *Let G' be an open set of X'^* with $G'\cap X'\in\mathcal{P}$, $\tilde{G}=\bar{\varphi}^{-1}(G')$, where $\bar{\varphi}$ is a continuous extension of φ on $X\cup\Delta_F$. If $\tilde{C}$ is a compact subset of $\tilde{G}\cap\Gamma^W$, then we have*

$$\hat{\varphi}(\pi(\tilde{x})) = \varphi^*(\tilde{x}) \qquad d\,\omega^W\text{-}a.e. \text{ on } \tilde{C} . \tag{3.3}$$

If there exists a closed set F with $\pi^{-1}(F)\cap\Gamma^W\subset\tilde{G}$, then

$$\hat{\varphi}(x)\subset G' \qquad d\,\omega^M\text{-}a.e. \text{ on } F.$$

Proof. We note that φ is a Fatou map on each component of $\tilde{G}\cap X$. Let g be a continuous function on X^W with the following properties:

$$\begin{cases} 1)\ \ 0\le g\le 1 , \\ 2)\ \ g = 1 \text{ on } \tilde{C} , \\ 3)\ \ \text{the carrier of } g \text{ is contained in } \tilde{G} . \end{cases} \tag{3.4}$$

Let $\{f'_n\}$ be a countable set of continuous functions on X'^* separating points of

11) If we denote by μ_u the canonical measure of $u=h^X_1$, i.e., $u(a)=\int_{\Delta_1} w_x(a)d\mu_u(w_x)$, where w_x is an extreme harmonic function corresponding to $x\in\Delta_1$, then $u_A=\int_A w_x d\mu_u(w_x)$. The result is an immediate consequence of [8], Cor. to Th. 1. Cf. [10].

12) Cf. [6], Th. 2.6.

X'^* and $0 \leq f'_n \leq 1$. Adding functions $1-f'_n$ to the set, if necessary, we may assume that for each pair of points $y'_1 \neq y'_2$ there exists an f'_n so that $f'_n(y'_1) > f'_n(y_2)$. Since the restrictions $f'_n|_{G' \cap X'}$ are bounded Wiener functions on $G' \cap X'$, $f_n = f'_n|_{G' \cap X'} \circ \varphi$ are Wiener functions on $\tilde{G} \cap X$. Set

$$g_n = \begin{cases} \min (f_n, g) & \text{on } \tilde{G} \cap X \\ 0 & \text{on } X - \tilde{G} . \end{cases}$$

The functions $g_n \in \mathcal{W}(X)$[13].

By Lemma 2, there exists a set $\tilde{N}$ of $d\,\omega^W$-measure zero such that

$$\lim_{a \to \tilde{x}} g_n(a) = \text{fine} \lim_{a \to \pi(\tilde{x})} g_n(a) \qquad \text{for each } \tilde{x} \in \Delta^W - \tilde{N} \text{ and } n = 1, 2, \cdots .$$

(we may suppose that $\tilde{N} \supset \Delta^W - \Gamma^W$) Therefore

$$\lim_{a \to \tilde{x}} f_n(a) = \text{fine} \lim_{a \to \pi(\tilde{x})} g_n(a) \qquad \text{for each } \tilde{x} \in \tilde{C} - \tilde{N} \text{ and } n = 1, 2, \cdots .$$

We shall show that

$$\Phi(\pi(\tilde{x})) = \bar{\varphi}(\tilde{x}) = \varphi^*(\tilde{x}) \qquad \text{for each } \tilde{x} \in \tilde{C} - \tilde{N} .$$

In fact, if $\Phi(\pi(\tilde{x}_0)) \neq \bar{\varphi}(\tilde{x}_0)$ for some point $\tilde{x}_0 \in \tilde{C} - \tilde{N}$, then there exists a point $z' \in \Phi(\pi(\tilde{x}_0))$ and a function f'_n satisfying

$$\alpha = f'_n[\bar{\varphi}(\tilde{x}_0)] > f'_n(z') = \beta .$$

Since

$$\alpha = \lim_{a \to \tilde{x}_0} f_n(a) = \text{fine} \lim_{a \to \pi(\tilde{x}_0)} g_n(a) ,$$

we have a set D such that

(3.5) $\qquad X - D$ is thin at $\pi(\tilde{x}_0)$ and $g_n > (\alpha + \beta)/2$ on D .

From (3.5) we have $D \subset \tilde{G} \cap X$ and $f_n > (\alpha + \beta)/2$ on D. However this is impossible since $z' \in \Phi(\pi(\tilde{x}_0)) \subset \overline{\varphi(D)}$ implies

$$f'_n(z') \geq (\alpha + \beta)/2 > \beta .$$

It is clear that the set

$$E = \{x \in \Delta ; \pi^{-1}(x) \subset \tilde{N}\}$$

is of $d\,\omega^M$-measure zero. Let $x \in F - E$. Then, $\pi^{-1}(x) \cap \Gamma^W \subset \tilde{G}$ and there exists a point $\tilde{x} \in \pi^{-1}(x) - \tilde{N}$. Applying the above result to the compact set $\pi^{-1}(F) \cap \Gamma^W$, we have

13) Cf. [6], Lemma 5.3..

$$\hat{\varphi}(x) = \bar{\varphi}(\tilde{x}) \subset G', \quad \text{q.e.d.}.$$

Theorem 1. $\omega^W(\pi^{-1}(\Delta-\hat{P}) \cap \Delta_P)=0.$
In particular, $\omega^W(\pi^{-1}(\hat{F}) \cap \Delta_P)=0.$

Proof. Let $\{G'_n\}$ be a countable basis of open sets for X'^*. We shall show that

$$\omega^W(\pi^{-1}(B_n) \cap \Delta_P) = 0 ,$$

where $B_n=\{x \in \Delta_1; \hat{\varphi}(x) \cap G'_n=\emptyset\}$. In fact, suppose for a moment that

$$\omega^W(\pi^{-1}(B_n) \cap \Delta_P)>0 .$$

Let f' be a finite continuous function on X'^* such that the carrier of f' is contained in an open set G' with $\bar{G}' \subset G'_n$, and let $f=f' \circ \varphi$. Since $\overline{\hat{\varphi}(B_n)} \cap \bar{G}'=\emptyset$, applying Lemma A, we have

$$\lim_{a \to \tilde{x}} f(a) = 0 \qquad d\,\omega^W\text{—}a.e. \text{ on } \pi^{-1}(B_n) \cap \Gamma^W .$$

Then, at some point $\tilde{x}_0 \in \pi^{-1}(B_n) \cap \Delta_P$

$$\lim_{a \to \tilde{x}_0} f(a) = 0 .$$

However this is impossible, since $\tilde{x}_0 \in \Delta_P$ and $\varphi^*(\tilde{x}_0)=X'^*$.

We have $\Delta_1-\hat{P}=\bigcup_{n=1}^{\infty} B_n$, therefore

$$\omega^W(\pi^{-1}(\Delta_1-\hat{P}) \cap \Delta_P) \le \sum_{n=1}^{\infty} \omega^W(\pi^{-1}(B_n) \cap \Delta_P) = 0 , \quad \text{q.e.d.}.$$

Theorem 2. $\omega^W(\Delta_F-\Delta'_F)=0$, *where* $\Delta'_F=\{\tilde{x} \in \Delta_F; \hat{\varphi}(\pi(\tilde{x}))=\varphi^*(\tilde{x})\}$.
In particular, $\omega^W(\Delta_F-\pi^{-1}(\hat{F}))=0.$

Proof. Let G' be an open set of X'^* with $G' \cap X' \in \mathcal{P}$, and let $\tilde{G}=\bar{\varphi}^{-1}(G')$. We shall show that

$$\omega^W(\tilde{G} \cap \Gamma^W-\Delta'_F) = 0 .$$

In fact, suppose for a moment that

$$\omega^W(\tilde{G} \cap \Gamma^W-\Delta'_F)>0 .$$

We choose a compact subset $\tilde{C}$ of $\tilde{G} \cap \Gamma^W-\Delta'_F$ with positive $d\,\omega^W$-harmonic measure. Applying Lemma B, we have

$$\hat{\varphi}(\pi(\tilde{x})) = \varphi^*(\tilde{x}) \qquad d\,\omega^W\text{—}a.e. \text{ on } \tilde{C} .$$

However, this is impossible since this occurs at some point $\tilde{x}_0 \in \tilde{C}$ and $\tilde{x}_0 \notin \Delta'_F$.

Let G_1' and G_2' be open sets of X'^* such that

$$G_1' \cup G_2' = X'^* \text{ and } G_i' \cap X' \in \mathcal{P} \text{ for } i=1 \text{ and } 2\,.$$

Since $\Delta_F \cap \Gamma^W = (\tilde{G}_1 \cap \Gamma^W) \cap (\tilde{G}_2 \cap \Gamma^W)$, where $\tilde{G}_i = \bar{\varphi}^{-1}(G_i')$ $(i=1, 2)$ we have

$$\omega^W(\Delta_F - \Delta_F') \leq \omega^W(\tilde{G}_1 \cap \Gamma^W - \Delta_F') + \omega^W(\tilde{G}_2 \cap \Gamma^W - \Delta_F') = 0\,, \quad \text{q.e.d.}\,.$$

Theorem 3. (*i*) $\omega^M(\hat{P} - \pi(\Delta_P)) = 0$,
(*ii*) $\omega^M(\hat{F} - \pi(\Delta_F' \cap \Gamma^W)) = 0$[14].

Proof. (i) $\omega^M(\hat{P} - \pi(\Delta_P)) \leq \omega^M(\Delta - \hat{F} - \pi(\Delta_P))$
$= \omega^W(\pi^{-1}[\Delta - \hat{F} - \pi(\Delta_P)]) \leq \omega^W(\Delta^W - \Delta_P - \pi^{-1}(\hat{F}))$
$= \omega^W(\Delta_F - \pi^{-1}(\hat{F})) = 0$. [by Lemma 1 and Theorem 2)
(ii) $\omega^M(\hat{F} - \pi(\Delta_F' \cap \Gamma^W)) = \omega^W(\pi^{-1}[\hat{F} - \pi(\Delta_F' \cap \Gamma^W)])$
$\leq \omega^W(\pi^{-1}(\hat{F}) - (\Delta_F' \cap \Gamma^W))$
$\leq \omega^W(\pi^{-1}(\hat{F}) \cap \Delta_P) + \omega^W(\Delta_F - \Delta_F') = 0$.
(by Theorem 1 and Theorem 2)

It is an open question to the author whether $\hat{P} = \pi(\Delta_P)$ except a set of $d\,\omega^M$-measure zero or not.

Corollary 1. $\omega^M(\hat{P} - P^*) = 0$.

Corollary 2.[15] *If φ is an open map, then*

$$X' - \bigcap_{r>0} \varphi(U_r(x) \cap X)$$

is a polar set for $d\,\omega^M$-almost all points x of $\hat{P}$, where $U_r(x)$ denotes an r-neighbourhood of x in a metric of X^M.

In fact, we shall show that $X' - \bigcap_{r>0} \varphi(U_r(x) \cap X)$ is a polar set for every $x \in \pi(\Delta_P)$. Let $x \in \pi(\Delta_P)$ and $\tilde{x} \in \pi^{-1}(x) \cap \Delta_P$. From the definition of Δ_P,

$$\overline{\varphi(\tilde{U} \cap X)} = X'^* \qquad \text{for every neighbourhood } \tilde{U} \text{ of } \tilde{x}.$$

There exists a sequence of open neighbourhoods $\{\tilde{U}_n\}$ of $\tilde{x}$ such that $\tilde{U}_n \subset \pi^{-1}(U_{1/n}(x))$. Since

$$X' - \bigcap_{n=1}^{\infty} \varphi(U_{1/n}(x) \cap X) \subset X' - \bigcap_{n=1}^{\infty} \varphi(\tilde{U}_n \cap X) = \bigcup_{n=1}^{\infty} [X' - \varphi(\tilde{U}_n \cap X)]\,,$$

in order to prove the assertion, it is enough to show that

$$X' - \varphi(\tilde{U}_n \cap X)$$

14) (ii) is pointed out by F–Y. Maeda.
15) Cf. [7], Th. 6..

is polar. Suppose, on the contrary, $X'-\varphi(\tilde{U}_n \cap X)$ is not polar, then there exists a positive potential on each component of $\varphi(\tilde{U}_n \cap X)$, therefore φ is a Fatou map on each component of $\tilde{U}_n \cap X$. However, this is impossible since, by Lemma 1, φ is not a Fatou map on some component of $\tilde{U}_n \cap X$.

T. Fuji'i'e considered cluster sets of analytic maps of Riemann surfaces and obtained some results [7]. Some of them are extended to harmonic maps between harmonic spaces. For example, let us consider the following. Let J be the family of boundary sets with vanishing $d\,\omega^M$-harmonic measure. Following Fuji'i'e, we shall define

$$C^*(\varphi, x) = \bigcap_{E\in J} \bigcap_{r>0} \left[\overline{\bigcup_{x\in(U_r \cap \Delta_1)-E} \phi(x)}\right],$$

where U_r is an r-neighbourhood of $x\in\Delta_1$ in a metric of X^M. Then we have

Theorem 4. $C^*(\varphi, x)=\tilde{\varphi}(x)$ *for* $x\in\Delta_1^*$.

Proof. First, we shall show that

$$\tilde{\varphi}(x)\subset\overline{\phi((U\cap\Delta_1)-E)} \tag{3.6}$$

for every open neighbourhood U of x in X^M and $E\in J$, where

$$\phi((U\cap\Delta_1)-E) = \bigcup_{x\in(U\cap\Delta_1)-E} \phi(x)\,.$$

Suppose $z'\in\tilde{\varphi}(x)-\overline{\phi((U\cap\Delta_1)-E)}$. Let G' be an open neighbourhood of z' with

$$\bar{G}'\cap\overline{\phi((U\cap\Delta_1)-E)}=\emptyset,$$

f' be a finite continuous function on X'^* with carrier contained in G' and $f'(z')\neq 0$. Then, by Lemma A, we have

$$\lim_{a\to\tilde{z}} f'[\varphi(a)] = 0 \qquad d\,\omega^W\text{-}a.e. \text{ on } \pi^{-1}[(U\cap\Delta_1)-E]\cap\Gamma^W,$$

from which we derive

$$\lim f'\circ\varphi = 0 \qquad \text{on } \pi^{-1}(x)\cap\Gamma^W,$$

and we have $f'(z')=0$, which is absurd. Thus we have (3.6).

Next, we show

$$C^*(\varphi, x)\subset\tilde{\varphi}(x)\,.$$

Suppose $z'\in C^*(\varphi, x)-\tilde{\varphi}(x)$. Then, we know that $\tilde{\varphi}(x)\neq X'^*$ and $\pi^{-1}(x)\cap\Delta_P=\emptyset$. Hence, $\bar{\varphi}$ is defined and continuous on $\pi^{-1}(x)\cap\Gamma^W$. It is easy to find an open subset G' of X'^* satisfying $\tilde{\varphi}(x)\subset G'$ and $z'\notin\bar{G}'$, and an open neighbourhood U of x in X^M such that

$$\tilde{\phi}(y) \subset G' \qquad \text{for every } y \in U \cap \Delta_1^* .$$

From this we may derive $\pi^{-1}(U) \cap \Gamma^W \subset \bar{\varphi}^{-1}(G') \cap \Gamma^W$. Let V be a compact neighbourhood of x satisfying $V \subset U$. Then, by Lemma B, there exists a set $E_1 \in J$ such that

$$\hat{\phi}((V \cap \Delta_1) - E_1) \subset G' .$$

Thus,

$$\bigcap_{r>0} \overline{\hat{\phi}[(U_r \cap \Delta_1) - E_1]} \subset \bar{G}'$$

and $z' \in C^*(\varphi, x) \subset \bar{G}'$ which is absurd. Hence we have proved the theorem.

4. Theorems of Fatou and Plessner type

Theorem 5. (Plessner) *Let X'^* be a metrizable and resolutive compactification of X', φ be a harmonic map of X into X'. Then, $\hat{\phi}(x)$ is either X'^* or a set of a single point for $d\,\omega^M$-almost all points of Δ, i.e.,*

$$\omega^M(\Delta - \hat{F} - \hat{P}) = 0 .$$

Proof. $\omega^M(\Delta - \hat{F} - \hat{P}) = \omega^W(\pi^{-1}(\Delta - \hat{F} - \hat{P}))$
$= \omega^W(\pi^{-1}(\Delta - \hat{P}) - \pi^{-1}(\hat{F}))$
$\leq \omega^W(\pi^{-1}(\Delta - \hat{P}) \cap \Delta_P) + \omega^W(\Delta_F - \pi^{-1}(\hat{F}))$
$= 0.$ (by Theorem 1 and Theorem 2)

Corollary 1. *$\hat{\phi}(x)$ is a set of a single point for $d\,\omega^M$-almost all points of $\Delta - \pi(\Delta_P)$, i.e.,*

$$\omega^M(\Delta - \pi(\Delta_P) - \hat{F}) = 0 .$$

In particular, $\hat{\phi}(x)$ is a set of a single point for $d\,\omega^M$-almost all points of $\{x \in \Delta_1^* ;\ \tilde{\phi}(x) \neq X'^*\}$.

This is an immediate consequence of Theorem 3 (i) and the above theorem.

Corollary 2. *For $d\,\omega^M$-almost every point x there exists a point $\tilde{x} \in \pi^{-1}(x) \cap \Gamma^W$ such that*

$$\hat{\phi}(x) = \varphi^*(\tilde{x}) .$$

In particular,

$$\hat{\phi}(x) \subset \tilde{\phi}(x) \qquad d\,\omega^M\text{-a.e. on } \Delta .$$

By the above theorem, for $d\,\omega^M$-almost every point we have (1) $x \in \hat{P}$ or (2) $x \in \hat{F}$. In case (1), we have, by Theorem 3 (i)

$$x \in \pi(\Delta_P) \qquad \text{for } d\,\omega^M\text{-almost all } x \text{ of } \hat{P}.$$

Thus, letting $\tilde{x}\in\pi^{-1}(x)\cap\Delta_P$ we have

$$X'^* = \varphi^*(\tilde{x}) = \phi(x) .$$

In case (2), we have, by Theorem 3 (ii)

$$x\in\pi(\Delta'_F\cap\Gamma^W) \qquad \text{for } d\,\omega^M\text{-almost all } x \text{ of } \hat{F}.$$

Thus, there exists a point $\tilde{x}\in\pi^{-1}(x)\cap\Gamma^W$ with $\phi(x)=\varphi^*(\tilde{x})$, q.e.d..

REMARK. As we have shown in [10], in the classical case of the unit disc: $X=\{z;\ |z|<1\}$, a Wiener boundary point $\tilde{x}\in\pi^{-1}(x)\cap\Gamma^W$, where $|x|=1$, has a neighbourhood whose trace (the intersection with X) lies outside a horocycle at x (a circle internally tangent to $|z|=1$ at x)[16]. By Corollary 2 to Theorem 5, we have *for $d\,\omega^M$-almost all points x a fine cluster set $\phi(x)$ is a cluster set concerning a tangential filter converging to x. Therefore the following facts hold for $d\,\omega^M$-almost all points of Δ: if a fine cluster set is total at x, i.e., $x\in\hat{P}$, then it is also true for a tangential cluster set $\tilde{\varphi}(x)$, and if a tangential cluster set $\tilde{\varphi}(x)$ at x is reduced to a single point, then it is also true for a fine cluster set.* This is analogous to the results of Bagemihl[17]: *almost every Plessner point is a horocyclic Plessner point and almost every horocyclic Fatou point is a Fatou point.* However, a defining filter of a cluster set $\tilde{\varphi}(x)$ would be more tangential than a horocycle.

In [10], the author obtained the following theorem of Riesz type concerning fine cluster sets. Under the additional condition:

(*) *when $X'\in\mathcal{H}\text{-}\mathcal{P}$ there exists a non-polar subset E' of X' each point of which is polar,*

if

$$\phi(x)\subset A' \qquad \textit{for every } x\in A\subset\Delta_1 \tag{4.1}$$

holds for a polar set A' of an arbitrary compactification X'^* and a boundary set A of outer $d\,\omega^M$-harmonic measure positive, then φ is a constant map.

Now, from Corollary 2, we have a theorem of Riesz type concerning a different sort of cluster set $\tilde{\varphi}(x)$. It is noteworthy that in the classical case of the unit disc, $\tilde{\varphi}(x)$ is a tangential cluster set, so that the following result is, so to speak, a theorem of Riesz type concerning tangential cluster sets.

Corollary 3. *Let φ be a non-constant harmonic map, When $X'\in\mathcal{H}\text{-}\mathcal{P}$, we assume further the existence of non-polar set each point of which is polar. Let X'^* be a metrizable and resolutive compactification of X' and A' be a polar set of X'^*. If we have*

$$\tilde{\varphi}(x)\subset A' \qquad \textit{for each } x\in A\subset\Delta_1^* ,$$

16) Cf. [9], Th. 5.2.
17) Cf. [1], Th. 1 and Th. 2.

then $\omega^M(A)=0$.

Theorem 6. (Fatou) *Let X'^* be a metrizable and resolutive compactification of X'. In order that a harmonic map φ of X into X' be a Fatou map, it is necessary and sufficient that $\hat{\varphi}$ be reduced to a single point d ω^M-almost everywhere on Δ.*

Proof. Let φ be Fatou map. Then we have $\Delta_P=\emptyset$ and by Theorem 3 (i), $\omega^M(\hat{P})=0$. By Theorem 5, we conclude that

$$\omega^M(\Delta-\hat{F}) = 0 .$$

Next, if $\omega^M(\Delta-\hat{F})=0$, then $\omega^W(\Delta^W-\pi^{-1}(\hat{F}))=0$. By Lemma 1 and Theorem 2, we have $\omega^W(\Delta_P-\pi^{-1}(\hat{F}))=0$. Since

$$\omega^W(\Delta_P\cap\pi^{-1}(\hat{F})) = 0 \qquad \text{(by Theorem 1)}$$

we have $\omega^W(\Delta_P)=0$ and $\Delta_P=\emptyset$, which implies that φ is a Fatou map. q.e.d..

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