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THE FUNDAMENTAL SOLUTION FOR PSEUDO-DIFFERENTIAL OPERATORS OF PARABOLIC TYPE

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Introduction

In this paper we shall construct the fundamental solution $E(t, s)$ for a degenerate pseudo-differential operator L of parabolic type only by symbol calculus and, as an application, we shall solve the Cauchy problem for L:

$$(0.1)\qquad \begin{cases} Lu(t)=f(t) & \text{in } t>s\,, \\ u(s)=u_0\,. \end{cases}$$

Another application of the present fundamental solution will be done in [12] in order to construct left parametrices for degenerate operators studied by Grushin in [2].

Now let us consider the operator L of the form

$$L=\frac{\partial}{\partial t}+p(t; x, D_x)\,,$$

where $p(t; x, D_x)$ is a pseudo-differential operator of class $S^m_{\lambda,\rho,\delta}$ with a parameter t $(\rho>\delta)$ (See §1). For the operator $p(t; x, D_x)$ we set the following conditions:

$$(0.2)\qquad \operatorname{Re} p(t; x, \xi)+c_0\geqslant c_1\lambda(x, \xi)^{m'}$$

$$0.3)\qquad |p^{(\alpha)}_{(\beta)}(t; x, \xi)/(\operatorname{Re} p(t; x, \xi)+c_0)|\leqslant C_{\alpha,\beta}\lambda(x, \xi)^{-(\rho,\alpha)+(\delta,\beta)}\,,$$

where $m\geqslant m'\geqslant 0$ and $\lambda=\lambda(x, \xi)$ is a basic weight function defined in §1. We note that $\lambda(x, \xi)$ in general varies even in x and increases in polynomial order.

We call $E(t, s)$ a fundamental solution for L when $E(t, s)$ satisfies

$$\begin{cases} LE(t, s)=0 & \text{in } t>s\,, \\ E(s, s)=I\,. \end{cases}$$

The main theorem of this paper is stated as follows.

Main theorem. *Under the conditions* (0, 2) *and* (0, 3) *we can construct the unique fundamental solution* $E(t, s)$ *for* L *as a pseudo-differential operator of* *class* $S^0_{\lambda,\rho,\delta}$ *with parameters t and s* (For the precise statement see Theorem 3.1).

Using the fundamental solution of this theorem the solution of the Cauchy problem (0. 1) is given in the form

$$u(t) = E(t, s)u_0 + \int_s^t E(t, \sigma)f(\sigma)d\sigma .$$

We note that Greiner [1] constructed the fundamental solution for parabolic differential operators on a compact C^∞-manifold by using pseudo-differential operators. But his method is different from ours and not applicable to our non-compact case R^n. We reduce the construction of the fundamental solution to solving the integral equation

$$\Phi(t, s) + K(t, s) + \int_s^t K(t, \sigma)\Phi(\sigma, s)d\sigma = 0 \tag{0.4}$$

for a known operator $K(t, s) \in S^0_{\lambda,\rho,\delta}$.

To solve the equation (0.4) the product formula of pseudo-differential operators plays an essential role. We also note that by the same method we can construct the fundamental solution for degenerate operators which have been considered by Helffer [3] and Matsuzawa [7]. On the other hand Shinkai [9] constructed the fundamental solution $E(t, s)$ when $p(x, \xi)$ is a system of pseudo-differential operator by our method and applied it to the proof of hypoellipticity of L.

In Section 1 we define pseudo-differential operators with symbol $S^m_{\lambda,\rho,\delta}$. In Section 2 main properties of pseudo-differential operators defined in Section 1 will be given. In Section 3 we shall construct the fundamental solution $E(t, s)$ under the conditions (0.2) and (0.3), and in Section 4 we study the behavior of $E(t, s)$ for large $(t-s)$.

The results of the present paper have been announced partly in [10] and [11].

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1. Definitions and notations

Let R^n be the n-dimensional Euclidean space. $\mathcal{S} = \mathcal{S}(R^n)$ is the space of all rapidly decreasing functions with semi-norms

$$|f|_{l,\mathcal{S}} = \max_{|\alpha|+|\beta|\leq l} \sup_{x\in R^n} |x^\alpha \partial_x^\beta f(x)| ,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial_x^\beta = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$. $\mathcal{S}'$ is its dual space. $\hat{f}(\xi) = \mathcal{F}[f](\xi)$ denotes the Fourier transform of $f(x)$ which is defined by

$$\hat{f}(\xi) = \int_{R^n} e^{-ix\cdot\xi} f(x)dx , \qquad f \in \mathcal{S} .$$

For a pair of real vectors $a=(a_1, \cdots, a_n)$ and $b=(b_1, \cdots, b_n)$ we denote $a>b$, if $a_j>b_j$ for any j and $a\geqslant b$, if $a_j\geqslant b_j$ for any j.

DEFINITION 1.1. We say that a C^∞-function $\lambda(x, \xi)$ defined in $R^n_x \times R^n_\xi$ is a basic weight function if there exists a pair of vectors $\tilde{\rho}=(\tilde{\rho}_1, \cdots, \tilde{\rho}_n)$ and $\delta=(\delta_1, \cdots, \delta_n)$ such that

$$(1.1)\qquad \begin{cases} \text{(i)} & \tilde{\rho}>\delta, \quad \tilde{\rho}_j>0 \ \ 1\leqslant j\leqslant n \\ \text{(ii)} & 1\leqslant \lambda(x+y, \xi)\leqslant A_0\langle y\rangle^\tau \lambda(x, \xi) \qquad \tau\geqslant 0\,, \ A_0\geqslant 1 \\ \text{(iii)} & |\lambda^{(\alpha)}_{(\beta)}(x, \xi)|\leqslant A_{\alpha,\beta}\lambda(x, \xi)^{1-(\tilde{\rho},\alpha)+(\delta,\beta)} \end{cases}$$

where $\lambda^{(\alpha)}_{(\beta)}(x, \xi)=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_n)^{\alpha_n}(-i\partial/\partial x_1)^{\beta_1}\cdots(-i\partial/\partial x_n)^{\beta_n}\lambda(x, \xi)$, $\langle y\rangle=(1+|y|^2)^{1/2}$, $(\tilde{\rho}, \alpha)=\sum_{j=1}^n \tilde{\rho}_j\alpha_j$ and A_0 and $A_{\alpha,\beta}$ are constants.

For a basic weight function $\lambda(x, \xi)$ and a vector $\rho=(\rho_1, \cdots, \rho_n)$ such that $\tilde{\rho}\geqslant\rho\geqslant\delta$, we define symbol class $S^m_{\lambda,\rho,\delta}$ as follows.

DEFINITION 1.2. $S^m_{\lambda,\rho,\delta}$ is the set of all C^∞-functions $p(x, \xi)$ defined in $R^n_x\times R^n_\xi$ which satisfy for any α and β

$$|p^{(\alpha)}_{(\beta)}(x, \xi)|\leqslant C_{\alpha,\beta}\lambda(x, \xi)^{m-(\rho,\alpha)+(\delta,\beta)}$$

for some constant $C_{\alpha,\beta}$. For $p\in S^m_{\lambda,\rho,\delta}$ we define semi-norms $|p|^{(m)}_l$ by

$$|p|^{(m)}_l = \max_{|\alpha|+|\beta|\leqslant l}\ \sup_{(x,\xi)\in R^n\times R^n}\{|p^{(\alpha)}_{(\beta)}(x, \xi)|\lambda(x, \xi)^{-m+(\rho,\alpha)-(\delta,\beta)}\}\,.$$

Set $S^{-\infty}_{\lambda,\rho,\delta}=\bigcap_m S^m_{\lambda,\rho,\delta}$ and $S^{\infty}_{\lambda,\rho,\delta}=\bigcup_m S^m_{\lambda,\rho,\delta}$.

For $p(x, \xi)\in S^m_{\lambda,\rho,\delta}$ we define *a pseudo-differential operator with the symbol* $\sigma(P)=p(x, \xi)$ by

$$Pu(x) = Os\text{-}\iint e^{-iy\cdot\xi}p(x, \xi)u(x+y)dy\,đ\xi$$

for $u\in\mathcal{S}$, where $đ\xi=(2\pi)^{-n}d\xi$ and '$Os-$' means the oscillatory integral defined in Definition 1.4 below.

Now let us mention the important properties about the oscillatory integral contained in [5].

DEFINITION 1.3. We say that a C^∞-function $q(\eta, y)$ in $R^n_\eta\times R^n_y$ belongs to a class $\mathcal{A}^m_{\delta,\tau}$ $(-\infty<m<\infty,\ \delta<1,\ \tau=(\tau_1, \cdots, \tau_k, \cdots),\ \tau_k\geqslant 0)$ if for any multiindex α and β there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial^\alpha_\eta\partial^\beta_y q(\eta, y)|\leqslant C_{\alpha,\beta}\langle\eta\rangle^{m+\delta|\beta|}\langle y\rangle^{\tau_{|\beta|}}\,.$$

We also define the semi-norms $|q|^{(m)}_l$ by

$$|q|^{(m)}_l = \max_{|\alpha|+|\beta|\leqslant l}\ \sup_{(\eta,y)\in R^n\times R^n}\{|\partial^\alpha_\eta\partial^\beta_y q(\eta, y)|\langle y\rangle^{-\tau_{|\beta|}}\langle\eta\rangle^{-m-\delta|\beta|}\}\,.$$

Definition 1.4. For $q(\eta, y)\in\mathcal{A}^m_{\delta,\tau}$ we define

$$Os-[e^{-iy\cdot\eta}q(\eta, y)] = Os-\iint e^{-iy\cdot\eta}q(\eta, y)dyd\eta$$
$$= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi_\varepsilon(\eta, y)q(\eta, y)dyd\eta ,$$

where $\chi_\varepsilon(\eta, y)=\chi(\varepsilon\eta, \varepsilon y)$ and $\chi(\eta, y)$ is a function such that $\chi\in\mathcal{S}(R^{2n})$ and $\chi(0, 0)=1$.

Proposition 1.5. *For $q(\eta, y)\in\mathcal{A}^m_{\delta,\tau}$ we can write*

$$Os-[e^{-iy\cdot\eta}q(\eta, y)]$$
$$= \iint e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{-2l'}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}q(\eta, y)\}\, dyd\eta ,$$

where l and l' are positive integers such that $-2l(1-\delta)<-n$ and $-2l'+\tau_{2l}<-n$.

Proposition 1.6. *Let $\{q_\varepsilon\}_{0<\varepsilon<1}$ be a subset of $\mathcal{A}^m_{\delta,\tau}$ such that $\sup_\varepsilon|q_\varepsilon|^{(m)}_l\leqslant M_l$ for any l. If there exists $q_0(\eta, y)\in\mathcal{A}^m_{\delta,\tau}$ such that $q_\varepsilon(\eta, y)\to q_0(\eta, y)$ as $\varepsilon\to 0$ uniformly on any compact set of $R^n_\eta\times R^n_y$, then we have $\lim_{\varepsilon\to 0} Os-[e^{-iy\cdot\eta}q_\varepsilon]=Os-[e^{-iy\cdot\eta}q_0]$.*

Definition 1.7. Let F be a Fréchet space. We define $\mathcal{E}^l_t(F)$ by

$\mathcal{E}^l_t(F) = \{l$-times continuously differentiable F-valued function $u(t)$ in the interval I$\}$.

Definition 1.8 ([6]). We say that $\{p_\varepsilon(x, \xi)\}_{0<\varepsilon<1}$ *converges to* $p_0(x, \xi)$ *weakly in* $S^m_{\lambda,\rho,\delta}$ if $\{p_\varepsilon(x, \xi)\}_{0<\varepsilon<1}$ is a bounded set in $S^m_{\lambda,\rho,\delta}$ and if $p_\varepsilon(x, \xi)$ converges to $p_0(x, \xi)$ as $\varepsilon\to 0$ uniformly on any compact set of $R^n_x\times R^n_\xi$. We define $w-\mathcal{E}^l_{t,s}(S^m_{\lambda,\rho,\delta})$ *in* $0\leqslant s\leqslant t\leqslant T$ by

$w-\mathcal{E}^l_{t,s}(S^m_{\lambda,\rho,\delta}) = \{S^m_{\lambda,\rho,\delta}$-valued functions $u(t, s)$ defined in $0\leqslant s\leqslant t\leqslant T$ which are l-times continuously differentiable with respect to t and s in the weak topology of $S^m_{\lambda,\rho,\delta}\}$.

2. Calculus of pseudo-differential operators in class $S^m_{\lambda,\rho,\delta}$

The main theorem of this section is the following

Theorem 2.1. *Let $P_j\in S^{m_j}_{\lambda,\rho,\delta}$ $(j=1, \cdots, \nu)$. Then the product operator $P=P_1\cdots P_\nu$ belongs to $S^{m_0}_{\lambda,\rho,\delta}$, where $m_0=\sum_{j=1}^{\nu} m_j$. Moreover for any l there exists l_0 such that*

$$|\sigma(P)|^{(m_0)}_l\leqslant(C_0)^\nu \prod_{j=1}^{\nu}|p_j|^{(m_j)}_{l_0} \tag{2.1}$$

where l_0 and C_0 are constants depending on $\sum_{j=1}^{\nu}|m_j|$ but independent of ν.

Proof. We can write

$$Pu(x) = Os-\int\cdots\int \exp\{-i\sum_{j=1}^{\nu} y^j\cdot\xi^j\}\, p_1(x, \xi^1)p_2(x+y^1, \xi^2)\cdots$$
$$\cdots p_\nu(x+\sum_{j=1}^{\nu-1} y^j, \xi^\nu)u(x+\sum_{j=1}^{\nu} y^j)dy^1dy^2\cdots dy^\nu d\xi^1 d\xi^2\cdots d\xi^\nu .$$

So the symbol of P is given by

$$p(x, \xi) = Os-\int\cdots\int \exp\{-i\sum_{j=1}^{\nu-1} y^j\cdot\eta^j\}\prod_{j=1}^{\nu} p_j(x+\sum_{k=0}^{j-1} y^k, \xi+\eta^j)dV, \tag{2.2}$$

where $y^0=0$, $\eta^\nu=0$ and $dV=dy^1dy^2\cdots dy^{\nu-1}d\eta^1 d\eta^2\cdots d\eta^{\nu-1}$.

By (2.2) it is sufficient to prove (2.1) for $l=0$.

For the proof we prepare the following

Lemma 2.2. *Let $q(x^1, \xi^1, \cdots, x^\nu, \xi^\nu)$ be a C^∞-function on $R^{2n\nu}$ such that*

$$|\partial_{x^1}^{\beta^1}\partial_{x^2}^{\beta^2}\cdots\partial_{x^\nu}^{\beta^\nu}\partial_{\xi^1}^{\alpha^1}\partial_{\xi^2}^{\alpha^2}\cdots\partial_{\xi^\nu}^{\alpha^\nu} q^1(x^1, \xi^1, x^2, \xi^2, \cdots, x^\nu, \xi^\nu)|$$
$$\leqslant M_{\alpha^1,\alpha^2,\cdots,\alpha^\nu,\beta^1,\beta^2,\cdots,\beta^\nu}\prod_{j=1}^{\nu}\lambda(x^j, \xi^j)^{m_j-(\rho,\alpha^j)+(\delta,\beta^j)} \tag{2.3}$$

for any sequence of multi-indices $\alpha^1, \alpha^2, \cdots, \alpha^\nu, \beta^1, \beta^2, \cdots, \beta^\nu$. Set

$$I_\theta = Os-\int\cdots\int \exp\{-i\sum_{j=1}^{\nu-1} y^j\cdot\eta^j\} \tag{2.4}$$
$$\times q(x, \xi+\theta\eta^1, x+y^1, \xi+\theta\eta^2, \cdots, \xi+\theta\eta^{\nu-1}, x+\sum_{j=1}^{\nu-1} y^j, \xi)dV$$
$$(0\leqslant\theta\leqslant 1).$$

Then we can find l_0 such that

$$|I_\theta|\leqslant (C_0)^\nu M_{l_0}\lambda(x, \xi)^{m_0}, \tag{2.5}$$

where $m_0=\sum_{j=1}^{\nu} m_j$, $M_{l_0}=\max_{|\alpha^j|+|\beta^j|\leq l_0}\{M_{\alpha^1,\alpha^2,\cdots,\alpha^\nu,\beta^1,\beta^2,\cdots,\beta^\nu}\}$ and C_0 is a constant depending on $\sum_{j=1}^{\nu}|m_j|$ but independent of ν and θ.

Apply the above Lemma 2.2 to (2.2) setting $q(x^1, \xi^1, x^2, \xi^2, \cdots, x^\nu, \xi^\nu) =\prod_{j=1}^{\nu} p_j(x^j, \xi^j)$ and $\theta=1$. Then we get

$$|p|_0^{(m_0)}\leqslant (C_0)^\nu\prod_{j=1}^{\nu}|p_j|_0^{(m_j)} .$$

Thus the proof is completed.

For the proof of Lemma 2.2 we prepare some propositions. For simplicity we may assume $\tilde{\rho}_j=\tilde{\rho}$, $\rho_j=\rho$ and $\delta_j=\delta$ for any j. Otherwise we have only to repeat the same argument for each variable.

Set

$$F(x, \eta; y) = (1+\lambda(x, \eta)^{2\bar{\delta}n_0}|y|^{2n_0})^{-1},$$

where $\bar{\delta}=\max(\delta, 0)$ and $n_0=[n/2]+1$. Then, by (1.1)-(iii) we have easily the following

Proposition 2.3. *$F(x, \eta; y)$ satisfies the inequality with constants $C_{\alpha,\beta,\gamma}$*

$$|\partial_x^\alpha \partial_y^\beta \partial_\eta^\gamma F(x, \eta; y)| \leqslant C_{\alpha,\beta,\gamma} F(x, \eta; y)\lambda(x, \eta)^{-\tilde{\rho}|\gamma|+\bar{\delta}|\alpha+\beta|}$$

for all α β, and γ.

Proof is omitted.

Proposition 2.4. *If $r_1\geqslant 0$ and $r_2-2\tau\bar{\delta}n_0\geqslant 0$, then we get for some constant C*

$$\begin{aligned}&\int F(z^1, \xi+\eta^1; z^1-z^0)F(z^2, \xi+\eta^2; z^2-z^1)\langle z^0-z^1\rangle^{-r_1}\langle z^2-z^1\rangle^{-r_2}dz^1\\ &\leqslant C\langle z^2-z^0\rangle^{-r_3}\{F(z^2, \xi+\eta^2; z^2-z^0)\lambda(z^2, \xi+\eta^1)^{-n\bar{\delta}}\\ &\quad +F(z^2, \xi+\eta^1; z^2-z^0)\lambda(z^2, \xi+\eta^2)^{-n\bar{\delta}}\}\ .\end{aligned}$$

where $r_3=\min(r_1, r_2-2\tau\bar{\delta}n_0)$.

Proof. We devide R^n into two parts $\Omega_1=\{z^1\in R^n;\ |z^1-z^2|\geqslant|z^0-z^2|/2\}$ and $\Omega_2=R^n\backslash\Omega_1$. For $z^1\in\Omega_1$ we have

$$F(z^2, \xi+\eta^2; z^2-z^1)\leqslant 2^{2n_0}F(z^2, \xi+\eta^2; z^2-z^0) \quad \text{in } \Omega_1 \tag{2.6}$$

and

$$\langle z^1-z^2\rangle^{-1}\leqslant 2\langle z^2-z^0\rangle^{-1} \quad \text{in } \Omega_1\,. \tag{2.7}$$

For $z^1\in\Omega_2$, we get

$$F(z^2, \xi+\eta^1; z^1-z^0)\leqslant 2^{2n_0}F(z^2, \xi+\eta^1; z^2-z^0) \quad \text{in } \Omega_2 \tag{2.8}$$

and

$$\langle z^1-z^0\rangle^{-1}\leqslant 2\langle z^2-z^0\rangle^{-1} \quad \text{in } \Omega_2\,. \tag{2.9}$$

Since $2n_0>n$, it is clear that

$$\int_{R^n} F(x, \eta; y)dy = c_1\lambda(x, \eta)^{-n\bar{\delta}}\,. \tag{2.10}$$

By (1.1)-(ii) we get

$$F(z^1, \xi+\eta^1; z^1-z^0)\leqslant(A_0)^{2\bar{\delta}n_0}\langle z^2-z^1\rangle^{2\tau\bar{\delta}n_0}F(z^2, \xi+\eta^1: z^1-z^0)\,. \tag{2.11}$$

Then by (2.6)~(2.11) we get the assertion. Q.E.D.

By (1.1)~(iii) there exists a constant $c_0>0$ such that

$$|\lambda(x, \xi+\eta)-\lambda(x, \xi)|\leqslant\lambda(x, \xi)/2$$

if $|\eta|\leqslant c_0\lambda(x, \xi)^{\tilde{\rho}}$.

Proposition 2.5. *Set*

$$\begin{aligned} I(K) = &|\eta|^{-2K}\lambda(x, \xi+\eta)^m\{\lambda(x, \xi+\eta)+\lambda(x, \xi)\}^{2K\bar{\delta}} \\ &\times\left\{\lambda(x, \xi+\eta)^{-n\bar{\delta}}+\frac{F(x, \xi+\eta; y)}{F(x, \xi; y)}\lambda(x, \xi)^{-n\bar{\delta}}\right\} \qquad (K\geqslant 0)\end{aligned}$$

and set

$$I_1 = \{\eta;\ |\eta|\leqslant c_0\lambda(x, \xi)^{\bar{\delta}}\}\ ,$$

$$I_2 = \{\eta;\ c_0\lambda(x, \xi)^{\bar{\delta}}\leqslant|\eta|\leqslant c_0\lambda(x, \xi)^{\tilde{\rho}}\}$$

and

$$I_3 = \{\eta;\ |\eta|\geqslant c_0\lambda(x, \xi)^{\tilde{\rho}}\}\ .$$

Then we have for a constant c

$$\int_{I_j} I(K_j)d\eta\leqslant c\lambda(x, \xi)^m \qquad (j=1, 2, 3)\,, \tag{2.12}$$

if $K_1=0$, $K_2>n/2$ *and* $K_3>(|m|+2\bar{\delta}n_0+n\tilde{\rho})/2(\tilde{\rho}-\delta)$.

Proof. If η belongs to I_1 or I_2, then we have for some constant c_2

$$I(K)\leqslant c_2|\eta|^{-2K}\lambda(x, \xi)^{(2K-n)\bar{\delta}+m}\,, \qquad K\geqslant 0\,.$$

Hence (2.12) is proved for $j=1$ and 2. If η belongs to I_3 we have

$$I(K)\leqslant c_3|\eta|^{-2K+(\bar{m}+2\bar{\delta}K+2\bar{\delta}n_0)/\tilde{\rho}}\,, \qquad \bar{m}=\max(m, 0)\,, \tag{2.13}$$

since it holds that

$$\begin{cases} \lambda(x, \xi+\eta)\leqslant c_4|\eta|^{1/\tilde{\rho}}, \qquad \eta\in I_3\,, \\ \left|\dfrac{F(x, \xi+\eta; y)}{F(x, \xi; y)}\lambda(x, \xi)^{-n\bar{\delta}}\right|\leqslant c_4|\eta|^{2\bar{\delta}n_0/\tilde{\rho}} \end{cases}$$

for some constant c_4. By (2.13) we get (2.12) for $j=3$ if K_3 is chosen as above. Q.E.D.

Proposition 2.6. *Set*

$$\begin{aligned} J_l = &|\eta|^{-2K_l}\{\lambda(z^2, \xi+\eta)+\lambda(z^2, \xi)\}^{2\bar{\delta}K_l}\lambda(z^1, \xi+\eta)^m\langle z^1-z^0\rangle^{-r_1} \\ &\times F(z^1, \xi+\eta^1; z^1-z^0)\langle z^2-z^1\rangle^{-r_2}F(z^2, \xi; z^2-z^0)\,, \\ &(l=1, 2, 3)\,.\end{aligned}$$

Then we have for $l=1, 2, 3$

$$\int_{I_l}\int_{R^n} J_l dz^1 d\eta^1 \leqslant B\langle z^2-z^0\rangle^{-r_3}\lambda(z^2, \xi)^m F(z^2, \xi; z^2-z^0)$$

with $B=Cc(A_0)^{|m|}$ *and* $r_3=\min(r_1, r_2-2\tau\bar{\delta}n_0-\tau|m|)$ *if* K_l *and* I_l *are defined as in Proposition* 2.5 *and* $n_0=[n/2]+1$, $r_1\geqslant 0$ *and* $r_2-2\tau\bar{\delta}n_0-\tau|m|\geqslant 0$.

Proof. By means of Proposition 2.4 for $\eta^1=\eta$, $\eta^2=0$ and (1.1)-(ii) we get

$$(2.14)\qquad \int_{R^n} J_l dz^1 \leqslant C(A_0)^{|m|}|\eta|^{-2K_l}\{\lambda(z^2, \xi+\eta)+\lambda(z^2, \xi)\}^{2\bar{\delta}K_l}$$
$$\times\left\{\lambda(z^2, \xi+\eta)^{-\bar{\delta}n}+\frac{F(z^2, \xi+\eta; z^2-z^0)}{F(z^2, \xi, z^2-z^0)}\lambda(z^2, \xi)^{-\bar{\delta}n}\right\}$$
$$\times\langle z^2-z^0\rangle^{-r_3}\lambda(z^2, \xi+\eta)^m F(z^2, \xi; z^2-z^0), \qquad l=1, 2, 3 .$$

Now by Proposition 2.5 and we get the assertion. Q.E.D.

Proof of Lemma 2.2. Set $n_0=[n/2]+1$, $M=\sum_{j=1}^{\nu}|m_j|$, $K=[M+2\bar{\delta}n_0+n\tilde{\rho}/2(\tilde{\rho}-\delta)]+1$, $N=[\tau(3\bar{\delta}n_0+3\bar{\delta}K+2M)]+1$ and functions $K_j=K_j(\eta^j, \eta^{j+1}, z^{j+1})$ $(j=1, \cdots, \nu-1)$ as follow: $K_j=0$ on $I_{j,1}$, $K_j=n_0$ on $I_{j,2}$ and $K_j=K$ on $I_{j,3}$, where

$$I_{j,1}=\{\eta^j\in R^m;\ |\eta^j-\eta^{j+1}|\leqslant c_0\lambda(z^{j+1}, \xi+\theta\eta^{j+1})^{\bar{\delta}}\},$$
$$I_{j,2}=\{\eta^j\in R^n;\ c_0\lambda(z^{j+1}, \xi+\theta\eta^{j+1})^{\bar{\delta}}<|\eta^j-\eta^{j+1}|\leqslant c_0\lambda(z^{j+1}, \xi+\theta\eta^{j+1})^{\tilde{\rho}}\}$$

and

$$I_{j,3}=\{\eta^j\in R^n;\ |\eta^j-\eta^{j+1}|>c_0\lambda(z^{j+1}, \xi+\theta\eta^{j+1})^{\tilde{\rho}}\}\qquad (z^\nu=x,\ \eta^\nu=0).$$

By integration by parts we obtain

$$I_\theta=Os-\int\cdots\int\exp\{-i\sum_{j=1}^{\nu=1}y^j\cdot\eta^j\}\prod_{j=1}^{\nu-1}\langle y^j\rangle^{-2N}$$
$$\times\{1+(-\Delta_{\eta j})^{n_0}\lambda(x+\sum_{k=0}^{j-1}y^k, \xi+\theta\eta^j)^{2\bar{\delta}n_0}\}\{1+\lambda(x+\sum_{k=0}^{j-1}y^k, \xi+\theta\eta^j)^{2\bar{\delta}n_0}$$
$$\times|y^j|^{2n_0}\}^{-1}(-\Delta_{\eta j})^N q(x, \xi+\theta\eta^1, \cdots, x+\sum_{k=1}^{j-1}y^k, \xi+\theta\eta^j, \cdots, x+\sum_{k=1}^{\nu-1}y^k, \xi)dV,$$

where $y^0=0$. Then by change of variables $x+\sum_{k=1}^{j}y^k=z^j$ $(j=1, \cdots, \nu-1)$ we get

$$I_\theta=\int\cdots\int\exp\{-i\sum_{j=1}^{\nu-1}z^j\cdot(\eta^j-\eta^{j+1})\}\prod_{j=1}^{\nu-1}|\eta^j-\eta^{j+1}|^{-2K_j}(-\Delta_{z^j})^{K_j}r\,dV,$$

where

$$r=\prod_{j=1}^{\nu-1}\{1+(-\Delta_{\eta^j})^{n_0}\cdot\lambda(z^{j-1}, \xi+\theta\eta^j)^{2\bar{\delta}n_0}\}\prod_{j=1}^{\nu-1}\langle z^j-z^{j-1}\rangle^{-2N}$$
$$\times F(z^{j-1}, \xi+\theta\eta^j; z^j-z^{j-1})\langle\Delta_{\eta^j}\rangle^N q(z^0, \xi+\theta\eta^1, z^1, \cdots, \xi+\theta\eta^{\nu-1}, z^{\nu-1}, \xi),$$

$z^0=x$ and $\eta^\nu=0$.

Then from Proposition 2.3 and (2.3) we have with a constant C_1

$$(2.15)\quad |\prod_{j=1}^{\nu-1}(-\Delta_{z^j})^{K_j}r| \leqslant (C_1)^\nu M_{2(K+N+n_0)}\prod_{j=1}^{\nu-1}\langle z^j-z^{j-1}\rangle^{-2N}$$

$$\times\{\lambda(z^{j-1},\xi+\theta\eta^j)+\lambda(z^j,\xi+\theta\eta_0^{j+1})\}^{2\bar\delta K_j}F(z^{j-1},\xi+\theta\eta^j;z^j-z^{j-1})$$

$$\times\prod_{j=1}^{\nu}\lambda(z^{j-1},\xi+\theta\eta^j)^{m_j}$$

$$\leqslant C_2^\nu M_{2(K+N+n_0)}\prod_{j=1}^{\nu-1}\{\lambda(z^{j+1},\xi+\theta\eta^j)+\lambda(z^{j+1},\xi+\theta\eta^{j+1})\}^{2\bar\delta K_j}$$

$$\times\langle z^j-z^{j-1}\rangle^{-2M+R}F(z^j,\xi+\theta\eta^j;z^j-z^{j-1})\lambda(z^j,\xi+\theta\eta^j)^{m_j}$$

$$\times\langle z^\nu-z^{\nu-1}\rangle^{R'}\lambda(z^\nu,\xi)^{m_\nu},$$

where $z^0=z^\nu=x$, $\eta^\nu=0$, $R=\tau(2\bar\delta n_0+4\bar\delta K+M)$, $R'=\tau(2\bar\delta K+M)$ and $C_2=C_1(2A_0)^{M+2\bar\delta(K+n_0)}$. We used (1.1)–(iii) and

$$\{1+\lambda(z^j,\xi+\theta\eta^j)^{\bar\delta}\lambda(z^{j-1},\xi+\eta^j)^{-\bar\rho}\}\leqslant(2A_0)^{\bar\delta}\langle z^j-z^{j-1}\rangle^{\bar\delta}$$
$$(j=1,\cdots,\nu-1)$$

in the last step. From (2.15) and Proposition 2.6 we get for $l=1, 2, 3$

$$\int_{I_{1,l}}|\eta^1-\eta^2|^{-2K_1}|\prod_{j=1}^{\nu-1}(-\Delta_{z^j})^{K_j}r|\,dz^1d\eta^1$$

$$\leqslant(C_2)^{\nu-1}C_3M_{2(K+N+n_0)}\prod_{j=2}^{\nu-1}\{\lambda(z^{j+1},\xi+\theta\eta^j)+\lambda(z^{j+1},\xi+\theta\eta^{j+1})\}^{2\bar\delta K_j}$$

$$\times\prod_{j=3}^{\nu-1}F(z^j,\xi+\theta\eta^j;z^j-z^{j-1})\langle z^j-z^{j-1}\rangle^{-2N+R}\lambda(z^j,\xi+\theta\eta^j)^{m_j}$$

$$\times F(z^2,\xi+\theta\eta^2;z^2-z^0)\langle z^2-z^0\rangle^{-2N+R''}\lambda(z^2,\xi+\theta\eta^2)^{\tilde m_2}$$

$$\times\langle z^\nu-z^{\nu-1}\rangle^{R'}\lambda(z^\nu,\xi)^{m_\nu},$$

where $C_3=C_2Cc(A_0)^M$, $\tilde m_2=m_1+m_2$, and $R''=R+\tau(2\bar\delta n_0+M)$. Since $-2N+R+\tau(2\bar\delta n_0+M)\leqslant 0$, we can repeat the same argument. Hence we obtain

$$\int\cdots\int\prod_{j=1}^{\nu-1}|\eta^j-\eta^{j+1}|^{-2K_j}(-\Delta_{z^j})^{K_j}r\,dz^1dz^2\cdots dz^{\nu-2}d\eta^1d\eta^2\cdots d\eta^{\nu-2}$$

$$\leqslant(C_2)^2(C_3)^{\nu-1}M_{2(K+N+n_0)}|\eta^\nu-\eta^{\nu-1}|^{-2K_{\nu-1}}\{\lambda(z^\nu,\xi+\theta\eta^{\nu-1})$$

$$+\lambda(z^\nu,\xi+\theta\eta^\nu)\}^{2\bar\delta K_{\nu-1}}F(z^{\nu-1},\xi+\theta\eta^{\nu-1};z^{\nu-1}-z^0)$$

$$\times\langle z^{\nu-1}-z^0\rangle^{-2N+R''}\lambda(z^{\nu-1},\xi+\theta\eta^{\nu-1})^{\tilde m_{\nu-1}}\langle z^\nu-z^{\nu-1}\rangle^{R'}\lambda(z^\nu,\xi)^{m_\nu}$$

$$(z^0=z^\nu=x,\ \eta^\nu=0),$$

where $\tilde m_{\nu-1}=\sum_{j=1}^{\nu-1}m_j$. Noting $-2N+R''+R'+2\tau\bar\delta n_0+M=-2N+\tau(6\bar\delta n_0+6\bar\delta K+4M)\leqslant 0$, we get, by (1.1)–(ii), (2.10), (2.11) and Proposition 2.5,

$$\int\cdots\int\prod_{j=1}^{\nu-1}|\eta^j-\eta^{j+1}|^{-2K_j}|(-\Delta_{z^j})^{K_j}r|\,dV\leqslant C_2(C_3)^{\nu-1}M_{2(K+N+n_0)}\lambda(x,\xi)^{m_0}.$$

Take $l_0=2(K+N+n_0)$ and $C_0=C_3$. Thus we get (2.5). Q.E.D.

We denote the symbol $\sigma(P_1P_2\cdots P_\nu)$ by

$$\sigma(P_1P_2\cdots P_\nu)=p_1\circ p_2\circ\cdots\circ p_\nu$$

as used in [9].

Now for an operator $P=p(x,D_x)\in S^m_{\lambda,\rho,\delta}$ we define the adjoint operator P^* by the relation

$$(Pu,v)=(u,P^*v)\qquad\text{for } u, v\in\mathcal{S}.$$

Then we have

$$\begin{aligned}P^*u(x)&=\iint e^{i(x-y)\cdot\xi}p(y,\xi)u(y)dy\,d\xi\\&=\int\cdots\int e^{-(y^1\cdot\xi^1+y^2\cdot\xi^2)}p(x+y^1,\xi^1)u(x+y^1+y^2)dy^1d\xi^1dy^2d\xi^2.\end{aligned}$$

It is clear that P^* is also a pseudo-differential operator with symbol

$$\sigma(P^*)(x,\xi)=Os-\iint e^{-iy\cdot\eta}p(x+y,\xi+\eta)dyd\eta.$$

Theorem 2.7. *If P belongs to $S^m_{\lambda,\rho,\delta}$, then P^* belongs to $S^m_{\lambda,\rho,\delta}$. Moreover for any l there exists l' such that*

$$|\sigma(P^*)|_l^{(m)}\leqslant C|\sigma(P)|_{l'}^{(m)}$$

with some constant C.

Proof. Set $n_0=[n/2]+1$. By integration by parts we obtain

$$\begin{aligned}\sigma(P^*)(x,\xi)=Os-\iint &e^{-iy\cdot\eta}\langle y\rangle^{-2N}\{1+\lambda(x,\xi)^{2\bar\delta n_0}|y|^{2n_0}\}^{-1}\\&\times\{1+\lambda(x,\xi)^{2\bar\delta}(-\Delta_\eta)^{n_0}\}\langle-\Delta_\eta\rangle^N p(x+y,\xi+\eta)dy\,d\eta.\end{aligned}$$

Choose K as follows: $K=0$ on I_1, $K=n_0$ on I_2 and $K=[(|m|+2\bar\delta n_0+n\tilde\rho)/2(\tilde\rho-\delta)]+1$ on I_3, where

$$\begin{aligned}I_1&=\{\eta\in R^n;\ |\eta|\leqslant c_0\lambda(x,\xi)^{\bar\delta}\},\\I_2&=\{\eta\in R^n;\ c_0\lambda(x,\xi)^{\bar\delta}<|\eta|\leqslant c_0\lambda(x,\xi)^{\tilde\rho}\}\end{aligned}$$

and

$$I_3=R^n\backslash(I_1\cup I_2).$$

Then we have

$$\sigma(P^*)(x, \xi) = \iint e^{-iy\cdot\eta} |\eta|^{-2K}(-\Delta_y)^K r\, dy\, d\eta ,$$

where r satisfies

$$|(-\Delta_y)^K r| \leqslant C\, |p|^{(m)}_{2(K+N+n_0)} \langle y\rangle^{-2N+\tau(|m|+2\bar{\delta}K)}$$
$$\times \lambda(x, \xi+\eta)^{m+2\bar{\delta}K}\{1+\lambda(x, \xi)^{2\bar{\delta}n_0}|y|^{2n_0}\}^{-1} .$$

Choose $2N \geqslant \tau(|m|+2\bar{\delta}K)$. Noting the above estimate, we get the assertion if we repeat the same argument as in the proof of Lemma 2.2. Q.E.D.

From Theorems 2.1 and 2.7 we get the L^2-boundedness theorem by the same argument in [5].

Theorem 2.8. *Let $P \in S^0_{\lambda,\rho,\delta}$. Then P is a bounded operator in $L^2(R^n)$ and there exist l_0 and C such that*

$$\|Pu\| \leqslant C\, |p|^{(0)}_{l_0}\|u\| , \qquad u \in L^2(R^n) .$$

For pseudo-differential operators of this class we get the following expansion theorem.

Theorem 2.9. *If $p_j(x, \xi)$ belongs to $S^{m_j}_{\lambda,\rho,\delta}$ $(j=1, 2)$, we can write for any N*

$$(p_1\circ p_2)(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} p_1^{(\alpha)}(x, \xi) p_{2(\alpha)}(x, \xi) + r_N(x, \xi) ,$$

where $r_N(x, \xi)$ belongs to $S^{m_1+m_2-\varepsilon_0 N}_{\lambda,\rho,\delta}$ and $\varepsilon_0 = \min_{1\leqslant j\leqslant n} (\rho_j - \delta_j)$.

Proof. By the Taylor expansion we can write

$$\begin{aligned}(p_1\circ p_2)(x, \xi) &= Os\text{-}\iint e^{-iy\cdot\eta} p_1(x, \xi+\eta) p_2(x+y, \xi)\, dy\, d\eta \\ &= Os\text{-}\iint e^{-iy\cdot\eta} \sum_{|\alpha|<N} \frac{1}{\alpha!} p_1^{(\alpha)}(x, \xi)\eta^\alpha p_2(x+y, \xi)\, dy\, d\eta \\ &\quad + Os\text{-}\iint e^{-iy\cdot\eta} \sum_{|\gamma|=N} \frac{\eta^\gamma}{\gamma!} \int_0^1 (1-\theta)^{N-1} p_1^{(\gamma)}(x, \xi+\theta\eta)\, d\theta\, p_2(x+y, \xi)\, dy\, d\eta \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} p_1^{(\alpha)}(x, \xi) p_{2(\alpha)}(x, \xi) + r(x, \xi) ,\end{aligned}$$

where $r(x, \xi) = N \sum_{|\gamma|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} \Big\{ Os\text{-}\iint e^{-iy\cdot\eta} p_1^{(\gamma)}(x, \xi+\theta\eta)$ $\times p_{2(\gamma)}(x+y, \xi)\, dy\, d\eta \Big\} d\theta$. Apply Lemma 2.2 for $r(x, \xi)$ setting $q(x^1, \xi^1, x^2, \xi^2) = \sum_{|\gamma|=N} p_1^{(\gamma)}(x^1, \xi^1) p_{2(\gamma)}(x^2, \xi^2)$. Then it is clear that $r(x, \xi)$ belongs to $S^{m_1+m_2-\varepsilon_0 N}_{\lambda,\rho,\delta}$. Q.E.D.

In what follows we assume that $\varepsilon_0=\min_{1\leqslant j\leqslant n}(\rho_j-\delta_j)$ is positive. Let $p(x,\xi)\in S^m_{\lambda,\rho,\delta}$ satisfy the following conditions (H.E)

$$(2.16)\qquad (\text{H.E})\begin{cases} |p(x,\xi)|\geqslant c\lambda(x,\xi)^{m'} & m\geqslant m'\geqslant 0\,,\\ |p^{(\alpha)}_{(\beta)}(x,\xi)/p(x,\xi)|\leqslant C_{\alpha,\beta}\lambda(x,\xi)^{-(\rho,\alpha)+(\delta,\beta)} & (\rho>\delta)\,.\end{cases}$$

Then we get the following theorems in the same way as in [6].

Theorem 2.10 (cf. [4], [6]). *If $p(x,\xi)$ satisfies Condition (H.E), then $p(x,D_x)$ has a parametrix $q(x,D_x)$, which belongs to $S^{-m'}_{\lambda,\rho,\delta}$, in the sense $p(x,D_x)q(x,D_x)\equiv q(x,D_x)p(x,D_x)\equiv I$ (mod $S^{-\infty}_{\lambda,\rho,\delta}$).*

Theorem 2.11 (cf. [6], [8]). *If $p(x,\xi)$ satisfies (H.E) and arg $p(x,\xi)$ is well defined, then we can construct the complex power $\{p_z(x,D_x)\}_{z\in C}$ of $p(x,D_x)$ such that $P_{z_1}P_{z_2}\equiv P_{z_1+z_2}$, $P_0=I$, $P_1\equiv P$, $P_z\in S^{mRez}_{\lambda,\rho,\delta}$(Re $z\geqslant 0$) and $P_z\in S^{m'Rez}_{\lambda,\rho,\delta}$ (Re $z<0$).*

Let $\Lambda(x,D_x)$ be a pseudo-differential operator with a symbol $\lambda(x,\xi)$. For any $s\geqslant 0$ we define $H_s=\{u\in L^2(R^n);\ \Lambda_s(x,D_x)u\in L^2(R^m)\}$ with the norm

$$||u||_s^2=||\Lambda_s u||^2+||u||^2\,.$$

Let $0\leqslant s_1<s_2$ and let $\lambda(x,\xi)$ satisfy that for any $\varepsilon>0$ there exists C_ε such that

$$(2.17)\qquad \lambda(x,\xi)^{s_1}\leqslant\varepsilon\lambda(x,\xi)^{s_2}+C_\varepsilon\,.$$

Proposition 2.12. *If $\lambda(x,\xi)$ satisfies (2.17), then we have for any $\varepsilon>0$*

$$||u||_{s_1}\leqslant\varepsilon||u||_{s_2}+C_\varepsilon||u||$$

with a constant C_ε.

Proposition 2.13. *Let $p(x,\xi)\in S^m_{\lambda,\rho,\delta}$ satisfy (H.E) and let $q(x,\xi)$ satisfy*

$$|q^{(\alpha)}_{(\beta)}(x,\xi)/p(x,\xi)|\leqslant C_{\alpha,\beta}\lambda(x,\xi)^{k-(\rho,\alpha)+(\delta,\beta)}$$

with a constant k. Then there exists $r(x,\xi)\in S^k_{\lambda,\rho,\delta}$ such that $q(x,D_x)=r(x,D_x)p(x,D_x)+k(x,D_x)$, with $k(x,\xi)\in S^{-\infty}_{\lambda,\rho,\delta}$.

Proof. Let $r_1(x,\xi)=q(x,\xi)/p(x,\xi)\in S^k_{\lambda,\rho,\delta}$. Then we have for any N

$$(r_1\circ p)(x,\xi)=q(x,\xi)+t_N(x,\xi)+k_N(x,\xi)\,,$$

where $t_N(x,\xi)=\sum_{|\alpha|\geqslant 1}^{N-1}\frac{1}{\alpha!}r_1^{(\alpha)}(x,\xi)p_{(\alpha)}(x,\xi)$ and $k_N(x,\xi)\in S^{m+k-\varepsilon_0 N}_{\lambda,\rho,\delta}$. We note that

$$|t_N{}^{(\alpha)}_{(\beta)}(x,\xi)/p(x,\xi)|\leqslant C'_{\alpha,\beta}\lambda(x,\xi)^{k-\varepsilon_0-(\rho,\alpha)+(\delta,\beta)}.$$

Set $r_2(x,\xi)=t_N(x,\xi)/p(x,\xi)(\in S^{k-\varepsilon_0}_{\lambda,\rho,\delta})$. Then we have

$$\sigma(\sum_{j=1}^{2} r_j(x, D_x)p(x, D_x)) = q(x, \xi)+t'_N(x, \xi)+k'_N(x, \xi),$$

where $t'_N(x, \xi)=\sum_{|\alpha|\geq 1}^{N-2}\frac{1}{\alpha!}r_2^{(\alpha)}(x, \xi)p_{(\alpha)}(x, \xi)$ and $k'_N(x, \xi)\in S^{m+k-\varepsilon_0 N}_{\lambda,\rho,\delta}$. If we repeat the same calculus, we get the assertion. Q.E.D.

Proposition 2.14. *If $p_\varepsilon(x, \xi)$ converges to $p_0(x, \xi)$ weakly in $S^m_{\lambda,\rho,\delta}$ as $\varepsilon\to 0$, then $(p_\varepsilon\circ q)(x, \xi)$ converges to $(p_0\circ q)(x, \xi)$ weakly in $S^{m+k}_{\lambda,\rho,\delta}$ for any $q(x, \xi)\in S^k_{\lambda,\rho,\delta}$. Moreover $P_\varepsilon u$ converges to $P_0 u$ in H_{s-m} for $u\in H_s$.*

Proof. For large l and l' we can write

$$(p_\varepsilon\circ q)(x, \xi)$$
$$= \int\cdots\int e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\{\langle\eta\rangle^{-2l}\langle D_\eta\rangle^{2l}p_\varepsilon(x, \xi+\eta)q(x+y,\xi)\}dyd\eta.$$

Then the first part of the Proposition is clear. Set $Q_\varepsilon=\Lambda_{-s-m}P_\varepsilon\Lambda_s$. Then Q_ε belongs to $S^0_{\lambda,\rho,\delta}$ and $q_\varepsilon(x, \xi)$ converges to $q(x, \xi)$ weakly in $S^0_{\lambda,\rho,\delta}$. It is sufficient to show that if $q_\varepsilon(x, \xi)$ converges to 0 weakly in $S^0_{\lambda,\rho,\delta}$, then $Q_\varepsilon u$ converges to 0 for $u\in L^2(R^n)$. Define $u_\varepsilon(x)=\varphi_\varepsilon(x)\varphi_\varepsilon(D_x)u$ where $\varphi_\varepsilon(x)=\varphi(\varepsilon x)$ and φ is a $C_0^\infty(R^n)$-function such that $\varphi(x)=1$ $(|x|\leqslant 1)$ and $\varphi(x)=0$ $(|x|\geqslant 2)$. We have

$$\|Q_\varepsilon u\|\leqq\|Q_\varepsilon(u_\varepsilon-u)\|+\|Q_\varepsilon u_\varepsilon\|$$
$$\leqslant C\|u_\varepsilon-u\|+\|Q_\varepsilon u_\varepsilon\|,$$

where we use Theorem 2.8. It is clear that u_ε converges to u in $L^2(R^n)$. We can write

$$Q_\varepsilon u_\varepsilon = \tilde{q}_\varepsilon(x, D_x)u,$$

where

$$\tilde{q}_\varepsilon(x, \xi) = \iint_{|x+y|\leqslant 2\varepsilon^{-1}} e^{-iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}(\langle\eta\rangle^{-2l}q(x, \xi+\eta))$$
$$\times\langle D_y\rangle^{2l}\varphi(\varepsilon(x+y))\varphi(\varepsilon\xi)dyd\eta.$$

Then $\tilde{q}_\varepsilon(x, \xi)$ converges to 0 in $S^0_{\lambda,\rho,\delta}$. So we get $\lim_{\varepsilon\to 0}\|Q_\varepsilon u_\varepsilon\|=0$ by Theorem 2.8. Q.E.D.

3. Fundamental solution of degenerate pseudo-differential operator of parabolic type and the Cauchy problem

In this section we consider the Cauchy problem for a pseudo-differential operator of parabolic type as follows.

$$(3.1)\qquad \begin{cases} Lu(t) = \left(\dfrac{d}{dt}+p(t; x, D_x)\right)u(t) = f(t) & \text{in } 0<t<T \\ u(0) = u_0 \end{cases}$$

where $p(t; x, D_x)$ is an operator in the class $\mathcal{E}_t^0(S^m_{\lambda,\rho,\delta})$ $(\delta<\rho)$ on $[0, T]$ which satisfies the following conditions

$$
(3.2)\quad \begin{cases} \text{(i)} & \text{There exist constants } c_1\geqslant 0 \text{ and } c_0>0 \text{ such that} \\ & \quad Re\, p(t; x, \xi)+c_1\geqslant c_0\lambda(x, \xi)^{m'} \text{ in } [0, T]\ m\geqslant m'\geqslant 0\,. \\ \text{(ii)} & \text{For any multi-index } \alpha=(\alpha_1, \cdots, \alpha_n) \text{ and } \beta=(\beta_1, \cdots, \beta_n) \\ & \text{there exists a constant } C_{\alpha,\beta} \text{ such that} \\ & \quad |p^{(\alpha)}_{(\beta)}(t; x, \xi)/(Re\, p(t; x, \xi)+c_1)|\leqslant C_{\alpha,\beta}\lambda(x, \xi)^{-(\rho,\alpha)+(\delta,\beta)} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0, T]. \end{cases}
$$

We call $E(t, s)$ the fundamental solution of L if $E(t, s)$ satisfies

$$
(3.3)\quad \begin{cases} LE(t, s) = 0 & \text{in } 0\leqslant s<t\leqslant T\,, \\ E(s, s) = I \end{cases}
$$

Theorem 3.1. *Under the assumptions* (3.2)-(i) *and* (3.2)-(ii) *there exists a fundamental solution $E(t, s)$ in the class $\omega-\mathcal{E}^0_{t,s}(S^0_{\lambda,\rho,\delta})$ in $0\leqslant s\leqslant t\leqslant T$. Moreover for any N such that $m-\varepsilon_0 N\leqslant 0$ $(\varepsilon_0=\min_{1\leqslant j\leqslant n}(\rho_j-\delta_j))$ $E(t, s)$ has the following expansion*

$$
e(t, s) = \sum_{j=0}^{N-1} e_j(t, s)+f_N(t, s)\,,
$$

where

$$
(3.4)\quad \begin{cases} \text{(i)} & e_j(t, s)\in\omega-\mathcal{E}^0_{t,s}(S^{-\varepsilon_0 j}_{\lambda,\rho,\delta}), \qquad j\geqslant 0 \\ \text{(ii)} & e_0(t, s)\to 1 \text{ as } t\to s \text{ weakly in } S^0_{\lambda,\rho,\delta}, \\ \text{(iii)} & e_j(t, s)\to 0 \text{ as } t\to s \text{ weakly in } S^{-\varepsilon_0 j}_{\lambda,\rho,\delta}, \\ \text{(iv)} & a_{j,\alpha}(t, s; x, \xi)=e_{j}{}^{(\alpha)}_{(\beta)}(t, s; x, \xi)/e_0(t, s; x, \xi)\ (j\geqslant 0) \text{ satisfies} \\ & \quad |a_{j,\alpha,\beta}(t, s; x, \xi)|\leqslant C_{\alpha,\beta}\lambda(x, \xi)^{-\varepsilon_0 j-(\rho,\alpha)+(\delta,\beta)} \\ & \qquad \times \sum_{k=2}^{|\alpha|+|\beta|+2j}\left\{Re\int_s^t p(\sigma; x, \xi)d\sigma+c_1(t-s)\right\}^k \\ \text{(v)} & f_N(t, s)\in\omega-\mathcal{E}^0_{t,s}(S^{m-\varepsilon_0 N}_{\lambda,\rho,\delta}) \text{ and satisfies} \\ & \quad |f_{N}{}^{(\alpha)}_{(\beta)}(t, s; x, \xi)|\leqslant C(t-s)^k\lambda(x, \xi)^{km-\varepsilon_0 N-(\rho,\alpha)+(\delta,\beta)} \\ & \qquad\qquad\qquad\qquad\qquad\qquad (k=1, 2). \end{cases}
$$

Proof. We may assume (3.2) for $c_1=0$. In fact let $E_{c_1}(t, s)$ be the fundamental solution for $L+c_1$. Then $E(t, s)=e^{c_1(t-s)}E_{c_1}(t, s)$ is the fundamental solution for L.

As in [10], [11] we construct $e_j(t, s; x, \xi)$ $(0\leqslant s\leqslant t\leqslant T)$ as the series of solutions of the following equations

$$
(3.5)\quad \begin{cases} \left(\dfrac{d}{dt}+p(t; x, \xi)\right)e_0(t, s; x, \xi) = 0 & \text{in } t>s\,, \\ e_0(s, s; x, \xi) = 1 \end{cases}
$$

and for $j \geqslant 1$

$$(3.6)\qquad \begin{cases} \left(\frac{d}{dt}+p(t;x,\xi)\right)e_j(t,s;x,\xi)=-q_j(t,s;x,\xi) & \text{in } t>s, \\ e_j(s,s;x,\xi)=0, \end{cases}$$

where $q_j(t, s; x, \xi)$ is defined by

$$(3.7)\qquad q_j(t,s;x,\xi)=\sum_{k=0}^{j-1}\sum_{|\alpha|+k=j}\frac{1}{\alpha!}p^{(\alpha)}(t;x,\xi)e_{k(\alpha)}(t,s;x,\xi).$$

Set

$$(3.8)\qquad b_{j,\alpha,\beta}(t,s;x,\xi)=q_{j(\beta)}^{(\alpha)}(t,s;x,\xi)/e_0(t,s;x,\xi)\qquad j\geqslant 1.$$

Then, by (3.5)~(3.7) and (3.2)-(ii) we have the following proposition, which derives (3.4)-(i)~(3.4)-(iv).

Proposition 3.2. *For any α and β there exists a constant $C_{j,\alpha,\beta}$ such that*

$$(3.9)_j\qquad |a_{j,\alpha,\beta}(t,s;x,\xi)|\leqslant C_{j,\alpha,\beta}\lambda(x,\xi)^{-\varepsilon_0 j-(\rho,\alpha)+(\delta,\beta)}\omega_{j,\alpha,\beta}\qquad (j\geqslant 0),$$

$$(3.10)_j\qquad |b_{j,\alpha,\beta}(t,s;x,\xi)|\leqslant C_{j,\alpha,\beta}\operatorname{Re}p(t;x,\xi)\lambda(x,\xi)^{-\varepsilon_0 j-(\rho,\alpha)+(\delta,\beta)}\omega'_{j,\alpha,\beta}\qquad (j\geqslant 1),$$

where $\omega_{j,\alpha,\beta}$ and $\omega'_{j,\alpha,\beta}$ are defined by

$$\omega_{0,0,0}=1,\qquad \omega_{0,\alpha,\beta}=\max\{\omega,\omega^{|\alpha|+|\beta|}\}\qquad |\alpha|+|\beta|\neq 0$$

$$\omega_{j,\alpha,\beta}=\max\{\omega^2,\omega^{2+|\alpha|+|\beta|}\}\qquad (j\geqslant 1),$$

$$\omega'_{j,\alpha,\beta}=\max\{\omega,\omega^{2j-1+|\alpha|+|\beta|}\}\qquad (j\geqslant 1)$$

and $\omega=\int_s^t \operatorname{Re}p(\sigma;x,\xi)\,d\sigma$.

Proof. By (3.7) we have

$$q_{j(\beta)}^{(\alpha)}(t,s;x,\xi)=\sum_{k=0}^{j-1}\sum_{|\gamma|+k=j}\sum_{\substack{\alpha_k\leqslant\alpha\\ \beta_k\leqslant\beta}}C_{j,\alpha,\beta,\gamma}p_{(\beta_k)}^{(\gamma+\alpha_k)}(t)e_{k(\gamma+\beta-\beta_k)}^{(\alpha-\alpha_k)}(t,s)$$

with some positive constants $C_{\gamma,\alpha,\beta}$. Then it follows that

$$(3.11)\qquad b_{j,\alpha,\beta}(t,s)=\sum_{k=0}^{j-1}\sum_{|\gamma|+k=j}\sum_{\substack{\alpha_k+\alpha_k'=\alpha\\ \beta_k+\beta_k'=\beta}}C_{\gamma,\alpha,\beta}p_{(\beta_k)}^{(\gamma+\alpha_k)}(t)a_{k,\alpha_k',\gamma+\beta_k'}(t,s).$$

From (3.6) we can write

$$e_j(t,s;x,\xi)=\int_s^t -e_0(t,\sigma;x,\xi)q_j(\sigma,s;x,\xi)d\sigma.$$

Thus we have for any α β, and $j\geqslant 1$

$$(3.12)_j \qquad a_{j,\alpha,\beta}(t, s) = -\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \alpha!\beta!/(\alpha_1!\alpha_2!\beta_1!\beta_2!)\int_s^t a_{0,\alpha_1,\beta_1}(t, \sigma)b_{j,\alpha_2,\beta_2}(\sigma, s)d\sigma .$$

We shall prove $(3.9)_j$ and $(3.10)_j$ inductively. By (3.5) we get

$$(3.13) \qquad e_0(t, s; x, \xi) = \exp\left(-\int_s^t p(\sigma; x, \xi)\, d\sigma\right).$$

Then $a_{0,\alpha,\beta}(t, s)$ is a linear summation of

$$\int_s^t p_{(\beta_1)}^{(\alpha_1)}(\sigma; x, \xi)d\sigma \cdots \int_s^t p_{(\beta_l)}^{(\alpha_l)}(\sigma; x, \xi)d\sigma$$

with $\alpha_1+\cdots+\alpha_l=\alpha$, $\beta_l+\cdots+\beta_l=\beta$. Hence we get $(3.9)_0$ from the assumption (3.2)-(ii). By (3.11), $(3.9)_0$ and (3.2)-(ii) we get $(3.10)_1$. Now assume $(3.9)_j$ for $j\leqslant k-1$ and $(3.10)_j$ for $j\leqslant k$. Then we get $(3.9)_k$ and $(3.10)_{k+1}$ in the following way. From $(3.9)_0$, $(3.10)_k$ and (3.12) it follows that

$$\begin{aligned} |a_{k,\alpha,\beta}(t, s)| &\leqslant C'_{k,\alpha,\beta}\lambda^{-\varepsilon_0 k-(\rho,\alpha)+(\delta,\beta)}\omega \sum_{\substack{\alpha_2+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \omega'_{k,\alpha_1,\beta_1}\omega_{0,\alpha_2,\beta_2} \\ &\leqslant C_{k,\alpha,\beta}\lambda^{-\varepsilon_0 k-(\rho,\alpha)+(\delta,\beta)}\,\omega_{k,\alpha,\beta} . \end{aligned}$$

By (3.11) and $(3.9)_j$ for $j\leqslant k$, it is clear that

$$|b_{k+1,\alpha,\beta}(t, s)| \leqslant C'_{k,\alpha,\beta}\lambda^{-\varepsilon_0(k+1)-(\rho,\alpha)+(\delta,\beta)}\operatorname{Re} p(t)\sum_{j=0}^{k}\sum_{|\gamma|+j=k+1}\sum_{\substack{\alpha_j\leqslant\alpha \\ \beta_j\leqslant\beta}}\omega_{j,\alpha_j,\beta_j+\gamma}$$

with some constant $C'_{k,\alpha,\beta}$. Also it is easy to show

$$\max_{\substack{\alpha'\leqslant\alpha,\beta'\leqslant\beta \\ 0\leqslant j\leqslant k \\ |\gamma|+j=k+1}} \omega_{j,\alpha',\beta'+\gamma}\leqslant\omega'_{k+1,\alpha,\beta} .$$

Then $(3.10)_{k+1}$ is proved. Q.E.D.

Now by Theorem 2.9, we can write for any $N\geqslant 1$

$$\begin{aligned} (3.14) \qquad \sigma(P(t)E_j(t, s; x, D_x))(x, \xi) &= p(t; x, \xi)e_j(t, s; x, \xi) \\ &+\sum_{0<|\alpha|\leqslant N-j-1}\frac{1}{\alpha!}p^{(\alpha)}(t; x, \xi)e_{j(\alpha)}(t, s; x, \xi)+r_{N,j}(t, s; x, \xi) . \end{aligned}$$

Taking a summation in j, it is clear by (3.5)$\sim$(3.7) that

$$\begin{aligned} (3.15) \quad \left(\frac{d}{dt}+P(t)\right)\left(\sum_{j=0}^{N-1}E_j(t, s)\right) &= \sum_{j=0}^{N-1}\left(\left(\frac{d}{dt}+p(t)\right)e_j\right)(t, s; x, D_x) \\ &+\sum_{j=1}^{N-1}q_j(t, s; x, D_x)+\sum_{j=0}^{N-1}r_{N,j}(t, s; x, D_x) = \sum_{j=0}^{N-1}r_{N,j}(t, s; x, D_x) . \end{aligned}$$

Proposition 3.3. *We have* $r_{N,j}(t,s;x,\xi)\in\omega-\mathcal{E}^0_{t,s}(S^{m-\varepsilon_0 N}_{\lambda,\rho,\delta})$ *and for any* α,β

$$|r_{N,j}{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqslant C_{\alpha,\beta}(t-s)^k\lambda(x,\xi)^{(k+1)m-\varepsilon_0 N-(\rho,\alpha)+(\delta,\beta)}, \qquad k=0,1. \tag{3.16}$$

Proof. From (3.4)-(i) and (3.14) we have $r_{N,j}(t,s;x,\xi)\in\omega-\mathcal{E}^0_{t,s}(S^{m-\varepsilon_0 N}_{\lambda,\rho,\delta})$. From $(3.9)_j$ and $\omega\leqslant C(t-s)\lambda(x,\xi)^m$, we get (3.16). Q.E.D.

Put $\sum_{j=0}^{N} r_{N,j}(t,s;x,\xi)=r_N(t,s;x,\xi)$ and $\sum_{j=0}^{N} e_j(t,s;x,\xi)=k_N(t,s;x,\xi)$. Then we can write by (3.15)

$$\begin{cases} LK_N(t,s)=R_N(t,s) & \text{in } t>s\ (0\leqslant s<t\leqslant T)\\ K_N(s,s)=I. \end{cases} \tag{3.17}$$

Now we construct $e(t,s;x,\xi)$ in the form

$$e(t,s;x,D_x)=k_N(t,s;x,D_x)+\int_s^t k_N(t,\sigma;x,D_x)\varphi(\sigma,s;x,D_x)d\sigma.$$

Then $\varphi(t,s;x,D_x)=\Phi(t,s)$ must satisfy a Volterra's integral equation

$$R_N(t,s)+\Phi(t,s)+\int_s^t R_N(t,\sigma)\Phi(\sigma,s)d\sigma=0. \tag{3.18}$$

Set $\Phi_1(t,s)=-R_N(t,s)$ and define $\Phi_j(t,s)$ for $j\geqslant 2$

$$\begin{aligned}\Phi_j(t,s)&=\int_s^t \Phi_1(t,\sigma)\Phi_{j-1}(\sigma,s)d\sigma\\ &=\int_s^t\int_s^{s_1}\cdots\int_s^{s_{j-2}}\Phi_1(t,s_1)\Phi_1(s_1,s_2)\cdots\Phi_1(s_{j-1},s)ds_{j-1}\cdots ds_1.\end{aligned} \tag{3.19}$$

Then we have

$$\begin{aligned}\sum_{j=1}^{l}\Phi_j(t,s)&=\Phi_1(t,s)+\sum_{j=2}^{l}\Phi_j(t,s)\\ &=-R_N(t,s)-\int_s^t R_N(t,\sigma)\sum_{j=1}^{l-1}\Phi_j(\sigma,s)d\sigma.\end{aligned} \tag{3.20}$$

For $\sigma(\Phi_j(t,s))=\varphi_j(t,s;x,\xi)$ we have the following estimates.

Proposition 3.4. *We have some constants* $B_{\alpha,\beta}$ *and* $B'_{\alpha,\beta}$ *independent of* j *such that*

$$|\varphi_j{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqslant(B_{\alpha,\beta})^j\frac{(t-s)^{j-1}}{(j-1)!}\lambda(x,\xi)^{m-\varepsilon_0 N-(\rho,\alpha)+(\delta,\beta)} \tag{3.21}$$

$$|\varphi_j{}^{(\alpha)}_{(\beta)}(t,s;x,\xi)|\leqslant(B'_{\alpha,\beta})^j\frac{(t-s)^{j-1}}{j!}(t-s)\lambda(x,\xi)^{2m-\varepsilon_0 N-(\rho,\alpha)+(\delta,\beta)}. \tag{3.22}$$

Proof. Note that $r(t,s;x,\xi)=-\varphi_1(t,s;x,\xi)$ satisfies (3.16). Take N

such that $m-\varepsilon_0 N\leqslant 0$. Then we can apply Theorem 2.1 to $\Phi_1(s_{j-1}, s_j)$. For any l, α and β there exists l_0 such that

$$
\begin{aligned}
&|\varphi_{j(\beta)}^{(\alpha)}(t, s; x, \xi)|_l^{(m-\varepsilon_0 N)} \\
&\leqslant C^j |\varphi_1|_{l_0}^{(m-\varepsilon_0 N)} (|\varphi_1|_{l_0}^{(0)})^{j-1} \int_s^t \cdots \int_s^{s_{j-2}} ds_{j-1} \cdots ds_1 \\
&\leqslant (B_{\alpha,\beta})^j \frac{(t-s)^{j-1}}{(j-1)!} .
\end{aligned}
$$

If we use (3.16) for $k=1$ instead of (3.16) for $k=0$, we get

$$
\begin{aligned}
&|\varphi_{j(\beta)}^{(\alpha)}(t, s; x, \xi)|_l^{(2m-\varepsilon_0 N)} \\
&\leqslant C^j |\varphi_1|_{l_0}^{(2m-\varepsilon_0 N)} (|\varphi_1|_{l_0}^{(0)})^{j-1} \int_s^t \cdots \int_s^{s_{j-2}} (s_{j-1}-s) ds_{j-1} \cdots ds_1 \\
&\leqslant (B'_{\alpha,\beta})^j \frac{(t-s)^j}{j!}
\end{aligned}
$$

Q.E.D.

Set $\varphi(t, s; x, \xi)=\sum_{j=1}^{\infty} \varphi_j(t, s; x, \xi)$. In view of (3.21) $\varphi(t, s; x, \xi)$ belongs to $\omega-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^{m-\varepsilon_0 N})$ and satisfies (3.18) and

(3.23) $|\varphi_{(\beta)}^{(\alpha)}(t, s; x, \xi)| \leqslant \lambda(x, \xi)^{(k+1)m-\varepsilon_0 N-(\rho,\alpha)+(\delta,\beta)} \exp\{B_{\alpha,\beta}(t-s)\} \quad (k=0, 1)$.

Note that $K_N(t, s)$ belongs to $\omega-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^0)$. Then by (3.23) we get (3.4)-(v). Q.E.D.

REMARK. 1. By the same method we can construct the fundamental solution for $L=\frac{\partial}{\partial t}+p(t; x, D_x)+q(t; x, D_x)$ under the following conditions:

(i) $p(t; x, \xi)$ satisfies (3.2).

(ii) There exist $\varepsilon_1>0$ and $k\geqslant 0$ such that

$$
\left| \int_s^t q_{(\beta)}^{(\alpha)}(\sigma; x, \xi) d\sigma \right| \leqslant C'_{\alpha,\beta} \lambda(x, \xi)^{-\varepsilon_1-(\rho,\alpha)+(\delta,\beta)} \left\{ \int_s^t |p(\sigma; x, \xi)| d\sigma \right\}^k
$$

In this case $e_0(t, s; x, \xi)$ is defined by (3.5) and $e_j(t, s; x, \xi)$ is defined by (3.6) setting

$$
q_j(t, s; x, \xi) = \sum_{k=0}^{j-1} \sum_{|\alpha|+k=j} \frac{1}{\alpha!} p^{(\alpha)}(t; x, \xi) e_{k(\alpha)}(t, s; x, \xi) + q(t; x, \xi) e_{j-1}(t, s; x, \xi) .
$$

REMARK. 2. If $p(t; x, \xi)$ belongs to $\mathcal{E}_t^\infty(S_{\lambda,\rho,\delta}^m)$, the fundamental solution $e(t, s; x, \xi)$ belongs to $\bigcap_{l=0}^{\infty} \mathcal{E}_t^l(S_{\lambda,\rho,\delta}^{ml})$.

We note that $P^*(t)$ also satisfies the assumptions of Theorem 3.1. So we can construct $V(t, s)\in\omega-\mathcal{E}_{t,s}^0(S_{\lambda,\rho,\delta}^0)$ which satisfies

$$(3.24)\qquad \begin{cases} -\dfrac{\partial}{\partial s}V(t, s)+p^*(s; x, D_x)V(t, s)=0 & 0\leqslant s<t\leqslant T \\ V(t, t)=I \end{cases}$$

Theorem 3.5. *Let $V(t, s)$ and $E(t, s)$ satisfy* (3.24) *and* (3.3) *respectively. Then we get*

$$(3.25)\qquad E^*(t, s)=V(t, s) \qquad 0\leqslant s\leqslant t\leqslant T$$

and

$$(3.26)\qquad -\frac{\partial}{\partial s}E(t, s)+E(t, s)p(s; x, D_x)=0\,.$$

Proof. Let f and g be any function of $\mathcal{S}(R^n)$. For any r such that $s<r<t$ it is easy to see that

$$\begin{aligned} &\frac{\partial}{\partial r}(E(r, s)f,\ V(t, r)g) \\ &= -(P(r)E(r, s)f,\ V(t, r)g)+(E(r, s)f,\ P^*(r)V(t, r)g) \\ &= 0\,. \end{aligned}$$

If we use that $E(t, s)\to I$, $V(t, s)\to I$ in $L^2(R^n)$ as $t\to s$, we get (3.25). Considering the adjoint of (3.24), we get (3.26) if we use (3.25). Q.E.D.

Corollary. *If $p(t; x, D_x)$ is independent of t and self-adjoint then $E(t, s)=E(t-s)$ is also self-adjoint.*

Theorem 3.6. *Under the condition* (3.2) *the fundamental solution $E(t, s)$ is uniquely determined in the class $w-\mathcal{E}^0_{t,s}(S^{\infty}_{\lambda,\rho,\delta})$.*

In order to prove the above theorem we prepare the following

Proposition 3.7. *Under the condition* (3.2) *there exists a constant $c>0$ such that*

$$\mathrm{Re}\,(p(t; x, D_x)u, u)+c(u, u)\geqslant 0 \qquad u\in\mathcal{S}(R^n)\,.$$

Proof of Theorem 3.6. Let $E(t, s)(\in w-\mathcal{E}^0_{t,s}(S^{\infty}_{\lambda,\rho,\delta}))$ satisfy $LE(t, s)=0$ in $t>s$ and $E(s, s)=0$. Then $e^{-ct}E(t, s)=E_c(t, s)$ satisfies

$$(3.27)\qquad \begin{cases} (L+c)E_c(t, s)=0 & \text{in } t>s\,, \\ E_c(s, s)=0 \end{cases}$$

For any $u\in\mathcal{S}(R^n)$ we get by the above proposition

$$\frac{d}{dt}(E_c(t, s)u,\ E_c(t, s)u)$$

$$= 2\,\mathrm{Re}\Big(\frac{d}{dt}E_c(t, s)u,\ E_c(t, s)u\Big)$$
$$= -2\,\mathrm{Re}\,((P(t)+c)E_c(t, s)u,\ E_c(t, s)u) \leqslant 0\,.$$

Then we have

$$\|E_c(t, s)u\| \leqslant \|E_c(s, s)u\| = 0\,.$$

This means for any $x \in R^n$ and $\xi \in R^n$

$$e_c(t, s;\, x, \xi) = 0 \qquad \text{in } t \geqslant s\,.$$

Hence we get $e(t, s;\, x, \xi)=0$. Q.E.D.

Theorem 3.8. *Let* $p(t;\, x, \xi)$ *belong to* $\mathcal{E}_t^\infty(S^m_{\lambda,\rho,\delta})$ *and satisfy* (3.2). *Then for any* $f(t) \in \mathcal{E}_t^0(H_s)$ *and* $u_0 \in H_s$ *the solution* $u(t) \in \mathcal{E}_t^k(H_{s-km})$ *of* (3.1) *is given by*

$$u(t) = E(t, 0)u_0 + \int_0^t E(t, s)f(s)ds\,. \tag{3.28}$$

This is the unique solution of (3.1) *and* $u(t) \to u_0$ *in* H_s *as* $t \to 0$. *Moreover we get*

$$\Big\|\frac{d^k}{dt^k}u(t)\Big\|_{s-km} \leqslant C\|u_0\|_s + \int_0^t \|f(\sigma)\|_s d\sigma\,. \tag{3.29}$$

Proof. It is easy to show that $u(t)$ given by (3.28) is a solution of (3.1). Let $u(t)$ satisfy (3.1). Then

$$E(t, s)P(s)u(s) = E(t, s)\Big(-\frac{\partial}{\partial s}\Big)u(s) + E(t, s)f(s)\,.$$

Integrating with respect to s, we get

$$\int_0^t E(t, s)P(s)u(s)ds = \int_0^t E(t, s)f(s)ds + \int_0^t \frac{d}{ds}E(t, s)u(s)ds - [E(t, s)u(s)]_0^t\,.$$

By (3.28) we have

$$u(t) = \int_0^t E(t, s)f(s)ds + E(t, 0)u(0)\,.$$

The inequality (3.29) is clear if we note that $E(t, s)$ belongs to $w-\mathcal{E}^l_{t,s}(S^{ml}_{\lambda,\rho,\delta})$ $(l=1, 2, \cdots,)$.

Proof of Proposition 3.7. Set $Q(t) = (P(t)+P^*(t))/2$. Then $q(t;\, x, \xi)$ satisfies

$$\mathrm{Re}\, q(t;\, x, \xi) + c_1 \geqslant c_0\lambda(x, \xi)^{m'},$$
$$|q^{(\alpha)}_{(\beta)}(t;\, x, \xi)/(\mathrm{Re}\, q(t;\, x, \xi)+c_1)| \leqslant C_{\alpha,\beta}\lambda(x, \xi)^{-(\rho,\alpha)+(\delta,\beta)}$$

with constants c_0 and c_1. Apply Theorem 2.11. Then we can construct the complex power $\{\tilde{Q}_z(t)\}$ for $Q(t)+c_1$. Note that $Q(t)$ is self-adjoint. Then we have $\tilde{Q}_s^*(t) \equiv \tilde{Q}_s(t)$ for real s (See Lemma 4.2 in [6]). We obtain

$$\operatorname{Re}((P(t)+c_1)u, u) = (\tilde{Q}(t)u, u) = (\tilde{Q}_{1/2}(t)u, \tilde{Q}_{1/2}(t)u)+(K(t)u, u),$$

for some $K(t)\in\mathcal{E}_t^0(S^{-\infty}_{\lambda,\rho,\delta})$. Then we have

$$\operatorname{Re}((P(t)+c_1)u, u)\geqslant ||\tilde{Q}_{1/2}u||^2-c_2||u||^2.$$

Take $C=c_1+c_2$. Then we get the assertion. Q.E.D.

4. Behavior of $E(t, s)$ as $(t-s)\to\infty$

In this section we assume for the basic weight function $\lambda(x, \xi)$ to satisfy

$$\lambda(x, \xi)\geqslant A_0(1+|x|+|\xi|)^{\sigma} \tag{4.1}$$

with a positive constant σ and for $p(t; x, \xi)\in\mathcal{E}_t^\infty(S^m_{\lambda,\rho,\delta})$ to satisfy (3.2) with a positive constant m' and assume that there exist a positive constant c_2 and $t_0\geqslant 0$ such that

$$\operatorname{Re}(P(t)u, u)\geqslant c_2||u||^2 \qquad t_0<t<\infty \tag{4.2}$$

for $u\in\mathcal{S}(R^n)$.

Theorem 4.1. *Let $u(t)\in\mathcal{E}_t^\infty(\mathcal{S}(R^n))$ satisfy $Lu(t)=g(t)$ in $t>t_0$. Then for $b\geqslant 0$ and any $c_3<c_2$ there exists a constant B independent of t such that*

$$||u(t)||_b\leqslant B\left(e^{-c_3(t-t_0)}||u(t_0)||_b+\int_{t_0}^t e^{-c_3(t-s)}||g(s)||_b ds\right).$$

For the proof of the above theorem we prepare the following

Lemma 4.2. *Let v and w belong to $\mathcal{S}(R^n)$. Then we have with a constant C*

$$|(Av, Bw)|\leqslant C||v||\,||w|| \qquad \text{if } A\in S^{-m}_{\lambda,\rho,\delta} \text{ and } B\in S^m_{\lambda,\rho,\delta}, \tag{4.3}$$

$$|(Av, Bw)-(A_1v, B_1w)|\leqslant C||v||\,||w|| \qquad \text{if } A, A_1, B, B_1\in S^\infty_{\lambda,\rho,\delta},\ A\equiv A_1 \text{ and } B\equiv B_1, \tag{4.4}$$

$$\operatorname{Re}(P(t)\Lambda_s v, \Lambda_s v)\geqslant 1/2||Q_{1/2}\Lambda_s v||^2-C||v||^2 \tag{4.5}$$

and

$$|([\Lambda_s, P(t)]v, \Lambda_s v)|\leqslant\varepsilon||Q_{1/2}\Lambda_s v||^2+C_\varepsilon||v||^2 \qquad \text{for any } \varepsilon>0 \tag{4.6}$$

where $\{Q_z(t)\}$ is the complex power of $Q(t)=(P(t)+P^*(t))/2+c_1$

Proof. Set $R=(\Lambda+\Lambda^*)/2+d$ for large number d such that $\sigma(R)$ satisfies (H.E) (see (2.16)). Let $\{R_z\}$ be the complex power for R constructed in §2. We can write $R_{-m}R_m+K_1=I$, where K_1 belongs to $S^{-\infty}_{\lambda,\rho,\delta}$. Then we have

$$\begin{aligned}(Av, Bw) &= (R_mAv, R_{-m}Bw)+(K_1Av, Bw)\\ &= (R_mAv, R_{-m}Bw)+(R_mK_1Av, R_{-m}Bw)+(K_1Av, K_1^*Bw).\end{aligned}$$

Noting that R_mA, $R_{-m}B$, R_mK_1A, K_1A and K_1^*B belong to $S^0_{\lambda,\rho,\delta}$, we get (4.3). The estimate (4.4) is clear by (4.3).
For (4.5) we write

$$\mathrm{Re}\,(P(t)\Lambda_s v,\ \Lambda_s v) = (Q_{1/2}(t)\Lambda_s v,\ Q_{1/2}(t)\Lambda_s v)+(K_2(t)\Lambda_s v,\ \Lambda_s v)-c_1(\Lambda_s v,\ \Lambda_s v)\,,$$

where

$$(4.7) \qquad Q^*_{1/2}(t)Q_{1/2}(t)+K_2(t) = Q(t)\,, \qquad K_2\in\mathcal{E}^\infty_t(S^{-\infty}_{\lambda,\rho,\delta})\,.$$

We can write by Proposition 2.13 $c_1\equiv G_1(t)Q_{1/2}(t)$ where $G_1(t)$ belongs to $\mathcal{E}^\infty_t(S^{-m'/2}_{\lambda,\rho,\delta})$. Then we get

$$\mathrm{Re}\,(P(t)\Lambda_s v,\ \Lambda_s v)\geqslant ||Q_{1/2}(t)\Lambda_s v||^2-||G_1(t)Q_{1/2}(t)\Lambda_s v||^2-C'||v||^2\,.$$

by (4.4). Now applying Proposition 2.12, we get

$$\mathrm{Re}\,(P(t)\Lambda_s v,\ \Lambda_s v)\geqslant 1/2||Q_{1/2}(t)\Lambda_s v||^2-C''||v||^2\,.$$

By Proposition 2.13 we can write $[\Lambda_s,\ P(t)]\equiv G_2Q(t)$, where $G_2(t)\in\mathcal{E}^\infty_t(S^{-\varepsilon_0}_{\lambda,\rho,\delta})$. By (4.7) and $Q_{1/2}G_2^*\equiv G_3Q_{1/2}$ with $G_3\in\mathcal{E}^\infty_t(S^{-\varepsilon_0}_{\lambda,\rho,\delta})$ we get for any $\varepsilon>0$ the estimate (4.6). Q.E.D.

Proof of Theorem 4.1. Note that $\Lambda_b u(t)$ satisfies

$$\Big(\frac{\partial}{\partial t}+P(t)\Big)\Lambda_b u(t) = \Lambda_b g(t)-[\Lambda_b,\ P(t)]u(t) \qquad \text{for } b\geqslant 0\,.$$

Then we have

$$\frac{\partial}{\partial t}(\Lambda_b u(t),\ \Lambda_b u(t)) = -2\mathrm{Re}\,(P(t)\Lambda_b u(t),\ \Lambda_b u(t))$$
$$+2\mathrm{Re}\,(\Lambda_b g(t),\ \Lambda_b u(t))+2\mathrm{Re}([\Lambda_b,\ P(t)]u(t),\ \Lambda_b u(t))\,.$$

By Lemma 4.2 and (4.2) we get for any $c_3<c_2$

$$(4.9) \qquad \frac{d}{dt}||\Lambda_b u(t)||^2\leqslant -2c_3||\Lambda_b u(t)||^2+2||\Lambda_b g(t)||\ ||\Lambda_b u(t)||+C||u(t)||^2$$

with some constant C. Integrating (4.9) from t_0 to t, we get

$$(4.10) \qquad ||\Lambda_b u(t)||\leqslant e^{-c_3(t-t_0)}||\Lambda_b u(t_0)||+\int_{t_0}^t e^{-c_3(t-s)}\{||\Lambda_b g(s)||+C||u(s)||\}\,ds\,.$$

On the other hand it is clear that

$$(4.11) \qquad ||u(t)||\leqslant e^{-c_2(t-t_0)}||u(t_0)||+\int_{t_0}^t e^{-c_2(t-s)}||g(s)||\,ds\,.$$

Then from (4.10) and (4.11) we get the assertion. Q.E.D.

Lemma 4.3. *For any b such that* $\sigma b-(n+1)/2\geqslant 0$ *we have*

$$C_b^{-1}|u|_{b_1,\mathcal{S}}\leqslant\|u\|_b\leqslant C_b|u|_{b_2,\mathcal{S}}, \quad b_1=[\sigma b-(n+1)/2], \quad b_2=\tilde{\tau}(b+1)+(n+1)/2$$

for $u\in\mathcal{S}(R^n)$, *where* $\tilde{\tau}=\max(1/\tilde{\rho}_j, \tau)$.

Proof. For $l\geqslant 0$ we have

$$|u|_{l,\mathcal{S}}\leqslant C_l\|u\|_k, \quad k=l/\sigma+(n+1)/2\sigma .$$

Note that $\lambda(x, \xi)\leqslant(|x|+|\xi|+1)^{\tilde{\tau}}$. Then we get Lemma 4.4. Q.E.D.

Theorem 4.4. *Let* $E(t, s)$ *be the fundamental solution which is constructed in* §3. *Then for any fixed* $t_0>s_0\geqslant 0$ *and any integers* l_j $(j=1, 2, 3)$ *there exists a constant C independent of t such that*

$$|\partial_t^{l_1}e(t, s_0)|_{l_3}^{(-l_2)}\leqslant C\exp\{-c_3(t-t_0)\} \qquad t\geqslant t_0$$

where c_3 *is any constant such that* $c_3<c_2$.

Proof. Let $f(t, s; x, \xi)=e^{ix\cdot\xi}e(t, s; x, \xi)$. Then we get

$$\sigma(P(t)E(t, s))(x, \xi)=e^{-ix\cdot\xi}p(t; x, D_x)f(t, s; x, \xi).$$

From the above equation we get the following equations for f

$$\text{(4.12)} \qquad \begin{cases} \dfrac{\partial}{\partial t}f(t, s; x, \xi)+p(t; x, D_x)f(t, s; x, \xi)=0 & \text{in } t>s \\ f(s, s; x, \xi)=e^{ix\cdot\xi}. \end{cases}$$

Then $f(t, s; x, \xi)$ is a solution of (0.1) with the intial data $e^{ix\cdot\xi}$. We see that $f(t, s_0; x, \xi)$ for $t>s_0$ belongs to $\mathcal{S}(R^{2n}_{x,\xi})$ from Theorem 3.1 and the assumption (4.1) for $\lambda(x, \xi)$. Apply Theorem 4.1 for $g=0$ and $u=f$. Then we get

$$\|f(t, s_0; \cdot; \xi)\|_b\leqslant Be^{-c_3(t-t_0)}\|f(t_0, s_0; \cdot, \xi)\|_b .$$

Lemma 4.3 means that for any l there exists l' such that

$$|f(t, s_0; \cdot, \xi)|_{l,\mathcal{S}}\leqslant B'e^{-c_3(t-t_0)}|f(t, s; \cdot, \xi)|_{l',\mathcal{S}} .$$

From (4.12) we get

$$\begin{cases} \dfrac{\partial}{\partial t}\Big(\dfrac{\partial f}{\partial\xi_j}\Big)(t, s; x, \xi)+p(t; x, D_x)\dfrac{\partial}{\partial\xi_j}f(t, s; x, \xi)=0 \\ \dfrac{\partial}{\partial\xi_j}f(s, s; x, \xi)=ix_je^{ix\cdot\xi} . \qquad j=1, 2, \cdots, n . \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} f(t, s; x, \xi) + p(t; x, D_x) \dfrac{\partial}{\partial t} f(t, s; x, \xi) = -\dfrac{\partial}{\partial t} p(t; x, D_x) f(t, s; x, \xi) \\ \dfrac{\partial}{\partial t} f(s, s; x, \xi) = -p(s; x, D_x) e^{ix\cdot\xi} . \end{cases}$$

By the same argument we get

$$\left| \frac{\partial}{\partial \xi_j} f(t, s_0; \cdot, \xi) \right|_{l, \mathcal{S}} \leqslant B' e^{-c_3(t-t_0)} \left| \frac{\partial}{\partial \xi_j} f(t_0, s_0; \cdot, \xi) \right|_{l', \mathcal{S}}$$

and

$$\left| \frac{\partial}{\partial t} f(t, s_0; \cdot, \xi) \right|_{l, \mathcal{S}} \leqslant B' e^{-c_3(t-t_0)} \left| \frac{\partial}{\partial t} f(t_0, s_0; \cdot, \xi) \right|_{l' \mathcal{S}} .$$

$\partial_t^{l_1} e(t_0, s_0; x, \xi) \in S_{\lambda,\rho,\delta}^{-\infty}$ for $t_0 > s_0$ means that $\partial_t^{l_1} f(t_0, s_0; x, \xi)$ belongs to $\mathcal{S}(R_x^n \times R_\xi^n)$ for $t_0 > s_0$ by the assumption (4.1) for $\lambda(x, \xi)$. Hence we get the assertion.

Q.E.D.

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