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A PRIORI UPPER BOUNDS OF SOLUTIONS SATISFYING A CERTAIN DIFFERENTIAL INEQUALITY ON COMPLETE MANIFOLDS

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Abstract

In this article we study a priori upper bounds of subsolutions satisfying a certain differential inequality (*) below on a non-compact complete Riemannian manifold \((M, g)\) without any Ricci curvature condition. Our method depends on a volume estimate of open subsets where those solutions satisfy a certain strong subharmonicity. Several applications in conformal deformation of metrics and value distribution of harmonic maps are given.

1. Introduction

Let \((M, g)\) be a connected Riemannian manifold of dimension \(m\) and \(\Delta_g\) the Laplacian defined by \(\Delta_g u := \text{Trace}_g \nabla \nabla u\) for a smooth function \(u\) on \(M\). Throughout this article \((M, g)\) is always assumed to be non-compact complete and connected unless otherwise stated. We are interested in a priori upper bounds of a non-negative smooth function \(u\) satisfying the following differential inequality:

\[
\Delta_g u + ku - lu^{a+1} \geq 0
\]

on \(M\) where \(k\) and \(l\) are continuous functions on \(M\), and \(a > 0\) is a constant respectively. A differential geometric interpretation of a priori upper bounds of such a subsolution appears in conformal deformation of metrics and value distribution of harmonic maps and has been studied under a certain curvature condition of \(g\). Nevertheless our method does not depend on any curvature condition of \(g\) and depends only on a volume estimate of an open subset where \(u\) satisfies a certain strong subharmonicity. It can be stated as follows.

**Theorem 1.1.** Let a smooth function \(u\) on \(M\) satisfy the following differential inequality

\[
\Delta_g u \geq \frac{C u^{a+1}}{(1 + r_u)^b}
\]

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on an open subset $[u > \delta] \neq \emptyset$ for certain constants $C > 0$, $a > 0$ and $\delta > 0$, where $r_a$ is the distance function from a fixed point $x_a$ of $M$. If $b < 2$ (resp. $b = 2$), then

$$\liminf_{r \to +\infty} \frac{\log V_{x_a}(r)}{r^{2-b}} = +\infty \quad \left(\text{resp.} \quad \liminf_{r \to +\infty} \frac{\log V_{x_a}(r)}{\log r} = +\infty\right),$$

where $V_{x_a}(r)$ is the volume of the geodesic ball $B_{x_a}(r)$ centered at a fixed point $x_a \in M$ and of radius $r > 0$.

This result is a refinement of Theorem 1.1 in [18] and plays a crucial role to show the following a priori upper estimate of $u$ satisfying (*).

**Theorem 1.2.** Let $u$ be a non-negative smooth function satisfying the differential inequality (*) on $M$ and the functions $k$ and $l$ in (*) satisfy the following condition

$$k \leq Hl \quad \text{for some constant} \quad H \geq 0$$

and

$$l \geq \frac{L}{(1 + r_a)^b} \quad \text{for certain constants} \quad L > 0 \quad \text{and} \quad b \in \mathbb{R}$$

on $M$ respectively. Suppose the following volume growth condition either

(1) $$\liminf_{r \to +\infty} \frac{\log V_{x_a}(r)}{r^{2-b}} < +\infty \quad \text{if} \quad b < 2,$$

or

(2) $$\liminf_{r \to +\infty} \frac{\log V_{x_a}(r)}{\log r} < +\infty \quad \text{if} \quad b = 2.$$

Then

$$\sup_M u \leq H^{1/a}.$$

Especially $u$ vanishes identically if $H = 0$.

**Remark 1.1.** The above volume growth condition is weaker than a decay condition of Ricci curvature studied in [14, 15] (cf. [14], Theorem A and [15], Theorem 0.2). Actually in view of the Laplacian comparison theorem (cf. [6]), if there exist constants $C \geq 0$ and $b \leq 2$ such that

$$\text{Ric}_g(x) \geq -C(1 + r_a(x))^{2(1-b)}$$
for any $x \in M$, then one can see that
\[
\limsup_{r \to +\infty} \frac{\log V_x(r)}{r^{2-b}} < +\infty \quad \text{if} \quad b < 2 \quad \left(\text{resp. } \limsup_{r \to +\infty} \frac{\log V_x(r)}{\log r} < +\infty \quad \text{if} \quad b = 2\right).
\]

In case $b = 2$ the above pointwise lower bound condition of Ricci curvature can be replaced by the following weaker condition. Namely if the negative part $R_{M,-}$ of the Ricci curvature of $(M, g)$ satisfies the following
\[
\int_{B_{\alpha}(x)} R_{M,-}^p d\nu_g = O(r^k)
\]
for any $r > 0$ and positive integers $p, k$ with $p > m - 1$ and $p/(2p + k)(m - 1) \ll 1$, then the condition (2) is satisfied (cf. [7], Theorem 1.1 and Corollary 1.2).

As an applications of Theorem 1.2, we can show the following.

**Theorem 1.3.** Under the condition either (1) or (2) of Theorem 1.2 for $b \leq 2$, suppose $(M, g)$ has dimension $m \geq 2$ and the scalar curvature $s_g$ of $g$ satisfies the following inequality
\[
s_g \leq -\frac{L}{(1 + r)^b} \quad \text{for some constant} \quad L > 0
\]
on $M$. Then any conformal transformation $f$ of $(M, g)$ which preserves $s_g$ i.e., the scalar curvature $K_{f^*g}$ of $f^*g$ coincides with $s_g$ is an isometry (cf. Corollary 3.1 and [14], Corollary 1).

Theorem 1.1 is deeply related to a generalized maximum principle for the Laplacian $\Delta_g$ on a complete manifold $(M, g)$. In fact we can show the following in terms of our formulation.

**Theorem 1.4.** Suppose the condition either (1) or (2) of Theorem 1.2 is satisfied for $b \leq 2$, and a smooth function $u$ is bounded from above on $M$. Then for any $\varepsilon > 0$ and $x \in M$, there exists a point $x_\varepsilon$ of $M$ such that
\[
(i) \quad u(x) \leq u(x_\varepsilon),
\]
\[
(ii) \quad \Delta_g u(x_\varepsilon) < \frac{\varepsilon}{(1 + r_\varepsilon(x_\varepsilon))^b}.
\]

Furthermore if $0 \leq b \leq 2$ and there exists a continuous function $\lambda$ on a real line such that $\Delta_g u \geq \lambda(u)$ on $M$, then one can take the above point $x_\varepsilon$ which satisfies $|\nabla u|(x_\varepsilon) < \varepsilon$ simultaneously.
REMARK 1.2. In [18], Theorem 2.3 we have announced that a generalized maximum principle for $\Delta_g$ can be induced under the condition (1) for $b = 0$ in Theorem 1.2. However its proof is incomplete and so the problem is still unsolved except the above case (cf. [10]).

By Theorem 1.4 we can restore several results stated in [18], §3, Applications without proof. For instance we get the following.

**Theorem 1.5.** Under the condition either (1) or (2) of Theorem 1.2 for $b \leq 2$, suppose that $f : (M, g) \to (N, h)$ is a harmonic map to an Hadamard manifold $(N, h)$ and the energy density $e(f)$ of $f$ satisfies the following inequality

$$e(f) \geq \frac{C}{(1 + r_s)^b}$$

for some constant $C > 0$ on $M$. Then the image of $f$ is unbounded. In particular if $(N, h)$ is an $n$-dimensional Euclidean space $(\mathbb{R}^n, g_e)$ provided with Euclidean metric $g_e$ and the condition either (1) or (2) of Theorem 1.2 is satisfied for $0 \leq b \leq 2$, then the image of $f$ can not be contained in any non-degenerate cone of $\mathbb{R}^n$ (cf. Corollary 3.6, Theorem 3.7, and [9], Remark B).

In the second section we give the proof of the above results except Theorems 1.3 and 1.5. Their applications including those theorems are given in the third section.

**REMARK 1.3.** In preparation of this work, an article [11] has been published by S. Pigola, M. Rigoli and A.S. Setti. In the paper they study a priori upper bounds of $u$ satisfying $(\ast)$ from a view of volume growth condition of complete manifolds and give certain applications related to our results. However their method can not allow us to study the case $b = 2$, i.e., $(M, g)$ has a polynomial volume growth. The upper bound 2 of $b$ is originated from the fact that $\Delta_g$ is the 2-Laplacian which is a special case of the $p$-Laplacian $\Delta_{g,p}$ defined by $\Delta_{g,p}u := \text{div}(|\nabla u|^{p-2}\nabla u)$ for $u \in C^\infty(M)$.

**2. A volume estimate for a strong subharmonicity of solutions**

Let $(M, g)$ be a complete non-compact Riemannian manifold of dimension $m$ as in the introduction and $r_s$ the distance function from a fixed point $x_s \in M$. We restate Theorem 1.1 in the introduction.

**Theorem 2.1.** Let $u$ be a smooth function on $(M, g)$ satisfying the inequality

$$\Delta_g u \geq \frac{C_1 u^{a+1}}{(1 + r_s)^b} \quad \text{on} \quad \{u > \delta\} \neq \emptyset \quad \text{for} \quad C_1 > 0, \quad a > 0 \quad \text{and} \quad \delta > 0.$$
If $b < 2$ (resp. $b = 2$), then

$$\lim_{r \to +\infty} \frac{\log V_{x_1}(r)}{r^{2-b}} = +\infty \quad \text{(resp.} \lim_{r \to +\infty} \frac{\log V_{x_1}(r)}{\log r} = +\infty) .$$

Proof. We may assume $\sup_M u = +\infty$. If $u^* := \sup_M u < +\infty$, then putting $v := 1/(u^* - u)$ ($u$ does not attain $u^*$ on $[u > \delta]$ by (2.1)) one can verify that $\Delta_s v \geq C_1 \delta^{a+1} v^2/(1 + r_s)^b$ on $[v > \delta_s]$ with $\delta_s := 1/(u^* - \delta) > 0$. We have only to discuss by replacing $u$ by $v$ for $a = 1$. Since we can assume that $[u > \delta + C_2] \neq \emptyset$ for any $C_2 > 0$, we replace $u$ by $u/(\delta + C_2)$ and $C_1$ by $C := C_1(\delta + C_2)^a > 0$ respectively, and set $k_s := 1/(1 + r_s)^b$ in (2.1). The inequality (2.1) can be modified into the following form:

$$\Delta_s u \geq C k_s u^{a+1} \quad \text{on} \quad M^* := [u > 1] \neq \emptyset.$$

From the above observation, we can take the constant $C$ arbitrarily large in (2.2). We choose a non-negative smooth convex function $\lambda$ on a real line $\mathbb{R}$ such that $\lambda(t) = 0$ if $t \leq 1$, $\lambda(t) > 0$, $\lambda'(t) > 0$, $\lambda''(t) \geq 0$ if $t > 1$ and $\lambda(t) \equiv 1$ if $t > 1 + \eta$ for a sufficiently small $\eta > 0$ and a Lipschitz continuous function $\omega$ on $M$ such that $0 \leq \omega \leq 1$, $\text{Supp}(\omega) \subset B_{x_1}(2r)$, $\omega \equiv 1$ on $B_{x_1}(r)$, and $|\nabla \omega| \leq 1/r$. By using (2.2), a direct calculation shows the following for any $p$ and $q > 0$:

$$\text{div}(\omega^q \nabla \lambda(u^p)) \geq p \lambda'(u^p) \{(p-1)\omega^q u^{p-2} |\nabla u|^2 + \omega^q u^{p-1} \Delta_s u + 2q \omega^q u^{p-1} |\nabla \omega, \nabla u|\} \geq p \lambda'(u^p) \{(p-1)\omega^q u^{p-2} |\nabla u|^2 + C \omega^q k_s u^{p+a} - 2q \omega^q u^{p-1} |\nabla u| |\nabla \omega| \} .$$

By integrating the both sides and hypothesis, for any $\varepsilon > 0$ we get

$$(p-1) \int \omega^q \lambda'(u^p) u^{p-2} |\nabla u|^2 \, dv_g + C \int \omega^q k_s \lambda'(u^p) u^{p+a} \, dv_g \leq 2q \int \omega^{q-1} \lambda'(u^p) u^{p-1} |\nabla u| |\nabla \omega| \, dv_g \leq \varepsilon \int \omega^q \lambda'(u^p) u^{p-2} |\nabla u|^2 \, dv_g + q^2 \int \omega^{q-1} \lambda'(u^p) u^{p} |\nabla \omega|^2 \, dv_g .$$

For $\varepsilon = (p-1)/2 > 0$ we obtain

$$\int \omega^q k_s \lambda'(u^p) u^{p+a} \, dv_g \leq \frac{2q^2}{C(p-1)} \int \omega^{q-1} \lambda'(u^p) u^{p} |\nabla \omega|^2 \, dv_g .$$

By setting $q = (p+a)/a > 1$ in (2.3), the following holds:

$$\int \omega^{2(p+a)/a} k_s \lambda'(u^p) u^{p+a} \, dv_g \leq \frac{2(p+a)^2}{a^2(p-1)C} \int \omega^{2p/a} u^p \lambda'(u^p) |\nabla \omega|^2 \, dv_g .$$
The Hölder inequality yields the following:
\[
\int \omega^{2p/a} u^p \lambda'(u^p) |\nabla \omega|^2 \, dv_g \\
\leq \left( \int \omega^{2(2p+a)/a} k_s \lambda'(u^p) u^{p+a} \, dv_g \right)^{p/(p+a)} \left( \int k_s^{-p/a} \lambda'(u^p) |\nabla \omega|^{2(p+a)/a} \, dv_g \right)^{a/(p+a)}.
\]

Since there exists \( r_0 \geq 2 \) depending on \( C \) such that \( B_{x_i}(r) \cap M^* \neq \emptyset \) for any \( r \geq r_0 \), by substituting the above inequality into the right hand side of (2.4), we get
\[
(2.5) \\
\int_{B_{x_i}(r)} k_s \lambda'(u^p) u^{p+a} \, dv_g \leq \left( \frac{2(p+a)^2}{a^2(p-1)C} \right)^{(p+a)/a} \int_{B_{x_i}(2r,r)} k_s^{-p/a} \lambda'(u^p) |\nabla \omega|^{2(p+a)/a} \, dv_g,
\]
where \( B_{x_i}(2r,r) := B_{x_i}(2r) \setminus B_{x_i}(r) \) for any \( r \geq r_0 \). We set
\[
F(r, p) := \int_{B_{x_i}(r)} \lambda'(u^p) \, dv_g > 0
\]
for any \( r \geq r_0 \) and \( p > 1 \). Since \( r^{-2}(1+rs)^b \leq 2^4(1+r)^{b-2} \) for 1 \( \leq r \leq r_* \leq 2r \) and \( b \leq 2 \), the right-hand side of (2.5) can be estimated as follows:
\[
\int_{B_{x_i}(2r,r)} k_s^{-p/a} \lambda'(u^p) |\nabla \omega|^{2(p+a)/a} \, dv_g \\
\leq \alpha_b(r) F(2r, p) \left( \frac{2^5(p+a)^2}{a^2(p-1)C(1+r)^{2-b}} \right)^{(p+a)/a},
\]
where \( \alpha_b(r) = (1+r)^{-b} \) if \( b \geq 0 \) and \( \alpha_b(r) = (1+2r)^{-b} \) if \( b < 0 \). Since \( \lambda'(u^p) > 0 \) if and only if \( u > 1 \), by combining this estimate with (2.5), we get for any \( r \geq r_0 \) and \( p > 1 \)
\[
(2.6) \\
F(r, p) \leq \beta_b(r) F(2r, p) \left( \frac{2^5(p+a)^2}{a^2(p-1)C(1+r)^{2-b}} \right)^{(p+a)/a},
\]
where \( \beta_b(r) = 1 \) if \( b \geq 0 \) and \( \beta_b(r) = (1+2r)^{-b} \) if \( b < 0 \). If \( r \geq 1 \) and \( b \leq 2 \), then we set
\[
p(r) := \frac{a^2C(1+r)^{2-b}}{2^9}.
\]
Since we may assume that \( p(r) \geq a + 2 \) for any \( r \geq 1 \) by taking \( C \) arbitrarily large, we get
\[
\frac{2^5(p(r) + a)^2}{a^2(p(r)-1)C(1+r)^{2-b}} \leq \frac{1}{2} \quad \text{for any} \quad r \geq 1.
\]
By putting \( p = p(r) \) and \( F(r) := F(r, p(r)) \) in (2.6), we have

\[
F(r) \leq \beta_b(r) F(2r) \left( \frac{1}{2} \right) \left( \frac{(p(r) + a)}{a} \right) \quad \text{for any } r \geq r_0.
\]  

We fix \( r \) with \( r > 2r_0 \geq 4 \) and assume \( b < 2 \). Since there exists an integer \( k \geq 1 \) with \( 2^{-(k+1)} < r_0/r \leq 2^{-k} \), by putting \( r_j = 2^j r_0 \) and using (2.7), we can see

\[
F(r_0) \leq \beta_b(r) F\left( \frac{r_0}{r} \right) \left( \frac{1}{2} \right) \left( \frac{(p(r) + a)}{a} \right) F(r_j)
\]

\[\leq \frac{r_1 \beta_b(r) F\left( \frac{r_0}{r} \right)}{r} \left( \frac{1}{2} \right) a C r^{2-b}/2^{13-2b} F(r),\]

which implies

\[
\frac{aC \log 2}{2^{13-2b}} - \frac{\max\{-b, 0\} (\log(1 + 2r))^2}{r^{2-b} \log 2} \leq \frac{\log F(r)}{r^{2-b}}
\]

for any \( r \geq r(C, a, b) \) with a sufficiently large \( r(C, a, b) > r_0 \). By taking \( r(C, a, b) \) so large again we get

\[
\frac{aC \log 2}{2^{14-2b}} \leq \frac{\log F(r)}{r^{2-b}}
\]

for any \( r \geq r(C, a, b) \). Since we can take \( C \) arbitrarily large and \( F(r) \leq V_s(r) \) by \( \sup \lambda' = 1 \), we attain the conclusion. If \( b = 2 \), then by \( \beta_2(r) \equiv 1 \) we get the following by the same argument as above:

\[
F(r_0) \leq \left( \frac{r_1}{r} \right) (aC/2^b)^+ F(r),
\]

which implies

\[
\frac{aC}{2^b} \leq \frac{\log F(r)}{\log r}
\]

for any \( r \) with \( r \geq r(C, a, 2) \gg 0 \). Therefore we attain the conclusion similarly.

Now we are in a position to show Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. If \( \{u > H^{1/a}\} \neq \emptyset \), then taking \( \epsilon > 0 \) with \( \{u > (H + \epsilon)^{1/a}\} \neq \emptyset \), \( u \) satisfies \( \Delta_g u \geq C_1 u^{r+1}/(1 + r_0)^b \) on \( \{u > \delta\} \) for \( C_1 = \epsilon L/(H + \epsilon) \) and \( \delta = (H + \epsilon)^{1/a} \). However this contradicts the volume growth condition in view of Theorem 2.1.

As a corollary of Theorem 1.2 we get the following.
Corollary 2.2. Let \( u \) be a non-negative smooth function \( u \) satisfying the differential inequality (*) and the function \( k \) (resp. \( l \)) in (*) satisfy the following

\[
k \leq \frac{K}{(1 + r_s)^c} \left( \text{resp. } l \geq \frac{L}{(1 + r_s)^b} \right) \quad \text{for } K \geq 0 \text{ (resp. } L > 0) \text{ and } c \text{ (resp. } b) \in \mathbb{R}
\]
on \( M \). If the condition either (1) or (2) of Theorem 1.2 is satisfied for \( b \leq \min[2, c] \), then

\[
\sup_M u \leq \left( \frac{K}{L} \right)^{1/a}.
\]

Especially \( u \) vanishes identically on \( M \) if \( K = 0 \).

Proof. Since \( k \leq (K/L)l \) if \( b \leq c \), the assertion follows from Theorem 1.2 immediately.

The difference of two solutions of (*) can be estimated as follows (cf. [19], Theorem 4.9).

Corollary 2.3. Let \( u_1 \) and \( u_2 \) be non-negative solutions of the equality

\[
\Delta_g u + ku - lu^{a+1} = 0
\]
on \( M \), where \( h \) and \( k \) satisfy the assumption of Theorem 1.2 respectively. If the condition of either (1) or (2) of Theorem 1.2 is satisfied for \( b \leq 2 \), then \( \sup_M |u_1 - u_2| \leq H^{1/a} \).

Proof. By setting \( w := (u_1 - u_2)^2 \), one can verify that \( w \) satisfies the inequality \( \Delta_g w \geq -2kw + 2lw^{(a/2)+1} \) on \( M \). Hence the conclusion follows from Theorem 1.2 immediately.

Here we show Theorem 1.4 stated in the introduction.

Proof of Theorem 1.4. We may assume that \( u \) does not attain \( u^* := \sup_M u < +\infty \) on \( M \). We put \( \varepsilon_a := \min\{\varepsilon, u^* - u(x)\}/(1 + \min\{\varepsilon, u^* - u(x)\}) > 0 \) for a fixed constant \( \varepsilon > 0 \) and point \( x \in M \) respectively. We set \( \varepsilon_a := 1/(1 + u^* - u) > 0 \) and

\[
M_p := \{ y \in M; w^p(y) > 1 - \varepsilon_a \} \quad \text{and} \quad \Gamma_p := \left\{ y \in M; \Delta_g w^p(y) < \frac{\varepsilon_a w^{2p}(y)}{(1 + r_s(y))^b} \right\}
\]

for any positive integer \( p \). One can verify that \( M_p \subset M_q \) and \( \Gamma_p \subset \Gamma_q \) for any \( p > q \geq 1 \) in view of the equality \( \Delta_g w^p = (p/q)w^{p-q} \Delta_g w^q + p(p - q)w^{p+2}[\nabla u]^2 \). By Theorem 2.1 the volume growth condition implies that \( \Sigma_p := M_p \cap \Gamma_p \) is a non-empty,
and unbounded subset of \( M \) for any \( p \) otherwise \( w^p \) satisfies \( 1 - \varepsilon < \sup_{\Sigma_p} w^p < 1 \) and \( \Delta_g w^p \geq \varepsilon w^{2p}/(1 + r_s)^p \) on \( \{ w^p > \sup_{\Sigma_p} w^p \} \neq \emptyset \). Moreover if \( y \in \Sigma_p \) for \( p \geq 1 \), then one can see that \( u(x) \leq u(y) \) and

\[
\Delta_g u(y) < \frac{\varepsilon}{p(1 + r_s(y))^p} - (p + 1)(1 - \varepsilon)^{1/p}\|\nabla u\|^2(y).
\]

The estimate 2.8 implies that any point of \( \Sigma_1 \) is the desired one. To show the latter half assertion, suppose \( |\nabla u| \geq \eta \) on \( \Sigma_1 \) for a constant \( \eta > 0 \). Clearly we can verify that \( \Sigma_p \subset \Sigma_q \) for \( p > q \geq 1 \), and \( \bigcap_{p=1}^{\infty} \Sigma_p = \emptyset \) because \( u < u^* \) on \( M \). Hence for each point \( y_p \in \Sigma_p \) we get \( \lambda(u^*) = \lim_{p \to +\infty} \lambda(u(y_p)) \leq \lim_{p \to +\infty} \Delta_g u(y_p) = -\infty \) by (2.8). This is a contradiction.

As a direct consequence of Theorems 1.4 and 2.1 we can obtain the following similarly to an aspect by Cheng and Yau (cf. [3], Corollary).

**Corollary 2.4.** Let \( u \) be a smooth function satisfying the inequality

\[
\Delta_g u \geq \frac{\lambda(u)}{(1 + r_s)^b}
\]
on \( M \), where \( \lambda \) is a continuous function on \( \mathbb{R} \) such that

\[
\lambda(t) \geq C_\varepsilon t^{a+1} \quad \text{for any} \quad t \geq \varepsilon \quad \text{with certain constants} \quad a > 0, \varepsilon > 0 \quad \text{and} \quad C_\varepsilon > 0.
\]

If the condition either (1) or (2) of Theorem 1.2 is satisfied for \( b \leq 2 \), then \( \sup_M u \leq \varepsilon \) and \( \lambda(\sup_M u) \leq 0 \). Especially if \( u \geq 0 \) and \( \lambda \) satisfies the above property for any small \( \varepsilon > 0 \), then \( u \equiv 0 \). Moreover \( \inf_M |\nabla u| = 0 \) if \( 0 \leq b \leq 2 \).

**Remark 2.1.** As a related topic, Tachikawa showed a non-existence theorem of harmonic maps from \( \mathbb{R}^m \) to an Hadamard manifold with negative sectional curvature under a certain non-degenerate condition which is similar to the condition (3.2) below (cf. [17], Theorem 1). His result can be also induced by applying Corollary 2.4 to \( \lambda(t) = \sinh \kappa t \) (\( \kappa > 0 \)) and \( b = 2 \) (see the inequality (2.2) in [17], p.152).

We can also get the following theorem which is related to a priori bound estimates of solutions for a certain Poisson equation (cf. [19], Corollary 4.3).

**Corollary 2.5.** Let \( u \) be a smooth solution of the following equation

\[
\Delta_g u = \frac{\lambda(u)}{(1 + r_s)^b}
\]
on \( M \), where \( \lambda \) is a continuous function on a real line such that \( \lambda(t) \geq C_+ t^{a+1} \) (resp. \( \lambda(t) \leq C_- t^3 \)) for \( t \geq \alpha_+ \geq 0, a > 0 \) and \( C_+ > 0 \) (resp. \( t \leq \alpha_- \leq 0 \) and
If the condition either (1) or (2) of Theorem 1.2 is satisfied for \( b \leq 2 \), then 
\[ \alpha_- \leq \inf_M u \leq \sup_M u \leq \alpha_+. \] 
Especially if \( \alpha_+ = \alpha_- = 0 \), then \( u \equiv 0 \). Moreover \( \inf_M |\nabla u| = 0 \) if \( 0 \leq b \leq 2 \).

3. Applications in differential geometry

Let \((M, g)\) be a complete non-compact Riemannian manifold of dimension \( m \geq 2 \) and \( f: (M, g) \to (N, h) \) a smooth map to a Riemannian manifold \((N, h)\). \( f: (M, g) \to (N, h) \) is said to be a conformal immersion if there exists a smooth function \( u \neq 0 \) on \( M \) satisfying 
\[ f^* h = u^{4/(m-2)} g \] 
(resp. \( f^* h = u g \)) if \( m \geq 3 \) (resp. \( m = 2 \)). It is known that \( u \) satisfies the following equality on \( M \):

\[
\begin{align*}
\text{if } m \geq 3 & \quad \Rightarrow \quad \Delta g u - s_g u + K_{f^* h} u^{(m+2)/(m-2)} = 0, \\
\text{if } m = 2 & \quad \Rightarrow \quad \Delta g \log u - s_g + K_{f^* h} u = 0,
\end{align*}
\]

where \( s_g \) (resp. \( K_{f^* h} \)) is the scalar curvature of \( g \) (resp. the pull back \( f^* h \) of \( h \) by \( f \)). First we state the following theorem (cf. [14], Theorem 1).

**Theorem 3.1.** Suppose \( f: (M, g) \to (N, h) \) is a conformal immersion such that

\[
K_{f^* h} \leq \min\{s_g, 0\} \quad \text{and} \quad K_{f^* h} \leq -\frac{L}{(1 + r_g)^b} \quad \text{for some constant } L > 0
\]

on \( M \). If the condition either (1) or (2) of Theorem 1.1 is satisfied for \( b \leq 2 \), then \( f \) is distance decreasing, i.e., \( \sup_M u \leq 1 \).

**Proof.** By applying Theorem 1.2 to \( k = -\min\{0, s_g\}/c_m, l = -K_{f^* h}/c_m \), \( H = 1 \) and \( a = 4/(m-2) \) (resp. \( a = 1 \)) for \( m \geq 3 \) (resp. \( m = 2 \)), we can get the conclusion. \( \square \)

We get the following from Theorem 3.1 immediately (cf. [14], Corollary 1 & the references, and [19], Theorem 4.7).

**Corollary 3.2.** Under the condition either (1) or (2) of Theorem 1.2 for \( b \leq 2 \), suppose \((M, g)\) has dimension \( m \geq 2 \) and the scalar curvature \( s_g \) of \( g \) satisfies the following:

\[
s_g \leq -\frac{L}{(1 + r_g)^b} \quad \text{for some constant } L > 0
\]

on \( M \). Then any conformal transformation \( f \) of \((M, g)\) which preserves \( s_g \) i.e., the scalar curvature \( K_{f^* g} \) of \( f^* g \) coincides with \( s_g \) is an isometry.

By applying this to the identity map of \( M \) we get the following (cf. [14], Corollary 2).
Corollary 3.3. Under the same hypothesis as Corollary 3.2 suppose \( h \) is a conformal metric of \( g \) whose scalar curvature coincides with \( s_g \). Then \( h = g \).

Corollary 2.2 yields the following (cf. [13], Corollary 4.2 and [15], Theorem 0.2 & Corollary 0.1).

Corollary 3.4. Under the condition either (1) or (2) of Theorem 1.2 for \( b \leq 2 \), suppose the scalar curvature of \( g \) is non-negative on \( M \) and \( S \) is a smooth function satisfying

\[
S \leq -\frac{L}{(1+r_s)^b} \quad \text{for some constant } \quad L > 0
\]

on \( M \). Then the metric \( g \) cannot be conformally deformed to any metric of scalar curvature \( S \).

Remark 3.1. In the above results it is not necessary to control the lower bound of \( s_g \). The reader should see [11] (resp. [14, 15]) which studies the case \(-C_1 \leq s_g \leq -C_2/(1 + r_s)^b \) (resp. \(-C_1/(1 + r_s)^{(2b-1)} \leq -C_2/(1 + r_s)^b \)) for certain constants \( C_1, C_2 \) and \( b \) with \( C_1 \geq C_2 > 0 \) and \( b \leq 2 \) respectively.

Remark 3.2. If \( l \) asymptotically behaves like \(-1/(1 + r_s)^b \) for \( b > 2 \) and \( k \equiv 0 \), then an existence theorem of non-trivial solutions \( u \) satisfying the equation \( \Delta_{ge}u = lu^{p+1} \) is known on an \( m \geq 3 \) dimensional Euclidean space \( \mathbb{R}^m \) provided with Euclidean metric \( ge \) (cf. [2], Theorem II).

The rest of this section is devoted to give several applications of Theorem 1.1 related to value distribution of maps. First we begin with the following (cf. [4], Theorem 3.1, [12], Theorem 2.17, and [18], Theorem 3.5).

Theorem 3.5. Let \( f: (M, g) \rightarrow (N, h) \) be a smooth map to an Hadamard manifold \( (N, h) \) whose sectional curvature is bounded from above by a non-positive constant \( K \). Suppose the energy density \( e(f) \) and tension field \( \tau(f) \) of \( f \) satisfy the following

\[
e(f) \geq \frac{C_1}{(1 + r_s)^b} \quad \text{and} \quad \|\tau(f)\| \leq \frac{C_2}{(1 + r_s)^c} \quad \text{for certain constants} \quad C_1 > 0 \quad \text{and} \quad C_2 \geq 0
\]

on \( M \) respectively. If the condition either (1) or (2) of Theorem 1.2 is satisfied for \( b \leq \min\{2, c\} \) and \( 2\sqrt{-K}C_1 > C_2 \geq 0 \) (resp. \( 2C_1 > C_2 \geq 0 \)) for \( K < 0 \) (resp. \( K = 0 \)), then \( f \) is unbounded, i.e., the image \( f(M) \) of \( M \) can not be relatively compact in \( N \).

Proof. By letting \( r_y \) be the distance function from a point \( y \in N \setminus \overline{f(M)} \neq \emptyset \), we set \( u(x) := f^*\lambda(r_y) \) with \( \lambda(t) = \cosh(C_3t)/2 \) for \( C_3 = \sqrt{-K} \) if \( K < 0 \) and \( C_3 = 1 \) if
\( K = 0 \). By combining the composition law of maps (cf. [5], (2.20), Proposition) with the Hessian comparison theorem (cf. [6], §2), the following estimate holds (cf. [12], (2.22)):

\[
\Delta_g u \geq 2C_3u \left( C_3 e(f) - \frac{1}{2} \| \tau(f) \| \tanh \left( C_3 f^s r_a \right) \right).
\]

By hypothesis we can see

\[
\Delta_g u \geq \frac{C_3(2C_3C_1 - C_2)u}{2(1 + r_a)^b} > 0
\]
on \( M \). If \( f \) is bounded, then \( u \) is bounded from above and \( \inf_M u > 0 \). However \( u \) does not attain its supremum on \( M \) by the above inequality. By putting \( w = 1/\left( \sup_M u - u \right) > 0 \), a direct calculation shows the following:

\[
\Delta_g w \geq \frac{C_4w^2}{(1 + r_a)^b} \quad \text{for some constant } \ C_4 > 0
\]
on \( M \). On the other hand Corollary 2.4 implies that \( w \) should vanish identically. This is a contradiction. \( \square \)

Especially by letting \( C_2 = 0 \) we get the following immediately (cf. [12], Theorem 2.12).

**Corollary 3.6.** Under the condition either (1) or (2) of Theorem 1.2 for \( b \leq 2 \), suppose that \( f : (M, g) \rightarrow (N, h) \) is a harmonic map to an Hadamard manifold \((N, h)\) and the energy density \( e(f) \) of \( f \) satisfies

\[
e(f) \geq \frac{C}{(1 + r_a)^b} \quad \text{for some constant } \ C > 0
\]
on \( M \). Then \( f \) is unbounded.

In case \((N, h) = (\mathbb{R}^n, g_e)\), we can show the following which is a more precise result than Corollary 3.6 (cf. [9], Theorem B, [1], Theorem 3 and [18], Theorem 3.3).

**Theorem 3.7.** Let \( f : (M, g) \rightarrow (\mathbb{R}^n, g_e) \) be a harmonic map satisfying the condition (3.2) for \( 0 \leq b \leq 2 \). If the condition either (1) or (2) of Theorem 1.2 is satisfied for \( 0 \leq b \leq 2 \), then the image of \( f \) can not be contained in any non-degenerate cone of \( \mathbb{R}^n \).

Proof. The idea of proof is due to [9], Theorem B (see also [1]). Assume there exists a unit vector \( v \) at the origin of \( \mathbb{R}^n \) such that \( \langle f(x), v \rangle / \| f(x) \| \geq \delta \) for a fixed constant \( \delta > 0 \) and any \( x \in M \). Here \( \langle \cdot, \cdot \rangle \) (resp. \( \| \cdot \| \)) is the inner product (resp. the
norm) relative to $g_e$. Let $\mathbb{R}^{m-1}$ be the subspace of $\mathbb{R}^m$ which is orthogonal to $v$ and $\overline{\mathcal{F}}$ the $\mathbb{R}^{m-1}$-component of the position vector $f$, i.e. $\overline{\mathcal{F}} := f - (f, v)v$. We may assume that $(f, v)^2 - \delta^2(\overline{\mathcal{F}}, \overline{\mathcal{F}}) \geq 1$ on $M$. For a constant $a$ with $\delta > a > 0$, we set

$$F_a := -(f, v) + \sqrt{a^2(\overline{\mathcal{F}}, \overline{\mathcal{F}}) + 1} \leq 0.$$ 

Since the set $\{f(x); F_a(x) \geq F_a(x_0)\}$ is contained in a compact set for any $a$ with $0 < a < \delta - \delta'$ and a fixed point $x_0 \in M$, there exists a small constant $a > 0$ such that

$$a^2(f(x), f(x)) \leq 1$$

for any $x \in M_a := \{x; F_a(x) \geq F_a(x_0)\}$. We fix such a constant $a$ and put $F := F_a - F_a(x_0)$. Clearly $F$ is bounded from above and $F(x_0) = 0$. A direct calculation shows

$$\langle f_a X, v \rangle^2 \leq 2 \left\{ \|F_a X\|^2 + a^2 (\|f_a X - (f_a X, v)v\|^2) \|f\|^2 \right\}$$

for any $X \in TM_x$ and $x \in M$. The harmonicity of $f$ implies

$$\Delta_g F \geq \frac{a^2 \sum_{i=1}^n \|f_a X_i - (f_a X_i, v)v\|^2}{(a^2(\overline{\mathcal{F}}, \overline{\mathcal{F}}) + 1)^{3/2}}$$

for an orthogonal basis $\{X_i\}$ in $TM_x$ and $x \in M$. By applying Theorem 1.4 to $\lambda \equiv \inf_M \Delta_g F \geq 0$ (see (3.5)) and putting $u = F$ in (2.8) there exists a sequence $\{x_n\}$ of points of $M$ such that

(i) $F(x_n) > 0$,  
(ii) $|\nabla F|^2(x_n) < \frac{1}{n(1 + r_s(x_n))^b}$  
and  
(iii) $\Delta_g F(x_n) < \frac{1}{n(1 + r_s(x_n))^b}$. 

By putting $k_n := (1 + r_s(x_n))^b$, if $k_n \sum_{i=1}^n \|f_a X_i - (f_a X_i, v)v\|^2(x_n)$ tends to zero, then $k_n \sum_{i=1}^m \|f_a X_i \|^2(x_n)$ also tends to zero by the conditions (3.3), (3.4) and (ii) of (3.6), and so $k_n \sum_{i=1}^m \|f_a X_i \|^2(x_n) = 2k_n \|e(f)(x_n)\|$ tends to zero. However this contradicts (3.2). Hence there exists a constant $C_5 > 0$ such that $k_n \sum_{i=1}^m \|f_a X_i - (f_a X_i, v)v\|^2(x_n) \geq C_5 > 0$ for any $n$. However this again contradicts the condition (iii) of (3.6) in view of (3.3) and (3.5).

We can show the following distance decreasing property of holomorphic maps of complex manifolds (cf. [20], Theorem 2, [16], Theorem 1, and [14], Theorem 3).

**Theorem 3.8.** Let $f: (M, g) \to (N, h)$ be a holomorphic map from an $m$-dimensional complete non-compact Kähler manifold $(M, g)$ to a complex hermitian manifold $(N, h)$. Let $R_{M,-}$ (resp. $HS_N$) be the negative part of the pointwise lower
bound of the Ricci curvature of $g$ (resp. the pointwise upper bound of the holomorphic sectional curvature of $h$). Suppose

$$R_{M,-} \leq \frac{K}{(1 + r_s)^c} \quad \text{and} \quad HS_N(f) \leq -\frac{L}{(1 + r_s)^b}$$

on $M$ for certain constants $K \geq 0$, $L > 0$, $b$ and $c$. If $b \leq \min\{2, c\}$, then $\sup_M e(f) \leq 2vK/(v+1)L$, where $v$ is the maximal rank of $df$. Especially $f$ is constant if the Ricci curvature of $g$ is non-negative.

Proof. Since $b \geq 2(b - 1)$ for $b \leq 2$, by hypothesis the Ricci curvature of $g$ can be supported from below by $-K/(1 + r_s)^{2(b-1)}$. Hence the condition either (1) or (2) of Theorem 1.2 is satisfied as stated in the introduction. On the other hand since the energy density $e(f)$ of $f$ satisfies the inequality

$$\Delta_g \log e(f) \geq -2R_{M,-} - \frac{v + 1}{v} HS_N(f)e(f),$$

where $e(f) \neq 0$ (cf. [16], Proposition 4), the conclusion follows by applying Corollary 2.2 to $k = 2H/(1 + r_s)^c$ and $l = K(v + 1)/v(1 + r_s)^b$ respectively.

We can also show the following volume decreasing property of holomorphic maps of complex manifolds (cf. [8], §1, [7], Theorem 3.5 & Corollary 3.6, and [18], Theorem 3.7).

**Theorem 3.9.** Let $f : (M, g) \to (N, h)$ be a holomorphic map from an $m$-dimensional complete non-compact Kähler manifold $(M, g)$ to a complex hermitian manifold $(N, h)$ of the same dimension. Let $S_{M,-}$ be the negative part of the scalar curvature $S_M$ of $g$. Let $u_f$ denote the ratio $f^*V_N/V_M$ of the volume forms $V_M$ relative to $g$ and $V_N$ relative to $h$ respectively. Suppose

$$S_{M,-} \leq \frac{K}{(1 + r_s)^c} \quad \text{and} \quad \text{Ric}_N(f) \leq -\frac{L}{(1 + r_s)^b}$$

on $M$ for certain constants $K \geq 0$, $L > 0$, $b$ and $c$. If the condition either (1) or (2) of Theorem 1.2 is satisfied for $b \leq \min\{2, c\}$, then $\sup_M u_f \leq (2K/mL)^{2m}$.

Proof. By letting $u := u_f^{1/2m}$, $u$ satisfies the following inequality on $[u > 0]$ (see [7], the proof of Theorem 3.5):

$$\Delta_g \log u \geq \frac{1}{2m} S_M - \frac{1}{4} \text{Ric}_N(f)u^2.$$

To get the conclusion we have only to apply Corollary 2.2 to $k = -(1/2m)S_{M,-}$ and $l = -(1/4)\text{Ric}_N(f)$ respectively.
If the scalar curvature is non-negative, then the Ricci curvature is bounded from below and so the condition (1) of Theorem 1.2 is satisfied for $b = 1$. By setting $K = 0$ in Theorem 3.9 we obtain the following immediately.

**Corollary 3.10.** Let $(M, g)$ be a complete Kähler manifold whose scalar curvature is non-negative. Let $f : (M, g) \to (N, h)$ be a holomorphic map of complex manifolds with the same dimension such that

$$\text{Ric}_N(f) \leq \frac{L}{1 + r_*}$$

on $M$ for some constant $L > 0$. Then $f$ degenerates everywhere on $M$.

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