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ON SOME NEW CLASSES OF SEMIFIELD PLANES

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1. Introduction

In [9] Hiramine, Matsumoto and Oyama introduced a construction method that associates with any translation plane of order q^2 (q odd) and kernel $K \cong GF(q)$, translation planes of order q^4 and kernel $K' \cong GF(q^2)$. In this article we study the class of semifield planes of order q^4 obtained from this method and show that with a few exceptions, the members of this class are new semifield planes. This class includes some recently constructed classes of planes; namely the class presented by Boerner-Lantz in [4] and the one by Cordero in [6].

2. Notation and preliminary results

Let $\mathcal{S}=(\mathcal{S}, +, \cdot)$ be a finite semifield which is not a field. We denote by $\pi(\mathcal{S})$ the semifield plane coordinatized by \mathcal{S} with respect to the points $(0), (\infty), (0, 0)$ and $(1, 1)$. The dual translation plane of $\pi(\mathcal{S})$ is also a semifield plane and it is coordinatized by the semifield $\mathcal{S}^*=(\mathcal{S}, +, *)$, where $a*b=b \cdot a$. Let q be an odd prime power, $\mathcal{F}=GF(q^2)$ and $x^r=\bar{x}=x^q$ for $x \in \mathcal{F}$. Let π be a semifield plane obtained by the construction method of Hiramine, Matsumoto and Oyama. Then π admits a matrix spread set of the form

$$\mathcal{M} = \left\{ M(u, v) = \begin{bmatrix} u & v \\ f(v) & \bar{u} \end{bmatrix} : u, v \in \mathcal{F} \right\}$$

where $f: \mathcal{F} \rightarrow \mathcal{F}$ is an additive function. π is coordinatized by the semifield $\mathcal{P}=\mathcal{P}_f=(\mathcal{P}, +, \cdot)$, where $\mathcal{P}=\mathcal{F} \times \mathcal{F}$ and

$$(x, y) \cdot (u, v) = (x, y) \begin{bmatrix} u & v \\ f(v) & \bar{u} \end{bmatrix}.$$

We shall denote this plane by π_f . We define the following classes:

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$$\begin{aligned} \Omega(\mathcal{F}) &= \{f: \mathcal{F} \rightarrow \mathcal{F}: f \text{ is an additive function and } \mathcal{P}_f \text{ is a proper semifield}\}. \\ \Lambda(\mathcal{F}) &= \{f \in \Omega(\mathcal{F}): \text{either } f(v) = av \text{ for some } a \in \mathcal{F} - GF(q), \text{ or } f(v) = av^\theta \text{ for} \\ &\quad \text{some nonsquare } a \in \mathcal{F} \text{ and } \theta \in \text{Aut}(\mathcal{F}), \theta \neq \tau\}. \\ \Pi(\mathcal{F}) &= \{\pi_f: f \in \Omega(\mathcal{F})\}. \\ \Sigma(\mathcal{F}) &= \{\mathcal{P}_f: \pi(\mathcal{P}_f) \in \Pi(\mathcal{F})\}. \end{aligned}$$

Notice that $\Pi(\mathcal{F})$ is the class of semifield planes of order q^4 which are obtained from the construction method of Hiramane, Matsumoto and Oyama applied to translation planes of order q^2 .

Among the known classes of proper finite semifields we have the following:

- (i) Cohen and Ganley commutative semifields [5]
- (ii) Kantor semifields [13]
- (iii) Knuth semifields of characteristic 2 [14]
- (iv) Twisted fields [1] and Generalized twisted fields [2]
- (v) Sandler semifields [15]
- (vi) Knuth four-type semifields [14], these include the Hughes-Kleinfeld semifields [10]
- (vii) Generalized Dickson semifields [8]
- (viii) Boerner-Lantz semifields [4]
- (ix) p -primitive type IV and type V semifields [6]

The semifield planes coordinatized by the semifields on class (viii) belong to the class $\Pi(\mathcal{F})$, see [12], Theorem 4.3, and those coordinatized by semifields on class (ix) belong to $\Pi(F)$ where $F = GF(p^2)$ and p is a prime number, see [6]. The two main results on this paper state that the only known semifields (from classes (i) to (vii)) which belong to $\Sigma(\mathcal{F})$ are the Knuth semifields which are of all four types and the Generalized Dickson semifields.

We now state some properties of \mathcal{P}_f and π_f .

Lemma 1. *Let $f \in \Omega(\mathcal{F})$ and $\mathcal{P} = \mathcal{P}_f$. The nuclei of \mathcal{P} are:*

- (i) $\mathcal{N}_l(\mathcal{P}) = \{(x, 0) : x \in \mathcal{F}\}$,
- (ii) $\mathcal{N}_m(\mathcal{P}) = \mathcal{N}_r(\mathcal{P}) = \{(x, 0) : f(xy) = \bar{x}f(y), \text{ for any } y \in \mathcal{F}\}$

Proof. For $a = (x, y)$, $b = (u, v)$ and $c = (r, s)$ in \mathcal{P} the condition $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ is equivalent to the two equations

$$y(rf(v) + \bar{u}f(s)) = yf(us + v\bar{r}) \tag{2.1}$$

and

$$ysf(v) = yv\overline{f(s)}. \tag{2.2}$$

Clearly, from 2.1 and 2.2, $(x, 0) \in \mathcal{N}_l(\mathcal{P})$ for $x \in \mathcal{F}$ and since \mathcal{P} is not a field, (i) follows.

Assume now that $(u, v) \in \mathcal{N}_m(\mathcal{P})$. If $v \neq 0$, then from 2.2 with $y=1$ we

have that $f(\bar{s}) = \frac{sf(v)}{v}$, for any $s \in \mathcal{F}$, which implies that $c = \frac{f(v)}{v} \in GF(q)$ and $f(s) = c\bar{s}$. This implies that \mathcal{P} is a field, which is not the case. Thus, $v=0$ and from 2.1 we get that $\mathcal{N}_m(\mathcal{P}) = \{(u, 0) : f(us) = uf(s), \text{ for any } s \in \mathcal{F}\}$.

Let $(r, s) \in \mathcal{N}_r(\mathcal{P})$. Then, as above, $s=0$ and from 2.1 we get that $rf(v) = f(v\bar{r})$, for any $v \in \mathcal{F}$. By taking $x = \bar{r}$ (so $\bar{x} = r$), we have $\bar{x}f(v) = f(vx)$, for any $v \in \mathcal{F}$. This completes the proof of (ii).

The following lemma is a consequence of the previous one.

Lemma 2. *Let $f \in \Omega(\mathcal{F})$. Then $f(v) = av$ for some $a \in \mathcal{F} - GF(q)$ if and only if $\mathcal{N}_l(\mathcal{P}) = \mathcal{N}_m(\mathcal{P}) = \mathcal{N}_r(\mathcal{P}) \cong \mathcal{F}$.*

3. On the class $\Pi(\mathcal{F})$

Let $f \in \Omega(\mathcal{F})$ and let π_f^* denote the dual translation plane of π_f with respect to (∞) . We begin this section by showing that the semifields on classes (i)-(v) above do not coordinatize planes in $\Pi(\mathcal{F})$.

Lemma 3. *Let $f \in \Omega(\mathcal{F})$ and let S be a semifield belonging to any one of the classes (i)-(v) above. Then neither π_f nor π_f^* is isomorphic to $\pi(S)$.*

Proof. If $\mathcal{P} (\mathcal{P}^*)$ is a semifield which coordinatizes $\pi_f (\pi_f^*)$, then $\mathcal{P} (\mathcal{P}^*)$ has characteristic $\neq 2$. On the other hand, if S belongs to classes (ii) or (iii), then the characteristic of S is 2 and therefore S is not isotopic to \mathcal{P} (or \mathcal{P}^*). If S belongs to class (i), then S is commutative and by using Exercise 8.10 in [11] we conclude that \mathcal{P} (or \mathcal{P}^*) is not isotopic to S . Thus in these cases $\pi_f \not\cong \pi(S) \cong \pi_f^*$. In [3] it is shown that a generalized twisted field plane of order p^n , p an odd prime, $n \geq 3$, admits an autotopism g whose order is a p -primitive divisor of $p^n - 1$, i.e. $|g| \mid p^n - 1$ but $|g| \nmid p^i - 1$ for $1 \leq i \leq n - 1$. From Propositions 6.3 and 6.4 in [9] it follows that if g is an autotopism of π_f then $|g| \mid 4(q^2 - 1)$. Therefore if S is a generalized twisted field plane then $\pi_f \cong \pi(S) \cong \pi_f^*$. (Recall that every twisted field plane is a generalized twisted field plane, [2].)

Assume now that S belongs to class (v) above. Then the dimension of S over $\mathcal{N}_l(S)$ is ≥ 4 and $\mathcal{N}_m(S) = \mathcal{N}_r(S)$ ([15], Theorem 1). Since \mathcal{P} is a 2-dimensional vector space over $\mathcal{N}_l(\mathcal{P})$, we have that $\pi_f \cong \pi(S)$. If $\pi_f^* \cong \pi(S)$, then by Theorem 8.2 in [11] we would have $\mathcal{F} \cong \mathcal{N}_l(\mathcal{P}) \cong \mathcal{N}_r(S) = \mathcal{N}_m(S) \cong \mathcal{N}_m(\mathcal{P}) = \mathcal{N}_r(\mathcal{P}) \cong \mathcal{N}_l(S)$. From here we conclude that S is a 2-dimensional vector space over $\mathcal{N}_l(S)$ which is a contradiction. Thus $\pi_f^* \not\cong \pi(S)$.

Next we deal with the Knuth four-type semifields. These semifields were defined in [14]. The semifields of type II, III and IV are characterized by their nuclei; type II: $\mathcal{N}_r = \mathcal{N}_m \cong \mathcal{F}$; type III: $\mathcal{N}_l = \mathcal{N}_m \cong \mathcal{F}$ and type IV: $\mathcal{N}_l = \mathcal{N}_r \cong \mathcal{F}$. A semifield of type I has multiplication given by:

$$(x, y) \cdot (u, v) = (xu + y^{\sigma^{-2}}v^\sigma h, xv + yu^\sigma + y^{\sigma^{-1}}v^\sigma g) \tag{3.3}$$

where $(x, y), (u, v) \in \mathcal{F} \times \mathcal{F}$, $1 \neq \sigma \in \text{Aut}(\mathcal{F})$ and h and g are elements in \mathcal{F} such that the polynomial $x^{\sigma+1} + gx - h$ is irreducible in \mathcal{F} . The next lemma gives the condition under which a semifield plane coordinatized by a Knuth semifield plane belongs to the class $\Pi(\mathcal{F})$.

Lemma 4. *Let $f \in \Omega(\mathcal{F})$ and let \mathbf{K} be a Knuth four-type semifield. Then π_f or π_f^* is isomorphic to $\pi(\mathbf{K})$ if and only if $f(v) = av$ for some $a \in \mathcal{F} - GF(q)$.*

Proof. Assume that $f(v) = av$. Then by Lemma 2 and Corollary 7.4.2 in [14] we have that \mathcal{P}_f is of all four types I, II, III, IV where $\sigma^2 = 1$ and $g = 0$.

Let \mathbf{K} be a Knuth four-type semifield. If \mathbf{K} is of type II, III, or IV and if $\pi_f \cong \pi(\mathbf{K})$ or $\pi_f^* \cong \pi(\mathbf{K})$ then by ([11], Theorem 8.2) and Lemmas 1 and 2 it follows that $f(v) = av$. Suppose that \mathbf{K} is of type I. If $g = 0$ and $\sigma^2 = 1$ then from 3.3 we get that $\mathbf{K} = \mathcal{P}_{f_1}$ where $f_1(v) = hv^\sigma = hv$. Hence, by Lemma 2, $\mathcal{F} \cong \mathcal{N}_I(\mathbf{K}) = \mathcal{N}_m(\mathbf{K}) = \mathcal{N}_r(\mathbf{K})$. Now if $\pi_f \cong \pi(\mathbf{K})$ or $\pi_f^* \cong \pi(\mathbf{K})$, then by ([11], Theorem 8.2) and Lemma 2 we have that $f(v) = av$. We now show that the case when $g = 0$ and $\sigma^2 \neq 1$ and the case when $g \neq 0$ are not possible.

Let $\mathcal{P} = \mathcal{P}_f$ and suppose that $\pi_f \cong \pi(\mathbf{K})$. Then $\mathcal{F} \cong \mathcal{N}_I(\mathcal{P}) \cong \mathcal{N}_I(\mathbf{K})$. Let $(x, y) \in \mathcal{N}_I(\mathbf{K})$. The condition $((x, y) \cdot (0, 1)) \cdot (0, s) = (x, y) \cdot ((0, 1) \cdot (0, s))$, for all s in \mathcal{F} is equivalent to

$$(x + y^{\sigma^{-1}}g)^{\sigma^{-2}}s^\sigma h = xs^\sigma h + y^{\sigma^{-2}}s^{\sigma^2}g^\sigma h, \tag{3.4}$$

and

$$y^{\sigma^{-2}}hs + (x + y^{\sigma^{-1}}g)^{\sigma^{-1}}s^\sigma g = xs^\sigma g + ys^{\sigma^2}h^\sigma + y^{\sigma^{-1}}s^{\sigma^2}g^\sigma g, \tag{3.5}$$

for any s in \mathcal{F} . If $g = 0$ and $\sigma^2 \neq 1 \neq \sigma$, then 3.5 implies that $y = 0$ and from 3.4 we get that $x^{\sigma^{-2}} = x$. Therefore $\mathcal{F} \cong \mathcal{N}_I(\mathbf{K}) \subset \{(x, 0) : x \in \mathcal{F} \text{ and } x^{\sigma^2} = x\}$ which implies that $\sigma^2 = 1$, but $\sigma^2 \neq 1$. If $g \neq 0$ then from 3.4 we get that $y = 0$, and from 3.5 we have that $x^{\sigma^{-1}}s^\sigma g = xs^\sigma g$. Hence $\mathcal{F} \cong \mathcal{N}_I(\mathbf{K}) \subset \{(x, 0) : x \in \mathcal{F} \text{ and } x^\sigma = x\}$ and therefore $\sigma = 1$, which is a contradiction. A similar argument shows that $\pi_f^* \cong \pi(\mathbf{K})$ is not possible.

The last class to consider is the class of generalized Dickson semifields. Let $\pi(\mathcal{D})$ be a generalized Dickson semifield plane of order q^4 which is coordinatized by the semifield $\mathcal{D} = (\mathcal{D}, +, \cdot)$ where $\mathcal{D} = \mathcal{F} \times \mathcal{F}$ and the product is given by (cf [8])

$$(x, y) \cdot (r, s) = (xr + y^\alpha s^\beta \omega, xs + yr^\sigma) \tag{3.6}$$

where α, β, σ are arbitrary automorphisms of \mathcal{F} but not all the identity, and ω is a nonsquare in \mathcal{F} . If $(u, v) \cdot ((x, y) \cdot (r, s)) = ((u, v) \cdot (x, y)) \cdot (r, s)$ then the following two conditions must be satisfied:

$$uy^\alpha s^\beta \omega + v^\alpha (xs + yr^\sigma)^\beta \omega = v^\alpha y^\beta r \omega + (uy + vx^\sigma)^\alpha s^\beta \omega, \tag{3.7}$$

and

$$vy^{\alpha\sigma} s^{\beta\sigma} \omega^\sigma = v^\alpha y^\beta s \omega \tag{3.8}$$

From now on \mathcal{D} will denote a generalized Dickson semifield plane of order q^4 with multiplication given by (3.6).

Under certain conditions a generalized Dickson semifield is a Knuth four-type semifield. In the next lemma we give the necessary conditions on the automorphisms α, β, σ under which \mathcal{D} is a Knuth four-type semifield.

Lemma 5. *If any of the following conditions are satisfied :*

- (i) $\beta = \alpha\sigma$ and $\beta\sigma = 1$, or
- (ii) $\alpha = 1$ and $\sigma = \beta$, or
- (iii) $\alpha = 1$ and $\sigma\beta = 1$

then \mathcal{D} is a Knuth four-type semifield.

Proof. Assume that (i) is true. Then 3.7 and 3.8 become, respectively,

$$uy^\alpha s^\beta \omega = u^\alpha y^\alpha s^\beta \omega, \tag{3.9}$$

and,

$$vy^\beta s \omega^\sigma = v^\alpha y^\beta s \omega \tag{3.10}$$

From these equations we get that $(x, 0) \in \mathcal{N}_m(\mathcal{D})$ for any $x \in \mathcal{F}$ and $(r, 0) \in \mathcal{N}_r(\mathcal{D})$ for any $r \in \mathcal{F}$. Since \mathcal{D} is not a field we have that $\mathcal{N}_m(\mathcal{D}) = \mathcal{N}_r(\mathcal{D}) \cong \mathcal{F}$ and \mathcal{D} is a Knuth semifield of type II. In a similar way if (ii) or (iii) occur then \mathcal{D} is a Knuth semifield of type III or IV, respectively.

In the following lemma the nuclei of \mathcal{D} are given.

Lemma 6. *Assume that \mathcal{D} is not a Knuth four-type semifield. Then the nuclei of \mathcal{D} are :*

- (i) $\mathcal{N}_i(\mathcal{D}) = \{(u, 0) \in \mathcal{D} : u^\alpha = u\}$,
- (ii) $\mathcal{N}_m(\mathcal{D}) = \{(x, 0) \in \mathcal{D} : x^\beta = x^{\alpha\sigma}\}$, and
- (iii) $\mathcal{N}_r(\mathcal{D}) = \{(r, 0) \in \mathcal{D} : r^{\sigma\beta} = r\}$.

Proof. Let $(u, v) \in \mathcal{N}_i(\mathcal{D})$ and suppose that $v \neq 0$. Then from 3.8 we get that $v\omega^\sigma = v^\alpha \omega$, $y^{\alpha\sigma} = y^\beta$ and $s^{\beta\sigma} = s$, for all $y, s \in \mathcal{F}$. Hence, $\alpha\sigma = \beta$ and $\beta\sigma = 1$, which is a contradiction by Lemma 5 (i). Thus, $v = 0$ and from 3.8 we have that $uy^\alpha s^\beta \omega = u^\alpha y^\alpha s^\beta \omega$ for all $y, s \in \mathcal{F}$; from this (i) follows. (ii) and (iii) are proved similarly.

In the next two lemmas the question of when a generalized Dickson semifield plane belongs to the class $\Pi(\mathcal{F})$ is answered.

Lemma 7. *Let $f \in \Omega(\mathcal{F})$ and $\mathcal{P} = \mathcal{P}_f$. Assume that \mathcal{U} is a non-*

desarguesian semifield plane that admits a matrix spread set of the form

$$\mathcal{M}_1 = \left\{ Q(x, y) = \begin{pmatrix} x & y \\ ky^\theta & x^\varphi \end{pmatrix} : x, y \in \mathcal{F} \right\}$$

where θ, φ are automorphisms of \mathcal{F} and k is a nonsquare in \mathcal{F} . Then, if $\pi_f \cong \mathcal{U}$, one of the following must be true :

- (i) $\theta = \varphi = \tau$, where $x^\tau = \bar{x}$, and $f(v) = cv$ for some $c \in \mathcal{F} - GF(q)$.
- (ii) $f(v) = cv^\psi$, for some $\psi \in \text{Aut}(\mathcal{F})$ and some nonsquare c in \mathcal{F} .

Proof. Let $\mathcal{X} = \mathcal{F} \times \mathcal{F}$. Then $\mathcal{M}^* = \{(X, XM(u, v)) : M(u, v) \in \mathcal{M}\} \cup \{(0, X)\}$ is a spread for π_f in $\mathcal{X} \oplus \mathcal{X}$. Let \mathcal{M}_1^* be the spread for \mathcal{U} in $\mathcal{X} \oplus \mathcal{X}$ associated with \mathcal{M}_1 . Since $\pi_f \cong \mathcal{U}$, there is a semilinear transformation T from the \mathcal{F} -vector space $\mathcal{X} \oplus \mathcal{X}$ into itself that maps \mathcal{M}^* onto \mathcal{M}_1^* . We may assume that $(X, 0)^T = (X, 0)$ and $(0, X)^T = (0, X)$, so the linear part of T has the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, for some $A, B \in GL(2, q^2)$. Let δ be the automorphism of \mathcal{F} associated with T . Since T maps $(X, XM(u, v)) \in \mathcal{M}^*$ onto $(X, XA^{-1}M(u, v)^\delta B) \in \mathcal{M}_1^*$, where $(a_{ij})^\delta = (a_{ij}^\delta)$, we have that for each $M(u, v) \in \mathcal{M}$ there is a unique $Q(x, y) \in \mathcal{M}_1$ such that

$$A^{-1}M(u, v)^\delta B = Q(x, y). \tag{3.11}$$

Let $Q(a, b) = A^{-1}M(1, 0)^\delta B = A^{-1}B$, $u \in GF(q) - \{0\}$ and $u' = u^\delta$. Then $A^{-1}M(u, 0)^\delta B = u' A^{-1}B = u' Q(a, b) \in \mathcal{M}_1$, for all $u' \in GF(q) - \{0\}$. Thus, if $a \neq 0$, then $u' = (u')^\varphi$, which implies that $\varphi \in \{1, \tau\}$. Similarly, if $b \neq 1$, then $\theta \in \{1, \tau\}$. Since $A^{-1} = Q(a, b)B^{-1}$, 3.11 becomes

$$B^{-1}M(u, v)^\delta B = Q(a, b)^{-1}Q(x, y). \tag{3.12}$$

Let $\Delta = \det Q(a, b)^{-1}$ and $\text{tr}(N) = \text{trace of a matrix } N$. Since $\text{tr}(B^{-1}M(u, v)^\delta B) = (u + \bar{u})^\delta \in GF(q)$, from 3.12 we have that $\text{tr}(Q(a, b)^{-1}Q(x, 0)) = \Delta(a^\varphi x + ax^\varphi) \in GF(q)$, for all $x \in \mathcal{F}$. If $\varphi = 1$, then we have that $2ax\Delta \in GF(q)$, for any $x \in \mathcal{F}$, which implies that $a = 0$. Therefore if $a \neq 0$ then $\varphi = \tau$. Likewise, considering $Q(0, y)$ we get that if $b \neq 0$ then $\theta = \tau$.

First we assume that $a \neq 0$ and $b \neq 0$. Then $\theta = \varphi = \tau$ and $\mathcal{U} = \pi(\mathcal{P}_g)$, where $g(y) = k\bar{y}$. By Lemma 2 the three nuclei of \mathcal{P}_g are equal and isomorphic to \mathcal{F} . Now (i) follows from Lemma 2. Assume that $a = 0$ and $b \neq 0$. Then $\theta = \tau$ and we may assume that $\varphi \neq \tau$. Letting $r = (yb^{-1})^\tau$, $s = x^\varphi(kb^\tau)$, $g(s) = ds^{\varphi-1}$ where $d = (k\bar{b})^{\varphi-1}b^{-1}$ is a nonsquare in \mathcal{F} and $Q_1(r, s) = \begin{pmatrix} r & s \\ g(s) & \bar{r} \end{pmatrix}$ we have that $Q(0, b)^{-1}Q(x, y) = Q_1(r, s)$. Now 3.12 becomes

$$M(u, v)^\delta = BQ_1(r, s)B^{-1}. \tag{3.13}$$

Let $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ and $e = \det B$. Then $u^\delta = e^{-1}(b_1 b_4 r + b_4 b_2 g(s) - b_1 b_3 s - b_2 b_3 \bar{r})$ and $u^\delta = e^{-1}(-b_2 b_3 r - b_2 b_4 g(s) + b_1 b_3 s + b_1 b_4 \bar{r})$. Since $\bar{u}^\delta = u^\delta$, with $s=0$ we get $\overline{b_1 b_4 e^{-1}} = b_1 b_4 e^{-1}$ and $\overline{b_2 b_3 e^{-1}} = b_2 b_3 e^{-1}$. Thus $b_1 b_4 e^{-1}$ and $b_2 b_3 e^{-1}$ are in $GF(q)$. Taking $r=0$ we get $b_4 b_2 e^{-1} d \bar{s}^{\varphi^{-1}} - (b_1 b_3 e^{-1}) \bar{s} = b_1 b_3 e^{-1} s - b_2 b_4 e^{-1} d s^{\varphi^{-1}}$. If $\varphi^{-1} \neq 1$ (also $\varphi^{-1} \neq \tau$), then $b_1 b_3 = 0$ and $b_2 b_4 = 0$. If $\varphi^{-1} = 1$ then $b_1 b_3 = b_2 b_4 d$ and $(b_1 b_3 e^{-1})^2 = z d$ where $z = b_1 b_2 b_3 b_4 e^{-2}$. Since $z \in GF(q)$, z is a square in $GF(q^2)$, then since d is a non-square in $GF(q^2)$ we must have $b_1 b_3 = 0$ and $b_2 b_4 = 0$. So for any $\varphi \neq \tau$ we conclude that $b_1 b_3 = 0$ and $b_2 b_4 = 0$. Since $e \neq 0$, then $b_2 = b_3 = 0$ or $b_1 = b_4 = 0$. If $b_2 = b_3 = 0$, then from 3.13 we have $v^\delta = b_1 b_4^{-1} s$ and $f(v)^\delta = e^{-1} b_4^2 g(s)$. From these equations it follows that $f(v) = c v^{\varphi^{-1}}$ where c is a nonsquare in \mathcal{F} . If $b_1 = b_4 = 0$, then a similar argument shows that $f(v) = c v^\varphi$ where again c is a nonsquare in \mathcal{F} . Thus in either case (ii) follows. The case when $a \neq 0$ and $b = 0$ is handled similarly.

Lemma 8. *Let $f \in \Omega(\mathcal{F})$ and assume that \mathcal{D} is not a Knuth four-type semifield. If either π_f or π_f^* is isomorphic to $\pi(\mathcal{D})$, then $f(v) = c v^\psi$ for some nonsquare c in \mathcal{F} and $\psi \in \text{Aut}(\mathcal{F})$, $\psi \neq \tau$.*

Proof. Assume that $\pi_f \cong \pi(\mathcal{D})$. Then from Lemmas 1 (i) and 6 (i) we have that $\mathcal{F} \cong \mathcal{N}_l(\mathcal{P}_f) \cong \mathcal{N}_l(\mathcal{D})$; this implies that $u^\alpha = u$ for all $u \in \mathcal{F}$. Hence $\alpha = 1$ and 3.6 becomes $(x, y) \cdot (r, s) = (x, y) \begin{pmatrix} r & s \\ s^\beta \omega & r^\sigma \end{pmatrix}$. Let $Q(r, s) = \begin{pmatrix} r & s \\ s^\beta \omega & r^\sigma \end{pmatrix}$. Then $\{Q(r, s) : r, s \in \mathcal{F}\}$ is a matrix spread set for $\pi(\mathcal{D})$. Suppose now that $\pi_f^* \cong \pi(\mathcal{D})$. Then $\pi_f \cong \pi(\mathcal{D}^*)$ and $\mathcal{F} \cong \mathcal{N}_l(\mathcal{P}_f) \cong \mathcal{N}_l(\mathcal{D}^*)$, so \mathcal{D}^* is a 2-dimensional vector space over $\mathcal{N}_l(\mathcal{D}^*)$. Since $\mathcal{N}_l(\mathcal{D}^*) = \mathcal{N}_r(\mathcal{D})$, from Lemma 6 (iii) we get that $\sigma\beta = 1$. Let $z \in \mathcal{D}^*$ and let $(u, v)'$ be the coordinates of z with respect to the basis $(0, 1), (1, 0)$ of \mathcal{D}^* over $\mathcal{N}_l(\mathcal{D}^*)$, i.e. $(u, v)' = (u, 0) * (1, 0) + (v, 0) * (0, 1)$ where $*$ is the product in \mathcal{D}^* . Then $(u, v)' = (u, v^\sigma)$. Now $(r, s)' * (x, y)' = (r, s^\sigma) * (x, y^\sigma) = (x, y^\sigma) \cdot (r, s^\sigma) = (x r + y^{\alpha\sigma} s^\beta \omega, x s^\sigma + y^\sigma r^\sigma) = (x r + y^{\alpha\sigma} s \omega, x^{\sigma^{-1}} s + y r)'$. Letting $Q'(x, y) = \begin{pmatrix} x & y \\ y^{\alpha\sigma} \omega & x^{\sigma^{-1}} \end{pmatrix}$ we have that $(r, s)' * (x, y)' = (r, s) Q'(x, y)$. Hence, $\{Q'(x, y) : x, y \in \mathcal{F}\}$ is a matrix spread set for $\pi(\mathcal{D}^*)$. Therefore in either case ($\pi_f \cong \pi(\mathcal{D})$ or $\pi_f^* \cong \pi(\mathcal{D}^*)$) we may apply Lemma 7. Since \mathcal{D} (and therefore \mathcal{D}^*) is not a Knuth four-type semifield, by Lemmas 2 and 4, case (i) of Lemma 7 does not occur; therefore the proof is complete.

We can now state our main results; their proofs follow from the lemmas.

Theorem 3.1. *Let $f \in \Omega(\mathcal{F}) - \Lambda(\mathcal{F})$. Then neither π_f nor π_f^* is isomorphic to a semifield plane coordinatized by a semifield belonging to any one of the classes (i)-(vii).*

Theorem 3.2. *Let $f \in \Lambda(\mathcal{F})$. Then*

- (i) *$f(v) = av$ for some $a \in \mathcal{F} - GF(q)$ if and only if π_f or π_f^* is isomorphic to a semifield plane coordinatized by a Knuth four-type semifield.*
- (ii) *$f(v) = av^\theta$ for some nonsquare $a \in \mathcal{F}$ and $\theta \in \text{Aut}(\mathcal{F})$, $\theta \neq \tau$ if and only if π_f or π_f^* is isomorphic to a semifield plane coordinatized by a generalized Dickson semifield.*

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