ON SOME NEW CLASSES OF SEMIFIELD PLANES

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1. Introduction

In [9] Hiramine, Matsumoto and Oyama introduced a construction method that associates with any translation plane of order \( q^2 \) (\( q \) odd) and kernel \( K \cong GF(q) \), translation planes of order \( q^4 \) and kernel \( K' \cong GF(q^2) \). In this article we study the class of semifield planes of order \( q^4 \) obtained from this method and show that with a few exceptions, the members of this class are new semifield planes. This class includes some recently constructed classes of planes; namely the class presented by Boerner-Lantz in [4] and the one by Cordero in [6].

2. Notation and preliminary results

Let \( S = (S, +, \cdot) \) be a finite semifield which is not a field. We denote by \( \pi(S) \) the semifield plane coordinatized by \( S \) with respect to the points \( (0), (\infty), (0, 0) \) and \( (1, 1) \). The dual translation plane of \( \pi(S) \) is also a semifield plane and it is coordinatized by the semifield \( S^* = (S, +, \cdot) \), where \( a \cdot b = b \cdot a \). Let \( q \) be an odd prime power, \( F = GF(q^2) \) and \( x^q = x = x^q \) for \( x \in F \). Let \( \pi \) be a semifield plane obtained by the construction method of Hiramine, Matsumoto and Oyama. Then \( \pi \) admits a matrix spread set of the form

\[
\mathcal{M} = \left\{ M(u, v) = \begin{bmatrix} u & v \\ f(v) & u \end{bmatrix} : u, v \in F \right\}
\]

where \( f: F \to F \) is an additive function. \( \pi \) is coordinatized by the semifield \( \mathcal{P} = \mathcal{P}_f = (\mathcal{P}, +, \cdot) \), where \( \mathcal{P} = \mathcal{F} \times \mathcal{F} \) and

\[
(x, y) \cdot (u, v) = (x, y) \begin{bmatrix} u & v \\ f(v) & u \end{bmatrix}.
\]

We shall denote this plane by \( \pi_f \). We define the following classes:

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\( \Omega(\mathcal{F}) = \{ f: \mathcal{F} \to \mathcal{F} : f \text{ is an additive function and } \mathcal{P}_f \text{ is a proper semifield} \} \).

\( \Lambda(\mathcal{F}) = \{ f \in \Omega(\mathcal{F}) : \text{ either } f(v) = \alpha \theta \text{ for some } \alpha \in \mathcal{F} - GF(q), \text{ or } f(v) = av^\theta \text{ for some nonsquare } a \in \mathcal{F} \text{ and } \theta \in \text{Aut}(\mathcal{F}), \theta \neq \tau \} \).

\( \Pi(\mathcal{F}) = \{ \pi_f : f \in \Omega(\mathcal{F}) \} \).

\( \Sigma(\mathcal{F}) = \{ \pi_f : \pi(\mathcal{P}_f) \in \Pi(\mathcal{F}) \} \).

Notice that \( \Pi(\mathcal{F}) \) is the class of semifield planes of order \( q^4 \) which are obtained from the construction method of Hiramine, Matsumoto and Oyama applied to translation planes of order \( q^2 \).

Among the known classes of proper finite semifields we have the following:

(i) Cohen and Ganley commutative semifields [5]

(ii) Kantor semifields [13]

(iii) Knuth semifields of characteristic 2 [14]


(v) Sandler semifields [15]

(vi) Knuth four-type semifields [14], these include the Hughes-Kleinfeld semifields [10]

(vii) Generalized Dickson semifields [8]

(viii) Boerner-Lantz semifields [4]

(ix) \( p \)-primitive type IV and type V semifields [6]

The semifield planes coordinatized by the semifields on class (viii) belong to the class \( \Pi(\mathcal{F}) \), see [12], Theorem 4.3, and those coordinatized by semifields on class (ix) belong to \( \Pi(F) \) where \( F = GF(p^2) \) and \( p \) is a prime number, see [6]. The two main results on this paper state that the only known semifields (from classes (i) to (vii)) which belong to \( \Sigma(\mathcal{F}) \) are the Knuth semifields which are of all four types and the Generalized Dickson semifields.

We now state some properties of \( \mathcal{P}_f \) and \( \pi_f \).

**Lemma 1.** Let \( f \in \Omega(\mathcal{F}) \) and \( \mathcal{P} = \mathcal{P}_f \). The nuclei of \( \mathcal{P} \) are:

(i) \( \mathcal{N}_1(\mathcal{P}) = \{ (x, 0) : x \in \mathcal{F} \} \),

(ii) \( \mathcal{N}_m(\mathcal{P}) = \mathcal{N}_r(\mathcal{P}) = \{ (x, 0) : f(xy) = xf(y), \text{ for any } y \in \mathcal{F} \} \)

Proof. For \( a = (x, y), b = (u, v) \) and \( c = (r, s) \) in \( \mathcal{P} \) the condition \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \) is equivalent to the two equations

\[ y(rf(v) + uf(s)) = yf(us + vr) \]  (2.1)

and

\[ ys f(v) = yf(s) \]  (2.2)

Clearly, from 2.1 and 2.2, \( (x, 0) \in \mathcal{N}_1(\mathcal{P}) \) for \( x \in \mathcal{F} \) and since \( \mathcal{P} \) is not a field, (i) follows.

Assume now that \( (u, v) \in \mathcal{N}_m(\mathcal{P}) \). If \( v \neq 0 \), then from 2.2 with \( y = 1 \) we
have that \( f(s) = f(v) \), for any \( s \in \mathcal{F} \), which implies that \( c = f(v) \in GF(q) \) and \( f(s) = c^s \). This implies that \( \mathcal{P} \) is a field, which is not the case. Thus, \( v = 0 \) and from 2.1 we get that \( \mathcal{N}_m(\mathcal{P}) = \{(u, 0): f(us) = uf(s), \text{ for any } s \in \mathcal{F}\} \).

Let \((r, s) \in \mathcal{N}_m(\mathcal{P})\). Then, as above, \( s = 0 \) and from 2.1 we get that \( rf(v) = f(vr) \), for any \( v \in \mathcal{F} \). By taking \( x = r \) (so \( s = r \)), we have \( sf(v) = f(vx) \), for any \( v \in \mathcal{F} \). This completes the proof of (ii).

The following lemma is a consequence of the previous one.

**Lemma 2.** Let \( f \in \Omega(\mathcal{F}) \). Then \( f(v) = av \) for some \( a \in \mathcal{F} - GF(q) \) if and only if \( \mathcal{N}_m(\mathcal{P}) = \mathcal{N}_m(\mathcal{P}) \approx \mathcal{F} \).

### 3. On the class \( \Pi(\mathcal{F}) \)

Let \( f \in \Omega(\mathcal{F}) \) and let \( \pi_f^* \) denote the dual translation plane of \( \pi_f \) with respect to \((\infty)\). We begin this section by showing that the semifields on classes (i)-(v) above do not coordinatize planes in \( \Pi(\mathcal{F}) \).

**Lemma 3.** Let \( f \in \Omega(\mathcal{F}) \) and let \( S \) be a semifield belonging to any one of the classes (i)-(v) above. Then neither \( \pi_f \) nor \( \pi_f^* \) is isomorphic to \( \pi(S) \).

Proof. If \( \mathcal{P} \) (or \( \mathcal{P}^* \)) is a semifield which coordinatizes \( \pi_f \), then \( \mathcal{P} \) (or \( \mathcal{P}^* \)) has characteristic \( \neq 2 \). On the other hand, if \( S \) belongs to classes (ii) or (iii), then the characteristic of \( S \) is 2 and therefore \( S \) is not isotopic to \( \mathcal{P} \) (or \( \mathcal{P}^* \)). If \( S \) belongs to class (i), then \( S \) is commutative and by using Exercise 8.10 in [11] we conclude that \( \mathcal{P} \) (or \( \mathcal{P}^* \)) is not isotopic to \( S \). Thus in these cases \( \pi_f \cong \pi(S) \) \( \cong \pi^* \). In [3] it is shown that a generalized twisted field plane of order \( p^n \), \( p \) an odd prime, \( n \geq 3 \), admits an autotopism \( g \) whose order is a \( p \)-primitive divisor of \( p^n - 1 \), i.e. \( |g| \mid p^n - 1 \) but \( |g| \mid p^i \) for \( 1 \leq i \leq n - 1 \). From Propositions 6.3 and 6.4 in [9] it follows that if \( g \) is an autotopism of \( \pi_f \) then \( |g| \mid 4(q^2 - 1) \). Therefore if \( S \) is a generalized twisted field plane then \( \pi_f \cong \pi(S) \cong \pi^* \). (Recall that every twisted field palne is a generalized twisted field plane, [2].)

Assume now that \( S \) belongs to class (v) above. Then the dimension of \( S \) over \( \mathcal{N}_h(S) \) is \( \geq 4 \) and \( \mathcal{N}_m(S) = \mathcal{N}_r(S) \) ([15], Theorem 1). Since \( \mathcal{P} \) is a 2-dimensional vector space over \( \mathcal{N}_h(\mathcal{P}) \), we have that \( \pi_f \cong \pi(S) \). If \( \pi_f^* \cong \pi(S) \), then by Theorem 8.2 in [11] we would have \( \mathcal{F} \cong \mathcal{N}_h(\mathcal{P}) \cong \mathcal{N}_r(S) = \mathcal{N}_m(\mathcal{P}) = \mathcal{N}_r(\mathcal{P}) \cong \mathcal{N}_r(\mathcal{P}) \cong \mathcal{N}_h(S) \). From here we conclude that \( S \) is a 2-dimensional vector space over \( \mathcal{N}_h(S) \) which is a contradiction. Thus \( \pi_f^* \cong \pi(S) \).

Next we deal with the Knuth four-type semifields. These semifields were defined in [14]. The semifields of type II, III and IV are characterized by their nuclei; type II: \( \mathcal{N}_r = \mathcal{N}_m \approx \mathcal{F} \); type III: \( \mathcal{N}_r = \mathcal{N}_m \approx \mathcal{F} \) and type IV: \( \mathcal{N}_r = \mathcal{N}_m \approx \mathcal{F} \). A semifield of type I has multiplication given by:
(x, y)·(u, v) = (xu + y^{σ^2}v^{σ^2}h, xv + yu^{σ^2} + y^{σ^2-1}v^{σ^2}g) \quad (3.3)

where (x, y), (u, v) ∈ \mathcal{D} \times \mathcal{D}, 1 = σ ∈ \text{Aut}(\mathcal{D}) and h and g are elements in \mathcal{D} such that the polynomial \(x^{σ^2+1} + gx - h\) is irreducible in \mathcal{D}. The next lemma gives the condition under which a semifield plane coordinatized by a Knuth semifield plane belongs to the class \(\Pi(\mathcal{D})\).

**Lemma 4.** Let \(f \in \Omega(\mathcal{D})\) and let \(K\) be a Knuth four-type semifield. Then \(\pi_f\) or \(\pi^*\) is isomorphic to \(\pi(K)\) if and only if \(f(v) = av\) for some \(a \in \mathcal{D} - GF(q)\).

Proof. Assume that \(f(v) = av\). Then by Lemma 2 and Corollary 7.4.2 in [14] we have that \(\mathcal{D}_f\) is of all four types I, II, III, IV where \(σ^2 = 1\) and \(g = 0\).

Let \(K\) be a Knuth four-type semifield. If \(K\) is of type II, III, or IV and if \(\pi_f \simeq \pi(K)\) or \(\pi^* \simeq \pi(K)\) then by ([11], Theorem 8.2) and Lemmas 1 and 2 it follows that \(f(v) = av\). Suppose that \(K\) is of type I. If \(g = 0\) and \(σ^2 = 1\) then from 3.3 we get that \(K = D_f,\) where \(f_j(v) = hv^{σ^2} = hv\). Hence, by Lemma 2, \(\mathcal{D} \cong \mathcal{N}_f(K) \cong \mathcal{N}_f(K) = \mathcal{I}_l(K)\). Now if \(\pi_f \simeq \pi(K)\) or \(\pi^* \simeq \pi(K)\), then by ([11], Theorem 8.2) and Lemma 2 we have that \(f(v) = av\). We now show that the case when \(g = 0\) and \(σ^2 = 1\) and the case when \(g \neq 0\) are not possible.

Let \(\mathcal{D} = \mathcal{D}_f\) and suppose that \(\pi_f \simeq \pi(K)\). Then \(\mathcal{F} \cong \mathcal{N}_f(\mathcal{D}) \cong \mathcal{N}_f(K)\). Let \((x, y) \in \mathcal{N}_f(K)\). The condition \(((x, y) \cdot (0, 1)) \cdot (0, s) = (x, y) \cdot ((0, 1) \cdot (0, s))\), for all \(s\) is \(\mathcal{F}\) is equivalent to

\[
(x + y^{σ^2}g)σ^2 = xσ^2h + y^{σ^2}σ^2g^σh, \quad (3.4)
\]

and

\[
y^{σ^2}hσ^2 + (x + y^{σ^2}g)σ^2 = xσ^2g + yσ^2σ^2h + y^{σ^2}σ^2g^σg, \quad (3.5)
\]

for any \(s\) in \(\mathcal{D}\). If \(g = 0\) and \(σ^2 = 1\), then 3.5 implies that \(y = 0\) and from 3.4 we get that \(x^{σ^2} = x\). Therefore \(\mathcal{D} \cong \mathcal{N}_f(K) \subset \{(x, 0) : x ∈ \mathcal{D}\} \text{ and } x^{σ^2} = x\) which implies that \(σ^2 = 1\), but \(σ^2 = 1\). If \(g = 0\) then from 3.4 we get that \(y = 0\), and from 3.5 we have that \(x^{σ^2}σ^2g = xσ^2g\). Hence \(\mathcal{D} \cong \mathcal{N}_f(K) \subset \{(x, 0) : x ∈ \mathcal{D} \text{ and } x^{σ^2} = x\}\) and therefore \(σ = 1\), which is a contradiction. Similar argument shows that \(\pi^* \simeq \pi(K)\) is not possible.

The last class to consider is the class of generalized Dickson semifields. Let \(\pi(D)\) be a generalized Dickson semifield plane of order \(q^4\) which is coordinatized by the semifield \(D = (D, +, \cdot)\) where \(D = \mathcal{D} \times \mathcal{D}\) and the product is given by (cf [8])

\[
(x, y) \cdot (r, s) = (xr + y^αs^βω, xs + yr^σ) \quad (3.6)
\]

where \(α, β, σ\) are arbitrary automorphisms of \(\mathcal{D}\) but not all the identity, and \(ω\) is a nonsquare in \(\mathcal{D}\). If \((u, v) \cdot ((x, y) \cdot (r, s)) = (u, v) \cdot ((x, y)) \cdot (r, s)\) then the following two conditions must be satisfied:
and
\[ u\gamma^s\beta\omega + v\gamma((xy + yr)\beta)\omega = v\gamma y^\beta r\omega + (uy + vx)\gamma^s\beta\omega, \quad (3.7) \]

From now on \( D \) will denote a generalized Dickson semifield plane of order \( q^t \) with multiplication given by (3.6).

Under certain conditions a generalized Dickson semifield is a Knuth four-type semifield. In the next lemma we give the necessary conditions on the automorphisms \( \alpha, \beta, \sigma \) under which \( D \) is a Knuth four-type semifield.

**Lemma 5.** If any of the following conditions are satisfied:

1. \( \beta = a\sigma \) and \( \beta\sigma = 1 \), or
2. \( a = 1 \) and \( \sigma = \beta \), or
3. \( a = 1 \) and \( \sigma \beta = 1 \)

then \( D \) is a Knuth four-type semifield.

**Proof.** Assume that (i) is true. Then 3.7 and 3.8 become, respectively,
\[ u\gamma^\beta s\omega = u\gamma^s\beta\omega, \quad (3.9) \]

and,
\[ v\gamma^s\beta\omega = v\gamma y\beta^s\omega \quad (3.10) \]

From these equations we get that \((x, 0) \in \mathcal{N}_m(D)\) for any \( x \in \mathcal{F} \) and \((r, 0) \in \mathcal{N}_r(D)\) for any \( r \in \mathcal{F} \). Since \( D \) is not a field we have that \( \mathcal{N}_m(D) = \mathcal{N}_r(D) = \mathcal{F} \) and \( D \) is a Knuth semifield of type II. In a similar way if (ii) or (iii) occur then \( D \) is a Knuth semifield of type III or IV, respectively.

In the following lemma the nuclei of \( D \) are given.

**Lemma 6.** Assume that \( D \) is not a Knuth four-type semifield. Then the nuclei of \( D \) are:

1. \( \mathcal{N}_f(D) = \{(u, 0) \in D : u = u\} \),
2. \( \mathcal{N}_m(D) = \{(x, 0) \in D : x^\beta = x^{s\sigma}\} \), and
3. \( \mathcal{N}_r(D) = \{(r, 0) \in D : \sigma^r = r\} \).

**Proof.** Let \((u, v) \in \mathcal{N}_f(D)\) and suppose that \( v \neq 0 \). Then from 3.8 we get that \( v\gamma^\sigma = v\gamma^\sigma, \gamma^s = \gamma^s \) and \( s^\sigma = s \), for all \( y, s \in \mathcal{F} \). Hence, \( \alpha\sigma = \beta \) and \( \beta\sigma = 1 \), which is a contradiction by Lemma 5 (i). Thus, \( v = 0 \) and from 3.8 we have that \( u\gamma^s\beta\omega = u\gamma y^\beta s\omega \) for all \( y, s \in \mathcal{F} \); from this (i) follows. (ii) and (iii) are proved similarly.

In the next two lemmas the question of when a generalized Dickson semifield plane belongs to the class \( \Pi(\mathcal{F}) \) is answered.

**Lemma 7.** Let \( f \in \Omega(\mathcal{F}) \) and \( \mathcal{P} = \mathcal{P}_f \). Assume that \( \mathcal{U} \) is a non-
desarguesian semifield plane that admits a matrix spread set of the form

\[ \mathcal{M}_1 = \left\{ Q(x, y) = \left( \begin{array}{cc} x & y \\ ky & x^\theta \end{array} \right) : x, y \in \mathcal{F} \right\} \]

where \( \theta, \varphi \) are automorphisms of \( \mathcal{F} \) and \( k \) is a nonsquare in \( \mathcal{F} \). Then, if \( \pi \cong U \), one of the following must be true:

(i) \( \theta = \varphi = \tau \), where \( x^\tau = x \), and \( f(v) = cv \) for some \( c \in \mathcal{F} - GF(q) \).

(ii) \( f(v) = cv^\varphi \), for some \( \psi \in \text{Aut}(\mathcal{F}) \) and some nonsquare \( c \) in \( \mathcal{F} \).

Proof. Let \( \mathcal{L} = \mathcal{F} \times \mathcal{F} \). Then \( \mathcal{M}^* = \{(X, XM(u, v)): M(u, v) \in \mathcal{M}_0 \cup \{(0, X)\} \) is a spread for \( \pi \), in \( \mathcal{L} \oplus \mathcal{L} \). Let \( \mathcal{M}^* \) be the spread for \( U \) in \( \mathcal{L} \oplus \mathcal{L} \) associated with \( \mathcal{M}_1 \). Since \( \pi \cong U \), there is a semilinear transformation \( T \) from the \( \mathcal{F} \)-vector space \( \mathcal{L} \oplus \mathcal{L} \) into itself that maps \( \mathcal{M}^* \) onto \( \mathcal{M}_1^* \). We may assume that \( (X, 0)^T = (X, 0) \) and \( (0, X)^T = (0, X) \), so the linear part of \( T \) has the form \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \), for some \( A, B \in GL(2, \mathcal{F}) \). Let \( \delta \) be the automorphism of \( \mathcal{F} \) associated with \( T \). Since \( T \) maps \( (X, XM(u, v)) \in \mathcal{M}^* \) onto \( (X, XA^{-1}M(u, v)^sB) \in \mathcal{M}_1^* \), where \( (a_{ij})^s = (a_{ij}) \), we have that for each \( M(u, v) \in \mathcal{M} \) there is a unique \( Q(x, y) \in \mathcal{M}_1 \) such that

\[ A^{-1}M(u, v)^sB = Q(x, y) \] (3.11)

Let \( Q(a, b) = A^{-1}M(1, 0)^sB = A^{-1}B, u \in GF(q) - \{0\} \) and \( u' = u^s \). Then

\[ A^{-1}M(u, 0)^sB = u'A^{-1}B = u'Q(a, b) \in \mathcal{M}_1, \] for all \( u' \in GF(q) - \{0\} \). Thus, if \( a \neq 0 \), then \( u' = (u')^\tau \), which implies that \( \varphi \in \{1, \tau\} \). Similarly, if \( b \neq 0 \), then \( \theta \in \{1, \tau\} \).

Since \( A^{-1} = Q(a, b)B^{-1} \), 3.11 becomes

\[ B^{-1}M(u, v)^sB = Q(a, b)^{-1}Q(x, y) \] (3.12)

Let \( \Delta = \det Q(a, b)^{-1} \) and \( \text{tr}(N) = \text{trace of a matrix } N \). Since \( \text{tr}(B^{-1}M(u, v)^sB) = (u + v)^s \in GF(q) \), from 3.12 we have that \( \text{tr}(Q(a, b)^{-1}Q(x, 0)) = \Delta(a^s + ax^\theta) \in GF(q) \), for any \( x \in \mathcal{F} \). If \( \phi = 1 \), then we have that \( 2ax_\Delta \in GF(q) \), which implies that \( a = 0 \). Therefore if \( a \neq 0 \) then \( \varphi = a \). Likewise, considering \( Q(0, y) \) we get that if \( b \neq 0 \) then \( \theta = a \).

First we assume that \( a \neq 0 \) and \( b \neq 0 \). Then \( \theta = a = \tau \) and \( U = \pi(D) \). Letting \( r = (yb^{-1}), s = x^e(kb^s), g(s) = ds^{e^{-1}} \), where \( d = (kb)^{e^{-1}}b^{-1} \) is a nonsquare in \( \mathcal{F} \) and \( Q(r, s) = \begin{pmatrix} r \\ g(s) \end{pmatrix} \) we have that \( Q(0, b)^{-1}Q(x, y) = Q_1(r, s) \). Now 3.12 becomes

\[ M(u, v)^s = BQ_1(r, s)B^{-1} \] (3.13)
Let \( B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \) and \( e = \det B \). Then \( u^s = e^{-1}\overline{b_1b_4r + b_2b_3g(s) - b_3b_4s - b_2b_4r} \) and \( u^s = e^{-1}\overline{-b_1b_4r - b_2b_3g(s) + b_3b_4s + b_2b_4r} \). Since \( \overline{u} = u^s \), with \( s = 0 \) we get \( \overline{b_1b_4e^{-1}} = b_1b_4e^{-1} \) and \( \overline{b_2b_3e^{-1}} = b_2b_3e^{-1} \). Thus \( b_1b_4e^{-1} \) and \( b_2b_3e^{-1} \) are in \( GF(q) \). Taking \( r = 0 \) we get \( b_1b_4e^{-1} = b_1b_4e^{-1} \) and \( b_2b_3e^{-1} = b_2b_3e^{-1} \). Thus \( b_1b_4e^{-1} \) and \( b_2b_3e^{-1} \) are in \( GF(q) \).

Since \( \overline{\tau} = \tau \), with \( s = 0 \) we get \( b_1b_4e^{-1} = b_1b_4e^{-1} \) and \( b_2b_3e^{-1} = b_2b_3e^{-1} \). Thus \( b_1b_4e^{-1} \) and \( b_2b_3e^{-1} \) are in \( GF(q) \).

Thus in either case (ii) follows. The case when \( a \neq 0 \) and \( b = 0 \) is handled similarly.

**Lemma 8.** Let \( f \in \Omega(\mathbb{F}) \) and assume that \( \mathcal{D} \) is not a Knuth four-type semifield. If either \( \pi_\tau \) or \( \pi^*_\tau \) is isomorphic to \( \pi(\mathcal{D}) \), then \( f(v) = cv^\psi \) for some nonsquare \( c \) in \( \mathbb{F} \) and \( \psi \in \text{Aut}(\mathbb{F}) \), \( \psi \neq \tau \).

**Proof.** Assume that \( \pi_\tau \approx \pi(\mathcal{D}) \). Then from Lemmas 1 (i) and 6 (i) we have that \( \mathbb{F} \cong \mathcal{P}_l(\mathcal{D}_l) \cong \mathcal{P}_l(\mathcal{D}) \); this implies that \( u^s = u \) for all \( u \in \mathbb{F} \). Hence \( \alpha = 1 \) and 3.6 becomes \((x, y) \cdot (r, s) = (x, y) \begin{pmatrix} r \\ s \end{pmatrix} \). Let \( Q(r, s) = \begin{pmatrix} r \\ s \end{pmatrix} \begin{pmatrix} \sigma \omega \\ \omega \sigma \end{pmatrix} \).

Then \( \{Q(r, s): r, s \in \mathbb{F}\} \) is a matrix spread set for \( \pi(\mathcal{D}) \). Suppose now that \( \pi^*_\tau \approx \pi(\mathcal{D}) \). Then \( \pi_\tau \approx \pi(\mathcal{D}^*) \) and \( \mathbb{F} \cong \mathcal{P}_l(\mathcal{D}_l) \cong \mathcal{P}_l(\mathcal{D}^*) \), so \( \mathcal{D}^* \) is a 2-dimensional vector space over \( \mathcal{P}_l(\mathcal{D}^*) \). Since \( \mathcal{P}_l(\mathcal{D}^*) = \mathcal{P}_l(\mathcal{D}) \), from Lemma 6 (iii) we get that \( \sigma \beta = 1 \). Let \( \mathcal{X} \subset \mathcal{D}^* \) and let \( (u, v) \) be the coordinates of \( x \) with respect to the basis \((0, 1), (1, 0) \) of \( \mathcal{D}^* \) over \( \mathcal{P}_l(\mathcal{D}^*) \), i.e. \((u, v) = (u, 0)* (1, 0) + (v, 0)* (0, 1) \) where \(* \) is the product in \( \mathcal{D}^* \). Then \((u, v) = (u, v^\sigma \omega) \), \((u, v^\sigma) = (u, v^\sigma \omega) \), \((r, s) = (r, s)^* \), \((r, s)^* = (x + y^\sigma \omega, x^\sigma + y^\sigma s^\sigma \omega) \), \((x + y^\sigma \omega, x^\sigma + y^\sigma s^\sigma \omega) = (x + y^\sigma \omega, x^\sigma + y^\sigma s^\sigma \omega) \).

Hence, \( \{Q(x, y): x, y \in \mathbb{F}\} \) is a matrix spread set for \( \pi(\mathcal{D}^*) \). Therefore in either case \((\pi_\tau \approx \pi(\mathcal{D}) \) or \( \pi^*_\tau \approx \pi(\mathcal{D}^*) \) we may apply Lemma 7. Since \( \mathcal{D} \) (and therefore \( \mathcal{D}^* \)) is not a Knuth four-type semifield, by Lemmas 2 and 4, case (i) of Lemma 7 does not occur; therefore the proof is complete.

We can now state our main results; their proofs follow from the lemmas.

**Theorem 3.1.** Let \( f \in \Omega(\mathbb{F}) - \Lambda(\mathbb{F}) \). Then neither \( \pi_\tau \) nor \( \pi^*_\tau \) is isomorphic to a semifield plane coordinatized by a semifield belonging to any one of the classes (i)-(vii).
Theorem 3.2. Let $f \in \Lambda(\mathcal{F})$. Then

(i) $f(v) = av$ for some $a \in \mathcal{F} - GF(q)$ if and only if $\pi_f$ or $\pi_f^*$ is isomorphic to a semifield plane coordinatized by a Knuth four-type semifield.

(ii) $f(v) = av^\theta$ for some nonsquare $a \in \mathcal{F}$ and $\theta \in \text{Aut}(\mathcal{F})$, $\theta \neq \tau$ if and only if $\pi_f$ or $\pi_f^\theta$ is isomorphic to a semifield plane coordinatized by a generalized Dickson semifield.

References

[7] M. Cordero and R. Figueroa: *On semifield planes of order $q^n$ that admit a collineation whose order is a $p$-primitive divisor of $q^n - 1$*, submitted.