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Osaka University
POLYNOMIAL REPRESENTATIONS ASSOCIATED WITH
SYMMETRIC BOUNDED DOMAINS

MASARU TAKEUCHI

(Received November 1, 1972)

Introduction. In this note we want to construct a complete orthonormal system of the Hilbert space $H^2(D)$ of square integrable holomorphic functions on an irreducible symmetric bounded domain $D$. A symmetric bounded domain $D$ is canonically realizable as a circular starlike bounded domain with the center $0$ in a complex cartesian space by means of Harish-Chandra’s imbedding (Harish-Chandra [3]), which is constructed as follows. The largest connected group $G$ of holomorphic automorphisms of $D$ is a connected semi-simple Lie group without center, which is transitive on $D$. Thus denoting the stablizer in $G$ of a point $o \in D$ by $K$, $D$ is identified with the quotient space $G/K$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $G$ (resp. $K$) and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Then there exists uniquely an element $H$ of the center of $\mathfrak{k}$ such that $\text{ad}H$ restricted to $\mathfrak{p}$ coincides with the complex structure tensor on the tangent space $T_o(D)$ at the origin $o$, identifying as usual $\mathfrak{p}$ with $T_o(D)$. Let $\mathfrak{g}^c$ be the Lie algebra of the complexification $G^c$ of $G$ and put $Z=\sqrt{-1}H \in \mathfrak{g}^c$. Let $(\mathfrak{p}^c)^\pm$ be the $(\pm 1)$-eigenspace in $\mathfrak{g}^c$ of $\text{ad}Z$. Then they are invariant under the adjoint action of $K$ and the complexification $\mathfrak{p}^c$ of $\mathfrak{p}$ is the direct sum of $(\mathfrak{p}^c)^+$ and $(\mathfrak{p}^c)^-$. Let $U^c$ denote the normalizer of $(\mathfrak{p}^c)^+$ in $G^c$. Then $D=G/K$ is holomorphically imbedded as an open submanifold into the quotient space $G^c/U^c$ in the natural way. For any point $z \in D$, there exists uniquely a vector $X \in (\mathfrak{p}^c)^-$ such that

$$\exp X \mod U^c = z.$$ 

The map $z \mapsto X$ of $D$ into $(\mathfrak{p}^c)^-$ is the desired imbedding. Note that the natural action of $K$ on $D$ can be extended to the adjoint action of $K$ on the ambient space $(\mathfrak{p}^c)^-$. 

Henceforth we assume that $D$ is a bounded domain in $(\mathfrak{p}^c)^-$ realized in the above manner. Let $\langle , \rangle$ denote the Killing form of $\mathfrak{g}^c$ and $\tau$ the complex conjugation of $\mathfrak{g}^c$ with respect to the compact real form $\mathfrak{k}+\sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}^c$. We define a $K$-invariant hermitian inner product $\langle , \rangle$, on $\mathfrak{g}^c$ by

$$\langle X, Y \rangle = -(X, \tau Y) \quad \text{for} \quad X, Y \in \mathfrak{g}^c.$$
This defines a $K$-invariant Euclidean measure $d\mu(X)$ on $(\mathfrak{p}\mathfrak{c})^-$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(X)$. The inner product of $H^2(D)$ will be denoted by $\langle \, , \rangle$. $K$ acts on $H^2(D)$ as unitary operators by
\[(kf)(X) = f(k^{-1}X) \quad \text{for} \quad k \in K, \, X \in D.\]

Let $S^*((\mathfrak{p}\mathfrak{c})^-)$ denote the graded space of polynomial functions on $(\mathfrak{p}\mathfrak{c})^-$. It has the natural hermitian inner product $(\, , \rangle_\tau$, induced from the inner product $(\, , \rangle$, on $(\mathfrak{p}\mathfrak{c})^-$. $K$ acts on $S^*((\mathfrak{p}\mathfrak{c})^-)$ as unitary operators by
\[(kf)(X) = f(\text{Ad} \, k^{-1}X) \quad \text{for} \quad k \in K, \, X \in (\mathfrak{p}\mathfrak{c})^- .\]

Now let $S$ denote the Shilov boundary of $D$. It is known (Korányi-Wolf [7]) that $K$ acts transitively on $S$. Thus denoting by $L$ the stabilizer in $K$ of a point $X_0 \in S$, $S$ is identified with the quotient space $K/L$. Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure of $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle \, , \rangle$. $K$ acts on $L^2(S)$ as unitary operators by
\[(kf)(X) = f(\text{Ad} \, k^{-1}X) \quad \text{for} \quad k \in K, \, X \in S.\]

The space $C^\omega(S)$ of $C$-valued $C^\omega$-functions on $S$ is a $K$-submodule of $L^2(S)$. The restrictions $S^*((\mathfrak{p}\mathfrak{c})^-) \rightarrow H^2(D)$ and $S^*((\mathfrak{p}\mathfrak{c})^-) \rightarrow L^2(S)$ are both $K$-equivariant monomorphisms. Their images will be denoted by $S^*(D)$ and $S^*(S)$, respectively. They have natural gradings induced from that of $S^*((\mathfrak{p}\mathfrak{c})^-)$. Then the “restriction” $S^*(D) \rightarrow S^*(S)$ is defined in the natural manner and it is a $K$-equivariant isomorphism. Since $D$ is a circular starlike bounded domain, a theorem of H. Cartan [2] yields that the subspace $S^*(D)$ of $H^2(D)$ is dense in $H^2(D)$ (cf. 1).

We decompose first the $K$-module $S^*(D)$ into irreducible components. We take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and identify the real part $\sqrt{-1} \mathfrak{t}$ of the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$ with its dual space by means of Killing form of $\mathfrak{g}^c$. Let $\sum_{\alpha \in \sqrt{-1} \mathfrak{t}}$ denote the set of roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$. We choose root vectors $X_\alpha \in \mathfrak{g}^c$ for $\alpha \in \sum_{\alpha}$ such that
\[[X_\alpha, X_{-\alpha}] = -\frac{2}{(\alpha, \alpha)} \alpha, \quad \tau X_\alpha = X_{-\alpha}.\]

A root is called compact if it is also a root of the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$, otherwise it is called non-compact. $\sum_{\text{c}}$ (resp. $\sum_{\text{p}}$) denotes the set of compact roots (resp. of non-compact roots). We choose and fix once for all a linear order $>$ on $\sqrt{-1} \mathfrak{t}$ such that $(\mathfrak{p}\mathfrak{c})^+$ is spanned by the root spaces for non-compact positive
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roots \( \Sigma^+_p \). Two roots \( \alpha, \beta \in \Sigma \) are called strongly orthogonal if \( \alpha \pm \beta \) is not a root. We define a maximal strongly orthogonal subsystem

\[
\Delta = \{ \gamma_1, \ldots, \gamma_p \}, \quad \gamma_1 > \gamma_2 > \cdots > \gamma_p > 0, \quad p = \text{rank } D
\]
of \( \Sigma^+_p \) as follows (cf. Harish-Chandra [3]). Let \( \gamma_i \) be the highest root of \( \Sigma \) and for each \( j \), \( \gamma_{j+1} \) be the highest positive non-compact root that is strongly orthogonal to \( \gamma_1, \ldots, \gamma_j \). We put

\[
X_0 = -\sum_{\gamma \in \Delta} X_\gamma.
\]

Then it is known (Körányi-Wolf [7]) that \( X_0 \) is on the Shilov boundary \( S \) of \( D \). Henceforth we shall take the above point \( X_0 \) as the origin of \( S \). We put for \( \nu \in \mathbb{Z}, \nu \geq 0 \)

\[
S'(K, L) = \{ \sum_{i=1}^{p} n_i \gamma_i; \ n_i \in \mathbb{Z}, \ n_1 \geq n_2 \geq \cdots \geq n_p \geq 0, \ \sum_{i=1}^{p} n_i = \nu \},
\]
and

\[
S^*(K, L) = \sum_{\nu \geq 0} S'(K, L).
\]

We shall prove the following

**Theorem A.** Any irreducible \( K \)-submodule of \( S^*(D) \) is contained exactly once in \( S^*(D) \). The set \( S'(D) \) of highest weights (with respect to \( t^c \) of irreducible \( K \)-submodules contained in \( S'(D) \) coincides with \( S'(K, L) \). Denoting by \( S^*_*(D) \) (resp. \( S^*_*(S) \)) the irreducible \( K \)-submodule of \( S^*(D) \) (resp. of \( S^*(S) \)) with the highest weight \( \lambda \in S^*(K, L) \),

\[
S^*(D) = \sum_{\lambda \in S^*(K, L)} \bigoplus S^*_*(D)
\]

and

\[
S^*(S) = \sum_{\lambda \in S^*(K, L)} \bigoplus S^*_*(S)
\]

are the orthogonal sum relative to the inner product \( \langle , \rangle \) and \( < , > \), respectively. The restriction \( f \mapsto f' \) of \( S^*_*(D) \rightarrow S^*_*(S) \) is a similitude for each \( \lambda \in S^*(K, L) \), i.e. there exists a constant \( h_\lambda > 0 \) such that

\[
\langle f, g \rangle = h_\lambda \langle f', g' \rangle \quad \text{for any } f, g \in S^*_*(D).
\]

Thus, if

\[
\{ f_{\lambda, i}; 1 \leq i \leq d_\lambda \}, \quad \lambda \in S^*(K, L)
\]
is an orthonormal basis of \( S^*_*(S) \), then

\[
\{ \sqrt{h_\lambda^{-1}} f_{\lambda, i}; \lambda \in S^*(K, L), 1 \leq i \leq d_\lambda \}
\]
is a complete orthonormal system of \( H^2(D) \).
A basis \( \{ f_{\lambda,i}; 1 \leq i \leq d_{\lambda} \} \) is, for instance, constructed as follows. Take an irreducible \( K \)-module \((\rho, V)\) with the highest weight \( \lambda \), carrying a \( K \)-invariant hermitian inner product \((\ , \)\). Choose an orthonormal basis \( \{ u_i; 1 \leq i \leq d_{\lambda} \} \) of \( V \) such that the first vector \( u_i \) is \( L \)-invariant. This can be done in view of Frobenius’ reciprocity since the \( K \)-module \( V \) is \( K \)-isomorphic with a \( K \)-submodule of \( C^\infty(S) \). Then the functions \( f_{\lambda,i}(1 \leq i \leq d_{\lambda}) \) defined by

\[
 f_{\lambda,i}(kX_u) = \sqrt{d_{\lambda}}(u_i, \rho(k)u_i) \quad \text{for} \quad k \in K
\]

form an orthonormal basis of \( S^*_\lambda(S) \) (cf. 2).

We compute next the normalizing factor \( h_{\lambda} \). Let

\[
 a = \{ \sqrt{-1}\Delta \}_{\mathcal{R}}
\]

be the \( \mathcal{R} \)-span of \( \sqrt{-1}\Delta \) in \( t \) and

\[
 \varpi: \sqrt{-1}t \to \sqrt{-1}a
\]

denote the orthogonal projection of \( \sqrt{-1}t \) onto \( \sqrt{-1}a \). For \( \gamma \in \varpi \sum - \{0\} \), the number of roots \( \alpha \in \Sigma \) such that \( \varpi \alpha = \gamma \) is called the multiplicity of \( \gamma \). Let \( r \) (resp. 2s) be the multiplicity of \( \frac{1}{2}(\gamma_1 - \gamma_2) \) (resp. of \( \frac{1}{2}\gamma_1 \)). If follows from Theorem A and Frobenius’ reciprocity that for each \( \lambda \in S^*(K, L) \) there exists uniquely an \( L \)-invariant polynomial \( \Omega_{\lambda} \) in \( S^*_\Lambda((\mathfrak{p}^0)^-)) \) such that \( \Omega_{\lambda}(X_\alpha) = 1 \), where \( S^*_\Lambda((\mathfrak{p}^0)^-) \) denotes the irreducible \( K \)-submodule of \( S^*(\mathfrak{p}^0)^-)) \) with the highest weight \( \lambda \). The polynomial \( \Omega_{\lambda} \) is called the zonal spherical polynomial for \( D \) belonging to \( \lambda \). Let

\[
 (a^-)^{\mathcal{C}} = \{ X_{-\gamma}; \gamma \in \Delta \}_{\mathcal{C}}
\]

be the \( \mathcal{C} \)-span of \( \{ X_{-\gamma}; \gamma \in \Delta \} \) in \((\mathfrak{p}^0)^-\). It is identified with the complex cartesian space \( \mathcal{C}^p \) by the map

\[
 - \sum_{i=1}^p z_i X_{-\gamma_i} \mapsto \left( \begin{array}{c} z_1 \\ \vdots \\ z_p \end{array} \right).
\]

Thus the zonal spherical polynomial \( \Omega_{\lambda} \) restricted to \( (a^-)^{\mathcal{C}} \) is a polynomial \( \Omega_{\lambda}(Y_1, \cdots, Y_p) \) in \( \mathfrak{p} \)-variables. Let \( \mu(D) \) denote the volume of \( D \) with respect to the measure \( d\mu(X) \). We shall prove the following

**Theorem B.** For \( \lambda \in S^*(K, L) \), the normalizing factor \( h_{\lambda} \) is given by

\[
 h_{\lambda} = c(D) \int_{0 < y_i < 1 \ (1 \leq i \leq p)} \Omega_{\lambda}(y_1, \cdots, y_p) \prod_{1 \leq i < j \leq p} (y_i - y_j)^r \prod_{i=1}^p y_i^s \, dy_1 \cdots dy_p
\]

where

\[
 c(D) = \mu(D) \left( \int_{0 < y_i < 1 \ (1 \leq i \leq p)} \prod_{1 \leq i < j \leq p} (y_i - y_j)^r \prod_{i=1}^p y_i^s \, dy_1 \cdots dy_p \right)^{-1}.
\]
Hua [6] proved Theorem A for classical domains by decomposing the character of the $K$-module $S^*((\mathfrak{p}^c)^{-})$ into the sum of irreducible characters of $K$, while Schmid [11] proved it for general domain $D$. Schmid proved

(a) \[ S^v(D) \subseteq S^v(K, L) \]

by seeing the character of the $K$-module $S^*((\mathfrak{p}^c)^{-})$ and by making use of E. Cartan's theory on spherical representations of a compact symmetric pair. But his proof of

(b) \[ S^v(K, L) \subseteq S^v(D) \]

is complicated and was done after nine successive lemmas. In this note we give another proof of (a) by means of a lemma of Murakami and Cartan's theory, and give a relatively short proof of (b) by means of a theorem of Harish-Chandra on invariant polynomials for a symmetric pair.

Hua [6] computed the factors $h_\lambda$ for certain classical domains by integrating certain polynomials. Our integral formula in Theorem B will clarify the meaning of integrals of Hua.

1. Circular domains

A domain $D \subset \mathbb{C}^n$ containing the origin 0 is said to be a circular domain with the center 0 if together with any point $z \in D$ the point $e^{\sqrt{-1} \theta} z$ is in $D$ for any real $\theta \in \mathbb{R}$. $D$ is said to be a starlike domain with the center 0 if together with any point $z \in D$ the point $rz$ is in $D$ for any real $r \in \mathbb{R}$ with $0 < r < 1$.

**Theorem 1.1.** (H. Cartan [2]) Let $D \subset \mathbb{C}^n$ be a circular domain with the center 0. Then any holomorphic function $f$ on $D$ can be developed in the sum of homogeneous polynomials $P_\nu$ in $n$-variables with degree $\nu$ ($\nu=0, 1, 2, \cdots$):

\[ f(z) = \sum_{\nu=0} P_\nu(z) \quad \text{for} \quad z \in D. \]

The sum converges uniformly on any compact subset of $D$. The homogeneous polynomials $P_\nu$ are uniquely determined for $f$.

Let $D$ be a bounded domain in $\mathbb{C}^n$, $d\mu(z)$ the Euclidean measure on $\mathbb{C}^n$, induced from the standard hermitian inner product of $\mathbb{C}^n$. Let $H^s(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(z)$. The inner product of $H^s(D)$ will be denoted by $\langle , \rangle$. Let $S^*(\mathbb{C}^n)$ be the graded space of polynomials in $n$-variables and $S^*(D)$ the subspace of $H^s(D)$ consisting of all functions on $D$ obtained by the restriction of polynomials in $S^*(\mathbb{C}^n)$. Then Theorem 1.1 yields the following

**Corollary.** Let $D \subset \mathbb{C}^n$ be a circular starlike bounded domain with the center 0. Then the subspace $S^*(D)$ of $H^s(D)$ is dense in $H^s(D)$. 
Proof. If suffices to show that if \( f \in H'(D) \) with \( \langle f, S^*(D) \rangle = \{0\} \), then \( f = 0 \). Theorem 1.1 implies that \( f \) can be developed as

\[
f = \sum_{\nu_0} P_{\nu} \quad P_{\nu} \in S^v(D),
\]

uniformly convergent on any compact subset of \( D \). Choose an orthonormal basis \( \{P_{\nu,j}\} \) of \( S^v(D) \) with respect to \( \langle \cdot, \cdot \rangle \) for each \( \nu \). Then we have

\[
\langle P_{\nu,j}, P_{\mu,i} \rangle = \delta_{\nu\mu} \delta_{ji}.
\]

In fact, since \( d\mu(e^{\sqrt{-1}\theta z}) = d\mu(z) \) for any \( \theta \in \mathbb{R} \), we have \( \langle P_{\nu,j}, P_{\nu,i} \rangle = e^{\sqrt{-1}\theta \mu} \langle P_{\nu,j}, P_{\mu, i} \rangle \) for any \( \theta \in \mathbb{R} \). Then \( f \) can be developed as

\[
f = \sum_{\nu,j} a_{\nu,j} P_{\nu,j} \quad \text{with } a_{\nu,j} \in C,
\]

uniformly convergent on any compact subset of \( D \). Since \( D \) is a starlike domain, the closure \( \overline{rD} \) of \( rD \) is a compact subset of \( D \) for any \( r \in \mathbb{R} \) with \( 0 < r < 1 \), so that the above series converges uniformly on \( \overline{rD} \). Therefore for any \( P_{\mu,i} \) we have

\[
\int_{\overline{rD}} f(z) \overline{P_{\mu,i}(z)} d\mu(z) = \sum_{\nu,j} a_{\nu,j} \int_{\overline{rD}} P_{\nu,j}(z) \overline{P_{\mu,i}(z)} d\mu(z).
\]

If we put

\[
z' = \frac{1}{r} z \quad \text{for } z \in rD,
\]

then \( z = rz' \), \( d\mu(z) = r^2 d\mu(z') \) so that

\[
\int_{rD} P_{\nu,j}(z) \overline{P_{\mu,i}(z)} d\mu(z) = r^{2n+\nu+\mu} \int_{D} P_{\nu,j}(z') \overline{P_{\mu,i}(z')} d\mu(z')
\]

\[
= r^{2n+\nu+\mu} \langle P_{\nu,j}, P_{\mu,i} \rangle = r^{2n+2\mu} \delta_{\nu\mu} \delta_{ji}.
\]

Hence we have

\[
\int_{rD} f(z) \overline{P_{\mu,i}(z)} d\mu(z) = a_{\mu,i} r^{2n+2\mu}
\]

and

\[
a_{\mu,i} = \lim_{r \to 1} a_{\mu,i} r^{2n+2\mu} = \lim_{r \to 1} \int_{rD} f(z) \overline{P_{\mu,i}(z)} d\mu(z)
\]

\[
= \langle f, P_{\mu,i} \rangle = 0 \quad (\text{from the assumption}).
\]

This implies that \( f = 0 \). q.e.d.
2. Spherical representations of a compact symmetric pair

Let $K$ be a compact connected Lie group, $L$ a closed subgroup of $K$ and $S$ be the quotient space $K/L$. The space of $C$-valued $C^\infty$-functions on $S$ will be denoted by $C^\infty(S)$. We shall often identify $C^\infty(S)$ with the space of $C^\infty$-functions $f$ on $K$ such that

$$ f(kl) = f(k) \quad \text{for any} \quad k \in K, \ l \in L. $$

Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure on $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle \ , \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$ (kf)(x) = f(k^{-1}x) \quad \text{for} \quad k \in K, \ x \in S. $$

Then $C^\infty(S)$ is a $K$-submodule of $L^2(S)$. A (continuous finite dimensional complex) representation

$$ \rho: K \to \text{GL}(V) $$

of $K$ is said to be spherical relative to $L$ if the $K$-module $V$ is equivalent to a $K$-submodule of $C^\infty(S)$, which amounts to the same from Frobenius' reciprocity that the $K$-module $V$ has a non-zero $L$-invariant vector. We denote by $\mathcal{D}(K, L)$ the set of equivalence classes of irreducible spherical representations of $K$ relative to $L$. The totality of $f \in C^\infty(S)$ contained in a finite dimensional $K$-submodule of $C^\infty(S)$, which will be denoted by $\mathfrak{o}(K, L)$, is a $K$-submodule of $C^\infty(S)$. A function in $\mathfrak{o}(K, L)$ is called a spherical function for the pair $(K, L)$. For $\rho \in \mathcal{D}(K, L)$, the totality of $f \in \mathfrak{o}(K, L)$ that transforms according to $\rho$, which will be denoted by $\mathfrak{o}_\rho(K, L)$, is a finite dimensional $K$-submodule of $\mathfrak{o}(K, L)$. Then

$$ \mathfrak{o}(K, L) = \sum_{\rho \in \mathcal{D}(K, L)} \oplus \mathfrak{o}_\rho(K, L) $$

is the orthogonal sum with respect to the inner product $\langle \ , \rangle$. Peter-Weyl approximation theorem implies that the subspace $\mathfrak{o}(K, L)$ of $L^2(S)$ is dense in $L^2(S)$. We assume furthermore that the pair $(K, L)$ satisfies the condition

$$(*) \quad \text{any} \ \rho \in \mathcal{D}(K, L) \ \text{is contained exactly once in} \ \mathfrak{o}(K, L),$$

which is by Frobenius' reciprocity equivalent to that for any spherical representation

$$ \rho: K \to \text{GL}(V) $$

of $K$ relative to $L$, an $L$-invariant vector of $V$ is unique up to scalar multiplication. Then for each $\rho \in \mathcal{D}(K, L)$, there exists uniquely an $L$-invariant function $\omega_\rho \in \mathfrak{o}_\rho(K, L)$ such that $\omega_\rho(\varepsilon) = 1$. $\omega_\rho$ is called the zonal spherical function for $(K, L)$ belonging to $\rho$. Let

$$ \rho: K \to \text{GL}(V) $$
be a spherical representation of $K$ relative to $L$. Choose a $K$-invariant hermitian inner product $(\ ,\ )$ on $V$. The equivalence class containing $\rho$ will be denoted by the same letter $\rho$. Choose an orthonormal basis $\{u_i; 1 \leq i \leq d_\rho\}$ of $V$ such that $u_i$ is $L$-invariant. Define $\varphi_i \in C^\infty(S)$ $(1 \leq i \leq d_\rho)$ by

$$\varphi_i(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.$$ 

We know that they are linearly independent, in view of orthogonality relations of matrix elements $(u_i, \rho(k)u_j)$. For any $k' \in K$ we have

$$\varphi_i(k'^{-1}k) = (u_i, \rho(k'^{-1}k)u_i) = (\rho(k')u_i, \rho(k)u_i)$$

$$= \sum_j (\rho(k')u_i, u_j)(u_j, \rho(k)u_i)$$

$$= \sum_j (\rho(k')u_i, u_j) \varphi_j(k),$$

i.e.

$$k'\varphi_i = \sum_j (\rho(k')u_i, u_j)\varphi_j \quad (1 \leq i \leq d_\rho).$$

In particular

$$l\varphi_i = \varphi_1 \quad \text{for any} \quad l \in L,$$

and

$$\varphi_i(e) = 1.$$ 

Therefore the system $\{\varphi_i; 1 \leq i \leq d_\rho\}$ forms a basis of $\mathfrak{o}_\rho(K, L)$ and the zonal spherical function $\omega_\rho$ is given by

$$\omega_\rho(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.$$ 

Furthermore orthogonality relations implies that the system

$$\{\sqrt{d_\rho}\varphi_i; 1 \leq i \leq d_\rho\}$$

forms an orthonormal basis of $\mathfrak{o}_\rho(K, L)$ and that

$$\langle\omega_\rho, \omega_\rho'\rangle = \delta_{\rho\rho'} \frac{1}{d_\rho}.$$ 

Henceforth we assume that the pair $(K, L)$ is a symmetric pair, i.e. there exists an involutive automorphism $\theta$ of $K$ such that if we put

$$K_\theta = \{k \in K; \theta(k) = k\},$$

$L$ lies between $K_\theta$ and the connected component $K_\theta^0$ of $K_\theta$. Then the pair $(K, L)$ satisfies the condition $(\ast)$ (E. Cartan [1]). For example, a compact connected Lie group $S$ admits a symmetric pair $(K, L)$ such that $S = K/L$. In fact,

$$K = S \times S,$$

$$L = \{(x, x); x \in S\}.$$
and
\[ \theta: (x, y) \mapsto (y, x) \quad \text{for} \quad x, y \in S \]

have desired properties.

In the following we summarize some known facts on a symmetric pair (cf. Helgason [4]).

Let \( \mathfrak{f} \) (resp. \( \mathfrak{l} \)) be the Lie algebra of \( K \) (resp. of \( L \)). The involutive automorphism of \( \mathfrak{f} \) obtained by differentiating the automorphism \( \theta \) of \( K \) will be also denoted by the same letter \( \theta \).

Choose and fix once for all a \( \mathbb{C} \)-bilinear symmetric form \( (\ , \) \) on the complexification \( \mathfrak{f}^\mathbb{C} \) of \( \mathfrak{f} \), which is invariant under both the \( \mathbb{C} \)-linear extension to \( \mathfrak{f}^\mathbb{C} \) of \( \theta \) and the adjoint action of \( \mathfrak{f}^\mathbb{C} \) and furthermore is negative definite on \( \mathfrak{f} \times \mathfrak{f} \). Then \( S \) is a Riemannian symmetric space with respect to the \( K \)-invariant Riemannian metric on \( S \) defined by \(- (\ , \) \). We put
\[ \mathfrak{s} = \{ X \in \mathfrak{f}; \theta X = -X \} = \{ X \in \mathfrak{f}; (X, I) = \{0\} \} . \]

Then we have orthogonal decompositions
\[ \mathfrak{f} = \mathfrak{l} + \mathfrak{s} = \mathfrak{c} \oplus \mathfrak{f}' , \]
where \( \mathfrak{c} \) is the center of \( \mathfrak{f} \) and \( \mathfrak{f}' \) is the derived algebra \([\mathfrak{f}, \mathfrak{f}] \) of \( \mathfrak{f} \). We choose a maximal abelian subalgebra \( \mathfrak{a} \) in \( \mathfrak{s} \). Such \( \mathfrak{a} \) are mutually conjugate under the adjoint action of \( L \). \( \dim \mathfrak{a} \) is the rank of the symmetric pair \((K, L)\). Extend \( \mathfrak{a} \) to a maximal abelian subalgebra \( \mathfrak{t} \) of \( \mathfrak{f} \) containing \( \mathfrak{a} \). Then we have the decomposition
\[ \mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a} \quad \text{where} \quad \mathfrak{b} = \mathfrak{t} \cap \mathfrak{l} . \]

Let \( \mathfrak{t}' = \mathfrak{t} \cap \mathfrak{l}' \) and \( \mathfrak{a}' = \mathfrak{a} \cap \mathfrak{l}' \). The real vector space \( \sqrt{-1} \mathfrak{t} \) has the natural inner product \((\ , \) \) induced from the bilinear form \((\ , \) \) on \( \mathfrak{f}^\mathbb{C} \). We shall identify \( \sqrt{-1} \mathfrak{t} \) with the dual space of \( \sqrt{-1} \mathfrak{t} \) by means of the inner product \((\ , \) \). We have the orthogonal decomposition
\[ \sqrt{-1} \mathfrak{t} = \sqrt{-1} \mathfrak{b} \oplus \sqrt{-1} \mathfrak{a} . \]

Let \( \sigma \) be the orthogonal transformation on \( \sqrt{-1} \mathfrak{t} \) defined by
\[ \sigma|\sqrt{-1} \mathfrak{b} = -1 \quad \text{and} \quad \sigma|\sqrt{-1} \mathfrak{a} = 1 \]
and
\[ \varpi = \frac{1}{2}(1+\sigma): \sqrt{-1} \mathfrak{t} \to \sqrt{-1} \mathfrak{a} \]
be the orthogonal projection of \( \sqrt{-1} \mathfrak{t} \) onto \( \sqrt{-1} \mathfrak{a} \). Let \( \Sigma^\mathfrak{t} \) denote the set of roots of \( \mathfrak{f}^\mathbb{C} \) with respect to the complexification \( \mathfrak{t}^\mathbb{C} \) of \( \mathfrak{t} \). Let \( W_\mathfrak{t} = N_K(T)/T \) be the Weyl group of \( \mathfrak{t} \), where \( T \) is the connected subgroup of \( K \) generated by \( \mathfrak{t} \) and \( N_K(T) \) is the normalizer of \( T \) in \( K \). \( \Sigma^\mathfrak{t} \) is a \( \sigma \)-invariant reduced root system in
\[ \sqrt{-1} t'. \] As a group of orthogonal transformations of \( \sqrt{-1} t \), \( W_t \) is generated by reflections with respect to roots in \( \Sigma_t \). Put

\[
\Sigma^0_t = \Sigma_t \cap \sqrt{-1} b = \{ \alpha \in \Sigma_t; \sigma \alpha = 0 \},
\]
\[
\Sigma_s = \{ \sigma \alpha; \alpha \in \Sigma_t - \Sigma^0_t \} = \sigma \Sigma_t - \{0\},
\]
\[
W_s = N_L(A)/Z_L(A),
\]
where \( A \) is the connected subgroup of \( K \) generated by \( a \) and \( N_L(A) \) (resp. \( Z_L(A) \)) the normalizer (resp. the centralizer) of \( A \) in \( L \). An element of \( \Sigma_s \) is a restricted root of the symmetric space \( S \) and \( W_s \) is the Weyl group of \( S \). \( \Sigma_s \) is a (not necessarily reduced) root system in \( \sqrt{-1} a' \). As a group of orthogonal transformations of \( \sqrt{-1} a \), \( W_s \) is generated by reflections with respect to roots in \( \Sigma_s \). A linear order \( \sigma \) on \( \sqrt{-1} t \) is said to be compatible for \( \Sigma_t \) with respect to \( \sigma \) (or with respect to the orthogonal decomposition \( \sqrt{-1} t = \sqrt{-1} b \oplus \sqrt{-1} a \)) if \( \alpha \in \Sigma_t \), \( \alpha > 0 \) and \( \sigma \alpha = -\alpha \) imply \( \sigma \alpha > 0 \). Take a compatible order \( \sigma \) on \( \sqrt{-1} t \) and fix it once and for all. Let

\[
\Pi_t = \{ \alpha_1, \ldots, \alpha_t \}
\]
be the fundamental root system of \( \Sigma_t \) with respect to the order \( \sigma \) and put

\[
\Pi^0_t = \Pi_t \cap \Sigma^0_t.
\]
\( W_t \) is also generated by reflections with respect to roots in \( \Pi_t \). We have the decomposition

\[
\sigma = sp \quad \text{where} \quad s \in W_t, \quad p \Pi_t = \Pi_t
\]
of \( \sigma \) in such a way that \( p^2 = 1 \), \( p(\Pi_t - \Pi^0_t) = \Pi_t - \Pi^0_t \) and \( \sigma \alpha_i \equiv p \alpha_i \mod \{ \Pi^0_t \} \) for any \( \alpha_i \in \Pi_t - \Pi^0_t \) (Satake [10]). We put

\[
\Pi_s = \{ \sigma \alpha_i; \alpha_i \in \Pi_t - \Pi^0_t \} = \sigma \Pi_t - \{0\}.
\]
We may assume that \( \Pi_s = \{ \gamma_1, \ldots, \gamma_p \} \) with \( \sigma \alpha_i = \gamma_i \) (1 \leq i \leq p), changing indices of the \( \alpha_i \)'s if necessary. \( \Pi_s \) is the fundamental root system of \( \Sigma_s \) with respect to the order \( \sigma \). We put

\[
\Sigma^*_s = \{ \gamma \in \Sigma_s; 2 \gamma \in \Sigma_s \}.
\]
Then \( \Sigma^*_s \) is a reduced root system in \( \sqrt{-1} a' \). The fundamental root system \( \Pi^*_s \) of \( \Sigma^*_s \) with respect to the order \( \sigma \) is given by

\[
\Pi^*_s = \{ \beta_1, \ldots, \beta_p \}
\]
where

\[
\beta_i = \begin{cases} \gamma_i & \text{if} \quad 2 \gamma_i \in \Sigma_s, \\ 2 \gamma_i & \text{if} \quad 2 \gamma_i \in \Sigma_s. \end{cases}
\]
\( W_s \) is also generated by reflections with respect to roots of \( \Pi_s \) or of \( \Pi^*_s \). Let
\( \Sigma^*_t \) (resp. \( \Sigma^*_s, (\Sigma^*_s)^* \)) denote the set of positive roots in \( \Sigma_t \) (resp. \( \Sigma_s, \Sigma^*_s \)). Then

\[
\Sigma^*_s = \omega (\Sigma^*_t - \Sigma^*_p) = \omega \Sigma^*_t - \{0\}.
\]

For \( \lambda \in \sqrt{-1} \mathbb{R} \), \( \lambda \neq 0 \), we define

\[
\lambda^* = \frac{2}{\langle \lambda, \lambda \rangle} \lambda.
\]

**Theorem 2.1.** (E. Cartan) Assume that \( K \) is simply connected. Then
1) \( K_\gamma \) is connected.
2) The kernel of \( \exp: \mathfrak{a} \rightarrow K \) is the subgroup of \( \mathfrak{a} \) generated by \( \{2\pi \sqrt{-1} \gamma^*; \gamma \in \Sigma_s\} \).

**Theorem 2.2.** (Harish-Chandra) Let \( S^*_L(\mathfrak{g}) \) (resp. \( S^*_W(\mathfrak{a}) \)) be the space of polynomial functions on \( \mathfrak{g} \) (resp. on \( \mathfrak{a} \)), which are invariant under the adjoint actions of \( L \) (resp. of \( W_s \)). Then the restriction map

\[
S^*_L(\mathfrak{g}) \rightarrow S^*_W(\mathfrak{a})
\]

is an isomorphism.

Now we shall consider \( W_s \)-invariant characters of a maximal torus of \( S \). Put

\[
\Gamma = \Gamma(K, L) = \{ H \in \mathfrak{a}; \exp H \in L \}
\]

and

\[
\Gamma_c = \Gamma \cap \mathfrak{c}_a \text{ where } \mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}.
\]

Then \( \Gamma \) is a \( W_s \)-invariant lattice in \( \mathfrak{a} \) and \( \Gamma_c \) is a lattice in \( \mathfrak{c}_a \). Let \( \mathcal{C}_a \) be the connected subgroup of \( K \) generated by \( \mathfrak{c}_a \). Then the \( A \)-orbit \( \hat{A} \) in \( S \) through the origin \( x_o \) of \( S \) and the \( C_a \)-orbit \( \hat{C}_a \) in \( S \) through the origin have identifications

\[
\hat{A} = \mathfrak{a}/\Gamma
\]

and

\[
\hat{C}_a = \mathfrak{c}_a/\Gamma_c.
\]

Hence both \( \hat{A} \) and \( \hat{C}_a \) have structures of toral groups. The toral group \( \hat{A} \) is said to be a maximal torus of the symmetric space \( S \). The adjoint action of \( W_s \) on \( A \) induces the action of \( W_s \) on \( A \). This action is compatible with the natural action of \( W_s \) on \( \mathfrak{a}/\Gamma \) relative to the identification: \( \hat{A} = \mathfrak{a}/\Gamma \). Put

\[
Z = Z(K, L) = \{ \lambda \in \sqrt{-1} \mathbb{R}; (\lambda, H) \in 2\pi \sqrt{-1} Z \text{ for any } H \in \mathfrak{a} \}.
\]

\( Z \) is isomorphic with the group \( \mathcal{D}(\hat{A}) \) of characters of \( \hat{A} \) by the correspondence \( \lambda \mapsto \varepsilon^\lambda \), where \( \varepsilon^\lambda \in \mathcal{D}(\hat{A}) \) is defined by \( \varepsilon^\lambda((\exp H)x_o) = \exp (\lambda, H) \) for \( H \in \mathfrak{a} \). Put
\[ D = D(K, L) = \{ \lambda \in Z; (\lambda, \gamma_i) \geq 0 \text{ for any } \gamma_i \in \Pi \} \]
\[ = \{ \lambda \in Z; (\lambda, \gamma) \geq 0 \text{ for any } \gamma \in \sum \} \]

Then we have
\[ D = \{ \lambda \in Z; s\lambda \leq \lambda \text{ for any } s \in W_s \} \]

An element of \( D \) is called a \textit{dominant integral form} on \( \alpha \). We define a lattice \( \Gamma' \) in \( \alpha' \) to be the subgroup of \( \alpha' \) generated by \( \{ 2\pi \sqrt{-1} (\frac{1}{2} \gamma \ast); \gamma \in \sum \} \). We define a lattice \( \Gamma_0 \) in \( \alpha \) and a toral group \( \hat{A}_0 \) by
\[ \Gamma_0 = \Gamma \oplus \Gamma' \]
and
\[ \hat{A}_0 = \alpha/\Gamma_0 \]

Put
\[ Z_0 = \{ \lambda \in \sqrt{-1} \alpha; (\lambda, H) \in 2\pi \sqrt{-1} Z \text{ for any } H \in \Gamma_0 \} \]
and
\[ D_0 = D \cap Z_0 \]

\( Z_0 \) is isomorphic with the group \( \mathcal{O}(\hat{A}_0) \) of characters of \( \hat{A}_0 \). Put furthermore
\[ Z'_0 = Z_0 \cap \sqrt{-1} \alpha' = \{ \lambda \in \sqrt{-1} \alpha'; 2(\lambda, \gamma) \in \sum \text{ for any } \gamma \in \sum \} \]
and
\[ D'_0 = D_0 \cap \sqrt{-1} \alpha' = D \cap Z'_0 \]

\textbf{Lemma 1.} If \( L=K_0 \), then
\[ \Gamma = \{ \frac{1}{2} H; H \in \alpha, \exp H = e \} \]

Proof. For \( H \in \alpha, \exp H = e \Leftrightarrow \exp \frac{H}{2} \exp \frac{H}{2} = e \Leftrightarrow \exp \frac{H}{2} = \left( \exp \frac{H}{2} \right)^{-1} \Leftrightarrow \]
\[ \exp \frac{H}{2} = \theta \left( \exp \frac{H}{2} \right) \Leftrightarrow \exp \frac{H}{2} \in K_0, \text{ which yields Lemma 1.} \]

\textbf{Lemma 2.} 1) \( \Gamma'_0 = 2\pi \sqrt{-1} \sum \beta_i \in \sum \) \( \mathcal{O}(\frac{1}{2} \beta \ast) \)

and it is \( W_s \)-invariant. Therefore \( \Gamma_0 \) is \( W_s \)-invariant.

2) \( \Gamma_0 \subseteq \Gamma \). Therefore \( Z_0 \supseteq Z \) and \( D_0 \supset D \).

3) If \( S \) is simply connected, then \( \Gamma = \Gamma_0 = \Gamma'_0 \) (thus \( Z = Z_0 = Z'_0, D = D_0 = D'_0 \)) and \( \hat{A}_0 \) can be identified with \( \hat{A} \).

Proof. 1) Denoting the reflection of \( \sqrt{-1} \alpha \) with respect to \( \beta_i \in \Pi \) by \( s_i \in W_s \), we have
It follows that $\Gamma_0'$ is $W_S$-invariant. Since we have

$$s_1\gamma^* = (s_1\gamma)^* = \gamma^* \frac{2(\beta, \gamma)}{(\gamma, \gamma)} \beta^*$$

for $\gamma \in \Sigma_S$.

$\Gamma_0'$ is the subgroup of $\alpha'$ generated by $2\pi \sqrt{-1}(\frac{1}{2} \gamma^*)$ for $\gamma \in \Sigma_S^*$. Thus it suffices to show that

$$\gamma^* \in \sum_{i=1}^{\ell} \mathbb{Z} \beta_i^*$$

for any $\gamma \in \Sigma_S^*$.

But this follows from the first equality since there exist $\beta_{i_1}, \ldots, \beta_{i_\ell} \in \Pi_S^*$ such that $s_1 \cdots s_{i_\ell} \gamma \in \Pi_S^*$.

2) Since $\Gamma \subset \Gamma'$, it suffices to show that $\Gamma_0' \subset \Gamma'$ for $\Gamma' = \Gamma \cap \alpha'$. Let $K'$ be the connected subgroup of $K$ generated by $\Gamma'$ and $L' = K' \cap L$. Then $(K', L')$ is also a symmetric pair with respect to $\theta$ and $S' = K'/L'$ can be identified with the $K'$-orbit in $S$ through the origin $x_0$ of $S$. Let

$$\pi': K_0' \to K'$$

be the covering homomorphism of the universal covering group $K_0'$ of $K'$ and put

$$L_0' = \{k \in K_0'; \theta_0(k) = k\},$$

where $\theta_0$ is the involutive automorphism of $K_0'$ covering the involutive automorphism $\theta$ of $K$. $K_0'$ is compact since $K'$ is semi-simple. $S'$ can be identified with $K_0'/\pi'^{-1}(L')$. It follows from Theorem 2.1 and Lemma 1 that $L_0'$ is connected and

$$\Gamma_0' = \{H \in \alpha'; \exp_{K_0'} H \in L_0'\}.$$

Let $A'$ (resp. $A_0'$) be the connected subgroup of $K'$ (resp. of $K_0'$) generated by $\alpha'$ and $\hat{A}'$ (resp. $\hat{A}_0'$) be the $A'$-orbit in $S'$ (resp. the $A_0'$-orbit in $S_0' = K_0'/L_0'$) through the origin. Then we have identifications

$$\hat{A}' = \alpha'//\Gamma'$$

and

$$\hat{A}_0' = \alpha'/\Gamma_0'.$$

On the other hand, since $\pi'^{-1}(L') \supset L_0'$, the covering homomorphism $\pi'$ induces the commutative diagram

$$\begin{array}{ccc}
S_0' & \xrightarrow{\pi'} & S' \\
\cup & & \cup \\
\hat{A}_0' & \xrightarrow{\pi'} & \hat{A}'.
\end{array}$$
It follows that
\[ \Gamma'_0 \subseteq \Gamma'. \]

3) Under the notation in 2), we have a covering map
\[ \hat{\mathcal{C}}_a \times S' \to S. \]

It follows from the assumption that \( \hat{\mathcal{C}}_a = \{ e \} \) and \( S' \) is simply connected. Thus the covering map \( \pi' \) is trivial and \( \Gamma' = \Gamma'_0 \). Moreover \( c_0 = \{ 0 \} \) implies that \( \Gamma = \Gamma' \) and \( \Gamma_0 = \Gamma'_0 \). q.e.d.

REMARK. Define \( \Lambda_1 \in \sqrt{-1} \alpha' \) \((1 \leq i \leq l)\) by
\[ (\Lambda_i, \alpha_i^\ast) = \delta_{ij} \quad (1 \leq i, j \leq l). \]

Then define \( M_i \) \((1 \leq i \leq p)\) by
\[ M_i = \begin{cases} \Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_i^\ast) = \{ 0 \} \\ 2\Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_i^\ast) \neq \{ 0 \} \\ \Lambda_i + \Lambda_i & \text{if } p\alpha_i = \alpha_i \neq \alpha_i. \end{cases} \]

Then it can be verified (cf. Sugiura [12]) that \( M_i \in \sqrt{-1} \alpha' \) \((1 \leq i \leq p)\) and
\[ (M_i, \frac{1}{2} \beta_i^\ast) = \delta_{ij} \quad (1 \leq i, j \leq p). \]

It follows that
\[ Z_0' = \sum_{i=1}^p ZM_i \]

and
\[ D_0' = \{ \sum_{i=1}^p m_i M_i; \ m_i \in \mathbb{Z}, \ m_i \geq 0 \ (1 \leq i \leq p) \}. \]

It follows from Lemma 2,1) that \( W_S \) acts on \( \hat{A}_s = \alpha _\Gamma_0 \) and from Lemma 2,2) that we have a \( W_S \)-equivariant homomorphism
\[ \pi_0 : \hat{A}_s \to \hat{A}. \]

Let \( \mathcal{R} (\hat{A}) \) denote the character ring of \( \hat{A} \). Then \( W_S \) acts on \( \mathcal{R} (\hat{A}) \) (or more generally on the space \( C^\infty (\hat{A}) \) of \( C \)-valued \( C^\infty \)-functions on \( \hat{A} \)) by
\[ (s \chi)(\hat{a}) = \chi(s^{-1} \hat{a}) \quad \text{for } s \in W_S, \ \hat{a} \in \hat{A}. \]

This action coincides on \( Z = \mathcal{D} (\hat{A}) \subseteq \mathcal{R} (\hat{A}) \) with the adjoint action of \( W_S \) on \( Z \). Let \( \mathcal{R}_{W_S} (\hat{A}) \) be the subring of \( W_S \)-invariant characters of \( \hat{A} \) and \( \mathcal{R}_{W_S} (\hat{A})^c \) the \( C \)-span of \( \mathcal{R}_{W_S} (\hat{A}) \) in \( C^\infty (\hat{A}) \). Let \( \mathcal{R} (\hat{A}_k) \), \( \mathcal{R}_{W_S} (\hat{A}_k) \), and \( \mathcal{R}_{W_S} (\hat{A}_k)^c \) denote the same objects for \( \hat{A}_k \). Then \( \pi_0 \) induces a \( W_S \)-equivariant monomorphism
\[ \pi_0^* : \mathcal{R} (\hat{A}) \to \mathcal{R} (\hat{A}_0). \]
and monomorphisms
\[ \pi^\#: \mathcal{R}_{\mathbb{W}_s}(\hat{A}) \to \mathcal{R}_{\mathbb{W}_s}(\hat{A}_0), \]
\[ \pi^\#: \mathcal{R}_{\mathbb{W}_s}(\hat{A}^c) \to \mathcal{R}_{\mathbb{W}_s}(\hat{A}_0^c). \]

Henceforth we shall identify \( \mathcal{R}_{\mathbb{W}_s}(\hat{A}) \) with a subring of \( \mathcal{R}_{\mathbb{W}_s}(\hat{A}_0) \) and \( \mathcal{R}_{\mathbb{W}_s}(\hat{A})^c \) with a subalgebra of \( \mathcal{R}_{\mathbb{W}_s}(\hat{A}_0)^c \) by means of these monomorphisms \( \pi^\# \).

For \( \lambda \in \sqrt{-1} \alpha \), we shall denote by \( \lambda_c \) the \( \sqrt{-1} e_0 \)-component of \( \lambda \) with respect to the orthogonal decomposition
\[ \sqrt{-1} \alpha = \sqrt{-1} e_0 \bigoplus \sqrt{-1} \alpha'. \]

The following facts can be proved in the same way as the classical results for a compact connected Lie group \( S \), so the proofs are omitted.

We define an element \( \delta \) in \( Z_\alpha \) by
\[ \delta = \sum_{\gamma \in \langle Z_\alpha \rangle^\vee} \gamma. \]
For \( \lambda \in Z_\alpha \), we define \( \xi_\lambda \in \mathcal{R}(\hat{A}_0) \) by
\[ \xi_\lambda = \sum_{\alpha \in \mathbb{W}_s} (\det z) e^{\alpha \lambda}. \]
For \( \lambda \in Z \), \( \xi_\lambda \) is divisible by \( \xi_\delta \) in the ring \( \mathcal{R}(\hat{A}_0) \) and
\[ \chi_\lambda = \frac{\xi_{\lambda+\delta}}{\xi_\delta} \]
is in \( \mathcal{R}_{\mathbb{W}_s}(\hat{A}) \). If \( \chi_\lambda \) has the expression
\[ \chi_\lambda = \sum m_\mu e^\mu \quad \text{with} \quad \mu \in Z, \ m_\mu \in Z, \ m_\mu \neq 0, \]
then \( m_\mu \) are the same for any \( \mu \). In particular, if \( \lambda \in D \), then the highest component in the above expression of \( \chi_\lambda \) is \( e^\lambda \) with \( m_\lambda = 1 \). Any \( W_\mathbb{S} \)-invariant character \( \chi \in \mathcal{R}_{\mathbb{W}_s}(\hat{A}) \) of \( \hat{A} \) has an expression
\[ \chi = \sum m_\lambda \chi_\lambda \quad \text{with} \quad \lambda \in D, \ m_\lambda \in Z. \]
The expression is unique for \( \chi \). In particular, the system \( \{ \chi_\lambda; \lambda \in D \} \) forms a basis of the space \( \mathcal{R}_{\mathbb{W}_s}(\hat{A})^c \).

Now we come back to spherical representations of a symmetric pair \((K, L)\).

**Theorem 2.3.** (E. Cartan [1]) Let \( \rho \in \mathcal{D}(K, L) \) have the highest weight \( \lambda \in \sqrt{-1} t \) and \( \omega_\lambda \) be the zonal spherical function for \((K, L)\) belonging to \( \rho \). Then
1) \( \lambda \in D \),
2) \( \omega_\lambda \) restricted to \( \hat{A} \) is in \( \mathcal{R}_{\mathbb{W}_s}(\hat{A})^c \) and has an expression
\[ \omega_\lambda = \sum a_\mu e^{-\mu} \quad \text{with} \quad \mu \in Z, \ a_\mu \in \mathbb{R}, \ a_\mu > 0, \ \sum a_\mu = 1, \]
with the lowest component \( a \lambda e^{-\lambda} \).

Proof. Proof of E. Cartan [1] was done in the case where \( K \) is semi-simple and \( L = K_\alpha \). His proof can be applied for our case without difficulties. But his proof of \( \lambda \in \sqrt{-1} a \) is not complete. A correct proof is seen, for example, in Schmid [11]. q.e.d.

**Lemma 3.** For any \( \lambda \in D \), there exists an irreducible representation \( \rho \) of \( K \) such that the highest weight of \( \rho \) on \( \mathfrak{t}^c \) is \( \lambda \).

Proof. Let \( H \in \mathfrak{t} \) with \( \exp H = e \). Decompose \( H \) as

\[
H = H' + H'' \quad \text{with} \quad H' \in \mathfrak{b}, \ H'' \in \mathfrak{a}.
\]

Then \( \exp H'' = (\exp H')^{-1} \in L \), i.e. \( H'' \in \Gamma \). It follows from \( \lambda \in Z \subset \sqrt{-1} a \) that \( (\lambda, H) = (\lambda, H') + (\lambda, H'') = (\lambda, H'') \in 2\pi \sqrt{-1} Z \). Moreover \( (\lambda, \alpha_i) = (\lambda, \sqrt{2} \alpha_i) \geq 0 \) for any \( \alpha_i \in \Pi \), since \( \lambda \in D \). Thus \( e^\lambda \) is a dominant character of the maximal torus \( T \) of \( K \). Then the classical representation theory of compact connected Lie groups assures the existence of \( \rho \). q.e.d.

**Lemma 4.** Let \( Z_L(A) \) be the centralizer in \( L \) of \( A \) and \( Z_L(A)^0 \) the connected component of \( Z_L(A) \). Then

\[
Z_L(A) = Z_L(A)^0 \exp \Gamma.
\]

Proof. The centralizer \( \mathfrak{z}_L(A) \) in \( L \) of \( A \) has the decomposition

\[
\mathfrak{z}_L(A) = \mathfrak{z}_L(A)^0 \oplus A,
\]

where \( \mathfrak{z}_L(A) \) is the centralizer in \( L \) of \( A \). Since the centralizer \( Z_K(A) \) in the compact connected Lie group \( K \) of the torus \( A \) is connected, we have the decomposition

\[
Z_K(A) = Z_L(A)^0 A.
\]

It follows that any element \( m \in Z_L(A) \) can be written as

\[
m = m'a \quad \text{with} \quad m' \in Z_L(A)^0, \ a \in A.
\]

Then \( a = m'^{-1} m \in L \) so that \( a \in \exp \Gamma \). Thus \( m \in Z_L(A)^0 \exp \Gamma \), which proves Lemma 4. q.e.d.

**Lemma 5.** Let \( K^c \) denote the Chevalley complexification of \( K \). Put

\[
K^* = L \exp \sqrt{-1} \mathfrak{s}
\]

and

\[
(K^*)_0 = L^0 \exp \sqrt{-1} \mathfrak{s},
\]

where \( L^0 \) denotes the connected component of \( L \). Then \( (K^*)_0 \) is a closed subgroup of
$K^c$ normalized by $K^*$ and

$$K^* = (K^*)^\circ \exp \Gamma.$$  

Therefore $K^*$ is a closed subgroup of $K^c$ with the connected component $(K^*)^\circ$.

Proof. The first statement is clear. Take any element $l \in L$. From the conjugteness of maximal abelian subalgebras in $\mathfrak{s}$ under the adjoint action of $L^\circ$, there exists $l_i \in L^i$ such that $l,l \in N_L(A)$. Since

$$N_L(A)/Z_L(A) = N_L^\circ(A)/Z_L^\circ(A) = W_S,$$

we can choose $l_z \in L^z$ such that $l_z l_i l \in Z_L(A)$. It follows from Lemma 4 that there exist $l_z \in Z_L(A)^\circ$ and $a \in \exp \Gamma$ such that $l_z l_i l = l_z a$. Therefore $l = l_z^{-1} l_z^{-1} l_z \in L^o \subset (K^*)^\circ$, i.e. $l \in (K^*)^\circ \exp \Gamma$. This completes the proof of Lemma 5. q.e.d.

Now we can prove the following

**Theorem 2.4.** (E. Cartan [1], Sugiura [12], Helgason [5]) For any $\lambda \in D$, there exists an irreducible spherical representation $\rho$ of $K$ relative to $L$ such that the highest weight of $\rho$ on $t^c$ is $\lambda$.

Together with Theorem 2.3 we have the following

**Corollary.** For $\rho \in D(K, L)$, let $\lambda(\rho)$ denote the highest weight of $\rho$ on $t^c$. Then the correspondence $\rho \mapsto \lambda(\rho)$ gives a bijection:

$$D(K, L) \rightarrow D(K, L).$$

Proof of Theorem 2.4. This theorem for the case where $K$ is semi-simple and $L=K_s$ was stated in E. Cartan [1] but its proof is not complete. It was stated for simply connected $K$ without proof in Sugiura [12]. It was proved in Helgason [5] for the case where $K$ is semi-simple and $L$ is connected. Helgason's proof can be applied for our case without difficulties, so we shall confine ourselves to point out necessary modifications.

Let

$$\rho: K \rightarrow GL(V)$$

be the irreducible representation of $K$ with the highest weight $\lambda$ (Lemma 3). By extending $\rho$ to the Chevalley complexification $K^c$ of $K$ and restricting it to the closed subgroup $K^*$ of $K^c$ (Lemma 5), we have an irreducible representation of $K^*$, which will be denoted by the same letter $\rho$. It suffices to show that $\rho$ has a non-zero $L$-invariant. Let $N$ be the connected subgroup of $K^*$ generated by the subalgebra

$$n = \mathfrak{t}^* \cap \sum_{a \in \Sigma^+_t} \mathfrak{t}^*_a.$$
where \( \mathfrak{k}^* \) is the Lie algebra of \( K^* \) and \( \mathfrak{k}_a^* \) is the root space of \( \mathfrak{k}^* \) for \( \alpha \). We shall first prove that the representation \( \rho \) of \( K^* \) is a conical representation of \( K^* \) in the sense of Helgason [5], i.e. if \( v_\lambda \in V, v_\lambda \neq 0 \), is a highest weight vector for \( \rho \) with respect to \( \mathfrak{k}^* \), we have
\[
\rho(mn) v_\lambda = v_\lambda \quad \text{for any } m \in Z_L(A), n \in N.
\]
Denoting the infinitesimal action of \( \mathfrak{k}^* \) on \( V \) by the same letter \( \rho \), we have
\[
\rho(n) v_\lambda = \rho(\delta_\lambda(a)) v_\lambda = \{0\}.
\]
In fact, \( \rho(n) v_\lambda = \{0\} \) since \( n \subseteq \sum_{\alpha \in \Sigma^0} \mathfrak{k}_a^* \). \( \rho(\mathfrak{b}^*) v_\lambda = \{0\} \) for the complexification \( \mathfrak{b}^c \) of \( \mathfrak{b} \) since \( (\sqrt{-1} \mathfrak{b}, \lambda) = \{0\} \). \( \rho(\mathfrak{t}_a^*) v_\lambda = \{0\} \) for \( \alpha \in \Sigma^0, \alpha > 0 \). It follows from \( (\alpha, \lambda) \in (\sqrt{-1} \mathfrak{b}, \lambda) = \{0\} \) for \( \alpha \in \Sigma_0^0 \) that \( \lambda - \alpha \) is not a weight of \( \rho \) for \( \alpha \in \Sigma_0^0, \alpha > 0 \). Since the complexification of \( \delta_\lambda(a) \) is spanned by \( \mathfrak{b}^c \) and the \( \mathfrak{t}_a^* \)'s for \( \alpha \in \Sigma_0^0 \), we have \( \rho(\delta_\lambda(a)) v_\lambda = \{0\} \). Therefore it suffices from Lemma 4 to show that
\[
\rho(\exp H) v_\lambda = v_\lambda \quad \text{for any } H \in \Gamma.
\]
But it is clear since \( \lambda \in Z \), i.e. \( (\lambda, H) \in 2\pi \sqrt{-1} Z \) for any \( H \in \Gamma \).

Thus we can prove in the same way as Helgason [5] that \( V \) has a non-zero \( L \)-invariant vector, by constructing a \( K^* \)-submodule \( V' \) of the \( K^* \)-module \( C^\omega(K^*) \) of \( C^\omega \)-functions on \( K^* \), having a non-zero \( L \)-invariant, and by constructing a \( K^* \)-equivariant isomorphism of \( V \) onto \( V' \).

Next we shall describe zonal spherical functions in terms of the basis \( \{x_\lambda; \lambda \in D\} \) of \( \mathcal{R}_{W_S}(\hat{A})^c \).

For \( \hat{a} = (\exp H)x_\lambda \in \hat{A}, H \in a \), we put
\[
D(\hat{a}) = \left| \prod_{\alpha \in \Sigma^0 - \Sigma^0_0} 2 \sin(\alpha, \sqrt{-1} H) \right|.
\]
Let \( d\hat{a} \) denote the normalized Haar measure of \( \hat{A} \) and \( |W_S| \) the order of the Weyl group \( W_S \). For \( W_S \)-invariant functions \( \chi, \chi' \) on \( \hat{A} \), we define
\[
\langle \chi, \chi' \rangle = \frac{c}{|W_S|} \int_{\hat{A}} \chi(\hat{a}) \overline{\chi'(\hat{a})} D(\hat{a}) d\hat{a},
\]
where
\[
c = \left( \frac{1}{|W_S|} \int_{\hat{A}} D(\hat{a}) d\hat{a} \right)^{-1}.
\]
\( c = 1 \) in the case where \( S \) is a compact connected Lie group. In particular, if \( \chi \) and \( \chi' \) can be extended to \( L \)-invariant functions \( f \) and \( f' \) on \( S \), then \( \langle \chi, \chi' \rangle \) coincides with the inner product \( \langle f, f' \rangle \) in \( L^2(S) \) (cf. Helgason [4]).

Fix a dominant integral form \( \lambda \in D \). We define a finite subset \( D_\lambda \) of \( D \) by
\[ D_\lambda = \{ \mu \in D; \mu_c = \lambda_c, \mu \preceq \lambda \} . \]

Since the system \( \{ X_\mu; \mu \in D \} \) forms a basis of \( \mathcal{R}_{W_\lambda}(\hat{A})^c \), the matrix
\[ \langle \langle X_\mu, X_\nu \rangle \rangle_{\mu, \nu \in D_\lambda} \]

is a positive definite hermitian matrix. Let
\[ (b^{\mu \nu})_{\mu, \nu \in D_\lambda} \]
be the inverse matrix of the above matrix. In particular \( b^{\lambda \lambda} > 0 \). For any \( \mu \in D_\lambda \), we put
\[ c^\mu_\lambda = \frac{b^{\lambda \mu}}{\sqrt{d_\lambda b^{\lambda \lambda}}} , \]
where \( d_\lambda \) is the degree of an irreducible representation of \( K \) with the highest weight \( \lambda \). Then we have

**Theorem 2.5.** Let \( \lambda \in D \) and \( \omega_\lambda \) be the zonal spherical function belonging to the class of an irreducible representation of \( K \) with the highest weight \( \lambda \). Then \( \omega_\lambda \) restricted to \( \hat{A} \) is given by
\[ \omega_\lambda = \sum_{\mu \in D_\lambda} c^\mu_\lambda X_\mu . \]

Proof. The idea of the following proof owes to Hua [6]. Let \( \mu \in D_\lambda \). Then \( \omega_\mu \) restricted to \( \hat{A} \) is in \( \mathcal{R}_{W_\lambda}(\hat{A})^c \) by Theorem 2.3. It follows by Theorem 2.3 and Corollary of Theorem 2.4 that \( \omega_\mu \) has an expression
\[ \omega_\mu = \sum_{\nu \in D_\lambda} c^\nu_\mu X_\nu \quad \text{with } c^\nu_\mu \in R, c^\nu_\mu > 0, c^\nu_\mu = 0 \text{ if } \nu > \mu . \]

We define an upper triangular matrix \( C' \) by
\[ C' = (c^\nu_\mu)_{\mu, \nu \in D_\lambda} . \]
Then we have
\[ \langle \langle \omega_\mu, \omega_\nu \rangle \rangle_{\mu, \nu \in D_\lambda} = 'C'(\langle X_\mu, X_\nu \rangle)_{\mu, \nu \in D_\lambda} C' . \]

Since \( \langle \omega_\mu, \omega_\nu \rangle = d_\mu^{-1} \delta_{\mu \nu} \), we have
\[ (d_\mu \delta_{\mu \nu})_{\mu, \nu \in D_\lambda} = C'^{-1} B'^{-1} C'^{-1} , \]
where
\[ B' = (b^{\mu \nu})_{\mu, \nu \in D_\lambda} = (\langle X_\mu, X_\nu \rangle)_{\mu, \nu \in D_\lambda}^{-1} . \]
It follows that
\[ C'(d_\mu \delta_{\mu \nu})_{\mu, \nu \in D_\lambda} 'C' = B' . \]
Comparing \( (\mu, \lambda) \)-components of both sides, we have
In particular

\[(c_\lambda^\mu)^2 d_\lambda = b^{\mu\lambda}, \quad \text{i.e.} \quad c_\lambda = \sqrt{\frac{b^{\mu\lambda}}{d_\lambda}},\]

hence

\[c_\lambda^{\mu\rho} = \frac{b^{\mu\lambda}}{d_\lambda c_\lambda^\rho} = \frac{b^{\rho\lambda}}{d_\lambda b^{\mu\lambda}}.\]

Since \(b^{\mu\nu} = b^{\nu\mu}\), we have

\[c_\lambda^{\mu\rho} = \frac{b^{\rho\mu}}{\sqrt{d_\lambda b^{\mu\lambda}}} = c_\lambda^{\rho\mu}. \quad \text{q.e.d.}\]

**Example.** If \(S\) is a compact connected Lie group and \((K, L)\) the symmetric pair with \(K/L = S\) as mentioned before, then the set \(\mathcal{D}(S)\) of equivalence classes of irreducible representations of \(S\) is in the bijective correspondence with \(\mathcal{D}(K, L)\) by the assignment \(\rho \mapsto \rho \otimes \rho^*\), where \(\rho^*\) denotes the contragredient representation of \(\rho\). \(\hat{A}\) is a maximal torus of the compact Lie group \(S\). Let \(\chi_{\rho}\) be the invariant character of \(\hat{A}\) for the dominant integral form in \(D(K, L)\) corresponding to \(\rho \otimes \rho^*\) by the bijection in Corollary of Theorem 2.4. Then it is nothing but the character of \(\rho\). It follows from orthogonality relations of irreducible characters that the matrix \((b^{\mu\nu})\) is the identity matrix. Thus the zonal spherical function \(\omega_{\rho \otimes \rho^*}\) belonging to \(\rho \otimes \rho^*\) is given by

\[\omega_{\rho \otimes \rho^*} = \frac{1}{d_\rho} \chi_{\rho},\]

where \(d_\rho\) is the degree of \(\rho\).

3. **Polynomial representations associated with symmetric bounded domains**

Let \(D\) be an irreducible symmetric bounded domain with rank \(p\) realized in \((p^c)^-\) as in Introduction. We shall use the same notation as in Introduction.

Let

\[\Pi = \{\alpha_1, \ldots, \alpha_t\}\]

be the fundamental root system of \(\sum\) with respect to the order \(>\) and let \(\Pi_f = \Pi \cap \sum_f\). It is known that \(\Pi_f\) is the fundamental root system of \(\sum_f\), \(\Pi - \Pi_f\) consists of one element, say \(\alpha_1\), which is the lowest root in \(\sum_f^+\), and for any \(\alpha = \sum \lambda_i \alpha_i \in \sum_f^+, \lambda_i = 1\). Let \(\sum_f^+\) denote the set of positive compact roots. Put

\[b = \{H \in a; (\sqrt{-1} H, \Delta) = \{0\}\}.\]
Then we have the orthogonal decomposition

$$\sqrt{-1}t = \sqrt{-1}b \oplus \sqrt{-1}a$$

with respect to \((\ , \ )\). We define an orthogonal transformation \(\sigma\) on \(\sqrt{-1}t\) by \(\sigma|b = -1\) and \(\sigma|\sqrt{-1}a = 1\). Let

$$\varpi = \frac{1}{2}(1+\sigma)\colon \sqrt{-1}t \to \sqrt{-1}a$$

be the orthogonal projection of \(\sqrt{-1}t\) onto \(\sqrt{-1}a\). Let \(\kappa\) be the unique involutive element of the Weyl group \(W_1\) of \(K\) such that \(\kappa \Pi_t = -\Pi_t\). Since \(\Sigma_p^+\) is the set of weights on \(t^c\) of the irreducible \(K\)-module \((p^c)^+\), we have \(\kappa \Sigma_p^+ = \Sigma_p^+\) and \(\kappa \gamma_i = \gamma_i\). Put

$$\Delta' = \kappa \Delta = \{\gamma_1', \ldots, \gamma_p'\}, \quad \gamma_i' = \kappa \gamma_i \ (1 \leq i \leq p), \quad \gamma_i' = \alpha_i.$$ 

It is the original maximal strongly orthogonal subsystem of \(\Sigma_p^+\) of Harish-Chandra [3]. For the system \(\Delta',\) the orthogonal projection

$$\varpi'\colon \sqrt{-1}t \to \sqrt{-1}a'$$

onto the \(R\)-span \(\sqrt{-1}a'\) of \(\Delta'\) is defined in the same way as for \(\Delta\). Put

$$P_1' = \{\alpha \in \Sigma_1^+; \varpi'(\alpha) = \frac{1}{2}(\gamma_i' + \gamma_j') \text{ for some } 1 \leq i \leq j \leq p\},$$

$$P_i' = \{\alpha \in \Sigma_i^+; \varpi'(\alpha) = \frac{1}{2} \gamma_i' \text{ for some } 1 \leq i \leq p\},$$

$$K_i' = \{\alpha \in \Sigma_i^+; \varpi'(\alpha) = \frac{1}{2} \gamma_i' \text{ for some } 1 \leq i \leq p\}.$$ 

Then (Harish-Chandra [3]) \(\Sigma\) is the disjoint union of \(P_1', -P_1', P_i', -P_i', K_i', -K_i'\) and we have

$$\varpi'P_i' = \{\frac{1}{2}(\gamma_i' + \gamma_j'); 1 \leq i \leq j \leq p\},$$

$$\varpi'P_i' = \{\frac{1}{2} \gamma_i'; 1 \leq i \leq p\} \quad \text{if } P_i' \neq \phi,$$

$$\varpi'K_i' - \{0\} = \{\pm \frac{1}{2}(\gamma_i' - \gamma_j'); 1 \leq i \leq j \leq p\},$$

$$\varpi'K_i' = \{\frac{1}{2} \gamma_i'; 1 \leq i \leq p\} \quad \text{if } P_i' \neq \phi.$$ 

Furthermore the multiplicity (with respect to \(\varpi'\)) of any \(\gamma_i'\) is 1 and that of any \(\frac{1}{2} \gamma_i'\) is even. It follows that

$$\varpi'\Sigma - \{0\} = \begin{cases} 
\{\pm \frac{1}{2}(\gamma_i' \pm \gamma_j'); 1 \leq i \leq j \leq p, \pm \gamma_i; 1 \leq i \leq p\} & \text{if } P_i' = \phi \\
\{\pm \frac{1}{2}(\gamma_i' \pm \gamma_j'); 1 \leq i \leq j \leq p, \pm \gamma_i', \pm \frac{1}{2} \gamma_i', 1 \leq i \leq p\} & \text{if } P_i' \neq \phi.
\end{cases}$$

Moreover we have (Moore [8])

$$\varpi'\Pi - \{0\} = \begin{cases} 
\{\gamma_1', \frac{1}{2}(\gamma_2' - \gamma_1'), \ldots, \frac{1}{2}(\gamma_p' - \gamma_{p-1}')\} & \text{if } P_i' = \phi \\
\{\gamma_1', \frac{1}{2}(\gamma_2' - \gamma_1'), \ldots, \frac{1}{2}(\gamma_p' - \gamma_{p-1}'), -\frac{1}{2} \gamma_p'\} & \text{if } P_i' \neq \phi.
\end{cases}$$
\[ \omega' \Pi - \{0\} = \begin{cases} \{ \frac{1}{2}(\gamma_i - \gamma'_{i-1}), \ldots, \frac{1}{2}(\gamma_p - \gamma'_{p-1}) \} & \text{if } P'_1 = \phi \\ \{ \frac{1}{2}(\gamma_i - \gamma'_{i-1}), \ldots, \frac{1}{2}(\gamma_p - \gamma'_{p-1}), -\frac{1}{2} \gamma'_{p} \} & \text{if } P'_1 = \phi \end{cases} \]

**Lemma 1.**

1) \[ \omega' \alpha_i = \begin{cases} \gamma_p & \text{if } P'_1 = \phi \\ \gamma_p & \text{if } P'_1 = \phi \end{cases} \]

2) (Schmid [11]) If \( P'_1 = \phi \) and

\[ \sum_{\beta \in \beta_1} m_{\beta} \beta \quad \text{with } m_{\beta} \geq 0 \]

is in the \( \mathbb{R} \)-span \( \{ P'_1 \}_\mathbb{R} \) of \( P'_1 \), then \( m_{\beta} = 0 \) for any \( \beta \).

**Proof.** For any \( \alpha \in \sum_{\mathbb{R}}^+ = P'_1 \cup P'_1 \), \( \omega' \alpha \) can be written as

\[
\omega' \alpha = \frac{1}{2} m_1 (\gamma_2 - \gamma'_1) + \frac{1}{2} m_2 (\gamma_3 - \gamma'_2) + \cdots + \frac{1}{2} m_{p-1} (\gamma_p - \gamma'_{p-1}) - \frac{1}{2} m_p \gamma'_p + m_{p+1} \gamma'_1
\]

\[
\frac{1}{2} (2m_{p+1} - m_i) \gamma'_1 + \frac{1}{2} (m_i - m_{p+1}) \gamma'_2 + \cdots + \frac{1}{2} (m_{p-2} - m_{p-1}) \gamma'_p
\]

where \( m_i \in \mathbb{Z} \), \( m_i \geq 0 \), \( m_{p+1} = 1 \). Since \( \omega' \alpha = \frac{1}{2} (\gamma_i' + \gamma'_j) \) or \( \frac{1}{2} \gamma'_i \) for some \( i, j \), we have

\[
2 \geq m_1 \geq m_2 \geq \cdots \geq m_{p-1} \geq m_p \geq 0
\]

Furthermore \( \alpha \in P'_1 \) (resp. \( \alpha \in P'_1 \)) if and only if \( m_p = 0 \) (resp. \( m_p = 1 \)).

1) If \( P'_1 = \phi \), then \( \gamma'_1 \in P'_1 \). For \( \alpha = \gamma'_1 \), the coefficients in the above expression are \( m_1 = \cdots = m_{p-1} = 2 \), \( m_p = 0 \) and \( \omega' \gamma'_1 = \gamma'_1 \). If \( P'_1 = \phi \), then for \( \alpha = \gamma'_1 \), the coefficients are \( m_1 = \cdots = m_{p-1} = 2 \), \( m_p = 1 \) and \( \omega' \gamma'_1 = \frac{1}{2} \gamma'_p \). Now the assertion 1) follows from \( \omega' \alpha_i = \kappa^{-1} \omega' \kappa \alpha_i = \kappa^{-1} \omega' \gamma'_1 \).

2) Let \[ \alpha = \sum_{i=1}^I n_i \alpha_i \quad \text{with } n_i \in \mathbb{Z}, n_i \geq 0 \]

be in \( \sum_{\mathbb{R}}^+ \). It follows from the first argument that

(a) if \( \alpha \in P'_1 \), \( \omega' \alpha_i = -\frac{1}{2} \gamma'_p \), then \( n_i = 0 \),
(b) if \( \alpha \in P'_1 \), then there exists \( \alpha_i \in \Pi_1 \) such that \( n_i > 0 \) and \( \omega' \alpha_i = -\frac{1}{2} \gamma'_p \).

This implies the assertion 2). q.e.d.
Now $P_1, P_1, K_i$ and $K_1$ are defined for $\Delta$ in the same way as for $\Delta'$. Then $\kappa$ transforms $P_1$ (resp. $P_1, K_1, K_1$) onto $P_1'$ (resp. $P_1', K_1', K_1'$). It follows that the above mentioned properties due to Harish-Chandra are also satisfied by our objects for $\Delta$. But Moore's results should be modified as follows.

$$\sigma \Pi - \{0\} = \begin{cases} \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p), \gamma_p \} & \text{if } P_1 = \phi \\ \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p), \frac{1}{2} \gamma_p \} & \text{if } P_1 \neq \phi. \end{cases}$$

$$\sigma \Pi_1 - \{0\} = \begin{cases} \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p) \} & \text{if } P_1 = \phi \\ \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p), \frac{1}{2} \gamma_p \} & \text{if } P_1 \neq \phi. \end{cases}$$

They follow from Lemma 1, 1) and

$$\sigma \Pi_t = \kappa^{-1} \sigma' \kappa \Pi_t = -\kappa^{-1} \sigma' \Pi_t.$$ 

Note that $K_1 \subset \Sigma_t^+$ while $K_1' \subset -\Sigma_t^+.$

**Lemma 2.** 1) The order $>$ is a compatible order for $\Sigma$ with respect to $\sigma$ in the sense of 2.

2) $\sigma K_0 - \{0\}$ is a root system with the fundamental root system

$$\{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p) \}$$

with respect to the order $>.$

3) If $P_1 \neq \phi$ and

$$\sum_{\beta \in P_1} m_\beta \beta \text{ with } m_\beta \geq 0$$

is in the $R$-span $\{P_1\}_R$ of $P_1$, then $m_\beta = 0$ for any $\beta$.

Proof. 1) is clear from the form of $\sigma \Pi - \{0\}$ above.

2) is clear since

$$\sigma K_0 - \{0\} = \{ \pm \frac{1}{2} (\gamma_i - \gamma_j); 1 \leq i < j \leq p \} .$$

3) follows from Lemma 1, 2) and $\kappa P_1 = P_1', \kappa P_1 = P_1'. q.e.d.$

For $\lambda \in \sqrt{-1}$, $\lambda \neq 0$, we define as in 2

$$\lambda^* = \frac{2}{(\lambda, \lambda)} \lambda.$$
and put

\[ Z_0 = \frac{1}{2} \sum_{\gamma \in \Delta} \gamma^*. \]

Since \((\frac{1}{2} \gamma_i, \gamma_j^* ) = \delta_{ij}\) for \(1 \leq i, j \leq p\), we have

\[
\begin{align*}
P_+ &= \{ \alpha \in \Gamma_p; (\alpha, Z_0) = 1 \}, \\
P_- &= \{ \alpha \in \Gamma_p; (\alpha, Z_0) = \frac{1}{2} \}, \\
K_0 &= \{ \alpha \in \Gamma^c; (\alpha, Z_0) = 0 \}, \\
K_+ &= \{ \alpha \in \Gamma^c; (\alpha, Z_0) = \frac{1}{2} \}.
\end{align*}
\]

Hence eigenvalues of \(\text{ad} \ Z_0\) are \(\pm 1, \pm \frac{1}{2}\) on \(\mathfrak{p}^c, 0, \pm \frac{1}{2}\) on \(\mathfrak{p}^c\). Let \(\mathfrak{p}^c_\pm, \mathfrak{p}^c_0, \mathfrak{k}^c_0, \mathfrak{k}^c_\pm\) denote the corresponding eigenspaces. Note that the origin \(X_0\) of the Shilov boundary \(S\) is in \(\mathfrak{p}^c\).

The following results are due to Korányi-Wolf [7]. We define an element \(c\) of \(G^c\), which is called Cayley transform, by

\[ c = \exp \left( -\frac{\pi}{4} \sum_{\gamma \in \Delta} (X_\gamma + X_{-\gamma}) \right) \]

and define an automorphism of \(G^c\) by

\[ \theta(x) = c^2 x c^{-2} \quad \text{for} \quad x \in G^c. \]

The automorphism \(\text{Ad } c^2\) of \(\mathfrak{g}^c\) obtained by differentiating \(\theta\) will be also denoted by the same letter \(\theta\). Then \(\theta' = 1\) and on \(\sqrt{-1} \mathfrak{t}\) it coincides with \(-\sigma\). Put

\[
\begin{align*}
g_0 &= \{ X \in \mathfrak{g}; \theta^* X = X \}, \\
\mathfrak{k}_0 &= \mathfrak{g}_0 \cap \mathfrak{k},
\end{align*}
\]

and

\[ \mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{p}. \]

Then \(\mathfrak{k}_0\) is \(\theta\)-invariant and

\[ \mathfrak{k}_0 = \{ X \in \mathfrak{k}; [Z_0, X] = 0 \}. \]

Hence \(\mathfrak{k}_0\) is a real form of \(\mathfrak{k}_0^c\) containing \(\mathfrak{t}\) as a maximal abelian subalgebra. \(K_0\) is nothing but the set of roots of \(\mathfrak{k}_0^c\) with respect to \(\mathfrak{t}^c\). The complexification \(\mathfrak{p}_0^c\) of \(\mathfrak{p}_0\) is the direct sum of \(\mathfrak{p}^c_+\) and \(\mathfrak{p}^c_-\). \(g_0\) is a reductive subalgebra of \(\mathfrak{g}\) with a Cartan decomposition

\[ g_0 = \mathfrak{k}_0 + \mathfrak{p}_0. \]
Let $G_0$ (resp. $K_0$) be the connected subgroup of $G$ generated by $g_0$ (resp. by $k_0$) and let

$$L_0 = \{ k \in K_0; \text{Ad} k X_0 = X_0 \} = K_0 \cap L.$$ 

Put

$$D_0 = D \cap \mathfrak{p}_{c_1}$$

and

$$S_0 = S \cap \mathfrak{p}_{c_1}.$$ 

Then $G_0$ acts on $D_0$ transitively and $K \cap G_0$ coincides with $K_0$, so that $D_0$ is identified with the quotient space $G_0/K_0$. Furthermore $K_0$ acts on $S_0$ transitively so that $S_0$ is identified with $K_0/L_0$. $D_0$ is totally geodesic in $D$ with respect to Bergmann metric of $D$ and it is also an irreducible symmetric bounded domain with the same rank as $D$. $S_0$ is the Shilov boundary of $D_0$. The complex structure of $D_0$ is given at the origin by $\text{ad} H_0$ with $\sqrt{-1} H_0 = Z_0$. We have

$$\text{ad} Z = Z_0.$$ 

The inclusion $D_0 \subset \mathfrak{p}_{c_1}$ is nothing but the Harish-Chandra's imbedding of $D_0 = G_0/K_0$. $(K_0, L_0)$ is a symmetric pair with respect to $\theta$, having the same rank as $D$. Hence

$$\mathfrak{l}_0 = \{ X \in \mathfrak{l}_0; \theta X = X \}$$

is the Lie algebra of $L_0$ and $\mathfrak{a}$ is a maximal abelian subalgebra of

$$\mathfrak{g}_0 = \{ X \in \mathfrak{l}_0; \theta X = -X \}.$$ 

We can define a semi-linear transformation $X \mapsto \bar{X}$ of $\mathfrak{p}_{c_1}$ by

$$\bar{X} = \tau \theta X = \theta \tau X \quad \text{for} \quad X \in \mathfrak{p}_{c_1}.$$ 

Put

$$\mathfrak{p}_{-1} = \{ X \in \mathfrak{p}_{c_1}; \bar{X} = X \}.$$ 

It is a real form of $\mathfrak{p}_{c_1}$ and is invariant under the adjoint action of $L_0$ on $\mathfrak{p}_{c_1}$. The correspondence $X \mapsto [X, X]$ gives an isomorphism

$$\psi: \sqrt{-1} \mathfrak{g}_0 \rightarrow \mathfrak{p}_{-1},$$

which is equivariant with respect to the adjoint actions of $L_0$.

Now we shall consider the polynomial representation $S^*((\mathfrak{p}_{c})^*)$ of $K$. Let $S_*((\mathfrak{p}_{c})^*)$ be the symmetric algebra over $(\mathfrak{p}_{c})^*$. $K$ acts on $S_*((\mathfrak{p}_{c})^*)$ by the natural extension $\text{Ad}$ of the adjoint action of $K$ on $(\mathfrak{p}_{c})^*$. On the other hand, the non-degenerate pairing

$$(\mathfrak{p}_{c})^* \times (\mathfrak{p}_{c})^* \rightarrow \mathbb{C}$$
by means of the Killing form $(\cdot, \cdot)$ induces the identification

$$S_\bullet((\mathfrak{g}^c)^+) = S^*((\mathfrak{g}^c)^-).$$

This identification is compatible with the actions of $K$, since the Killing form is invariant under the adjoint action of $K$. In the same way we have a $K_o$-equivariant identification

$$S_\bullet(\mathfrak{g}_1^c) = S^*(\mathfrak{g}_1^c).$$

$S_\bullet(\mathfrak{g}_1^c)$ can be considered as a $K_o$-submodule of $S_\bullet((\mathfrak{g}^c)^+)$ by means of the natural monomorphism $S_\bullet(\mathfrak{g}_1^c) \rightarrow S_\bullet((\mathfrak{g}^c)^+)$ induced from the inclusion $\mathfrak{g}_1^c \subseteq (\mathfrak{g}^c)^+$.

**Theorem 3.1.** (i) Any irreducible $K$-submodule of $S_\bullet((\mathfrak{g}^c)^+)$ (resp. $K_o$-submodule of $S_\bullet(\mathfrak{g}_1^c)$) is contained exactly once in $S_\bullet((\mathfrak{g}^c)^+)$ (resp. in $S_\bullet(\mathfrak{g}_1^c)$).

(ii) For an irreducible $K$-submodule $V$ of $S_\bullet((\mathfrak{g}^c)^+)$, we put

$$V_0 = V \cap S_\bullet(\mathfrak{g}_1^c).$$

Then $V \rightarrow V_0$ is the one to one correspondence between the set of irreducible $K$-submodules of $S_\bullet((\mathfrak{g}^c)^+)$ and the set of irreducible $K_o$-submodules of $S_\bullet(\mathfrak{g}_1^c)$ in such a way that

1) The highest weights on $\mathfrak{g}^c$ of $V$ and $V_0$ are the same.

2) The subspace of $L$-invariants in $V$ is 1-dimensional and contained in $V_0$.

(iii) The highest weight $\lambda$ of an irreducible $K$-submodule $V$ of $S_\bullet((\mathfrak{g}^c)^+)$ is of the form

$$\lambda = \sum n_i \gamma_i, \quad n_i \in \mathbb{Z}, \quad n_1 \geq n_2 \geq \cdots \geq n_\rho \geq 0.$$ 

If $\sum n_i = \nu$, then $V$ is contained in $S_\nu((\mathfrak{g}^c)^+)$. i.e. $S^\nu(D) \subseteq S^\nu(K, L)$ under the notation in Introduction.

For the proof of the theorem, we need the following

**Lemma 3.** (Murakami [9]) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$. Assume that there exists $Y \in \sqrt{-1} \mathfrak{g} \subset \mathfrak{g}^c$ such that $\mathfrak{g}^c$ is the direct sum of $0$-eigenspace $\mathfrak{g}_0^c$, $(+1)$-eigenspace $\mathfrak{g}_1^c$ and $(-1)$-eigenspace $\mathfrak{g}_-^c$ of $\text{ad} Y$, respectively. Let $(\rho, V)$ be a complex irreducible $\mathfrak{g}$-module with $\mathfrak{g}$-invariant hermitian inner product. Denoting the extension to $\mathfrak{g}^c$ of $\rho$ by the same letter $\rho$, let $a_1 > a_2 > \cdots > a_m (a_i \in \mathbb{R})$ be eigenvalues of $\rho(Y)$, and $S_t$ be $a_t$-eigenspace of $\rho(Y)$ ($1 \leq t \leq m$). Put $\mathfrak{g}_0 = \bigcap \mathfrak{g}_i$ (which is a real form of $\mathfrak{g}_0^c$). Then

1) $a_t = a_{t-1} + 1 \ (1 \leq t \leq m)$.

2) Each $S_t$ is a $\mathfrak{g}_c$-submodule of $V$ and

$$V = S_1 + \cdots + S_m$$

is the orthogonal direct sum.
3) \( S_i \) and \( S_m \) are irreducible \( \mathfrak{t}_e \)-submodules of \( V \) and characterized by

\[
S_i = \{ v \in V ; \rho(X)v = 0 \text{ for any } X \in \mathfrak{t}_e \}, \]
\[
S_m = \{ v \in V ; \rho(X)v = 0 \text{ for any } X \in \mathfrak{t}_e \}.
\]

Proof of Theorem 3.1. The infinitesimal action of \( \mathfrak{t}_e \) on \( S_*(\mathfrak{p}_e^c)^+ \) induced from the adjoint action \( \text{Ad} \) of \( K \) will be denoted by \( \text{ad} \).

Let \( V \) be an irreducible \( K \)-submodule of \( S_*(\mathfrak{p}_e^c)^+ \). Since \( Z \) is in the center of \( \mathfrak{k}_e \), it follows from Schur's lemma that \( V \) is contained in an eigenspace of \( \text{ad} \) \( Z \) in \( S_*(\mathfrak{p}_e^c)^+ \). But since \( \text{ad} \) \( Z \) is the scalar operator \( \nu \) on \( S_*(\mathfrak{p}_e^c)^+ \), \( V \) is contained in \( S_*(\mathfrak{p}_e^c)^+ \) for some \( \nu \). Let \( \lambda \in \sqrt{-1} \mathfrak{t} \) be the highest weight of \( V \). Put \( Y = 2Z \in \sqrt{-1} \mathfrak{t} \subset \mathfrak{t}_e \). Then the decomposition

\[
\mathfrak{t}_e = \mathfrak{t}_e^0 + \mathfrak{t}_e^1 + \mathfrak{t}_e^{-1}
\]

satisfies the assumption in Lemma 3. So we have a decomposition

\[
V = S_1 + \ldots + S_m
\]

into \( K_e \)-submodules, where \( S_i \) is an irreducible \( K_e \)-submodule and is the eigenspace for the maximum eigenvalue of \( \text{ad} \) \( Y \) in \( V \). It is characterized by

\[
S_i = \{ v \in V ; \text{ad} (X)v = 0 \text{ for any } X \in \mathfrak{t}_e \}.
\]

Thus a highest weight vector \( v_\lambda \) of the \( K \)-module \( V \) is contained in \( S_i \) because of \( K_i \subset \sum \mathfrak{t}^+ \). It follows that putting \( V_0 = S_1 \), \( V_0 \) is an irreducible \( K_e \)-submodule of \( S_*(\mathfrak{p}_e^c)^+ \) with the highest weight \( \lambda \).

We shall show that \( V_0 = V \cap S_*(\mathfrak{p}_e^c)^+ \). We have the decomposition

\[
S_*(\mathfrak{p}_e^c)^+ = \sum_{r, s} S_r(\mathfrak{p}_e^c) \otimes S_s(\mathfrak{p}_e^c)
\]

as \( K_e \)-modules. \( \text{ad} \) \( Z \) is the scalar operator \( r + \frac{1}{2}s = \frac{1}{2}(r + s) \) on \( S_r(\mathfrak{p}_e^c) \otimes S_s(\mathfrak{p}_e^c) \). In the same way as the first argument, we can get the decomposition

\[
V = V_1 + \ldots + V_h
\]

into irreducible \( K_e \)-submodules such that any \( V_i \) is contained in \( S_r(\mathfrak{p}_e^c) \otimes S_s(\mathfrak{p}_e^c) \) for some \( (r, s) \). Since \( S^*(\mathfrak{p}_e^c)^- \) is \( K \)-isomorphic with \( S^*(S) \subset C^*(S) \), \( V \) has an \( L \)-invariant \( w \neq 0 \). Decompose \( w \) as

\[
w = w_1 + \ldots + w_h, \quad w_i \in V_i \quad (1 \leq i \leq k).
\]

At least one of the \( w_i \)'s, say \( w_1 \), is not zero. Let \( \lambda_1 \in \sqrt{-1} \mathfrak{t} \) be the highest weight of the irreducible \( K_o \)-module \( V_1 \). Since \( w_1 \) is a non-zero \( L_o \)-invariant of \( V_1 \), \( V_1 \) is a spherical \( K_o \)-module relative to \( L_o \). \( (K_o, L_o) \) is a symmetric pair, \( \alpha \) is a maximal abelian subalgebra of \( \mathfrak{a}_o \) and the order \( > \) on \( \sqrt{-1} \mathfrak{t} \) is a compatible order for \( K_o \) with respect to \( \sigma \) by Lemma 1, 1), so we shall use the notations
\[ \Gamma(K_0, L_0), Z(K_0, L_0), D(K_0, L_0) \) in 2. Then it follows from Theorem 2.3 that 
\[ \lambda_i \in D(K_0, L_0). \]  
On the other hand, if \( V_1 \subset S_\alpha(p^f) \otimes S_\lambda(p^e) \), \( \lambda_i \) is of the form

\[ \lambda_i = \sum_{\alpha \in P_1} m_\alpha \alpha + \sum_{\beta \in P_2} m_\beta \beta, \quad m_\alpha, m_\beta \in Z, \quad m_\alpha \geq 0, \quad m_\beta \geq 0 \]

with \( \sum m_\alpha = r, \sum m_\beta = s. \) Since \( D(K_0, L_0) \subset \sqrt{-1} \Delta R \subset \{ P_1 \}_R \), we have

\[ \sum_{\beta \in P_2} m_\beta \beta \in \{ P_1 \}_R. \]

It follows from Lemma 2.3) that \( r = \nu, s = 0 \), i.e. \( V_1 \subset V \cap S_\lambda(p^e). \) On the other hand, \( \nabla S_\alpha(p^f) \subset V_0 \) since the possible maximum eigenvalue of \( \text{ad} Y \) on \( V \) is \( 2\nu. \) Thus we have that \( V_0 = V_1 = V \cap S_\lambda(p^e). \)

The above argument shows also that any \( L \)-invariant in \( V \) is contained in \( V_0. \) It is unique up to scalar since \( (K_0, L_0) \) is a symmetric pair.

Conversely, let \( V_0 \) be an irreducible \( K_0 \)-submodule of \( S_\lambda(p^e) \) with the highest weight \( \lambda \in \sqrt{-1} t. \) In the same way as the first argument, we know that \( V_0 \) is contained in \( S_\alpha(p^f) \) for some \( \nu. \) Let \( v_0 \in V_0 \) be a highest weight vector. Then \( \text{ad} t_i v_0 = 0 \) because of \( [t_i, p^f] = 0. \) Hence \( \text{ad} X_\alpha v_0 = 0 \) for any \( \alpha \in \sum \lambda. \) We define \( V \) to be the \( C \)-span of \( \{ \text{ad} k v_0; k \in K \} \) in \( S_\alpha(p^e). \) Then \( V \) is an irreducible \( K \)-submodule of \( S_\lambda(p^e) \) with the highest weight \( \lambda \in \sqrt{-1} t. \)

It is easy to see that each of the above correspondences \( V \mapsto V_0 \) and \( V_0 \mapsto V \) is the inverse of the other. This proves assertions (i) and (ii).

(iii) We have \( [\frac{1}{2} \gamma^f, X_{-\gamma_i}] = -\delta_{ij} X_{-\gamma_j} \) \((1 \leq i, j \leq p)\) because of \( (\frac{1}{2} \gamma^f, \gamma_i) = \delta_{ij} \) \((1 \leq i, j \leq p). \) It follows that for \( H = 2\pi \sqrt{-1} \sum_{i=1}^p x_i (\frac{1}{2} \gamma^f) \in \alpha \) we have

\[ \text{Ad}(\exp H)X_{-\gamma_i} = -\sum_{i=1}^p \exp(-2\pi \sqrt{-1} x_i) X_{-\gamma_i}. \]

Thus we have

\[ \Gamma(K_0, L_0) = 2\pi \sqrt{-1} \sum_{i=1}^p Z(\frac{1}{2} \gamma^f) \]

and

\[ Z(K_0, L_0) = \sum_{i=1}^p Z \gamma_i. \]

It follows from Lemma 2.2) that

\[ D(K_0, L_0) = \{ \sum_{i=1}^p n_i \gamma_i; n_i \in Z, n_1 \geq n_2 \geq \cdots \geq n_p \}. \]

Therefore \( \lambda \) is of the form

\[ \lambda = \sum_{i=1}^p n_i \gamma_i \quad \text{with} \quad n_i \in Z, \quad n_1 \geq \cdots \geq n_p. \]

On the other hand, \( \lambda \) is of the form
\[ \lambda = \sum_{\alpha \in \mathcal{P}} m_\alpha \alpha \quad \text{with } m_\alpha \in \mathbb{Z}, m_\alpha \geq 0, \]

which implies that \( n_1 \geq \cdots \geq n_p \geq 0 \). If \( V \subset S_i((\mathfrak{p}^\mathfrak{c})^+) \), then \( V_0 \subset S_i(\mathfrak{p}^\mathfrak{c}_1) \) and \( \text{ad } Z_0 \)
is the scalar operator \( \nu \) on \( V_0 \), which equals \( (\lambda, Z_0) = \sum_{i=1}^p n_i. \) q.e.d.

**Remark.** In terms of polynomial functions \( S^*((\mathfrak{p}^\mathfrak{c})^-) \), for an irreducible \( K \)-submodule \( V \) of \( S^*((p^\mathfrak{c})^-) \), \( V_0 \) is obtained by restriction to \( \mathfrak{p}^\mathfrak{c}_1 \) of functions in \( V \).

Proof of Theorem A. Orthogonality relations for the \( S^*_A(D)'s \) (resp. for the \( S^*_K(S)'s \) and the assertion that the restriction \( S^*_A(D) \rightarrow S^*_K(S) \) is a similitude follow from Schur's lemma. So it suffices to show that the cardinalities of \( S^*(D) \) and \( S^*(K, L) \) are the same.

From the first argument in the proof of Theorem 3.1 (iii), we see that \( \psi(\frac{1}{2} \gamma^*) = X_{\gamma} (\gamma \in \Delta) \) for the \( L_0 \)-equivariant isomorphism \( \psi: \sqrt{-1} \mathfrak{g}_0 \rightarrow \mathfrak{p}_-. \)

We put

\[ \alpha^- = \psi(\sqrt{-1} \alpha) = \{ X_{\gamma}; \gamma \in \Delta \} \subset \mathfrak{p}_-. \]

Since the Weyl group \( W_0 \) of \( S_0 \) is isomorphic with the group of permutations of \( \Delta \) by Lemma 2,2), the "Weyl group" \( W_0 = N_{L_0}(\alpha^-)/Z_{L_0}(\alpha^-) \), where \( N_{L_0}(\alpha^-) \) (resp. \( Z_{L_0}(\alpha^-) \)) is the normalized (resp. centralizer) of \( \alpha^- \) in \( L_0 \), is isomorphic with the group of permutations of \( \{ X_{\gamma}; \gamma \in \Delta \} \). On the other hand, since \( S^*_A(\mathfrak{g}_0) \) is isomorphic with \( S^*_K(\alpha) \) by Theorem 2.2, \( S^*_A(\mathfrak{p}_-) \) is isomorphic with \( S^*_W(\alpha^-) \). Hence \( S^*_W(\alpha) \) is isomorphic with \( S^*_W(\mathfrak{g}_0)((\alpha^-)^c) \). It follows from Theorem 3.1, (ii), 2) that the cardinality of \( S^*(D) \) is equal to \( \dim S^*_A(\mathfrak{p}_-) = \dim S^*_W(\alpha^-)^c = \text{the number of linearly independent symmetric polynomials in } p\text{-variables with degree } \nu \), which is known to be the cardinality of \( S^*(K, L) \).

q.e.d.

4. Normalizing factor \( h_\lambda \)

Let \( \hat{A} = \text{Ad}(A(X_0)) \), denoting by \( A \) the connected subgroup of \( K_0 \) generated by \( \alpha \). \( \hat{A} \) has a natural group structure induced from that of \( \alpha \). Let

\[ T = \{ t \in \mathbb{C}^*; |t| = 1 \} \]

be the 1-dimensional torus. Under the identification in Introduction of \( (\alpha^-)^c \) with \( \mathbb{C}^p \), \( \alpha^- \) is identified with \( R^p \) and \( \hat{A} \) with \( T^p \). We see that the latter identification is compatible with group structures and complex conjugations, in view of the expression of \( \text{Ad}(\exp H)X_0 \) in the proof of Theorem 3.1, (iii). Moreover, under the same identification we have (Moore [8])

\[ D \cap \alpha^- = \{ x \in \mathbb{R}^p; |x_i| < 1 \ (1 \leq i \leq p) \}, \]

denoting by \( z_i \ (1 \leq i \leq p) \) the \( i \)-th component of \( z \in \mathbb{C}^p \). By means of this
identification we define a measure on $\alpha^-$ by
$$dH = dx_1 \cdots dx_p$$
and a function $D(H)$ on $\alpha^-$ by
$$D(H) = \prod_{i=1}^p (2\pi i x_i)^{2s_i} \prod_{1 \leq i < j \leq p} ((x_i + x_j)(x_i - x_j))^{r_{ij}} \text{ for } H \in \alpha^-,$$
where $r$, $2s$ are multiplicities defined in Introduction. Then we have the following

**Lemma 1.** There exists a constant $c' > 0$ such that
$$\int_{D} f(X) d\mu(X) = c' \int_{D \cap \alpha^-} f(H) |D(H)| dH$$
for any integrable $K$-invariant function $f$ on $D$.

**Proof.** It is easy to see that $\text{Ad } cH = H$ for any $H \in \mathfrak{b}$ and $\text{Ad } c\gamma^* = X_\gamma - X_{-\gamma} \in \mathfrak{p}$ for any $\gamma \in \Delta$. Put
$$\alpha^0 = \text{Ad } c(\sqrt{-1}\alpha) = \{X_\gamma - X_{-\gamma}; \gamma \in \Delta\}_R,$$
$$\mathfrak{h} = \text{Ad } c(\mathfrak{b} \oplus \sqrt{-1}\alpha) = \mathfrak{b} \oplus \alpha^0$$
and
$$\mathfrak{h}_R = \sqrt{-1}\mathfrak{b} \oplus \alpha^0.$$Then $\alpha^0$ is a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\alpha^0$ and $\mathfrak{h}_R$ is the real part of the complexification $\mathfrak{h}^c$ of $\mathfrak{h}$. We define linear forms $h_i$ ($1 \leq i \leq p$) on $\alpha^0$ by
$$h_i(X_\gamma - X_{-\gamma}) = \delta_{ij} \quad (1 \leq i, j \leq p).$$
If $h_i$ is identified with an element of $\alpha^0$ by means of the Killing form, we have $\text{Ad } c(\frac{1}{2} \gamma_i) = h_i$ ($1 \leq i \leq p$). The linear order on $\mathfrak{h}_R$ induced by $\text{Ad } c$ from the order $>$ on $\sqrt{-1}t$ is a compatible order for $\text{Ad } c \sum$ with respect to the decomposition $\mathfrak{h}_R = \sqrt{-1}\mathfrak{b} \oplus \alpha^0$. This follows from 3, Lemma 2,1). Thus positive restricted roots on $\alpha^0$ of the symmetric space $D = G/K$ are
$$\{h_i \pm h_j; 1 \leq i < j \leq p, 2h_i; 1 \leq i \leq p\} \quad \text{ if } P_1 = \phi,$$
$$\{h_i \pm h_j; 1 \leq i < j \leq p, 2h_i, h_i; 1 \leq i \leq p\} \quad \text{ if } P_1 \neq \phi.$$The multiplicity of $h_i \pm h_j$ ($1 \leq i < j \leq p$), i.e. the number of roots in $\text{Ad } c \sum$ projecting to $h_i \pm h_j$, is the same as that of $\frac{1}{2}(\gamma_i \pm \gamma_j)$. Since the Weyl group $W_D$ on $\alpha^0$ of $D = G/K$ is generated by reflections with respect to $h_i - h_j$, $\cdots$, $h_{p-1} - h_p, h_p$, hence transitive on the set $\{\pm h_i \pm h_j; 1 \leq i < j \leq p\}$, it follows that
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multiplicities of these roots are the same \( r \). By the same reason, multiplicities of \( h_i \) \((1 \leq i \leq p)\) are the same \( 2r \), which is even from the results of Harish-Chandra mentioned in 3. In the same way we know that multiplicities of \( 2h_i \) \((1 \leq i \leq p)\) are 1. Thus the product \( D^\circ \) of positive restricted roots (multiplicity counted) is given by

\[
D^\circ (H^\circ) = \prod_{i=1}^{p} 2h_i (H^\circ) h_i (H^\circ)^r \prod_{1 \leq i < j \leq p} ((h_i + h_j) (H^\circ) (h_i - h_j) (H^\circ))^r \quad \text{for } H^\circ \in \alpha^\circ.
\]

Let \( dX \) (resp. \( dH^\circ \)) denote the Euclidean measure of \( \mathfrak{p} \) (resp. of \( \alpha^\circ \)) induced from the Killing form \((\ , \)\), and \( dk \) the normalized Haar measure of \( K \). Then (cf. Helgason [4]) under the surjective map \( K \times \alpha^\circ \to \mathfrak{p} \) defined by \((k, H^\circ) \mapsto \text{Ad } kH^\circ\), these measures are related as follows:

\[
dX = c'' \vert D^\circ (H^\circ) \vert dk dH^\circ \quad \text{with some constant } \ c'' > 0.
\]

Now we define a \( K \)-equivariant \( \mathcal{R} \)-isomorphism \( j: \mathfrak{p} \to (\mathfrak{p}^\circ)^- \) by

\[
j(X) = \frac{1}{2} (X - [Z, X]) \quad \text{for } X \in \mathfrak{p}.
\]

It is easy to see that \( j(X - X_{-\gamma}) = -X_{-\gamma} \) for any \( \gamma \in \Delta \), hence \( j\alpha^\circ = \alpha^- \). Since \( K \) acts irreducibly on \( \mathfrak{p} \), the map \( j \) is a similitude with respect to inner products \((\ , \)\) and the real part of \((\ , \)\). Therefore under the surjective map \( K \times \alpha^\circ \to (\mathfrak{p}^\circ)^- \) defined by \((k, H) \mapsto \text{Ad } kH\), we have

\[
d\mu(X) = c' \vert D(H) \vert dk dH \quad \text{with some constant } \ c'>0.
\]

Seeing \( \text{Ad } K(D \cap \alpha^-) = D \), we get the proof of Lemma 1. q.e.d.

Take a form \( \lambda \in S^* (K, L) \). Choose an orthonormal basis \( \{u_i; 1 \leq i \leq d_\lambda\} \) of \( S^*_\circ((\mathfrak{p}^\circ)^-\) with respect to \((\ , \)\), such that \( \{u_i; 1 \leq i \leq d_{\lambda, 0}\} \) spans \( S^*_\circ((\mathfrak{p}^\circ)^-) \cap S^* (\mathfrak{p}^\circ)\) and \( u_i \) is \( L \)-invariant. Put

\[
\rho_j^\circ (k) = (\text{Ad } k u_j, u_i), \quad \text{for } k \in K \quad (1 \leq i, j \leq d_\lambda),
\]

\[
\varphi^\circ_i (k) = \rho_i^j (k), \quad \text{for } k \in K \quad (1 \leq i \leq d_\lambda),
\]

\[
f_i = \sqrt{d_\lambda} \varphi^\circ_i \quad (1 \leq i \leq d_\lambda).
\]

The arguments in 2 show that \( \{f_i; 1 \leq i \leq d_\lambda\} \) form an orthonormal basis of \( S^*_\circ (S) \) with respect to \( \langle \ , \rangle \) and \( \varphi^\circ_i \) is the zonal spherical function \( \omega^\circ \) for \( (K, L) \) belonging to \( \lambda \), identifying \( C^\circ (S) \) with the space of right \( L \)-invariant \( C^\circ \)-functions on \( K \). The zonal spherical polynomial \( \Omega_\lambda \) for \( D \) belonging to \( \lambda \) defined in Introduction is characterized by that its restriction to \( S \) coincides with \( \omega^\circ_\lambda \). \( \Omega_\lambda \) restricted to \( \mathfrak{p} \) is the zonal spherical polynomial for \( D^\circ \) belonging to \( \lambda \) and \( \omega^\circ_\lambda \) restricted to \( S_0 \) is the zonal spherical function for \( (K_0, L_0) \) belonging to \( \lambda \). \( \Omega_\lambda \)
restricted to \((\alpha^-)^c\) is a symmetric polynomial since it is \(W_{\tilde{\alpha}}\)-invariant. Let \(f_i \in S^k((p^c)^-)(1 \leq i \leq d_\lambda)\) be the unique polynomial such that its restriction to \(S\) is \(f_i\). Then \(\{f_i; 1 \leq i \leq d_\lambda\}\) form an orthogonal basis of \(S^k((p^c)^-)(p^c)^{-}\). They satisfy relations

\[ f_i(\text{Ad} k^{-1} X) = \sum_{j=1}^{d_\lambda} \rho_1(k) f_j(X) \quad \text{for} \quad k \in \mathcal{K}, X \in (p^c)^{-} \quad (1 \leq i \leq d_\lambda). \]

We put

\[ \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \quad \text{for} \quad X \in (p^c)^-. \]

Then for any \(k \in \mathcal{K}\) we have

\[ \Phi_\lambda(\text{Ad} k^{-1} X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \left( \sum_j \rho_1(k) f_j(X) \right) \left( \sum_i \rho_1(k) f_i(X) \right) \]

\[ = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \left( \sum_j \rho_1(k) \rho_1(k) f_j(X) f_i(X) \right) \]

\[ = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \delta_{ij} f_j(X) \overline{f_i(X)} = \Phi_\lambda(X) \quad \text{for} \quad X \in (p^c)^-, \]

i.e. \(\Phi_\lambda\) is a \(K\)-invariant \(C^\infty\)-function on \((p^c)^-\). Note that

\[ \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{a=1}^{d_\lambda} |f_a(X)|^2 \quad \text{for} \quad X \in \mathfrak{p}_{c_1}. \]

**Lemma 2.**

\[ h_\lambda = c' \int_{\mathfrak{p}^\lambda} \Phi_\lambda(H) |D(H)| dH \]

**Proof.**

\[ \int_\mathfrak{p} \Phi_\lambda(X) d\mu(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \left< f_i, f_i \right> = \frac{1}{d_\lambda} \sum_{a=1}^{d_\lambda} h_\lambda \left< f_i, f_i' \right> = h_\lambda. \]

On the other hand, by Lemma 1 we have

\[ \int_\mathfrak{p} \Phi_\lambda(X) d\mu(X) = c' \int_{\mathfrak{p}^\lambda} \Phi_\lambda(H) |D(H)| dH. \]

q.e.d.

**Proof of Theorem B.** Making use of the complex conjugation \(X \mapsto \overline{X}\) of \(\mathfrak{p}_{c_1}\) defined in 3, we define \(\Phi_\lambda \in S^*(\mathfrak{p}_{c_1})\) by

\[ \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{a=1}^{d_\lambda} f_a(X) \overline{f_a(X)} \quad \text{for} \quad X \in \mathfrak{p}_{c_1}. \]

Then \(\Phi_\lambda = \Phi_\lambda\) on \(\mathfrak{p}_{-1}\) and we have for any \(k \in K_0\)
\[ \Phi_\lambda(\text{Ad } k X_0) = \frac{1}{d_\lambda} \sum_{a} f_\sigma(\text{Ad } k X_0) f_\sigma(\text{Ad } \theta(k) X_0) \]
\[ = \frac{1}{d_\lambda} \sum_{a} f_\sigma(\text{Ad } k X_0) f_\sigma(\text{Ad } \theta(k) X_0) \]
\[ = \frac{1}{d_\lambda} \sum_{a} f_\sigma'(k) f_\sigma'(\theta(k)) = \sum_{a} \varphi_\sigma'(k) \varphi_\sigma'(\theta(k)) \]
\[ = \sum_{a} \rho_\sigma^2(k) \rho_\sigma^2(\theta(k)) = \sum_{a} \rho_\sigma^2(k) \rho_\sigma^2(\theta(k)^{-1}) \]
\[ = \rho_\sigma^2(\theta(k)^{-1} k) = \omega_\lambda(\theta(k)^{-1} k). \]

In particular for any \( a \in A \)
\[ \Phi_\lambda(\text{Ad } a X_0) = \omega_\lambda(a^2), \]
i.e. for any \( a \in \hat{A} \)
\[ \Phi_\lambda(a) = \omega_\lambda(a^2) = \Omega_\lambda(a^2). \]

Since \( \hat{A} = T^p \) is a compact real form of \( C^*p \) and \( C^*p \) is open in \( C^p = (a^-)c \), we have
\[ \Phi_\lambda(z_1, \ldots, z_p) = \Omega_\lambda(z_1^2, \ldots, z_p^2) \quad \text{for any } z \in C^p = (a^-)c. \]

By Lemma 2 we have
\[ h_\lambda = c' \int_{D \cap a^-} \Phi_\lambda(H) |D(H)| dH \]
\[ = c' \int_{\langle \theta \rangle < \langle \omega \rangle} \Omega_\lambda(x_1^2, \ldots, x_p^2) \prod_{i<j} ((x_i-x_j)(x_i-x_j))^{r} dx_1 \cdots dx_p \]
\[ = c(D) \int_{\langle \theta \rangle < \langle \omega \rangle} \Omega_\lambda(y_1, \ldots, y_p) \prod_{i<j} (y_i-y_j)^{r} dy_1 \cdots dy_p \]
for some constant \( c(D) > 0 \), which does not depend on \( \lambda \). In particular, for \( \lambda = 0 \)
\[ \mu(D) = h_0 = c(D) \int_{\langle \omega \rangle} \prod_{i<j} (y_i-y_j)^{r} dy_1 \cdots dy_p, \]
since \( \Omega_0 \equiv 1 \). This completes the proof of Theorem B. q.e.d.

**Remark.** It can be proved that \( \Phi_\lambda \) is an \( L_\omega \)-invariant polynomial on \( F_x \).

The multiplicities \( r, s \) are given as follows.

<table>
<thead>
<tr>
<th>( D )</th>
<th>rank ( D )</th>
<th>( r )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I) ( p, s(p \leq q) )</td>
<td>( p )</td>
<td>2</td>
<td>( q-p )</td>
</tr>
<tr>
<td>(II) ( [n/2] )</td>
<td>4</td>
<td>( 2 ) if ( n ) odd( {0 ) if ( n ) even</td>
<td></td>
</tr>
<tr>
<td>(III) ( n )</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(IV) ( n \geq 3 )</td>
<td>2</td>
<td>( n-2 )</td>
<td>0</td>
</tr>
<tr>
<td>(EIII)</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>(EVII)</td>
<td>3</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>
The zonal spherical polynomial $\Omega_\lambda$ is given as follows.

For integers $n_1, \ldots, n_p$ we define the Schur function $\{n_1, \ldots, n_p\}$ on the $p$-dimensional torus $T^p$ by

$$\{n_1, \ldots, n_p\}(t) = \frac{\det(t_1^{n_1+i_1+p-1} \cdots t_p^{n_p+i_p})}{\det(t_1^{i_1+p-1} \cdots t_p^{i_p+p-1})} \quad \text{for} \quad t = \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \in T^p \subset \mathbb{C}^p.$$ 

$\{n_1, \ldots, n_p\}$ is symmetric in variables $t_1, \ldots, t_p$ and it is a polynomial in $t_1, \ldots, t_p$ if and only if $n_i \geq 0$ $(1 \leq i \leq p)$. For an element $\lambda = \sum \gamma_i = \sum \gamma_i = Z(K_o, L_o)$, the $i$-th coefficient $n_i$ will be denoted by $n_i(\lambda)$.

Then we have

**Theorem 4.1.** The zonal spherical polynomial $\Omega_\lambda$ for $D$ belonging to $\lambda \in S^* (K, L)$ is determined on $(\alpha^-)^c$ by the relation

$$\Omega_\lambda(t) = \sum_{\mu \in B_\lambda} c^\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for any} \quad t \in T^p = \hat{A} \subset (\alpha^-)^c,$$

where the $c^\mu$s are coefficients in Theorem 2.5 for the symmetric pair $(K_o, L_o)$.

**Proof.** As we have seen in the proof of Theorem B, $\Omega_\lambda$ is determined on $(\alpha^-)^c$ by

$$\Omega_\lambda(t) = \omega_\lambda(t) \quad \text{for any} \quad t \in T^p = \hat{A}.$$ 

By Theorem 2.5, $\omega_\lambda$ has an expression

$$\omega_\lambda(t) = \sum_{\mu \in B_\lambda} c^\mu \chi_\mu(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$ 

Since the Weyl group $W_{S_0}$ acts on $Z(K_o, L_o)$ by the group of permutations of $\gamma_1, \ldots, \gamma_p$, $W_{S_0}$-invariant characters $\chi_\lambda$ of $\hat{A}$ are nothing but Schur functions.

As we have seen in the proof of Theorem 3.1, (iii), the $i$-th component of $\text{Ad}(\exp H)X_\mu \in T^p = \hat{A}$ is $\exp (-\langle \gamma_i, H \rangle)$ for any $H \in \mathfrak{a}$. It follows that

$$\chi_\mu(t) = \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$ 

Hence we have

$$\Omega_\lambda(t) = \sum_{\mu \in B_\lambda} c^\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$ 

q.e.d.

In the case of the domain $D$ of type $(I)_{p, q} (p \leq q)$, $S_o$ is the unitary group $U(p)$ of degree $p$. We have in view of Example in 2 that

$$\Omega_\lambda(t) = \frac{1}{d_\lambda} \{n_1(\lambda), \ldots, n_p(\lambda)\}(t) \quad \text{for} \quad t \in T^p = \hat{A},$$
where $d_\lambda$ is the degree of the irreducible representation of $U(p)$ with the signature $(n_1(\lambda), \ldots, n_p(\lambda))$. In the case of the domain $D$ of type (IV)$_r$, $S_0$ is the Lie sphere and $\Omega_\lambda$ can be described in terms of Gegenbauer polynomials, which are zonal spherical functions for the sphere. So our integral formula in Theorem B clarifies the meaning of integrals of Hua [6].

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References
