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POLYNOMIAL REPRESENTATIONS ASSOCIATED WITH
SYMMETRIC BOUNDED DOMAINS

MASARU TAKEUCHI

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Introduction. In this note we want to construct a complete orthonormal
system of the Hilbert space $H^2(D)$ of square integrable holomorphic functions on
an irreducible symmetric bounded domain $D$. A symmetric bounded domain
$D$ is canonically realizable as a circular starlike bounded domain with the center
$0$ in a complex cartesian space by means of Harish-Chandra's imbedding (Harish-
Chandra [3]), which is constructed as follows. The largest connected group $G$
of holomorphic automorphisms of $D$ is a connected semi-simple Lie group without
center, which is transitive on $D$. Thus denoting the stabilizer in $G$ of a point
$o$$\in$$D$ by $K$, $D$ is identified with the quotient space $G/K$. Let $g$ (resp. $\mathfrak{k}$) be the
Lie algebra of $G$ (resp. $K$) and $g=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $g$ with
respect to $\mathfrak{k}$. Then there exists uniquely an element $H$ of the center of $\mathfrak{k}$ such
that $adH$ restricted to $\mathfrak{p}$ coincides with the complex structure tensor on the
tangent space $T_o(D)$ of $D$ at the origin $o$, identifying as usual $\mathfrak{p}$ with $T_o(D)$. Let $\mathfrak{g}^c$ be the Lie algebra of the complexification $G^c$ of $G$ and put $Z=\sqrt{-1}H$$\in$$\mathfrak{g}^c$.
Let $(\mathfrak{p}^c)^\pm$ be the $(\pm 1)$-eigenspace in $\mathfrak{g}^c$ of $ad Z$. Then they are invariant under
the adjoint action of $K$ and the complexification $\mathfrak{p}^c$ of $\mathfrak{p}$ is the direct sum of $(\mathfrak{p}^c)^+$
and $(\mathfrak{p}^c)^-$. Let $U^c$ denote the normalizer of $(\mathfrak{p}^c)^+$ in $G^c$. Then $D=G/K$ is
holomorphically imbedded as an open submanifold into the quotient space $G^c/U^c$
in the natural way. For any point $z$$\in$$D$, there exists uniquely a vector $X$$\in$$(\mathfrak{p}^c)^-$
such that

$$\exp X \mod U^c = z.$$ 

The map $z$$\mapsto$$X$ of $D$ into $(\mathfrak{p}^c)^-$ is the desired imbedding. Note that the natural
action of $K$ on $D$ can be extended to the adjoint action of $K$ on the ambient space
$(\mathfrak{p}^c)^-$.

Henceforth we assume that $D$ is a bounded domain in $(\mathfrak{p}^c)^-$ realized in the
above manner. Let $(\ , \ )$ denote the Killing form of $\mathfrak{g}^c$ and $\tau$ the complex con-
jugation of $\mathfrak{g}^c$ with respect to the compact real form $\mathfrak{k}+\sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}^c$. We define
a $K$-invariant hermitian inner product $(\ , \ )$, on $\mathfrak{g}^c$ by

$$(X, Y) = -(X, \tau Y) \quad \text{for} \quad X, Y \in \mathfrak{g}^c.$$
This defines a $K$-invariant Euclidean measure $d\mu(X)$ on $(\mathfrak{p}^c)^-$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(X)$. The inner product of $H^2(D)$ will be denoted by $\langle \ , \ \rangle$. $K$ acts on $H^2(D)$ as unitary operators by

$$(kf)(X) = f(k^{-1}X) \quad \text{for} \quad k \in K, \ X \in D.$$ 

Let $S^*((\mathfrak{p}^c)^-)$ denote the graded space of polynomial functions on $(\mathfrak{p}^c)^-$. It has the natural hermitian inner product $(\ , \ )$ induced from the inner product $(\ , \ )$, on $(\mathfrak{p}^c)^-$. $K$ acts on $S^*((\mathfrak{p}^c)^-)$ as unitary operators by

$$(kf)(X) = f(Ad k^{-1}X) \quad \text{for} \quad k \in K, \ X \in (\mathfrak{p}^c)^-.$$ 

Now let $S$ denote the Shilov boundary of $D$. It is known (Korányi-Wolf [7]) that $K$ acts transitively on $S$. Thus denoting by $L$ the stabilizer in $K$ of a point $X_0 \in S$, $S$ is identified with the quotient space $K/L$. Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure of $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle \ , \ \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$(kf)(X) = f(Ad k^{-1}X) \quad \text{for} \quad k \in K, \ X \in S.$$ 

The space $C^\infty(S)$ of $C$-valued $C^\infty$-functions on $S$ is a $K$-submodule of $L^2(S)$. The restrictions $S^*((\mathfrak{p}^c)^-) \to H^2(D)$ and $S^*((\mathfrak{p}^c)^-) \to L^2(S)$ are both $K$-equivariant monomorphisms. Their images will be denoted by $S^*(D)$ and $S^*(S)$, respectively. They have natural gradings induced from that of $S^*((\mathfrak{p}^c)^-)$. Then the “restriction” $S^*(D) \to S^*(S)$ is defined in the natural manner and it is a $K$-equivariant isomorphism. Since $D$ is a circular starlike bounded domain, a theorem of H. Cartan [2] yields that the subspace $S^*(D)$ of $H^2(D)$ is dense in $H^2(D)$ (cf. 1).

We decompose first the $K$-module $S^*(D)$ into irreducible components. We take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and identify the real part $\sqrt{-1}\mathfrak{t}$ of the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$ with its dual space by means of Killing form of $g^c$. Let $\Sigma \subset \sqrt{-1}\mathfrak{t}$ denote the set of roots of $g^c$ with respect to $\mathfrak{k}^c$. We choose root vectors $X_\alpha \in g^c$ for $\alpha \in \Sigma$ such that

$$[X_\alpha, X_{-\alpha}] = -\frac{2}{(\alpha, \alpha)} \alpha,$$

$$\tau X_\alpha = X_{-\alpha}.$$ 

A root is called compact if it is also a root of the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$, otherwise it is called non-compact. $\Sigma_+ \ (\text{resp.} \ \Sigma_-)$ denotes the set of compact roots (resp. of non-compact roots). We choose and fix once for all a linear order $\succ$ on $\sqrt{-1}\mathfrak{t}$ such that $(\mathfrak{p}^c)^+$ is spanned by the root spaces for non-compact positive
roots $\sum^+_p$. Two roots $\alpha, \beta \in \sum$ are called strongly orthogonal if $\alpha \pm \beta$ is not a root. We define a maximal strongly orthogonal subsystem

$$\Delta = \{\gamma_1, \ldots, \gamma_p\}, \quad \gamma_1 > \gamma_2 > \cdots > \gamma_p > 0, \quad p = \text{rank } D$$

of $\sum^+_p$ as follows (cf. Harish-Chandra [3]). Let $\gamma_i$ be the highest root of $\sum$ and for each $j$, $\gamma_{j+1}$ be the highest positive non-compact root that is strongly orthogonal to $\gamma_1, \ldots, \gamma_j$. We put

$$X_0 = -\sum_{\gamma \in \Delta} X_\gamma.$$

Then it is known (Korányi-Wolf [7]) that $X_0$ is on the Shilov boundary $S$ of $D$. Henceforth we shall take the above point $X_0$ as the origin of $S$. We put for $\nu \in \mathbb{Z}$, $\nu \geq 0$

$$\mathcal{S}'(K, L) = \{ \sum_{i=1}^p n_i \gamma_i; \ n_i \in \mathbb{Z}, \ n_1 \geq n_2 \geq \cdots \geq n_p \geq 0, \ \sum_{i=1}^p n_i = \nu \},$$

and

$$\mathcal{S}^*(K, L) = \sum_{\nu \in \mathbb{Z}} \mathcal{S}'(K, L).$$

We shall prove the following

**Theorem A.** Any irreducible $K$-submodule of $\mathcal{S}^*(D)$ is contained exactly once in $\mathcal{S}^*(D)$. The set $\mathcal{S}'(D)$ of highest weights (with respect to $\mathfrak{t}^0$) of irreducible $K$-submodules contained in $\mathcal{S}'(D)$ coincides with $\mathcal{S}'(K, L)$. Denoting by $\mathcal{S}^*_*(D)$ (resp. $\mathcal{S}^*_*(S)$) the irreducible $K$-submodule of $\mathcal{S}^*(D)$ (resp. of $\mathcal{S}^*(S)$) with the highest weight $\lambda \in \mathcal{S}^*(K, L)$,

$$\mathcal{S}^*(D) = \sum_{\lambda \in \mathcal{S}^*(K, L)} \mathcal{S}^*_*(D)$$

and

$$\mathcal{S}^*(S) = \sum_{\lambda \in \mathcal{S}^*(K, L)} \mathcal{S}^*_*(S)$$

are the orthogonal sum relative to the inner product $\langle \ , \rangle$ and $\langle \ , \ \rangle$, respectively. The restriction $f \mapsto f'$ of $\mathcal{S}^*_*(D) \mapsto \mathcal{S}^*_*(S)$ is a similitude for each $\lambda \in \mathcal{S}^*(K, L)$, i.e. there exists a constant $h_\lambda > 0$ such that

$$\langle f, g \rangle = h_\lambda \langle f', g' \rangle \quad \text{for any } f, g \in \mathcal{S}^*_*(D).$$

Thus, if

$$\{f_{\lambda, i}; 1 \leq i \leq d_\lambda\}, \quad \lambda \in \mathcal{S}^*(K, L)$$

is an orthonormal basis of $\mathcal{S}^*_*(S)$, then

$$\left\{ \sqrt{h^{-1}_\lambda} f_{\lambda, i}; \lambda \in \mathcal{S}^*(K, L), 1 \leq i \leq d_\lambda \right\}$$

is a complete orthonormal system of $H^2(D)$.
A basis \( \{ f_{\lambda,i}; 1 \leq i \leq d_\lambda \} \) is, for instance, constructed as follows. Take an irreducible \( K \)-module \((\rho, V)\) with the highest weight \( \lambda \), carrying a \( K \)-invariant hermitian inner product \( \langle , \rangle \). Choose an orthonormal basis \( \{ u_i; 1 \leq i \leq d_\lambda \} \) of \( V \) such that the first vector \( u_i \) is \( L \)-invariant. This can be done in view of Frobenius' reciprocity since the \( K \)-module \( V \) is \( K \)-isomorphic with a \( K \)-submodule of \( C^\infty(S) \). Then the functions \( f_{\lambda,i}(1 \leq i \leq d_\lambda) \) defined by

\[
 f_{\lambda,i}(kX_\alpha) = \sqrt{d_\lambda} (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K
\]

form an orthonormal basis of \( S^*_\lambda(\mathbb{S}) \) (cf. 2).

We compute next the normalizing factor \( h_\lambda \). Let

\[
 a = \{ \sqrt{-1}\Delta \}_R
\]

be the \( R \)-span of \( \sqrt{-1}\Delta \) in \( t \) and

\[
 \sigma: \sqrt{-1}t \to \sqrt{-1}a
\]

denote the orthogonal projection of \( \sqrt{-1}t \) onto \( \sqrt{-1}a \). For \( \gamma \in \sigma \Sigma - \{ 0 \} \), the number of roots \( \alpha \in \Sigma \) such that \( \sigma \alpha = \gamma \) is called the multiplicity of \( \gamma \). Let \( r \) (resp. \( 2s \)) be the multiplicity of \( \frac{1}{2}(\gamma_1 - \gamma_2) \) (resp. of \( \frac{1}{2}\gamma_i \)). If follows from Theorem A and Frobenius' reciprocity that for each \( \lambda \in S^*(K, L) \) there exists uniquely an \( L \)-invariant polynomial \( \Omega_\lambda \) in \( S^*_{\mathbb{C}}((\mathbb{P}^C)^{-}) \) such that \( \Omega_\lambda(X_\alpha) = 1 \), where \( S^*_{\mathbb{C}}((\mathbb{P}^C)^{-}) \) denotes the irreducible \( K \)-submodule of \( S^*_{\mathbb{C}}((\mathbb{P}^C)^{-}) \) with the highest weight \( \lambda \). The polynomial \( \Omega_\lambda \) is called the zonal spherical polynomial for \( D \) belonging to \( \lambda \). Let

\[
 (a^-)^C = \{ X_\gamma; \gamma \in \Delta \}_C
\]

be the \( C \)-span of \( \{ X_\gamma; \gamma \in \Delta \} \) in \( (\mathbb{P}^C)^{-} \). It is identified with the complex cartesian space \( C^p \) by the map

\[
 - \sum_{i=1}^p z_i X_{-\gamma_i} \mapsto \left( \begin{array}{c} z_1 \\ \vdots \\ z_p \end{array} \right).
\]

Thus the zonal spherical polynomial \( \Omega_\lambda \) restricted to \( (a^-)^C \) is a polynomial \( \Omega_\lambda(Y_1, \ldots, Y_p) \) in \( p \)-variables. Let \( \mu(D) \) denote the volume of \( D \) with respect to the measure \( d\mu(X) \). We shall prove the following

**Theorem B.** For \( \lambda \in S^*(K, L) \), the normalizing factor \( h_\lambda \) is given by

\[
 h_\lambda = c(D) \int_{0 < y_i < 1; i \leq \xi \leq p} \Omega_\lambda(y_1, \ldots, y_p) \prod_{1 \leq i < j \leq p} (y_i - y_j)^r | \prod_{i=1}^p y_i^s dy_1 \cdots dy_p
\]

where

\[
 c(D) = \mu(D) \left( \prod_{0 < y_i < 1; i \leq \xi \leq p} | \prod_{1 \leq i < j \leq p} (y_i - y_j)^r | \prod_{i=1}^p y_i^s dy_1 \cdots dy_p \right)^{-1}.
\]
Hua [6] proved Theorem A for classical domains by decomposing the character of the $K$-module $S^*((\mathfrak{p}^\mathfrak{c})^-)$ into the sum of irreducible characters of $K$, while Schmid [11] proved it for general domain $D$. Schmid proved

(a) \[ S^v(D) \subset S^v(K, L) \]

by seeing the character of the $K$-module $S^*((\mathfrak{p}^\mathfrak{c})^-)$ and by making use of E. Cartan's theory on spherical representations of a compact symmetric pair. But his proof of

(b) \[ S^v(K, L) \subset S^v(D) \]

is complicated and was done after nine successive lemmas. In this note we give another proof of (a) by means of a lemma of Murakami and Cartan's theory, and give a relatively short proof of (b) by means of a theorem of Harish-Chandra on invariant polynomials for a symmetric pair.

Hua [6] computed the factors $h_\lambda$ for certain classical domains by integrating certain polynomials. Our integral formula in Theorem B will clarify the meaning of integrals of Hua.

1. Circular domains

A domain $D \subset C^n$ containing the origin 0 is said to be a circular domain with the center 0 if together with any point $z \in D$ the point $e^{\sqrt{-1} \theta} z$ is in $D$ for any real $\theta \in R$. $D$ is said to be a starlike domain with the center 0 if together with any point $z \in D$ the point $rz$ is in $D$ for any real $r \in R$ with $0 \leq r < 1$.

**Theorem 1.1.** (H. Cartan [2]) Let $D \subset C^n$ be a circular domain with the center 0. Then any holomorphic function $f$ on $D$ can be developed in the sum of homogeneous polynomials $P_v$ in $n$-variables with degree $v$ ($v=0, 1, 2, \cdots$):

\[ f(z) = \sum_{v=0}^{\infty} P_v(z) \quad \text{for} \quad z \in D. \]

The sum converges uniformly on any compact subset of $D$. The homogeneous polynomials $P_v$ are uniquely determined for $f$.

Let $D$ be a bounded domain in $C^n$, $d\mu(z)$ the Euclidean measure on $C^n$, induced from the standard hermitian inner product of $C^n$. Let $H^q(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(z)$. The inner product of $H^q(D)$ will be denoted by $\langle \ , \ \rangle$. Let $S^*(C^n)$ be the graded space of polynomials in $n$-variables and $S^*(D)$ the subspace of $H^q(D)$ consisting of all functions on $D$ obtained by the restriction of polynomials in $S^*(C^n)$. Then Theorem 1.1 yields the following

**Corollary.** Let $D \subset C^n$ be a circular starlike bounded domain with the center 0. Then the subspace $S^*(D)$ of $H^q(D)$ is dense in $H^q(D)$. 

Proof. If suffices to show that if \( f \in H^2(D) \) with \( \langle f, S^*(D) \rangle = \{0\} \), then \( f = 0 \). Theorem 1.1 implies that \( f \) can be developed as

\[
f = \sum_{\nu=0}^{\infty} P_{\nu}, \quad P_{\nu} \in S^*(D),
\]

uniformly convergent on any compact subset of \( D \). Choose an orthonormal basis \( \{P_{\nu,j}\} \) of \( S^*(D) \) with respect to \( \langle \ , \rangle \) for each \( \nu \). Then we have

\[
\langle P_{\nu,j}, P_{\mu,i} \rangle = \delta_{\nu\mu} \delta_{ji}.
\]

In fact, since \( d\mu(e^{\sqrt{-1} \theta z}) = d\mu(z) \) for any \( \theta \in \mathbb{R} \), we have \( \langle P_{\nu,j}, P_{\nu,k} \rangle = e^{\sqrt{-1} \theta \nu - \mu} \) \( \langle P_{\nu,j}, P_{\nu,i} \rangle \) for any \( \theta \in \mathbb{R} \). Then \( f \) can be developed as

\[
f = \sum_{\nu,j} a_{\nu,j} P_{\nu,j} \quad \text{with} \quad a_{\nu,j} \in \mathbb{C},
\]

uniformly convergent on any compact subset of \( D \). Since \( D \) is a starlike domain, the closure \( rD \) of \( rD \) is a compact subset of \( D \) for any \( r \in \mathbb{R} \) with \( 0 < r < 1 \), so that the above series converges uniformly on \( rD \). Therefore for any \( P_{\mu,i} \) we have

\[
\int_{rD} f(x) P_{\mu,i}(x) d\mu(x) = \sum_{\nu,j} a_{\nu,j} \int_{rD} P_{\nu,j}(x) P_{\mu,i}(x) d\mu(x).
\]

If we put

\[
z' = \frac{1}{r} z \quad \text{for} \quad z \in rD,
\]

then \( z = rz' \), \( d\mu(z) = r^{2n} d\mu(z') \) so that

\[
\int_{rD} P_{\nu,j}(z) P_{\mu,i}(z) d\mu(z) = r^{2n+\nu+\mu} \int_{D} P_{\nu,j}(z') P_{\mu,i}(z') d\mu(z')
\]

\[
= r^{2n+\nu+\mu} \langle P_{\nu,j}, P_{\mu,i} \rangle = r^{2n+2\nu} \delta_{\nu\mu} \delta_{ji}.
\]

Hence we have

\[
\int_{rD} f(x) P_{\mu,i}(x) d\mu(x) = a_{\mu,i} r^{2n+2\nu}
\]

and

\[
a_{\mu,i} = \lim_{r \to 0} a_{\mu,i} r^{2n+2\nu} = \lim_{r \to 0} \int_{rD} f(x) P_{\mu,i}(x) d\mu(x)
\]

\[
= \langle f, P_{\mu,i} \rangle = 0 \quad \text{(from the assumption)}.
\]

This implies that \( f = 0 \). \( \quad \text{q.e.d.} \)
2. Spherical representations of a compact symmetric pair

Let $K$ be a compact connected Lie group, $L$ a closed subgroup of $K$ and $S$ be the quotient space $K/L$. The space of $C$-valued $C^\infty$-functions on $S$ will be denoted by $C^\infty(S)$. We shall often identify $C^\infty(S)$ with the space of $C^\infty$-functions $f$ on $K$ such that

$$f(kl) = f(k) \quad \text{for any } k \in K, l \in L.$$ 

Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure on $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle \cdot, \cdot \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$(kf)(x) = f(k^{-1}x) \quad \text{for } k \in K, x \in S.$$ 

Then $C^\infty(S)$ is a $K$-submodule of $L^2(S)$. A (continuous finite dimensional complex) representation

$$\rho: K \to GL(V)$$

of $K$ is said to be spherical relative to $L$ if the $K$-module $V$ is equivalent to a $K$-submodule of $C^\infty(S)$, which amounts to the same from Frobenius' reciprocity that the $K$-module $V$ has a non-zero $L$-invariant vector. We denote by $\mathcal{D}(K, L)$ the set of equivalence classes of irreducible spherical representations of $K$ relative to $L$. The totality of $f \in C^\infty(S)$ contained in a finite dimensional $K$-submodule of $C^\infty(S)$, which will be denoted by $\mathfrak{o}(K, L)$, is a $K$-submodule of $C^\infty(S)$. A function in $\mathfrak{o}(K, L)$ is called a spherical function for the pair $(K, L)$. For $\rho \in \mathcal{D}(K, L)$, the totality of $f \in \mathfrak{o}(K, L)$ that transforms according to $\rho$, which will be denoted by $\mathfrak{o}_\rho(K, L)$, is a finite dimensional $K$-submodule of $\mathfrak{o}(K, L)$. Then

$$\mathfrak{o}(K, L) = \bigoplus_{\rho \in \mathcal{D}(K, L)} \mathfrak{o}_\rho(K, L)$$

is the orthogonal sum with respect to the inner product $\langle \cdot, \cdot \rangle$. Peter-Weyl approximation theorem implies that the subspace $\mathfrak{o}(K, L)$ of $L^2(S)$ is dense in $L^2(S)$. We assume furthermore that the pair $(K, L)$ satisfies the condition

$$\text{(*) any } \rho \in \mathcal{D}(K, L) \text{ is contained exactly once in } \mathfrak{o}(K, L),$$

which is by Frobenius' reciprocity equivalent to that for any spherical representation

$$\rho: K \to GL(V)$$

of $K$ relative to $L$, an $L$-invariant vector of $V$ is unique up to scalar multiplication. Then for each $\rho \in \mathcal{D}(K, L)$, there exists uniquely an $L$-invariant function $\omega_\rho \in \mathfrak{o}_\rho(K, L)$ such that $\omega_\rho(e) = 1$. $\omega_\rho$ is called the zonal spherical function for $(K, L)$ belonging to $\rho$. Let

$$\rho: K \to GL(V)$$
be a spherical representation of $K$ relative to $L$. Choose a $K$-invariant hermitian inner product $(\cdot , \cdot)$ on $V$. The equivalence class containing $\rho$ will be denoted by the same letter $\rho$. Choose an orthonormal basis $\{u_i; 1 \leq i \leq d_\rho\}$ of $V$ such that $u_i$ is $L$-invariant. Define $\varphi_i \in C^\infty(S)$ $(1 \leq i \leq d_\rho)$ by

$$\varphi_i(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.$$ 

We know that they are linearly independent, in view of orthogonality relations of matrix elements $(u_i, \rho(k)u_j)$. For any $k' \in K$ we have

$$\varphi_i(k'^{-1}k) = (u_i, \rho(k'^{-1}k)u_i) = (\rho(k')u_i, \rho(k)u_i)$$

$$= \sum_j (\rho(k')u_i, u_j)(u_j, \rho(k)u_i)$$

$$= \sum_j (\rho(k')u_i, u_j) \varphi_j(k),$$

i.e.

$$k' \varphi_i = \sum_j (\rho(k')u_i, u_j) \varphi_j \quad (1 \leq i \leq d_\rho).$$

In particular

$$k \varphi_i = \varphi_i \quad \text{for any} \quad k \in L,$$

and

$$\varphi_i(e) = 1.$$ 

Therefore the system $\{\varphi_i; 1 \leq i \leq d_\rho\}$ forms a basis of $\mathfrak{o}_\rho(K, L)$ and the zonal spherical function $\omega_\rho$ is given by

$$\omega_\rho(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.$$ 

Furthermore orthogonality relations implies that the system

$$\{\sqrt{d_\rho} \varphi_i; 1 \leq i \leq d_\rho\}$$

forms an orthonormal basis of $\mathfrak{o}_\rho(K, L)$ and that

$$\langle \omega_\rho, \omega_{\rho'} \rangle = \delta_{\rho\rho'} \frac{1}{d_\rho}.$$

Henceforth we assume that the pair $(K, L)$ is a symmetric pair, i.e. there exists an involutive automorphism $\theta$ of $K$ such that if we put

$$K_\theta = \{k \in K; \theta(k) = k\},$$

$L$ lies between $K_\theta$ and the connected component $K_0$ of $K_\theta$. Then the pair $(K, L)$ satisfies the condition $(\ast)$ (E. Cartan [1]). For example, a compact connected Lie group $S$ admits a symmetric pair $(K, L)$ such that $S = K/L$. In fact,

$$K = S \times S,$$

$$L = \{(x, x); x \in S\}.$$
and

\[ \theta: (x, y) \mapsto (y, x) \quad \text{for} \quad x, y \in S \]

have desired properties.

In the following we summarize some known facts on a symmetric pair (cf. Helgason [4]).

Let \( \mathfrak{l} \) (resp. \( \mathfrak{l} \)) be the Lie algebra of \( K \) (resp. of \( L \)). The involutive automorphism of \( \mathfrak{l} \) obtained by differentiating the automorphism \( \theta \) of \( K \) will be also denoted by the same letter \( \theta \).

Choose and fix once for all a \( C \)-bilinear symmetric form \((\ , \)\) on the complexification \( \mathfrak{l}^C \) of \( \mathfrak{l} \), which is invariant under both the \( C \)-linear extension to \( \mathfrak{l}^C \) of \( \theta \) and the adjoint action of \( \mathfrak{l}^C \) and furthermore is negative definite on \( \mathfrak{l} \times \mathfrak{l} \). Then \( S \) is a Riemannian symmetric space with respect to the \( K \)-invariant Riemannian metric on \( S \) defined by \( -(\ , \)\). We put

\[ \mathfrak{s} = \{ X \in \mathfrak{l}; \, \theta X = -X \} = \{ X \in \mathfrak{l}; \, (X, l) = \{0\} \} . \]

Then we have orthogonal decompositions

\[ \mathfrak{l} = \mathfrak{c} \oplus \mathfrak{s} = \mathfrak{c} \oplus \mathfrak{l}' \],

where \( \mathfrak{c} \) is the center of \( \mathfrak{l} \) and \( \mathfrak{l}' \) is the derived algebra \([\mathfrak{l}, \mathfrak{l}]\) of \( \mathfrak{l} \). We choose a maximal abelian subalgebra \( \mathfrak{a} \) in \( \mathfrak{s} \). Such \( \mathfrak{a} \) are mutually conjugate under the adjoint action of \( L \). \( \dim \mathfrak{a} \) is the rank of the symmetric pair \((K, L)\). Extend \( \mathfrak{a} \) to a maximal abelian subalgebra \( \mathfrak{t} \) of \( \mathfrak{l} \) containing \( \mathfrak{a} \). Then we have the decomposition

\[ \mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a} \quad \text{where} \quad \mathfrak{b} = \mathfrak{t} \cap \mathfrak{l} . \]

Let \( \mathfrak{t}' = \mathfrak{t} \cap \mathfrak{l}' \) and \( \mathfrak{a}' = \mathfrak{a} \cap \mathfrak{l}' \). The real vector space \( \sqrt{-1} \mathfrak{t} \) has the natural inner product \((\ , \)\) induced from the bilinear form \((\ , \)\) on \( \mathfrak{l}^C \). We shall identify \( \sqrt{-1} \mathfrak{t} \) with the dual space of \( \sqrt{-1} \mathfrak{t} \) by means of the inner product \((\ , \)\). We have the orthogonal decomposition

\[ \sqrt{-1} \mathfrak{t} = \sqrt{-1} \mathfrak{b} \oplus \sqrt{-1} \mathfrak{a} . \]

Let \( \sigma \) be the orthogonal transformation on \( \sqrt{-1} \mathfrak{t} \) defined by

\[ \sigma|\sqrt{-1} \mathfrak{b} = -1 \quad \text{and} \quad \sigma|\sqrt{-1} \mathfrak{a} = 1 \]

and

\[ \varphi = \frac{1}{2}(1+\sigma): \sqrt{-1} \mathfrak{t} \rightarrow \sqrt{-1} \mathfrak{a} \]

be the orthogonal projection of \( \sqrt{-1} \mathfrak{t} \) onto \( \sqrt{-1} \mathfrak{a} \). Let \( \Sigma_{\mathfrak{t}} \) denote the set of roots of \( \mathfrak{l}^C \) with respect to the complexification \( \mathfrak{t}^C \) of \( \mathfrak{t} \). Let \( \mathcal{W}_t = N_K(T)/T \) be the Weyl group of \( \mathfrak{t} \), where \( T \) is the connected subgroup of \( K \) generated by \( \mathfrak{t} \) and \( N_K(T) \) is the normalizer of \( T \) in \( K \). \( \Sigma_{\mathfrak{t}} \) is a \( \sigma \)-invariant reduced root system in
\[ \sqrt{-1} t'. \] As a group of orthogonal transformations of \( \sqrt{-1} t \), \( W_t \) is generated by reflections with respect to roots in \( \Sigma_t \). Put
\[ \Sigma^0_t = \Sigma_t \cap \sqrt{-1} b = \{ \alpha \in \Sigma_t; \sigma \alpha = 0 \} , \]
\[ \Sigma_s = \{ \sigma \alpha; \alpha \in \Sigma_t - \Sigma^0_t \} = \sigma \Sigma_t - \{0\} , \]
\[ W_s = N_L(A)/Z_L(A) , \]
where \( A \) is the connected subgroup of \( K \) generated by \( a \) and \( N_L(A) \) (resp. \( Z_L(A) \)) the normalizer (resp. the centralizer) of \( A \) in \( L \). An element of \( \Sigma_s \) is a restricted root of the symmetric space \( S \) and \( W_s \) is the Weyl group of \( S \). \( \Sigma_s \) is a (not necessarily reduced) root system in \( \sqrt{-1} a' \). As a group of orthogonal transformations of \( \sqrt{-1} a \), \( W_s \) is generated by reflections with respect to roots in \( \Sigma_s \).

A linear order \( > \) on \( \sqrt{-1} t \) is said to be compatible for \( \Sigma_t \) with respect to \( \sigma \) (or with respect to the orthogonal decomposition \( \sqrt{-1} t = \sqrt{-1} b \oplus \sqrt{-1} a \)) if \( \alpha \in \Sigma_t, \alpha > 0 \) and \( \sigma \alpha = -\alpha \) imply \( \sigma \alpha > 0 \). Take a compatible order \( > \) on \( \sqrt{-1} t \) and fix it once and for all. Let
\[ \Pi_t = \{ \alpha_1, \ldots, \alpha_t \} \]
be the fundamental root system of \( \Sigma_t \) with respect to the order \( > \) and put
\[ \Pi^0_t = \Pi_t \cap \Sigma^0_t . \]

\( W_t \) is also generated by reflections with respect to roots in \( \Pi_t \). We have the decomposition
\[ \sigma = sp \quad \text{where} \quad s \in W_t, \quad p \Pi_t = \Pi_t \]
of \( \sigma \) in such a way that \( p^2 = 1, \quad p(\Pi_t - \Pi^0_t) = \Pi_t - \Pi^0_t \) and \( \sigma \alpha_i = p \alpha_i \mod \{ \Pi^0_t \} \) for any \( \alpha_i \in \Pi_t - \Pi^0_t \) (Satake [10]). We put
\[ \Pi_s = \{ \sigma \alpha_i; \alpha_i \in \Pi_t - \Pi^0_t \} = \sigma \Pi_t - \{0\} . \]

We may assume that \( \Pi_s = \{ \gamma_1, \ldots, \gamma_p \} \) with \( \sigma \alpha_i = \gamma_i (1 \leq i \leq p) \), changing indices of the \( \alpha_i \)’s if necessary. \( \Pi_s \) is the fundamental root system of \( \Sigma_s \) with respect to the order \( > \). We put
\[ \Sigma^*_s = \{ \gamma \in \Sigma_s; 2 \gamma \in \Sigma_s \} . \]

Then \( \Sigma^*_s \) is a reduced root system in \( \sqrt{-1} a' \). The fundamental root system \( \Pi^*_s \) of \( \Sigma^*_s \) with respect to the order \( > \) is given by
\[ \Pi^*_s = \{ \beta_1, \ldots, \beta_p \} \]
where
\[ \beta_i = \begin{cases} \gamma_i & \text{if } 2 \gamma_i \in \Sigma_s \\ 2 \gamma_i & \text{if } 2 \gamma_i \in \Sigma_s \end{cases} . \]

\( W_s \) is also generated by reflections with respect to roots of \( \Pi_s \) or of \( \Pi^*_s \). Let
\[ \Sigma^+_t \text{ (resp. } \Sigma^+_s, (\Sigma^*_s)^+) \text{ denote the set of positive roots in } \Sigma_t \text{ (resp. } \Sigma_s, \Sigma^*_s). \]

Then
\[ \Sigma^+_s = \omega (\Sigma^+_t - \Sigma^-_t) = \omega \Sigma^+_t - \{0\}. \]

For \( \lambda \in -1_\mathbb{Z} \), \( \lambda \neq 0 \), we define
\[ \lambda* = \frac{2}{(\lambda, \lambda)} \lambda. \]

**Theorem 2.1.** (E. Cartan) Assume that \( K \) is simply connected. Then
1) \( K_{\alpha} \) is connected.
2) The kernel of \( \exp: \alpha \to K \) is the subgroup of \( \alpha \) generated by \( \{2\pi \sqrt{-1} \gamma^*; \gamma \in \Sigma_s\} \).

**Theorem 2.2.** (Harish-Chandra) Let \( S^*_L(\mathfrak{g}) \) (resp. \( S^*_W(\alpha) \)) be the space of polynomial functions on \( \mathfrak{g} \) (resp. on \( \alpha \)), which are invariant under the adjoint actions of \( L \) (resp. of \( W_s \)). Then the restriction map
\[ S^*_L(\mathfrak{g}) \to S^*_W(\alpha) \]

is an isomorphism.

Now we shall consider \( W_s \)-invariant characters of a maximal torus of \( S \). Put
\[ \Gamma = \Gamma(K, L) = \{H \in \alpha; \exp H \in L\} \]
and
\[ \Gamma_c = \Gamma \cap c_\alpha \text{ where } c_\alpha = e \cap \alpha. \]

Then \( \Gamma \) is a \( W_s \)-invariant lattice in \( \alpha \) and \( \Gamma_c \) is a lattice in \( c_\alpha \). Let \( C_\alpha \) be the connected subgroup of \( K \) generated by \( c_\alpha \). Then the \( A \)-orbit \( \hat{A} \) in \( S \) through the origin \( x_0 \) of \( S \) and the \( C_\alpha \)-orbit \( \hat{C_\alpha} \) in \( S \) through the origin have identifications
\[ \hat{A} = \alpha/\Gamma \]
and
\[ \hat{C_\alpha} = c_\alpha/\Gamma_c. \]

Hence both \( \hat{A} \) and \( \hat{C_\alpha} \) have structures of toral groups. The toral group \( \hat{A} \) is said to be a **maximal torus** of the symmetric space \( S \). The adjoint action of \( W_s \) on \( A \) induces the action of \( W_s \) on \( A \). This action is compatible with the natural action of \( W_s \) on \( \alpha/\Gamma \) relative to the identification: \( \hat{A} = \alpha/\Gamma \). Put
\[ Z = Z(K, L) = \{\lambda \in -1_\mathbb{Z}; (\lambda, H) \in 2\pi \sqrt{-1} Z \text{ for any } H \in \Gamma\}. \]

\( Z \) is isomorphic with the group \( \mathcal{D}(\hat{A}) \) of characters of \( \hat{A} \) by the correspondence \( \lambda \mapsto e^\lambda \), where \( e^\lambda \in \mathcal{D}(\hat{A}) \) is defined by \( e^\lambda((\exp H)x_0) = \exp (\lambda, H) \) for \( H \in \alpha \). Put
Then we have

\[ D = \{ \lambda \in \mathbb{Z}; s \lambda \leq \lambda \text{ for any } s \in W_s \}. \]

An element of \( D \) is called a \textit{dominant integral form} on \( \alpha \). We define a lattice \( \Gamma_0' \) in \( \alpha' \) to be the subgroup of \( \alpha' \) generated by \( \{2\pi \sqrt{-1}(\frac{1}{2} \gamma^*); \gamma \in \sum_s\} \). We define a lattice \( \Gamma_0 \) in \( \alpha \) and a toral group \( \hat{A}_0 \) by

\[ \Gamma_0 = \Gamma \oplus \Gamma_0' \]

and

\[ \hat{A}_0 = \alpha/\Gamma_0. \]

Put

\[ Z_0 = \{ \lambda \in \sqrt{-1} \alpha; (\lambda, H) \in 2\pi \sqrt{-1} \mathbb{Z} \text{ for any } H \in \Gamma_0 \} \]

and

\[ D_0 = D \cap Z_0. \]

\( Z_0 \) is isomorphic with the group \( \mathcal{D}(\hat{A}_0) \) of characters of \( \hat{A}_0 \). Put furthermore

\[ Z_0' = Z_0 \cap \sqrt{-1} \alpha' = \left\{ \lambda \in \sqrt{-1} \alpha'; \frac{2(\lambda, \gamma)}{(\gamma, \gamma)} \in 2 \mathbb{Z} \text{ for any } \gamma \in \sum_s \right\} \]

and

\[ D_0' = D_0 \cap \sqrt{-1} \alpha' = D \cap Z_0'. \]

\textbf{Lemma 1.} If \( L = K_0 \), then

\[ \Gamma = \{ \frac{1}{2} H; H \in \alpha, \exp H = e \}. \]

\textbf{Proof.} For \( H \in \alpha, \exp H = e \iff \exp \frac{H}{2} \exp \frac{H}{2} = e \iff \exp \frac{H}{2} = \left( \exp \frac{H}{2} \right)^{-1} \implies \exp \frac{H}{2} = \theta \left( \exp \frac{H}{2} \right) \iff \exp \frac{H}{2} \in K_0, \text{ which yields Lemma 1.} \]

\textbf{Lemma 2.} 1) \( \Gamma_0' = 2\pi \sqrt{-1} \sum_{i=1}^s Z(\frac{1}{2} \beta_i^*) \) and it is \( W_s \)-invariant. Therefore \( \Gamma_0 \) is \( W_s \)-invariant.

2) \( \Gamma_0 \subseteq \Gamma \). Therefore \( Z_0 \supseteq Z \) and \( D_0 \supseteq D \).

3) If \( S \) is simply connected, then \( \Gamma = \Gamma_0 = \Gamma_0' \) (thus \( Z = Z_0 = Z_0', D = D_0 = D_0' \)) and \( \hat{A}_0 \) can be identified with \( \hat{A} \).

\textbf{Proof.} 1) Denoting the reflection of \( \sqrt{-1} \alpha \) with respect to \( \beta_i \in \Pi_s^* \) by \( s_i \in W_s \), we have
\[ s_i \gamma^* = (s_i \gamma)^* = \gamma^* \frac{2(\beta_\xi, \gamma)}{(\gamma, \gamma)} \beta_i^* \quad \text{for} \quad \gamma \in \Sigma_s. \]

It follows that \( \Gamma_o' \) is \( W_s \)-invariant. Since we have
\[ (2\lambda)^* = \frac{2 \cdot 2\lambda}{4(\lambda, \lambda)} = \frac{\lambda}{(\lambda, \lambda)} = \frac{1}{2} \lambda^* \quad \text{for} \quad \lambda \in \sqrt{-1} a, \lambda \neq 0, \]
\( \Gamma_o' \) is the subgroup of \( \alpha' \) generated by \( 2\tau \gamma^* \) for \( \gamma \in \Sigma_s^* \). Thus it suffices to show that
\[ \gamma^* \in \sum_{i=1}^l \mathbb{Z} \beta_i^* \quad \text{for any} \quad \gamma \in \Sigma_s^*. \]

But this follows from the first equality since there exist \( \beta_{i_1}, \ldots, \beta_{i_r} \in \Pi_s^* \) such that \( s_{i_1} \cdots s_{i_r} \gamma \in \Pi_s^* \).

2) Since \( \Gamma \subseteq \Gamma' \), it suffices to show that \( \Gamma_o' \subseteq \Gamma' \) for \( \Gamma' = \Gamma \cap \alpha' \). Let \( K' \) be the connected subgroup of \( K \) generated by \( \nu' \) and \( L' = K' \cap L \). Then \( (K', L') \) is also a symmetric pair with respect to \( \theta \) and \( S' = K' / L' \) can be identified with the \( K' \)-orbit in \( S \) through the origin \( x_0 \) of \( S \). Let
\[ \pi': K_o' \to K' \]
be the covering homomorphism of the universal covering group \( K_o' \) of \( K' \) and put
\[ L_o' = \{ k \in K_o'; \theta_0(k) = k \}, \]
where \( \theta_0 \) is the involutive automorphism of \( K_o' \) covering the involutive automorphism \( \theta \) of \( K' \). \( K_o' \) is compact since \( K' \) is semi-simple. \( S' \) can be identified with \( K_o' / \pi'^{-1}(L') \). It follows from Theorem 2.1 and Lemma 1 that \( L_o' \) is connected and
\[ \Gamma_o' = \{ H \in \alpha'; \exp_{K_o'} H \in L_o' \}. \]

Let \( A' \) (resp. \( A_o' \)) be the connected subgroup of \( K' \) (resp. of \( K_o' \)) generated by \( \alpha' \) and \( \hat{A}' \) (resp. \( \hat{A}_o' \)) be the \( A' \)-orbit in \( S' \) (resp. the \( A_o' \)-orbit in \( S_o' = K_o' / L_o' \)) through the origin. Then we have identifications
\[ \hat{A}' = \alpha' / \Gamma' \]
and
\[ \hat{A}_o' = \alpha' / \Gamma_o'. \]

On the other hand, since \( \pi'^{-1}(L') \supseteq L_o' \), the covering homomorphism \( \pi' \) induces the commutative diagram
\[ \begin{array}{ccc}
S_o' & \xrightarrow{\pi'} & S' \\
\cup & & \cup \\
\hat{A}_o' & \xrightarrow{\pi'} & \hat{A}'.
\end{array} \]
It follows that
\[ \Gamma_o' \subseteq \Gamma'. \]

3) Under the notation in 2), we have a covering map
\[ \hat{\mathcal{C}}_a \times S' \to S. \]

It follows from the assumption that \( \hat{\mathcal{C}}_a = \{ e \} \) and \( S' \) is simply connected. Thus the covering map \( \pi' \) is trivial and \( \Gamma = \Gamma_o' \). Moreover \( \mathfrak{c}_a = \{ 0 \} \) implies that \( \Gamma = \Gamma' \) and \( \Gamma_o = \Gamma_o' \). q.e.d.

REMARK. Define \((\alpha, \beta)\) by
\[ (\alpha, \beta) = \delta_{ij} \quad (1 \leq i, j \leq l). \]

Then define \( M_i \) \((1 \leq i \leq p)\) by
\[ M_i = \begin{cases} 2\Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \pi_i) = \{ 0 \} \\ \Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \pi_i) \neq \{ 0 \} \\ \Lambda_i + \Lambda_i & \text{if } p\alpha_i = \alpha_i + \alpha_i. \end{cases} \]

Then it can be verified (cf. Sugiura [12]) that \( M_i \in \sqrt{1-a'} \) \((1 \leq i \leq p)\) and
\[ (M_i, 1/2\beta_{ij}) = \delta_{ij} \quad (1 \leq i, j \leq p). \]

It follows that
\[ Z_0' = \sum_{i=1}^{\rho} ZM_i \]
and
\[ D_0' = \left\{ \sum_{i=1}^{\rho} m_i M_i ; m_i \in \mathbb{Z}, m_i \geq 0 \ (1 \leq i \leq \rho) \right\}. \]

It follows from Lemma 2,1) that \( W_s \) acts on \( \hat{A}_s = a/\Gamma_o \) and from Lemma 2,2) that we have a \( W_s \)-equivariant homomorphism
\[ \pi_o : \hat{A}_o \to \hat{A}. \]

Let \( \mathcal{R}(\hat{A}) \) denote the character ring of \( \hat{A} \). Then \( W_s \) acts on \( \mathcal{R}(\hat{A}) \) (or more generally on the space \( C^\infty(\hat{A}) \) of \( \mathbb{C} \)-valued \( C^\infty \)-functions on \( \hat{A} \)) by
\[ (s\chi)(\hat{a}) = \chi(s^{-1}\hat{a}) \quad \text{for } s \in W_s, \hat{a} \in \hat{A}. \]

This action coincides on \( Z = \mathcal{D}(\hat{A}) \subseteq \mathcal{R}(\hat{A}) \) with the adjoint action of \( W_s \) on \( Z \). Let \( \mathcal{R}_{\mathcal{W}_s}(\hat{A}) \) be the subring of \( W_s \)-invariant characters of \( \hat{A} \) and \( \mathcal{R}_{\mathcal{W}_s}(\hat{A})^c \) the \( \mathbb{C} \)-span of \( \mathcal{R}_{\mathcal{W}_s}(\hat{A}) \) in \( C^\infty(\hat{A}) \). Let \( \mathcal{R}(\hat{A}_o), \mathcal{R}_{\mathcal{W}_s}(\hat{A}_o) \) and \( \mathcal{R}_{\mathcal{W}_s}(\hat{A}_o)^c \) denote the same objects for \( \hat{A}_o \). Then \( \pi_o \) induces a \( W_s \)-equivariant monomorphism
\[ \pi_o^* : \mathcal{R}(\hat{A}) \to \mathcal{R}(\hat{A}_o). \]
and monomorphisms
\[ \pi^\# : R_{\bar{W}_S}(\hat{A}) \to R_{\bar{W}_S}(\hat{A}_0), \]
\[ \pi^\#: R_{\bar{W}_S}(\hat{A})^c \to R_{\bar{W}_S}(\hat{A}_0)^c. \]

Henceforth we shall identify \( R_{\bar{W}_S}(\hat{A}) \) with a subring of \( R_{\bar{W}_S}(\hat{A}_0) \) and \( R_{\bar{W}_S}(\hat{A})^c \) with a subalgebra of \( R_{\bar{W}_S}(\hat{A}_0)^c \) by means of these monomorphisms \( \pi^\# \).

For \( \lambda \in \sqrt{-1} \sigma \), we shall denote by \( \lambda_\sigma \) the \( \sqrt{-1} \sigma \)-component of \( \lambda \) with respect to the orthogonal decomposition
\[ \sqrt{-1} \sigma = \sqrt{-1} \epsilon \sigma \oplus \sqrt{-1} \sigma'. \]

The following facts can be proved in the same way as the classical results for a compact connected Lie group \( S \), so the proofs are omitted.

We define an element \( \delta \) in \( Z_\sigma \) by
\[ \delta = \sum_{\gamma \in \Sigma_\sigma} \gamma. \]

For \( \lambda \in Z_\sigma \), we define \( \xi_\lambda \in R(\hat{A}_0) \) by
\[ \xi_\lambda = \sum_{\gamma \in \Sigma_\sigma} (\det s)e^{\lambda}. \]

For \( \lambda \in Z \), \( \xi_\lambda \) is divisible by \( \xi_\delta \) in the ring \( R(\hat{A}_0) \) and
\[ \chi_\lambda = \frac{\xi_{\lambda+\delta}}{\xi_\delta} \]
is in \( R_{\bar{W}_S}(\hat{A}) \). If \( \chi_\lambda \) has the expression
\[ \chi_\lambda = \sum m_\mu e^\mu \quad \text{with} \quad \mu \in Z, \ m_\mu \in Z, \ m_\mu \neq 0, \]
then \( \mu_\epsilon \) are the same for any \( \mu \). In particular, if \( \lambda \in D \), then the highest component in the above expression of \( \chi_\lambda \) is \( e^\lambda \) with \( m_\lambda = 1 \). Any \( \bar{W}_S \)-invariant character \( \chi \in R_{\bar{W}_S}(\hat{A}) \) of \( \hat{A} \) has an expression
\[ \chi = \sum m_\lambda \chi_\lambda \quad \text{with} \quad \lambda \in D, \ m_\lambda \in Z. \]
The expression is unique for \( \chi \). In particular, the system \( \{ \chi_\lambda; \lambda \in D \} \) forms a basis of the space \( R_{\bar{W}_S}(\hat{A})^c \).

Now we come back to spherical representations of a symmetric pair \((K, L)\).

**Theorem 2.3.** (E. Cartan [1]) Let \( \rho \in D(K, L) \) have the highest weight \( \lambda \in \sqrt{-1} t \) and \( \omega_\lambda \) be the zonal spherical function for \((K, L)\) belonging to \( \rho \). Then
1) \( \lambda \in D \),
2) \( \omega_\lambda \) restricted to \( \hat{A} \) is in \( R_{\bar{W}_S}(\hat{A})^c \) and has an expression
\[ \omega_\lambda = \sum a_\mu e^{-\mu} \quad \text{with} \quad \mu \in Z, \ a_\mu \in R, \ a_\mu > 0, \ \sum a_\mu = 1, \]
with the lowest component $a_n e^{-\lambda}$.

Proof. Proof of E. Cartan [1] was done in the case where $K$ is semi-simple and $L = K_\mathbb{C}$. His proof can be applied for our case without difficulties. But his proof of $\lambda \in \sqrt{-1} a$ is not complete. A correct proof is seen, for example, in Schmid [11]. q.e.d.

Lemma 3. For any $\lambda \in D$, there exists an irreducible representation $\rho$ of $K$ such that the highest weight of $\rho$ on $\mathfrak{t}^\mathbb{C}$ is $\lambda$.

Proof. Let $H \in \mathfrak{t}$ with $\exp H = e$. Decompose $H$ as

$$H = H' + H'' \quad \text{with} \quad H' \in \mathfrak{b}, \ H'' \in \mathfrak{a}.$$  

Then $\exp H'' = (\exp H')^{-1} \in L$, i.e. $H'' \in \Gamma$. It follows from $\lambda \in Z \subset \sqrt{-1} a$ that $(\lambda, H) = (\lambda, H') + (\lambda, H'') = (\lambda, H'') \in 2\pi \sqrt{-1} \mathbb{Z}$. Moreover $(\lambda, \alpha_i) = (\lambda, \omega \alpha_i) \geq 0$ for any $\alpha_i \in \Pi$, since $\lambda \in D$. Thus $e^\lambda$ is a dominant character of the maximal torus $T$ of $K$. Then the classical representation theory of compact connected Lie groups assures the existence of $\rho$. q.e.d.

Lemma 4. Let $Z_L(A)$ be the centralizer in $L$ of $A$ and $Z_L(A)^\circ$ the connected component of $Z_L(A)$. Then

$$Z_L(A) = Z_L(A)^\circ \exp \Gamma.$$  

Proof. The centralizer $Z_\mathfrak{f}(\alpha)$ in $\mathfrak{f}$ of $\alpha$ has the decomposition

$$Z_\mathfrak{f}(\alpha) = Z_\mathfrak{f}(\alpha) \oplus \mathfrak{a},$$

where $Z_\mathfrak{f}(\alpha)$ is the centralizer in $\mathfrak{f}$ of $\alpha$. Since the centralizer $Z_K(A)$ in the compact connected Lie group $K$ of the torus $A$ is connected, we have the decomposition

$$Z_K(A) = Z_L(A)^\circ A.$$ 

It follows that any element $m \in Z_L(A)$ can be written as

$$m = m' a \quad \text{with} \quad m' \in Z_L(A)^\circ, \ a \in A.$$ 

Then $a = m'^{-1} m \in L$ so that $a \in \exp \Gamma$. Thus $m \in Z_L(A)^\circ \exp \Gamma$, which proves Lemma 4. q.e.d.

Lemma 5. Let $K^\mathbb{C}$ denote the Chevalley complexification of $K$. Put

$$K^* = L \exp \sqrt{-1} \mathfrak{s}$$

and

$$(K^*)^\circ = L^\circ \exp \sqrt{-1} \mathfrak{s},$$

where $L^\circ$ denotes the connected component of $L$. Then $(K^*)^\circ$ is a closed subgroup of
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\( K^c \) normalized by \( K^* \) and

\[
K^* = (K^*)^\circ \exp \Gamma.
\]

Therefore \( K^* \) is a closed subgroup of \( K^c \) with the connected component \( (K^*)^\circ \).

Proof. The first statement is clear. Take any element \( l \in L \). From the conjugateness of maximal abelian subalgebras in \( \mathfrak{g} \) under the adjoint action of \( L^\circ \), there exists \( l_i \in L^\circ \) such that \( l_i l \in N_L(A) \). Since

\[
N_L(A)/Z_L(A) = N_{L^\circ}(A)/Z_{L^\circ}(A) = W_S,
\]

we can choose \( l_2 \in L^\circ \) such that \( l_2 l_i l \in Z_L(A) \). It follows from Lemma 4 that there exist \( l_2, l_i \in Z_L(A)^\circ \) and \( a \in \exp \Gamma \) such that \( l_i l_i l_i = l_2 a \). Therefore \( l = l_2^{-1} l_i l_i \) with \( l_2^{-1} l_i l_i \in L^\circ \subset (K^*)^\circ \), i.e. \( l \in (K^*)^\circ \exp \Gamma \). This completes the proof of Lemma 5.

q.e.d.

Now we can prove the following

**Theorem 2.4.** (E. Cartan [1], Sugiura [12], Helgason [5]) For any \( \lambda \in \Delta \), there exists an irreducible spherical representation \( \rho \) of \( K \) relative to \( L \) such that the highest weight of \( \rho \) on \( t^c \) is \( \lambda \).

Together with Theorem 2.3 we have the following

**Corollary.** For \( \rho \in \mathcal{D}(K, L) \), let \( \lambda(\rho) \) denote the highest weight of \( \rho \) on \( t^c \). Then the correspondence \( \rho \mapsto \lambda(\rho) \) gives a bijection:

\[
\mathcal{D}(K, L) \rightarrow D(K, L).
\]

Proof of Theorem 2.4. This theorem for the case where \( K \) is semi-simple and \( L = K^\circ \) was stated in E. Cartan [1] but its proof is not complete. It was stated for simply connected \( K \) without proof in Sugiura [12]. It was proved in Helgason [5] for the case where \( K \) is semi-simple and \( L \) is connected. Helgason's proof can be applied for our case without difficulties, so we shall confine ourselves to point out necessary modifications.

Let

\[
\rho: K \rightarrow GL(V)
\]

be the irreducible representation of \( K \) with the highest weight \( \lambda \) (Lemma 3). By extending \( \rho \) to the Chevalley complexification \( K^c \) of \( K \) and restricting it to the closed subgroup \( K^* \) of \( K^c \) (Lemma 5), we have an irreducible representation of \( K^* \), which will be denoted by the same letter \( \rho \). It suffices to show that \( \rho \) has a non-zero \( L \)-invariant. Let \( N \) be the connected subgroup of \( K^* \) generated by the subalgebra

\[
n = \mathfrak{l}^* \cap \sum_{\alpha \in \Sigma^+_L} \mathfrak{h}^\alpha,
\]
where $\mathfrak{t}^*$ is the Lie algebra of $K^*$ and $\mathfrak{t}^*_\alpha$ is the root space of $\mathfrak{t}^*$ for $\alpha$. We shall first prove that the representation $\rho$ of $K^*$ is a conical representation of $K^*$ in the sense of Helgason [5], i.e. if $v_\lambda \in V$, $v_\lambda \neq 0$, is a highest weight vector for $\rho$ with respect to $\mathfrak{t}^*$, we have

$$\rho(mn)v_\lambda = v_\lambda \quad \text{for any} \quad m \in Z_L(A), \, n \in N.$$ 

Denoting the infinitesimal action of $\mathfrak{t}^*$ on $V$ by the same letter $\rho$, we have

$$\rho(n)v_\lambda = \rho(\delta_1(a))v_\lambda = \{0\}.$$ 

In fact, $\rho(n)v_\lambda = \{0\}$ since $n \in \sum t$ of the complexification $b^c$ of $b$ since $(\sqrt{-1} b, \lambda) = \{0\}$. $\rho(\mathfrak{t}^c)v_\lambda = \{0\}$ for $\alpha \in \sum^0_\Gamma$, $\alpha > 0$. It follows from $(\alpha, \lambda) \in (\sqrt{-1} b, \lambda) = \{0\}$ for $\alpha \in \sum^0_\Gamma$ that $\lambda - \alpha$ is not a weight of $\rho$ for $\alpha \in \sum^0_\Gamma$, $\alpha > 0$. Since the complexification of $\delta_1(a)$ is spanned by $b^c$ and the $\mathfrak{t}^c$'s for $\alpha \in \sum^0_\Gamma$, we have $\rho(\delta_1(a))v_\lambda = \{0\}$. Therefore it suffices from Lemma 4 to show that

$$\rho(\exp H)v_\lambda = v_\lambda \quad \text{for any} \quad H \in \Gamma.$$ 

But it is clear since $\lambda \in \mathbb{Z}$, i.e. $(\lambda, H) = 2\pi \sqrt{-1} \mathbb{Z}$ for any $H \in \Gamma$.

Thus we can prove in the same way as Helgason [5] that $V$ has a non-zero $L$-invariant vector, by constructing a $K^*$-submodule $V'$ of the $K^*$-module $C^\infty(K^*)$ of $C^\infty$-functions on $K^*$, having a non-zero $L$-invariant, and by constructing a $K^*$-equivariant isomorphism of $V$ onto $V'$.

Next we shall describe zonal spherical functions in terms of the basis $\{\chi_\lambda; \lambda \in \mathcal{D}\}$ of $\mathcal{D}_W(\hat{A})^c$.

For $\hat{a} = (\exp H)\chi_\lambda \in \hat{A}$, $H \in \mathfrak{a}$, we put

$$D(\hat{a}) = \left| \prod_{\alpha \in \sum^+_\Gamma - \sum^0_\Gamma} 2 \sin(\alpha, \sqrt{-1} H) \right|.$$ 

Let $d\hat{a}$ denote the normalized Haar measure of $\hat{A}$ and $|W_S|$ the order of the Weyl group $W_S$. For $W_S$-invariant functions $\chi$, $\chi'$ on $\hat{A}$, we define

$$\langle \chi, \chi' \rangle = \frac{c}{|W_S|} \int_{\hat{A}} \chi(\hat{a})\overline{\chi'}(\hat{a})D(\hat{a})d\hat{a},$$

where

$$c = \left( \frac{1}{|W_S|} \int_{\hat{A}} D(\hat{a})d\hat{a} \right)^{-1}.$$ 

c=1 in the case where $S$ is a compact connected Lie group. In particular, if $\chi$ and $\chi'$ can be extended to $L$-invariant functions $f$ and $f'$ on $S$, then $\langle \chi, \chi' \rangle$ coincides with the inner product $\langle f, f' \rangle$ in $L^2(S)$ (cf. Helgason [4]).

Fix a dominant integral form $\lambda \in \mathcal{D}$. We define a finite subset $D_\lambda$ of $D$ by
$D_\lambda = \{\mu \in D; \mu_c = \lambda_c, \mu \leq \lambda\}$.

Since the system $\{X_\mu; \mu \in D\}$ forms a basis of $R_{\mathbb{R}}(\hat{A})^c$, the matrix

$$(\langle X_\mu, X_\nu \rangle)_{\mu, \nu \in D_\lambda}$$

is a positive definite hermitian matrix. Let

$$(b^{\mu\nu})_{\mu, \nu \in D_\lambda}$$

be the inverse matrix of the above matrix. In particular $b^{\lambda\lambda} > 0$. For any $\mu \in D_\lambda$, we put

$$c_\lambda^\mu = \frac{b^{\lambda\mu}}{\sqrt{d_\lambda b^{\lambda\lambda}}},$$

where $d_\lambda$ is the degree of an irreducible representation of $K$ with the highest weight $\lambda$. Then we have

**Theorem 2.5.** Let $\lambda \in D$ and $\omega_\lambda$ be the zonal spherical function belonging to the class of an irreducible representation of $K$ with the highest weight $\lambda$. Then $\omega_\lambda$ restricted to $\hat{A}$ is given by

$$\omega_\lambda = \sum_{\mu \in D_\lambda} c_\lambda^\mu X_\mu.$$

Proof. The idea of the following proof owes to Hua [6]. Let $\mu \in D_\lambda$. Then $\omega_\mu$ restricted to $\hat{A}$ is in $R_{\mathbb{R}}(\hat{A})^c$ by Theorem 2.3. It follows by Theorem 2.3 and Corollary of Theorem 2.4 that $\omega_\mu$ has an expression

$$\omega_\mu = \sum_{\nu \in D_\lambda} c_\mu^\nu X_\nu,$$

with $c_\mu^\nu \in \mathbb{R}^c$, $c_\mu^\nu > 0$, $c_\mu^\nu = 0$ if $\nu > \mu$.

We define an upper triangular matrix $C'$ by

$$C' = (c_\mu^\nu)_{\mu, \nu \in D_\lambda}.$$

Then we have

$$(\langle \omega_\mu, \omega_\nu \rangle)_{\mu, \nu \in D_\lambda} = 'C'(\langle X_\mu, X_\nu \rangle)_{\mu, \nu \in D_\lambda} C'.$$

Since $\langle \omega_\mu, \omega_\nu \rangle = d_\mu^{-1} \delta_{\mu\nu}$, we have

$$(d_\mu \delta_{\mu\nu})_{\mu, \nu \in D_\lambda} = C'^{-1} B'^{-1} C'^{-1},$$

where

$$B' = (b^{\mu\nu})_{\mu, \nu \in D_\lambda} = (\langle X_\mu, X_\nu \rangle)_{\mu, \nu \in D_\lambda}^{-1}.$$

It follows that

$$C'(d_\mu \delta_{\mu\nu})_{\mu, \nu \in D_\lambda} 'C' = B'.$$

Comparing $(\mu, \lambda)$-components of both sides, we have
In particular
\[(c^\mu_\lambda)^2 d_\lambda = b^{\lambda\lambda}, \text{ i.e. } c^\mu_\lambda = \sqrt{\frac{b^{\lambda\lambda}}{d_\lambda}},\]
hence
\[c^\mu_\lambda = \frac{b^{\lambda\lambda}}{d_\lambda c^\lambda_\lambda} = \frac{b^{\lambda\lambda}}{\sqrt{d_\lambda} b^{\lambda\lambda}}.\]

Since \(b^{\mu\nu} = b^{\nu\mu}\), we have
\[c^\mu_\lambda = \frac{b^{\lambda\mu}}{\sqrt{d_\lambda} b^{\lambda\lambda}} = c^\mu_\lambda.\]

**Q.E.D.**

**Example.** If \(S\) is a compact connected Lie group and \((K, L)\) the symmetric pair with \(K/L=S\) as mentioned before, then the set \(D(S)\) of equivalence classes of irreducible representations of \(S\) is in the bijective correspondence with \(D(K, L)\) by the assignment \(\rho \mapsto \rho \otimes \rho^*\), where \(\rho^*\) denotes the contragredient representation of \(\rho\). \(\hat{A}\) is a maximal torus of the compact Lie group \(S\). Let \(\chi_\rho\) be the invariant character of \(\hat{A}\) for the dominant integral form in \(D(K, L)\) corresponding to \(\rho \otimes \rho^*\) by the bijection in Corollary of Theorem 2.4. Then it is nothing but the character of \(\rho\). It follows from orthogonality relations of irreducible characters that the matrix \((b^{\lambda\mu})\) is the identity matrix. Thus the zonal spherical function \(\omega_{\rho \otimes \rho^*}\) belonging to \(\rho \otimes \rho^*\) is given by
\[\omega_{\rho \otimes \rho^*} = \frac{1}{d_\rho} \chi_\rho,\]
where \(d_\rho\) is the degree of \(\rho\).

3. Polynomial representations associated with symmetric bounded domains

Let \(D\) be an irreducible symmetric bounded domain with rank \(p\) realized in \((\mathfrak{g}^c)^-\) as in Introduction. We shall use the same notation as in Introduction.

Let
\[\Pi = \{\alpha_1, \ldots, \alpha_l\}\]
be the fundamental root system of \(\Sigma\) with respect to the order \(>\) and let
\[\Pi_\perp = \Pi \cap \Sigma_\perp.\]
It is known that \(\Pi_\perp\) is the fundamental root system of \(\Sigma_\perp\), \(\Pi - \Pi_\perp\) consists of one element, say \(\alpha_1\), which is the lowest root in \(\Sigma_\perp^+\), and for any \(\alpha = \sum_{i \in \Pi_\perp} m_i \alpha_i \in \Sigma_\perp^+\), \(m_i = 1\). Let \(\Sigma_\perp^+\) denote the set of positive compact roots.

Put
\[b = \{H \in \mathfrak{a}; (\sqrt{-1} H, \Delta) = \{0\}\}.\]
Then we have the orthogonal decomposition

$$\sqrt{-1}t = \sqrt{-1}b \oplus \sqrt{-1}a$$

with respect to $\langle , \rangle$. We define an orthogonal transformation $\sigma$ on $\sqrt{-1}t$ by $\sigma|b = -1$ and $\sigma|\sqrt{-1}a = 1$. Let

$$\sigma = \frac{1}{2}(1+\sigma) : \sqrt{-1}t \to \sqrt{-1}a$$

be the orthogonal projection of $\sqrt{-1}t$ onto $\sqrt{-1}a$. Let $\kappa$ be the unique involutive element of the Weyl group $W_t$ of $K$ such that $\kappa \Pi_t = -\Pi_t$. Since $\sum^*_p$ is the set of weights on $t^c$ of the irreducible $K$-module $(p^c)^*$, we have $\kappa \sum^*_p = \sum^*_p$ and $\kappa \gamma_i = \gamma_i$. Put

$$\Delta' = \kappa \Delta = \{ \gamma_1', \ldots, \gamma_p' \}, \quad \gamma_i' = \kappa \gamma_i (1 \leq i \leq p), \quad \gamma_i' = \alpha_i.$$  

It is the original maximal strongly orthogonal subsystem of $\sum^*_p$ of Harish-Chandra [3]. For the system $\Delta'$, the orthogonal projection

$$\sigma' : \sqrt{-1}t \to \sqrt{-1}a'$$

onto the $R$-span $\sqrt{-1}a'$ of $\Delta'$ is defined in the same way as for $\Delta$. Put

$$P_i' = \{ \alpha \in \sum^*_p ; \quad \sigma'(\alpha) = \frac{1}{2} (\gamma_i' + \gamma_j') \text{ for some } 1 \leq i < j \leq p \},$$  

$$P_i' = \{ \alpha \in \sum^*_p ; \quad \sigma'(\alpha) = \frac{1}{2} \gamma_i' \text{ for some } 1 \leq i \leq p \},$$  

$$K_i' = \{ \alpha \in \sum^*_p ; \quad \sigma'(\alpha) = \frac{1}{2} \gamma_i' \text{ for some } 1 \leq i \leq p \}.$$  

Then (Harish-Chandra [3]) $\sum$ is the disjoint union of $P_i', -P_i', P_i', -P_i', K_i, K_i', -K_i'$ and we have

$$\sigma'P_i' = \left\{ \frac{1}{2} (\gamma_i' + \gamma_j') ; 1 \leq i \leq p \right\},$$  

$$\sigma'P_i' = \left\{ \frac{1}{2} \gamma_i' ; 1 \leq i \leq p \right\} \text{ if } P_i' = \phi,$$

$$\sigma'K_i' = \left\{ \pm \frac{1}{2} (\gamma_i' + \gamma_j') ; 1 \leq i < j \leq p \right\},$$  

$$\sigma'K_i' = \left\{ \frac{1}{2} \gamma_i' ; 1 \leq i \leq p \right\} \text{ if } P_i' = \phi.$$  

Furthermore the multiplicity (with respect to $\sigma'$) of any $\gamma_i'$ is 1 and that of any $\frac{1}{2} \gamma_i'$ is even. It follows that

$$\sigma' \sum - \{0\} = \left\{ \pm \frac{1}{2} (\gamma_i' + \gamma_j') ; 1 \leq i < j \leq p, \pm \gamma_i ; 1 \leq i \leq p \right\} \text{ if } P_i' = \phi$$  

$$\{ \pm \frac{1}{2} (\gamma_i' + \gamma_j') ; 1 \leq i < j \leq p, \pm \gamma_i' \text{ or } \pm \frac{1}{2} \gamma_i' ; 1 \leq i \leq p \right\} \text{ if } P_i' = \phi.$$  

Moreover we have (Moore [8])

$$\sigma' \Pi - \{0\} = \left\{ \gamma_1, \frac{1}{2} (\gamma_1' - \gamma_1), \ldots, \frac{1}{2} (\gamma_p' - \gamma_p'), \gamma_1', \frac{1}{2} (\gamma_1' - \gamma_1), \ldots, \frac{1}{2} (\gamma_p' - \gamma_p') \right\} \text{ if } P_i' = \phi.$$

$$\{ \gamma_1, \frac{1}{2} (\gamma_1' - \gamma_1), \ldots, \frac{1}{2} (\gamma_p' - \gamma_p'), -\frac{1}{2} \gamma_p' \} \text{ if } P_i' = \phi.$$
and
\[ \varpi' \Pi - \{0\} = \begin{cases} \{ \frac{1}{2}(\gamma'_s - \gamma'_t), \ldots, \frac{1}{2}(\gamma'_p - \gamma'_{p-1}) \} & \text{if } P'_1 = \phi \\ \{ \frac{1}{2}(\gamma'_s - \gamma'_t), \ldots, \frac{1}{2}(\gamma'_p - \gamma'_{p-1}), -\frac{1}{2} \gamma'_p \} & \text{if } P'_1 \neq \phi. \end{cases} \]

**Lemma 1.**

1. \[ \varpi \alpha_i = \begin{cases} \gamma'_p & \text{if } P'_1 = \phi \\ \frac{1}{2} \gamma'_p & \text{if } P'_1 \neq \phi. \end{cases} \]

2. (Schmid [11]) If \( P'_1 \neq \phi \) and
\[ \sum_{\beta \in P'_1} m_{\beta} \beta \]
is in the \( R \)-span \( \{ P'_1 \}_R \) of \( P'_1 \), then \( m_{\beta} = 0 \) for any \( \beta \).

**Proof.** For any \( \alpha \in \sum_{\gamma_i}^+ = P'_1 \cup P'_1 \), \( \varpi' \alpha \) can be written as
\[ \varpi' \alpha = \frac{1}{3} m_1 (\gamma'_s - \gamma'_t) + \frac{1}{3} m_2 (\gamma'_s - \gamma'_t) + \cdots + \frac{1}{3} m_{p-1} (\gamma'_p - \gamma'_{p-1}) \\
- \frac{1}{3} m_p \gamma'_p + m_{p+1} \gamma'_t \\
= \frac{1}{2} (2m_{p+1} - m_1) \gamma'_1 + \frac{1}{2} (m_1 - m_2) \gamma'_2 + \cdots + \frac{1}{2} (m_{p-1} - m_p) \gamma'_{p-1} \\
+ \frac{1}{3} (m_{p-1} - m_p) \gamma'_p \]
where \( m_i \in \mathbb{Z} \), \( m_i \geq 0 \), \( m_{p+1} = 1 \). Since \( \varpi' \alpha = \frac{1}{2} (\gamma'_i + \gamma'_j) \) or \( \frac{1}{2} \gamma'_i \) for some \( i, j \), we have
\[ 2 \geq m_1 \geq m_2 \geq \cdots \geq m_{p-1} \geq m_p \geq 0. \]
Furthermore \( \alpha \in P'_1 \) (resp. \( \alpha \in P'_1 \)) if and only if \( m_p = 0 \) (resp. \( m_p = 1 \)).

1) If \( P'_1 = \phi \), then \( \gamma'_i \in P'_1 \). For \( \alpha = \gamma_i \), the coefficients in the above expression are \( m_i = \cdots = m_{p-1} = 2 \), \( m_p = 0 \) and \( \varpi' \gamma_i = \gamma'_p \). If \( P'_1 = \phi \), then for \( \alpha = \gamma_i \), the coefficients are \( m_i = \cdots = m_{p-1} = 2 \), \( m_p = 1 \) and \( \varpi' \gamma_i = \frac{1}{2} \gamma'_p \). Now the assertion 1) follows from \( \varpi' \alpha = \kappa^{-1} \varpi' \kappa \alpha = \kappa^{-1} \varpi' \alpha \).

2) Let
\[ \alpha = \sum_{i=1}^t n_i \alpha_i \]
be in \( \sum_{\gamma_i}^+ \). It follows from the first argument that

(a) if \( \alpha \in P'_1 \), \( \varpi' \alpha_i = -\frac{1}{2} \gamma'_p \), then \( n_i = 0 \),
(b) if \( \alpha \in P'_1 \), then there exists \( \alpha_i \in \Pi_i \) such that \( n_i > 0 \) and \( \varpi' \alpha_i = -\frac{1}{2} \gamma'_p \).

This implies the assertion 2).

q.e.d.
Now $P_1, P_i, K_o$ and $K_i$ are defined for $\Delta$ in the same way as for $\Delta'$. Then $\kappa$ transforms $P_1$ (resp. $P_i, K_o, K_i$) onto $P_i'$ (resp. $P_i', K_o', K_i'$). It follows that the above mentioned properties due to Harish-Chandra are also satisfied by our objects for $\Delta$. But Moore's results should be modified as follows.

$$\varpi \Pi - \{0\} = \begin{cases} \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_p - \gamma_p), \gamma_p \} & \text{if } P_1 = \emptyset \\ \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_p - \gamma_p), \frac{1}{2} \gamma_p \} & \text{if } P_1 \neq \emptyset. \end{cases}$$

$$\varpi \Pi_1 - \{0\} = \begin{cases} \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_p - \gamma_p) \} & \text{if } P_i = \emptyset \\ \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_p - \gamma_p), \frac{1}{2} \gamma_p \} & \text{if } P_i \neq \emptyset. \end{cases}$$

They follows from Lemma 1, 1) and

$$\varpi \Pi_1 = \kappa^{-1} \varpi \kappa \Pi_1 = -\kappa^{-1} \varpi \kappa \Pi_1.$$

Note that $K_1 \subset \Sigma^+_t$ while $K_i' \subset -\Sigma^+_t$.

**Lemma 2.** 1) The order $> \text{ is a compatible order for } \Sigma \text{ with respect to } \sigma$ in the sense of 2.

2) $\varpi K_o - \{0\}$ is a root system with the fundamental root system

$$\{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_p - \gamma_p) \}$$

with respect to the order $>$. 3) If $P_i \neq \emptyset$ and

$$\sum_{\beta \in P_i} m_\beta \beta \text{ with } m_\beta \geq 0$$

is in the $R$-span $\{P_i\}_R$ of $P_i$, then $m_\beta = 0$ for any $\beta$.

Proof. 1) is clear from the form of $\varpi \Pi - \{0\}$ above.

2) is clear since

$$\varpi K_o - \{0\} = \{ \pm \frac{1}{2} (\gamma_i - \gamma_j); 1 \leq i < j \leq p \}.$$

3) follows from Lemma 1, 2) and $\kappa P_i = P_i', \kappa P_i = P_i'$. q.e.d.

For $\lambda \in \sqrt{-1}t$, $\lambda \neq 0$, we define as in 2

$$\lambda^* = \frac{2}{(\lambda, \lambda)} \lambda$$
and put
\[ Z_0 = \frac{1}{2} \sum_{\alpha \in \Delta} \gamma^* . \]

Since \((\frac{1}{2} \gamma^i, \gamma_j^*) = \delta_{ij}\) for \(1 \leq i, j \leq p\), we have
\[
\begin{align*}
P_+ &= \{ \alpha \in \sum_p ; (\alpha, Z_0) = 1 \}, \\
P_- &= \{ \alpha \in \sum_p ; (\alpha, Z_0) = \frac{1}{2} \}, \\
K_0 &= \{ \alpha \in \sum_p^r ; (\alpha, Z_0) = 0 \}, \\
K_i &= \{ \alpha \in \sum_p^r ; (\alpha, Z_0) = \frac{1}{2} \}.
\end{align*}
\]

Hence eigenvalues of \(\text{ad} Z_0\) are \(\pm 1, \pm \frac{1}{2}\) on \(p^c\), 0, \(\pm \frac{1}{2}\) on \(p_c\). Let \(p^c_{\pm, 1}, p^c_{\pm, 0}, p^c_0\), \(p^c_{\pm, \frac{1}{2}}\) denote the corresponding eigenspaces. Note that the origin \(X_0\) of the Shilov boundary \(S\) is in \(p^c_{\pm, \frac{1}{2}}\).

The following results are due to Korányi-Wolf [7]. We define an element \(c\) of \(G^c\), which is called Cayley transform, by
\[ c = \exp \left( -\frac{\pi}{4} \sum_{\gamma \in \Delta} (X_\gamma + X_{-\gamma}) \right) \]
and define an automorphism of \(G^c\) by
\[ \theta(x) = c^2 x c^{-2} \quad \text{for} \quad x \in G^c. \]

The automorphism \(\text{Ad} c^2\) of \(g^c\) obtained by differentiating \(\theta\) will be also denoted by the same letter \(\theta\). Then \(\theta^* = 1\) and on \(\sqrt{-1} t\) it coincides with \(-\sigma\). Put
\[ g_0 = \{ X \in g ; \theta^* X = X \}, \]
\[ k_0 = g_0 \cap k, \]
and
\[ p_0 = g_0 \cap p. \]

Then \(k_0\) is \(\theta\)-invariant and
\[ k_0 = \{ X \in k ; [Z_0, X] = 0 \}. \]

Hence \(k_0\) is a real form of \(k^c_0\) containing \(t\) as a maximal abelian subalgebra. \(K_0\) is nothing but the set of roots of \(k^c_0\) with respect to \(k^c\). The complexification \(p^c_0\) of \(p_0\) is the direct sum of \(p^c_{\pm, 1}\) and \(p^c_{\pm, \frac{1}{2}}\). \(g_0\) is a reductive subalgebra of \(g\) with a Cartan decomposition
\[ g_0 = k_0 + p_0. \]
Let $G_0$ (resp. $K_0$) be the connected subgroup of $G$ generated by $g_0$ (resp. by $I_0$) and let

$$L_0 = \{ k \in K_0; \text{Ad} X_0 = X_0 \} = K_0 \cap L.$$ 

Put

$$D_0 = D \cap \mathfrak{p} \mathfrak{c}_1$$

and

$$S_0 = S \cap \mathfrak{p} \mathfrak{c}_1.$$ 

Then $G_0$ acts on $D_0$ transitively and $K_0 \cap G_0$ coincides with $K_0$, so that $D_0$ is identified with the quotient space $G_0/K_0$. Furthermore $K_0$ acts on $S_0$ transitively so that $S_0$ is identified with $K_0/L_0$. $D_0$ is totally geodesic in $D$ with respect to Bergmann metric of $D$ and it is also an irreducible symmetric bounded domain with the same rank as $D$. $S_0$ is the Shilov boundary of $D_0$. The complex structure of $D_0$ is given at the origin by $\text{ad} H_0$ with $\sqrt{-1} H_0 = Z_0$. We have

$$\mathfrak{p} \mathfrak{c}_1 = Z.$$ 

The inclusion $D_0 \subset \mathfrak{p} \mathfrak{c}_1$ is nothing but the Harish-Chandra's imbedding of $D_0 = G_0/K_0$. $(K_0, L_0)$ is a symmetric pair with respect to $\theta$, having the same rank as $D$. Hence

$$I_0 = \{ X \in I_0; \theta X = X \}$$

is the Lie algebra of $L_0$ and $\alpha$ is a maximal abelian subalgebra of

$$\mathfrak{g}_0 = \{ X \in I_0; \theta X = -X \}.$$ 

We can define a semi-linear transformation $X \mapsto \overline{X}$ of $\mathfrak{p} \mathfrak{c}_1$ by

$$\overline{X} = \tau \theta X = \theta \tau X \quad \text{for} \quad X \in \mathfrak{p} \mathfrak{c}_1.$$ 

Put

$$\mathfrak{p}_{-1} = \{ X \in \mathfrak{p} \mathfrak{c}_1; \overline{X} = X \}.$$ 

It is a real form of $\mathfrak{p} \mathfrak{c}_1$ and is invariant under the adjoint action of $L_0$ on $\mathfrak{p} \mathfrak{c}_1$. The correspondence $X \mapsto [X, X]$ gives an isomorphism

$$\psi: \sqrt{-1} \mathfrak{g}_0 \rightarrow \mathfrak{p}_{-1},$$

which is equivariant with respect to the adjoint actions of $L_0$.

Now we shall consider the polynomial representation $S^*((\mathfrak{p} \mathfrak{c})^-)$ of $K$. Let $S_*((\mathfrak{p} \mathfrak{c})^+)$ be the symmetric algebra over $(\mathfrak{p} \mathfrak{c})^+$. $K$ acts on $S_*((\mathfrak{p} \mathfrak{c})^+)$ by the natural extension $\text{Ad}$ of the adjoint action of $K$ on $(\mathfrak{p} \mathfrak{c})^+$. On the other hand, the non-degenerate pairing

$$(\mathfrak{p} \mathfrak{c})^+ \times (\mathfrak{p} \mathfrak{c})^- \rightarrow \mathbb{C}$$
by means of the Killing form $(\ ,\ )$ induces the identification 
\[ S_\ast((\mathfrak{p}^C_\ast)^+) = S_\ast((\mathfrak{p}^C_\ast)^-) . \]

This identification is compatible with the actions of $K$, since the Killing form is invariant under the adjoint action of $K$. In the same way we have a $K_0$-equivariant identification 
\[ S_\ast(\mathfrak{p}^C_\ast_0) = S_\ast(\mathfrak{p}^C_\ast) . \]

$S_\ast(\mathfrak{p}^C_\ast_0)$ can be considered as a $K_0$-submodule of $S_\ast((\mathfrak{p}^C_\ast)^+)$ by means of the natural monomorphism $S_\ast(\mathfrak{p}^C_\ast_0) \rightarrow S_\ast((\mathfrak{p}^C_\ast)^+)$ induced from the inclusion $\mathfrak{p}^C_\ast_0 \subset (\mathfrak{p}^C_\ast)^+$.

**Theorem 3.1.**

(i) Any irreducible $K$-submodule of $S_\ast((\mathfrak{p}^C_\ast)^+)$ (resp. $K_0$-submodule of $S_\ast(\mathfrak{p}^C_\ast_0)$) is contained exactly once in $S_\ast((\mathfrak{p}^C_\ast)^+)$ (resp. in $S_\ast(\mathfrak{p}^C_\ast_0)$).

(ii) For an irreducible $K$-submodule $V$ of $S_\ast((\mathfrak{p}^C_\ast)^+)$, we put 
\[ V_0 = V \cap S_\ast(\mathfrak{p}^C_\ast_0) . \]

Then $V \mapsto V_0$ is the one to one correspondence between the set of irreducible $K$-submodules of $S_\ast((\mathfrak{p}^C_\ast)^+)$ and the set of irreducible $K_0$-submodules of $S_\ast(\mathfrak{p}^C_\ast_0)$ in such a way that

1) The highest weights on $\mathfrak{t}^C$ of $V$ and $V_0$ are the same.

2) The subspace of $L$-invariants in $V$ is 1-dimensional and contained in $V_0$.

(iii) The highest weight $\lambda \in \mathfrak{t}^C$ of an irreducible $K$-submodule $V$ of $S_\ast((\mathfrak{p}^C_\ast)^+)$ is of the form 
\[ \lambda = \sum_{i=1}^{r} n_i \gamma_i , \quad n_i \in \mathbb{Z}, \; n_i \geq n_i \geq \cdots \geq n_r \geq 0 . \]

If $\sum n_i = v$, then $V$ is contained in $S_\ast((\mathfrak{p}^C_\ast)^+)$. i.e. $S^\ast(D) \subset S^\ast(K, L)$ under the notation in Introduction.

For the proof of the theorem, we need the following

**Lemma 3.** (Murakami [9]) Let $\mathfrak{k}$ be a Lie algebra over $\mathbb{R}$ and $\mathfrak{k}^C$ the complexification of $\mathfrak{k}$. Assume that there exists $Y \in \sqrt{-1} \mathfrak{k} \subset \mathfrak{k}^C$ such that $\mathfrak{k}^C$ is the direct sum of 0-eigenspace $\mathfrak{k}_0^C$, (+1)-eigenspace $\mathfrak{k}_1^C$ and (-1)-eigenspace $\mathfrak{k}_2^C$ of ad $Y$, respectively. Let $(\rho, V)$ be a complex irreducible $\mathfrak{k}$-module with $\mathfrak{k}$-invariant hermitian inner product. Denoting the extension to $\mathfrak{k}^C$ of $\rho$ by the same letter $\rho$, let $a_1 > a_2 > \cdots > a_m (a_i \in \mathbb{R})$ be eigenvalues of $\rho(Y)$, and $S_t$ be $a_t$-eigenspace of $\rho(Y)$ $(1 \leq t \leq m)$. Put $\mathfrak{k}_0 = \mathfrak{k}_0^C \cap \mathfrak{k}$ (which is a real form of $\mathfrak{k}_0^C$). Then

1) $a_t = a_{t-1} - t + 1 (1 \leq t \leq m)$.

2) Each $S_t$ is a $\mathfrak{k}_t$-submodule of $V$ and 
\[ V = S_1 + \cdots + S_m \]

is the orthogonal direct sum.
3) $S_i$ and $S_m$ are irreducible $\mathfrak{g}_c$-submodules of $V$ and characterized by

$$S_i = \{ v \in V; \rho(X)v = 0 \text{ for any } X \in \mathfrak{g}_c \},$$

$$S_m = \{ v \in V; \rho(X)v = 0 \text{ for any } X \in \mathfrak{g}_c \}.$$

Proof of Theorem 3.1. The infinitesimal action of $\mathfrak{g}_c$ on $S_+((\mathfrak{g}^c)^\ast)$ induced from the adjoint action $\text{Ad}$ of $K$ will be denoted by $\text{ad}$.

Let $V$ be an irreducible $K$-submodule of $S_+((\mathfrak{g}^c)^\ast)$. Since $Z$ is in the center of $\mathfrak{g}_c$, it follows from Schur’s lemma that $V$ is contained in an eigenspace of $\text{ad} \ Z$ in $S_+((\mathfrak{g}^c)^\ast)$. But since $\text{ad} \ Z$ is the scalar operator $\nu$ on $S_+((\mathfrak{g}^c)^\ast)$, $V$ is contained in $S_+((\mathfrak{g}^c)^\ast)$ for some $\nu$. Let $\lambda \in \sqrt{-1}t$ be the highest weight of $V$. Put $Y = 2Z_{\nu} \in \sqrt{-1}t \subset \mathfrak{g}_c$. Then the decomposition

$$\mathfrak{g}_c = \mathfrak{g}_c^0 + \mathfrak{g}_c^i + \mathfrak{g}_c^j$$

satisfies the assumption in Lemma 3. So we have a decomposition

$$V = S_1 + \cdots + S_m$$

into $K$-submodules, where $S_i$ is an irreducible $K$-submodule and is the eigenspace for the maximum eigenvalue of $\text{ad} \ Y$ in $V$. It is characterized by

$$S_i = \{ v \in V; \text{ad} \ (X)v = 0 \text{ for any } X \in \mathfrak{g}_c \}.$$

Thus a highest weight vector $v_\lambda$ of the $K$-module $V$ is contained in $S_i$ because of $\mathfrak{g}_i \subset \mathfrak{g}_c^j$. It follows that putting $V_0 = S_i$, $V_0$ is an irreducible $K$-submodule of $S_+((\mathfrak{g}^c)^\ast)$ with the highest weight $\lambda$.

We shall show that $V_0 = V \cap S_+((\mathfrak{g}^c)^\ast)$. We have the decomposition

$$S_+((\mathfrak{g}^c)^\ast) = \sum_{r,s} S_r(\mathfrak{g}_r^c) \otimes S_s(\mathfrak{g}_s^c)$$

as $K$-modules. $\text{ad} \ Z_{\nu}$ is the scalar operator $r + \frac{1}{2}s = \frac{1}{2}(r + \nu)$ on $S_r(\mathfrak{g}_r^c) \otimes S_s(\mathfrak{g}_s^c)$. In the same way as the first argument, we can get the decomposition

$$V = V_1 + \cdots + V_h$$

into irreducible $K$-submodules such that any $V_i$ is contained in $S_r(\mathfrak{g}_r^c) \otimes S_s(\mathfrak{g}_s^c)$ for some $(r,s)$. Since $\mathcal{S}^*((\mathfrak{g}^c)^\ast)$ is $K$-isomorphic with $\mathcal{S}^*(\mathfrak{g}) \subset C^*(\mathfrak{g})$, $V$ has an $L$-invariant $w \neq 0$. Decompose $w$ as

$$w = w_1 + \cdots + w_h, \quad w_i \in V_i \ (1 \leq i \leq k).$$

At least one of the $w_i$'s, say $w_1$, is not zero. Let $\lambda_1 \in \sqrt{-1}t$ be the highest weight of the irreducible $K$-module $V_1$. Since $w_1$ is a non-zero $L_0$-invariant of $V_1$, $V_1$ is a spherical $K$-module relative to $L_0$. $(K_0, L_0)$ is a symmetric pair, $\alpha$ is a maximal abelian subalgebra of $\mathfrak{g}_0$ and the order $\prec$ on $\sqrt{-1}t$ is a compatible order for $K_0$ with respect to $\sigma$ by Lemma 1, 1), so we shall use the notations
\[ \Gamma(K_0, L_0), Z(K_0, L_0), D(K_0, L_0) \text{ in 2. Then it follows from Theorem 2.3 that} \]
\[ \lambda_1 \in D(K_0, L_0). \]
On the other hand, if \( V_1 \subset S_\nu(p^c) \otimes S_\nu(p^c) \), \( \lambda_1 \) is of the form
\[ \lambda_1 = \sum_{\alpha \in \mathbb{P}_1} m_\alpha \alpha + \sum_{\beta \in \mathbb{P}_2} m_\beta \beta, \quad m_\alpha, m_\beta \in \mathbb{Z}, \ m_\alpha \geq 0, \ m_\beta \geq 0 \]
with \( \sum m_\alpha = r, \sum m_\beta = s \). Since \( D(K_0, L_0) \subset \sqrt{-1} \alpha = \{ \Delta \}_R \subset \{ \mathbb{P}_1 \}_R \), we have
\[ \sum_{\beta \in \mathbb{P}_2} m_\beta \beta \in \{ \mathbb{P}_1 \}_R. \]
It follows from Lemma 2.3) that \( r = \nu, s = 0 \), i.e. \( V_1 \subset V \cap S_\nu(p^c) \). On the other hand, \( V \cap S_\nu(p^c) \subset V \) since the possible maximum eigenvalue of \( \text{ad} Y \) on \( V \) is \( 2\nu \). Thus we have that \( V = V_1 = V \cap S_\nu(p^c) \).

The above argument shows also that any \( L \)-invariant in \( V \) is contained in \( V_0 \). It is unique up to scalar since \((K_0, L_0)\) is a symmetric pair.

Conversely, let \( V_0 \) be an irreducible \( K_0 \)-submodule of \( S_\nu(p^c) \) with the highest weight \( \lambda \in \sqrt{-1} \mathbb{t} \). In the same way as the first argument, we know that \( V_0 \) is contained in \( S_\nu(p^c) \) for some \( \nu \). Let \( v_\lambda \in V_0 \) be a highest weight vector. Then \( \text{ad} t_i v_\lambda = \{ 0 \} \) because of \( \{ t_i, p^c \} = \{ 0 \} \). Hence \( \text{ad} X_\alpha v_\lambda = 0 \) for any \( \alpha \in \sum_1^+ \). We define \( V \) to be the \( \mathcal{C} \)-span of \( \{ \text{Ad} k v_\lambda ; k \in K \} \) in \( S_\nu(p^c) \). Then \( V \) is an irreducible \( K \)-submodule of \( S_\nu(p^c) \) with the highest weight \( \lambda \in \sqrt{-1} \mathbb{t} \).

It is easy to see that each of the above correspondences \( V \mapsto V_0 \) and \( V_0 \mapsto V \) is the inverse of the other. This proves assertions (i) and (iii).

(iii) We have \( \{ \frac{1}{2} \gamma^*_i, X_{-\gamma_i} \} = -\delta_{ij} X_{-\gamma_j} \) \((1 \leq i, j \leq p)\) because of \( \{ \frac{1}{2} \gamma^*_i, \gamma_i \} = \delta_{ij} \) \((1 \leq i, j \leq p)\). It follows that for \( H = 2\pi \sqrt{-1} \sum_{i=1}^p x_i (\frac{1}{2} \gamma^*_i) \in \mathfrak{a} \) we have
\[ \text{Ad}(\exp H) X_\gamma = -\sum_{i=1}^p \exp(-2\pi \sqrt{-1} x_i) X_{-\gamma_i}. \]
Thus we have
\[ \Gamma(K_0, L_0) = 2\pi \sqrt{-1} \sum_{i=1}^p Z(\frac{1}{2} \gamma^*_i) \]
and
\[ Z(K_0, L_0) = \sum_{i=1}^p Z \gamma_i. \]
It follows from Lemma 2.2) that
\[ D(K_0, L_0) = \{ \sum_{i=1}^p n_i \gamma_i ; n_i \in \mathbb{Z}, n_1 \geq n_2 \geq \cdots \geq n_p \}. \]

Therefore \( \lambda \) is of the form
\[ \lambda = \sum_{i=1}^p n_i \gamma_i \quad \text{with} \quad n_i \in \mathbb{Z}, n_1 \geq \cdots \geq n_p. \]
On the other hand, \( \lambda \) is of the form
\( \lambda = \sum_{\alpha \in P_1} m_\alpha \alpha \) with \( m_\alpha \in \mathbb{Z}, \, m_\alpha \geq 0 \),

which implies that \( n_1 \geq \cdots \geq n_p \geq 0 \). If \( V \subset S_\iota((p^C)^{+}) \), then \( V_0 \subset S_\iota(p^C_i) \) and \( \text{ad} \, Z_0 \) is the scalar operator \( \nu \) on \( V_0 \), which equals \( (\lambda, \, Z_0) = \sum_{i=1}^{p} n_i \).

**Remark.** In terms of polynomial functions \( S^*((p^C)^-), \) for an irreducible \( K \)-submodule \( V \) of \( S^*((p^C)^-) \), \( V_0 \) is obtained by restriction to \( p^C_i \) of functions in \( V \).

**Proof of Theorem A.** Orthogonality relations for the \( S^*_\iota(D)'s \) (resp. for the \( S^*_\iota(S)'s \) and the assertion that the restriction \( S^*_\iota(D) \rightarrow S^*_\iota(S) \) is a similitude follow from Schur's lemma. So it suffices to show that the cardinalities of \( S^\circ(D) \) and \( S^\circ(K, \, L) \) are the same.

From the first argument in the proof of Theorem 3.1 (iii), we see that \( \gamma \in \Delta \) for the \( L_\iota \)-equivariant isomorphism \( \psi: \sqrt{-1} \mathfrak{g}_\iota \rightarrow p_{-1} \).

We put

\[ \alpha^- = \psi(\sqrt{-1} \alpha) = \{X_{-\gamma}; \, \gamma \in \Delta \}_{R \subset p_{-1}}. \]

Since the Weyl group \( W_0 \) of \( S_0 \) is isomorphic with the group of permutations of \( \Delta \) by Lemma 2,2), the "Weyl group" \( W_{L_0} = N_{L_0}(\alpha^-)/Z_{L_0}(\alpha^-) \), where \( N_{L_0}(\alpha^-) \) (resp. \( Z_{L_0}(\alpha^-) \)) is the normalizer (resp. centralizer) of \( \alpha^- \) in \( L_0 \), is isomorphic with the group of permutations of \( \{X_{-\gamma}; \, \gamma \in \Delta \} \). On the other hand, since \( S^*_\iota(p_\iota) \) is isomorphic with \( S^*_\iota(a) \) by Theorem 2.2, \( S^*_\iota(p_{-1}) \) is isomorphic with \( S^*_{W_{L_0}}(\alpha^-) \). Hence \( S^*_\iota(p_\iota) \) is isomorphic with \( S^*_\iota((\alpha^-)^c) \). It follows from Theorem 3.1, (ii), 2) that the cardinality of \( S^\circ(D) \) is equal to \( \dim S^\circ_{L_0}(p_\iota) = \dim S^\circ_{W_{L_0}}((\alpha^-)^c) \) - the number of linearly independent symmetric polynomials in \( p \)-variables with degree \( \nu \), which is known to be the cardinality of \( S^\circ(K, \, L) \).

q.e.d.

**4. Normalizing factor \( h_\lambda \)**

Let \( \hat{A} = \text{Ad} \, A(X_0) \), denoting by \( A \) the connected subgroup of \( K_0 \) generated by \( \alpha \). \( \hat{A} \) has a natural group structure induced from that of \( \alpha \). Let

\[ T = \{t \in C^*; \, |t| = 1\} \]

be the 1-dimensional torus. Under the identification in Introduction of \( (\alpha^-)^c \) with \( C^p, \alpha^- \) is identified with \( R^p \) and \( \hat{A} \) with \( T^p \). We see that the latter identification is compatible with group structures and complex conjugations, in view of the expression of \( \text{Ad}(\exp H)X_\iota \) in the proof of Theorem 3.1, (iii). Moreover, under the same identification we have (Moore [8])

\[ D \cap \alpha^- = \{x \in R^p; \, |x_i| < 1 \, (1 \leq i \leq p)\}, \]

denoting by \( z_i \) \((1 \leq i \leq p)\) the \( i \)-th component of \( z \in C^p \). By means of this
identification we define a measure on $\alpha^-$ by
\[ dH = dx_1 \cdots dx_\rho \]
and a function $D(H)$ on $\alpha^-$ by
\[ D(H) = \prod_{i=1}^{\rho} (2x_i)x_i^{2s} \prod_{1 \leq i < j \leq \rho} ((x_i + x_j)(x_i - x_j))^r \quad \text{for } H \in \alpha^- , \]
where $r$, $2s$ are multiplicities defined in Introduction. Then we have the following

**Lemma 1.** There exists a constant $c' > 0$ such that
\[ \int_{\alpha^-} f(X)d\mu(X) = c' \int_{\alpha^+} f(H)D(H)dH \]
for any integrable $K$-invariant function $f$ on $D$.

**Proof.** It is easy to see that $\text{Ad} c H = H$ for any $H \in \mathfrak{b}$ and $\text{Ad} c \gamma^* = X_\gamma - X_{-\gamma} \in \mathfrak{p}$ for any $\gamma \in \Delta$. Put
\[ \alpha^0 = \text{Ad} c (\sqrt{-1} \alpha) = \{X_\gamma - X_{-\gamma} ; \gamma \in \Delta\} , \]
\[ \mathfrak{h} = \text{Ad} c (\mathfrak{b} \oplus \sqrt{-1} \alpha) = \mathfrak{b} \oplus \alpha^0 \]
and
\[ \mathfrak{h}_R = \sqrt{-1} \mathfrak{b} \oplus \alpha^0 . \]
Then $\alpha^0$ is a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\alpha^0$ and $\mathfrak{h}_R$ is the real part of the complexification $\mathfrak{h}^c$ of $\mathfrak{h}$. We define lienear forms $h_i$ $(1 \leq i \leq \rho)$ on $\alpha^0$ by
\[ h_i(X_{\gamma_j} - X_{-\gamma_j}) = \delta_{ij} \quad (1 \leq i, j \leq \rho) . \]
If $h_i$ is identified with an element of $\alpha^0$ by means of the Killing form, we have $\text{Ad} c (\frac{1}{2} \gamma_i) = h_i$ $(1 \leq i \leq \rho)$. The linear order on $\mathfrak{h}_R$ induced by $\text{Ad} c$ from the order $>$ on $\sqrt{-1} t$ is a compatible order for $\text{Ad} c \Sigma$ with respect to the decomposition $\mathfrak{h}_R = \sqrt{-1} \mathfrak{b} \oplus \mathfrak{a}^0$. This follows from 3, Lemma 2.1). Thus positive restricted roots on $\alpha^0$ of the symmetric space $D = G/K$ are
\[ \{h_i \pm h_j ; 1 \leq i < j \leq \rho , 2h_i ; 1 \leq i \leq \rho\} \quad \text{if } \mathbf{P}_1 = \phi , \]
\[ \{h_i \pm h_j ; 1 \leq i < j \leq \rho , 2h_i, h_i ; 1 \leq i \leq \rho\} \quad \text{if } \mathbf{P}_1 = \neq \phi . \]
The multiplicity of $h_i \pm h_j$ $(1 \leq i < j \leq \rho)$, i.e. the number of roots in $\text{Ad} c \Sigma$ projecting to $h_i \pm h_j$, is the same as that of $\frac{1}{2}(\gamma_i \pm \gamma_j)$. Since the Weyl group $W_D$ on $\alpha^0$ of $D = G/K$ is generated by reflections with respect to $h_i - h_j$, ..., $h_{\rho-1} - h_{\rho}, h_{\rho}$, hence transitive on the set $\{ \pm h_i \pm h_j ; 1 \leq i < j \leq \rho\}$, it follows that
multiplicities of these roots are the same \( r \). By the same reason, multiplicities of \( h_i \) (\( 1 \leq i \leq p \)) are the same \( 2s \), which is even from the results of Harish-Chandra mentioned in 3. In the same way we know that multiplicities of \( 2h_i \) (\( 1 \leq i \leq p \)) are 1. Thus the product \( D^\circ \) of positive restricted roots (multiplicity counted) is given by

\[
D^\circ(H^\circ) = \prod_{i=1}^{p} 2h_i(H^\circ) h_i(H^\circ)^s \prod_{1 \leq i < j \leq p} ((h_i + h_j)(H^\circ)(h_i - h_j)(H^\circ))^r \quad \text{for } H^\circ \in \alpha^\circ.
\]

Let \( dX \) (resp. \( dH^\circ \)) denote the Euclidean measure of \( \mathfrak{p} \) (resp. of \( \alpha^\circ \)) induced from the Killing form \((\ , \)\), and \( dk \) the normalized Haar measure of \( K \). Then (cf. Helgason [4]) under the surjective map \( K \times \alpha^\circ \rightarrow \mathfrak{p} \) defined by \((k, H^\circ) \mapsto \text{Ad } kH^\circ \), these measures are related as follows:

\[
dX = c'' |D^\circ(H^\circ)| \, dk \, dH^\circ \quad \text{with some constant } c'' > 0.
\]

Now we define a \( K \)-equivariant \( \mathcal{R} \)-isomorphism \( j: \mathfrak{p} \rightarrow (\mathfrak{p}^\circ)^{-} \) by

\[
j(X) = \frac{1}{2}(X - [Z, X]) \quad \text{for } X \in \mathfrak{p}.
\]

It is easy to see that \( j(X_\gamma - X_{-\gamma}) = -X_{-\gamma} \) for any \( \gamma \in \Delta \), hence \( j\alpha^\circ = \alpha^\circ \). Since \( K \) acts irreducibly on \( \mathfrak{p} \), the map \( j \) is a similitude with respect to inner products \((\ , \)\) and the real part of \((\ , \)\). Therefore under the surjective map \( K \times \alpha^\circ \rightarrow (\mathfrak{p}^\circ)^{-} \) defined by \((k, H) \mapsto \text{Ad } kH \), we have

\[
d\mu(X) = c' |D(H)| \, dk \, dH \quad \text{with some constant } c' > 0.
\]

Seeing \( \text{Ad } K(D \cap \alpha^-) = D \), we get the proof of Lemma 1.

q.e.d.

Take a form \( \lambda \in S^*(K, L) \). Choose an orthonormal basis \( \{u_i; \ 1 \leq i \leq d_\lambda\} \) of \( S^*_\mathfrak{p}(\mathfrak{p}^\circ)^{-} \) with respect to \((\ , \)\), such that \( \{u_i; \ 1 \leq i \leq d_{\lambda,0}\} \) spans \( S^*_\mathfrak{p}(\mathfrak{p}^\circ)^{-} \cap S^*_\mathfrak{p}(\mathfrak{p}_0^-) \) and \( u_i \) is \( L \)-invariant. Put

\[
\rho'_j(k) = (\text{Ad } ku_j, u_i), \quad \text{for } k \in K \quad (1 \leq i, j \leq d_\lambda),
\]

\[
\varphi'_\lambda(k) = \rho'_i(k), \quad \text{for } k \in K \quad (1 \leq i \leq d_\lambda),
\]

\[
f'_i = \sqrt{d_\lambda} \varphi'_i \quad (1 \leq i \leq d_\lambda).
\]

The arguments in 2 show that \( \{f'_i; \ 1 \leq i \leq d_\lambda\} \) form an orthonormal basis of \( S^*_\mathfrak{p}(S) \) with respect to \(< \ , \ >\) and \( \varphi'_\lambda \) is the zonal spherical function \( \omega_\lambda \) for \((K, L) \) belonging to \( \lambda \), identifying \( C_{\infty}(S) \) with the space of right \( L \)-invariant \( C_{\infty} \)-functions on \( K \). The zonal spherical polynomial \( \Omega_\lambda \) for \( D \) belonging to \( \lambda \) defined in Introduction is characterized by that its restriction to \( S \) coincides with \( \omega_\lambda \). \( \Omega_\lambda \) restricted to \( \mathfrak{p}_0^- \) is the zonal spherical polynomial for \( D_0 \) belonging to \( \lambda \) and \( \omega_\lambda \) restricted to \( S_0 \) is the zonal spherical function for \((K_0, L_0) \) belonging to \( \lambda \). \( \Omega_\lambda \)
restricted to \((\alpha^-)^c\) is a symmetric polynomial since it is \(W_{\delta_0}\)-invariant. Let 
\(f_i \in S_k^*((p^c)^-)\) (1 \(\leq i \leq d_\lambda\)) be the unique polynomial such that its restriction to \(S\) is \(f_i\). Then \(\{f_i; 1 \leq i \leq d_\lambda\}\) form an orthogonal basis of \(S_k^*((p^c)^-)\) with respect to \((\ , \ , \)\), such that \(\{f_i; 1 \leq i \leq d_{\lambda,0}\}\) form an orthogonal basis of \(S_k^*((p^c)^- \cap S^*(p^c))\). They satisfy relations

\[
 f_i(\Ad k^{-1} X) = \sum_{j=1}^{d_\lambda} \rho^j_k(k) f_j(X) \quad \text{for} \quad k \in K, X \in (p^c)^- (1 \leq i \leq d_\lambda).
\]

We put

\[
 \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \quad \text{for} \quad X \in (p^c)^-.
\]

Then for any \(k \in K\) we have

\[
 \Phi_\lambda(\Ad k^{-1} X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \left( \sum_{j=1}^{d_\lambda} \rho^j_k(k) f_j(X) \right) \left( \sum_{j=1}^{d_\lambda} \rho^j_k(k) \overline{f_j(X)} \right)
 = \frac{1}{d_\lambda} \sum_{j=1}^{d_\lambda} \delta_{jk} \overline{f_j(X)} f_j(X) = \Phi_\lambda(X) \quad \text{for} \quad X \in (p^c)^-,
\]

i.e. \(\Phi_\lambda\) is a \(K\)-invariant \(C^\infty\)-function on \((p^c)^-\). Note that

\[
 \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \quad \text{for} \quad X \in p^{c_1}.
\]

**Lemma 2.**

\[
 h_\lambda = c' \int_{p \cap a^-} \Phi_\lambda(H) |D(H)| dH
\]

**Proof.**

\[
 \int_{p} \Phi_\lambda(X) d\mu(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \langle f_i, f_i \rangle = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} h_\lambda \langle f_i, f_i \rangle = h_\lambda.
\]

On the other hand, by Lemma 1 we have

\[
 \int_{p} \Phi_\lambda(X) d\mu(X) = c' \int_{p \cap a^-} \Phi_\lambda(H) |D(H)| dH. \quad \text{q.e.d.}
\]

**Proof of Theorem B.** Making use of the complex conjugation \(X \mapsto \overline{X}\) of \(p^{c_1}\) defined in 3, we define \(\Phi_{\lambda} \in S^{*}(p^{c_1})\) by

\[
 \Phi_{\lambda}(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_{\lambda,0}} f_{\lambda}(X) \overline{f_{\lambda}(X)} \quad \text{for} \quad X \in p^{c_1}.
\]

Then \(\Phi_{\lambda} = \Phi_{\lambda}\) on \(p_{-1}\) and we have for any \(k \in K_0\)
\[ \Phi_\lambda(\text{Ad } k X_\alpha) = \frac{1}{d_\lambda} \sum_a f_a(\text{Ad } k X_\alpha) f_a(\text{Ad } \theta(k) X_\alpha) \]

\[ = \frac{1}{d_\lambda} \sum_a f_a(\text{Ad } k X_\alpha) f_a(\text{Ad } \theta(k) X_\alpha) \]

\[ = \frac{1}{d_\lambda} \sum_a f_a^{(1)}(k) f_a^{(1)}(\theta(k)) = \sum_a \varphi_a^{(1)}(k) \varphi_a^{(1)}(\theta(k)) \]

\[ = \sum_a \rho_\alpha(k) \rho_\alpha(\theta(k)) = \sum_a \rho_\alpha(k) \rho_\alpha(\theta(k)^{-1}) \]

\[ = \rho_\alpha(\theta(k)^{-1}k) = \omega_\alpha(\theta(k)^{-1}k). \]

In particular for any \( a \in A \)

\[ \Phi_\lambda(\text{Ad } a X_\alpha) = \omega_\lambda(a^2), \]

i.e. for any \( \tilde{a} \in \tilde{A} \)

\[ \Phi_\lambda(\tilde{a}) = \omega_\lambda(\tilde{a}^2) = \Omega_\lambda(\tilde{a}^2). \]

Since \( \tilde{A} = T^p \) is a compact real form of \( C^*_p \) and \( C^*_p \) is open in \( C^p = (\alpha^-)^c \), we have

\[ \Phi_\lambda(x_1, \ldots, x_p) = \Omega_\lambda(x_1^2, \ldots, x_p^2) \quad \text{for any } z \in C^p = (\alpha^-)^c. \]

By Lemma 2 we have

\[ h_\lambda = c' \int_{D \cap a^-} \Phi_\lambda(H) |D(H)| dH \]

\[ = c' \int_{\{y_1 < \cdots < y_p\}} \Omega_\lambda(y_1^2, \ldots, y_p^2) \prod_{i<j} (y_i - y_j)^r |dy_i \cdots dy_p| \]

\[ = c(D) \int_{\{y_1 < \cdots < y_p\}} \Omega_\lambda(y_1, \ldots, y_p) \prod_{i<j} (y_i - y_j)^r |dy_i \cdots dy_p| \]

for some constant \( c(D) > 0 \), which does not depend on \( \lambda \). In particular, for \( \lambda = 0 \)

\[ \mu(D) = h_0 = c(D) \int_{\{y_1 < \cdots < y_p\}} \prod_{i<j} (y_i - y_j)^r |dy_i \cdots dy_p|, \]

since \( \Omega_0 \equiv 1 \). This completes the proof of Theorem B. q.e.d.

**Remark.** It can be proved that \( \Phi_\lambda \) is an \( L_0 \)-invariant polynomial on \( \mathfrak{p}_c \).

The multiplicities \( r, s \) are given as follows.

<table>
<thead>
<tr>
<th>( D )</th>
<th>rank ( D )</th>
<th>( r )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>( p )</td>
<td>2</td>
<td>( q - p )</td>
</tr>
<tr>
<td>(II)_{p}</td>
<td>([n/2])</td>
<td>4</td>
<td>{ 2 \text{ if } n \text{ odd} } { 0 \text{ if } n \text{ even} }</td>
</tr>
<tr>
<td>(III)_{p}</td>
<td>( n )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(IV)_{n} (n \geq 3)</td>
<td>2</td>
<td>( n - 2 )</td>
<td>0</td>
</tr>
<tr>
<td>(EIII)</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>(EVII)</td>
<td>3</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>
The zonal spherical polynomial $\Omega_\lambda$ is given as follows.

For integers $n_1, \ldots, n_p$ we define the Schur function \( \{n_1, \ldots, n_p\} \) on the \( p \)-dimensional torus $T^p$ by

\[
\{n_1, \ldots, n_p\}(t) = \frac{\det(t_i^{n_{j+i+p-j}})}{\det(t_i^{p-j})_{i<j<p}} \quad \text{for} \quad t = \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \in T^p \subset \mathbb{C}^p .
\]

\( \{n_1, \ldots, n_p\} \) is symmetric in variables $t_1, \ldots, t_p$ and it is a polynomial in $t_1, \ldots, t_p$ if and only if $n_i \geq 0$ (1 $\leq i \leq p$). For an element $\lambda = \sum_{i=1}^p n_i \gamma_i \in \sum_{i=1}^p \mathbb{Z} \gamma_i = Z(K_o, L_0)$, the $i$-th coefficient $n_i$ will be denoted by $n_i(\lambda)$.

Then we have

**Theorem 4.1.** The zonal spherical polynomial $\Omega_\lambda$ for $D$ belonging to $\lambda \in S^\circ(K, L)$ is determined on $(a^-)^c$ by the relation

\[
\Omega_\lambda(t) = \sum_{\mu \in \mathcal{P}_\lambda} c_\lambda^\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for any} \quad t \in T^p = \hat{A} \subset (a^-)^c ,
\]

where the $c_\lambda^\mu$'s are coefficients in Theorem 2.5 for the symmetric pair $(K_o, L_0)$.

**Proof.** As we have seen in the proof of Theorem B, $\Omega_\lambda$ is determined on $(a^-)^c$ by

\[
\Omega_\lambda(t) = \omega_\lambda(t) \quad \text{for any} \quad t \in T^p = \hat{A} .
\]

By Theorem 2.5, $\omega_\lambda$ has an expression

\[
\omega_\lambda(t) = \sum_{\mu \in \mathcal{P}_\lambda} c_\lambda^\mu \chi_\mu(t) \quad \text{for} \quad t \in T^p = \hat{A} .
\]

Since the Weyl group $W_{S_0}$ acts on $Z(K_o, L_0)$ by the group of permutations of $\gamma_1, \ldots, \gamma_p$, $W_{S_0}$-invariant characters $\chi_\lambda$ of $\hat{A}$ are nothing but Schur functions. As we have seen in the proof of Theorem 3.1, (iii), the $i$-th component of $\text{Ad}(\exp H)X_\mu \in T^p = \hat{A}$ is $\exp(-\langle \gamma_i, H \rangle)$ for any $H \in a$. It follows that

\[
\chi_\mu(t) = \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A} .
\]

Hence we have

\[
\Omega_\lambda(t) = \sum_{\mu \in \mathcal{P}_\lambda} c_\lambda^\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t)
= \sum_{\mu \in \mathcal{P}_\lambda} c_\lambda^\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A} . \quad \text{q.e.d.}
\]

In the case of the domain $D$ of type (I)$_{p,q}$ ($p \leq q$), $S_o$ is the unitary group $U(p)$ of degree $p$. We have in view of Example in 2 that

\[
\Omega_\lambda(t) = \frac{1}{d_\lambda} \{n_1(\lambda), \ldots, n_p(\lambda)\}(t) \quad \text{for} \quad t \in T^p = \hat{A} ,
\]
where \( d_\lambda \) is the degree of the irreducible representation of \( U(p) \) with the signature \((n_1(\lambda), \ldots, n_p(\lambda))\). In the case of the domain \( D \) of type (IV), \( S_0 \) is the Lie sphere and \( \Omega_\lambda \) can be described in terms of Gegenbauer polynomials, which are zonal spherical functions for the sphere. So our integral formula in Theorem B clarifies the meaning of integrals of Hua [6].

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References


