Polynomial representations associated with symmetric bounded domains

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Osaka University
POLYNOMIAL REPRESENTATIONS ASSOCIATED WITH
SYMMETRIC BOUNDED DOMAINS

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Introduction. In this note we want to construct a complete orthonormal
system of the Hilbert space $H^2(D)$ of square integrable holomorphic functions on
an irreducible symmetric bounded domain $D$. A symmetric bounded domain
$D$ is canonically realizable as a circular starlike bounded domain with the center
0 in a complex cartesian space by means of Harish-Chandra's imbedding (Harish-
Chandra [3]), which is constructed as follows. The largest connected group $G$
of holomorphic automorphisms of $D$ is a connected semi-simple Lie group without
center, which is transitive on $D$. Thus denoting the stablizer in $G$ of a point
0 of $D$ by $K$, $D$ is identified with the quotient space $G/K$. Let $g$ (resp. $\mathfrak{t}$) be the
Lie algebra of $G$ (resp. $K$) and $g = \mathfrak{t} + \mathfrak{p}$ the Cartan decomposition of $g$ with
respect to $\mathfrak{t}$. Then there exists uniquely an element $H$ of the center of $\mathfrak{t}$ such
that $\text{ad} H$ restricted to $\mathfrak{p}$ coincides with the complex structure tensor on the
tangent space $T_0(D)$ of $D$ at the origin 0, identifying as usual $\mathfrak{p}$ with $T_0(D)$. Let $g^c$
be the Lie algebra of the complexification $G^c$ of $G$ and put $Z = i H \in g^c$.
Let $(\mathfrak{p}^c)^\pm$ be the $(\pm 1)$-eigenspace in $\mathfrak{p}^c$ of $\text{ad} Z$. Then they are invariant under
the adjoint action of $K$ and the complexification $\mathfrak{p}^c$ of $\mathfrak{p}$ is the direct sum of $(\mathfrak{p}^c)^+$
and $(\mathfrak{p}^c)^-$. Let $U^c$ denote the normalizer of $(\mathfrak{p}^c)^+$ in $G^c$. Then $D = G/K$ is
holomorphically imbedded as an open submanifold into the quotient space $G^c/U^c$
in the natural way. For any point $z \in D$, there exists uniquely a vector $X \in (\mathfrak{p}^c)^-$
such that

$$\exp X \mod U^c = z.$$ 

The map $z \mapsto X$ of $D$ into $(\mathfrak{p}^c)^-$ is the desired imbedding. Note that the natural
action of $K$ on $D$ can be extended to the adjoint action of $K$ on the ambient space
$(\mathfrak{p}^c)^-$. Henceforth we assume that $D$ is a bounded domain in $(\mathfrak{p}^c)^-$ realized in the
above manner. Let $(\ , \ )$ denote the Killing form of $g^c$ and $\tau$ the complex con-
jugation of $g^c$ with respect to the compact real form $\mathfrak{t} + \sqrt{-1} \mathfrak{p}$ of $g^c$. We define
a $K$-invariant hermitian inner product $(\ , \ )$, on $g^c$ by

$$(X, Y) = -(X, \tau Y) \quad \text{for} \quad X, Y \in g^c.$$
This defines a $K$-invariant Euclidean measure $d\mu(X)$ on $(p^C)^-$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(X)$. The inner product of $H^2(D)$ will be denoted by $\langle \cdot ,\cdot \rangle$. $K$ acts on $H^2(D)$ as unitary operators by
\[(kf)(X) = f(k^{-1}X) \quad \text{for} \quad k\in K, \quad X\in D.\]

Let $S*((p^c)^-)$ denote the graded space of polynomial functions on $(p^c)^-$. It has the natural hermitian inner product $(\cdot ,\cdot )$, induced from the inner product $(\cdot ,\cdot )$, on $(p^c)^-$. $K$ acts on $S*((p^c)^-)$ as unitary operators by
\[(kf)(X) = f(Ad k^{-1}X) \quad \text{for} \quad k\in K, \quad X\in (p^c)^-.\]

Now let $S$ denote the Shilov boundary of $D$. It is known (Korányi-Wolf [7]) that $K$ acts transitively on $S$. Thus denoting by $L$ the stabilizer in $K$ of a point $X_0 \in S$, $S$ is identified with the quotient space $K/L$. Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure of $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle \cdot ,\cdot \rangle$. $K$ acts on $L^2(S)$ as unitary operators by
\[(kf)(X) = f(Ad k^{-1}X) \quad \text{for} \quad k\in K, \quad X\in S.\]

The space $C^\omega (S)$ of $C$-valued $C^\omega $-functions on $S$ is a $K$-submodule of $L^2(S)$. The restrictions $S*((p^c)^-)\rightarrow H^2(D)$ and $S*((p^c)^-)\rightarrow L^2(S)$ are both $K$-equivariant monomorphisms. Their images will be denoted by $S*(D)$ and $S*(S)$, respectively. They have natural gradings induced from that of $S*((p^c)^-)$. Then the "restriction" $S*(D)\rightarrow S*(S)$ is defined in the natural manner and it is a $K$-equivariant isomorphism. Since $D$ is a circular starlike bounded domain, a theorem of H. Cartan [2] yields that the subspace $S*(D)$ of $H^2(D)$ is dense in $H^2(D)$ (cf. 1).

We decompose first the $K$-module $S*(D)$ into irreducible components. We take a maximal abelian subalgebra $t$ of $\mathfrak{k}$ and identify the real part $\sqrt{-1}t$ of the complexification $t^C$ of $t$ with its dual space by means of Killing form of $g^C$. Let $\Sigma \subset \sqrt{-1}t$ denote the set of roots of $g^C$ with respect to $t^C$. We choose root vectors $X_\alpha \in g^C$ for $\alpha \in \Sigma$ such that
\[
[X_\alpha , X_{-\alpha}] = -\frac{2}{\langle \alpha, \alpha \rangle} \alpha, \quad \tau X_\alpha = X_{-\alpha}.
\]

A root is called compact if it is also a root of the complexification $t^C$ of $t$, otherwise it is called non-compact. $\Sigma_\text{c}$ (resp. $\Sigma_\text{nc}$) denotes the set of compact roots (resp. of non-compact roots). We choose and fix once for all a linear order $>$ on $\sqrt{-1}t$ such that $(p^C)^+$ is spanned by the root spaces for non-compact positive
roots $\sum_{p}^{\pm}$. Two roots $\alpha, \beta \in \sum$ are called strongly orthogonal if $\alpha \pm \beta$ is not a root. We define a maximal strongly orthogonal subsystem

$$\Delta = \{\gamma_1, \ldots, \gamma_p\}, \quad \gamma_1 > \gamma_2 > \cdots > \gamma_p > 0, \quad p = \text{rank } D$$

of $\sum_{p}^{\pm}$ as follows (cf. Harish-Chandra [3]). Let $\gamma_1$ be the highest root of $\sum$ and for each $j$, $\gamma_{j+1}$ be the highest positive non-compact root that is strongly orthogonal to $\gamma_1, \ldots, \gamma_j$. We put

$$X_0 = -\sum_{\gamma \in \Delta} X_{-\gamma}.$$

Then it is known (Korányi-Wolf [7]) that $X_0$ is on the Shilov boundary $S$ of $D$. Henceforth we shall take the above point $X_0$ as the origin of $S$. We put for $\nu \in \mathbb{Z}, \nu \geq 0$

$$S'(K, L) = \left\{ \sum_{i=1}^{p} n_i \gamma_i; n_i \in \mathbb{Z}, n_1 \geq n_2 \geq \cdots \geq n_p \geq 0, \sum_{i=1}^{p} n_i = \nu \right\},$$

and

$$S^*(K, L) = \sum_{\nu \geq 0} S'(K, L).$$

We shall prove the following

**Theorem A.** Any irreducible $K$-submodule of $S^*(D)$ is contained exactly once in $S^*(D)$. The set $S'(D)$ of highest weights (with respect to $t^c$) of irreducible $K$-submodules contained in $S'(D)$ coincides with $S'(K, L)$. Denoting by $S^*_\lambda(D)$ (resp. $S^*_\lambda(S)$) the irreducible $K$-submodule of $S^*(D)$ (resp. of $S^*(S)$) with the highest weight $\lambda \in S^*(K, L)$,

$$S^*(D) = \sum_{\lambda \in \mathfrak{s}^*(K, L)} \oplus S^*_\lambda(D)$$

and

$$S^*(S) = \sum_{\lambda \in \mathfrak{s}^*(K, L)} \oplus S^*_\lambda(S)$$

are the orthogonal sum relative to the inner product $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$, respectively. The restriction $f \mapsto f'$ of $S^*_\lambda(D) \rightarrow S^*_\lambda(S)$ is a similitude for each $\lambda \in S^*(K, L)$, i.e. there exists a constant $h_\lambda > 0$ such that

$$\langle f, g \rangle = h_\lambda \langle f', g' \rangle \quad \text{for any } f, g \in S^*_\lambda(D).$$

Thus, if

$$\{f^\lambda_{\lambda, i}; 1 \leq i \leq d_\lambda\}, \quad \lambda \in S^*(K, L)$$

is an orthonormal basis of $S^*_\lambda(S)$, then

$$\{\sqrt{h_\lambda^{-1}} f^\lambda_{\lambda, i}; \lambda \in S^*(K, L), 1 \leq i \leq d_\lambda\}$$

is a complete orthonormal system of $H^2(D)$. 

A basis \( \{ f_{\lambda,i}; 1 \leq i \leq d_\lambda \} \) is, for instance, constructed as follows. Take an irreducible \( K \)-module \((\rho, V)\) with the highest weight \( \lambda \), carrying a \( K \)-invariant hermitian inner product \( (, ) \). Choose an orthonormal basis \( \{ u_i; 1 \leq i \leq d_\lambda \} \) of \( V \) such that the first vector \( u_1 \) is \( L \)-invariant. This can be done in view of Frobenius' reciprocity since the \( K \)-module \( V \) is \( K \)-isomorphic with a \( K \)-submodule of \( C^\infty(S) \). Then the functions \( f_{\lambda,i}(1 \leq i \leq d_\lambda) \) defined by

\[
f_{\lambda,i}(kX_\alpha) = \sqrt{d_\lambda} f_{\lambda,i}(\rho(k)u_1) \quad \text{for } k \in K
\]

form an orthonormal basis of \( S_\lambda^*(S) \) (cf. 2).

We compute next the normalizing factor \( h_\lambda \). Let

\[
a = \{ \sqrt{-1} \Delta \}_R
\]

be the \( R \)-span of \( \sqrt{-1} \Delta \) in \( t \) and

\[
\sigma: \sqrt{-1} t \to \sqrt{-1} a
\]

denote the orthogonal projection of \( \sqrt{-1} t \) onto \( \sqrt{-1} a \). For \( \gamma \in \sigma \Sigma - \{0\} \), the number of roots \( \alpha \in \Sigma \) such that \( \sigma \alpha = \gamma \) is called the multiplicity of \( \gamma \). Let \( r \) (resp. \( 2s \)) be the multiplicity of \( \frac{1}{2}(\gamma_1 - \gamma_2) \) (resp. of \( \frac{1}{2} \gamma_i \)). If follows from Theorem A and Frobenius' reciprocity that for each \( \lambda \in S^*(K, L) \) there exists uniquely an \( L \)-invariant polynomial \( \Omega_\lambda \) in \( S_\lambda^*((p^c)^-) \) such that \( \Omega_\lambda(X_\alpha) = 1 \), where \( S_\lambda^*((p^c)^-) \) denotes the irreducible \( K \)-submodule of \( S^*((p^c)^-) \) with the highest weight \( \lambda \). The polynomial \( \Omega_\lambda \) is called the zonal spherical polynomial for \( D \) belonging to \( \lambda \). Let

\[
(a^-)^c = \{ X_\gamma; \gamma \in \Delta \}_C
\]

be the \( C \)-span of \( \{ X_{-\gamma}; \gamma \in \Delta \} \) in \( (p^c)^- \). It is identified with the complex cartesian space \( C^p \) by the map

\[
-\sum_{i=1}^p z_i X_{-\gamma_i} \mapsto \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}.
\]

Thus the zonal spherical polynomial \( \Omega_\lambda \) restricted to \( (a^-)^c \) is a polynomial \( \Omega_\lambda(Y_1, \ldots, Y_p) \) in \( p \)-variables. Let \( \mu(D) \) denote the volume of \( D \) with respect to the measure \( d\mu(X) \). We shall prove the following

**Theorem B.** For \( \lambda \in S^*(K, L) \), the normalizing factor \( h_\lambda \) is given by

\[
h_\lambda = c(D) \int_{0 \leq y_i < 1; 1 \leq i \leq p} \Omega_\lambda(y_1, \ldots, y_p) \left| \prod_{i=1}^p (y_i - y_j)^r \right| \prod_{i=1}^p y_i^s dy_1 \cdots dy_p
\]

where

\[
c(D) = \mu(D) \left( \int_{0 \leq y_i < 1; 1 \leq i \leq p} \left| \prod_{i=1}^p (y_i - y_j)^r \right| \prod_{i=1}^p y_i^s dy_1 \cdots dy_p \right)^{-1}.
\]

(a) $S^v(D) \subset S^v(K, L)$

by seeing the character of the $K$-module $S^*((\mathfrak{p}c^-))$ and by making use of $E$. Cartan's theory on spherical representations of a compact symmetric pair. But his proof of

(b) $S^v(K, L) \subset S^v(D)$

is complicated and was done after nine successive lemmas. In this note we give another proof of (a) by means of a lemma of Murakami and Cartan's theory, and give a relatively short proof of (b) by means of a theorem of Harish-Chandra on invariant polynomials for a symmetric pair.

Hua [6] computed the factors $h_\lambda$ for certain classical domains by integrating certain polynomials. Our integral formula in Theorem B will clarify the meaning of integrals of Hua.

1. Circular domains

A domain $D \subset \mathbb{C}^n$ containing the origin 0 is said to be a circular domain with the center 0 if together with any point $z \in D$ the point $e^{\sqrt{-1} \theta} z$ is in $D$ for any real $\theta \in \mathbb{R}$. $D$ is said to be a starlike domain with the center 0 if together with any point $z \in D$ the point $rz$ is in $D$ for any real $r \in \mathbb{R}$ with $0 < r < 1$.

**Theorem 1.1.** (H. Cartan [2]) Let $D \subset \mathbb{C}^n$ be a circular domain with the center 0. Then any holomorphic function $f$ on $D$ can be developed in the sum of homogeneous polynomials $P_\nu$ in $n$-variables with degree $\nu$ ($\nu = 0, 1, 2, \cdots$):

$$f(z) = \sum_{\nu = 0}^{\infty} P_\nu(z) \quad \text{for} \quad z \in D.$$  

The sum converges uniformly on any compact subset of $D$. The homogeneous polynomials $P_\nu$ are uniquely determined for $f$.

Let $D$ be a bounded domain in $\mathbb{C}^n$, $d\mu(z)$ the Euclidean measure on $\mathbb{C}^n$, induced from the standard hermitian inner product of $\mathbb{C}^n$. Let $H^\nu(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(z)$. The inner product of $H^\nu(D)$ will be denoted by $\langle \ , \rangle$. Let $S^*(\mathbb{C}^n)$ be the graded space of polynomials in $n$-variables and $S^*(D)$ the subspace of $H^\nu(D)$ consisting of all functions on $D$ obtained by the restriction of polynomials in $S^*(\mathbb{C}^n)$. Then Theorem 1.1 yields the following

**Corollary.** Let $D \subset \mathbb{C}^n$ be a circular starlike bounded domain with the center 0. Then the subspace $S^*(D)$ of $H^\nu(D)$ is dense in $H^\nu(D)$. 

Proof. If suffices to show that if \( f \in H^r(D) \) with \( \langle f, S^*(D) \rangle = \{0\} \), then \( f = 0 \). Theorem 1.1 implies that \( f \) can be developed as

\[
f = \sum_{\nu} P_{\nu}, \quad P_{\nu} \in S^r(D),
\]
uniformly convergent on any compact subset of \( D \). Choose an orthonormal basis \( \{P_{\nu,j}\} \) of \( S^r(D) \) with respect to \( \langle \cdot, \cdot \rangle \) for each \( \nu \). Then we have

\[
\langle P_{\nu,j}, P_{\mu,i} \rangle = \delta_{\nu\mu} \delta_{ji}.
\]

In fact, since \( d\mu(e^{\sqrt{-1} \theta z}) = d\mu(z) \) for any \( \theta \in \mathbb{R} \), we have \( \langle P_{\nu,j}, P_{\mu,i} \rangle = e^{\sqrt{-1} \theta \mu \mu} \langle P_{\nu,j}, P_{\mu,i} \rangle \) for any \( \theta \in \mathbb{R} \). Then \( f \) can be developed as

\[
f = \sum_{\nu,j} a_{\nu,j} P_{\nu,j} \quad \text{with} \quad a_{\nu,j} \in C,
\]
uniformly convergent on any compact subset of \( D \). Since \( D \) is a starlike domain, the closure \( rD \) of \( rD \) is a compact subset of \( D \) for any \( r \in R \) with \( 0 < r < 1 \), so that the above series converges uniformly on \( rD \). Therefore for any \( P_{\mu,i} \) we have

\[
\int_{rD} f(z) P_{\mu,i}(z) d\mu(z) = \sum_{\nu,j} a_{\nu,j} \int_{rD} P_{\nu,j}(z) P_{\mu,i}(z) d\mu(z).
\]
If we put

\[
z' = \frac{1}{r} z \quad \text{for} \quad z \in rD,
\]
then \( z = rz' \), \( d\mu(z) = r^{2n} d\mu(z') \) so that

\[
\int_{rD} P_{\nu,j}(z) P_{\mu,i}(z) d\mu(z) = r^{2n+2\mu} \int_{D} P_{\nu,j}(z') P_{\mu,i}(z') d\mu(z')
\]

\[
= r^{2n+2\mu} \langle P_{\nu,j}, P_{\mu,i} \rangle = r^{2n+2\mu} \delta_{\nu\mu} \delta_{ji}.
\]
Hence we have

\[
\int_{rD} f(z) P_{\mu,i}(z) d\mu(z) = a_{\mu,i} r^{2n+2\mu}
\]
and

\[
a_{\mu,i} = \lim_{r \to 1} a_{\mu,i} r^{2n+2\mu} = \lim_{r \to 1} \int_{rD} f(z) P_{\mu,i}(z) d\mu(z)
\]

\[
= \langle f, P_{\mu,i} \rangle = 0 \quad \text{(from the assumption)}.
\]
This implies that \( f = 0 \).
2. **Spherical representations of a compact symmetric pair**

Let $K$ be a compact connected Lie group, $L$ a closed subgroup of $K$ and $S$ be the quotient space $K/L$. The space of $\mathbb{C}$-valued $C^\infty$-functions on $S$ will be denoted by $C^\infty(S)$. We shall often identify $C^\infty(S)$ with the space of $C^\infty$-functions $f$ on $K$ such that

$$f(kl) = f(k) \quad \text{for any} \quad k \in K, \ l \in L.$$  

Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure on $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle , \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$(kf)(x) = f(k^{-1}x) \quad \text{for} \quad k \in K, \ x \in S.$$  

Then $C^\infty(S)$ is a $K$-submodule of $L^2(S)$. A (continuous finite dimensional complex) representation

$$\rho : K \to GL(V)$$  

of $K$ is said to be **spherical** relative to $L$ if the $K$-module $V$ is equivalent to a $K$-submodule of $C^\infty(S)$, which amounts to the same from Frobenius' reciprocity that the $K$-module $V$ has a non-zero $L$-invariant vector. We denote by $\mathcal{D}(K, L)$ the set of equivalence classes of irreducible spherical representations of $K$ relative to $L$. The totality of $f \in C^\infty(S)$ contained in a finite dimensional $K$-submodule of $C^\infty(S)$, which will be denoted by $\mathfrak{o}(K, L)$, is a $K$-submodule of $C^\infty(S)$. A function in $\mathfrak{o}(K, L)$ is called a **spherical function** for the pair $(K, L)$. Then $\mathfrak{o}(K, L)$ is the orthogonal sum with respect to the inner product $\langle , \rangle$. Peter-Weyl approximation theorem implies that the subspace $\mathfrak{o}(K, L)$ of $L^2(S)$ is dense in $L^2(S)$. We assume furthermore that the pair $(K, L)$ satisfies the condition

(*) any $\rho \in \mathcal{D}(K, L)$ is contained exactly once in $\mathfrak{o}(K, L)$, which is by Frobenius' reciprocity equivalent to that for any spherical representation

$$\rho : K \to GL(V)$$  

of $K$ relative to $L$, an $L$-invariant vector of $V$ is unique up to scalar multiplication. Then for each $\rho \in \mathcal{D}(K, L)$, there exists uniquely an $L$-invariant function $\omega_\rho \in \mathfrak{o}_\rho(K, L)$ such that $\omega_\rho(e) = 1$. $\omega_\rho$ is called the **zonal spherical function** for $(K, L)$ belonging to $\rho$. Let

$$\rho : K \to GL(V)$$
be a spherical representation of $K$ relative to $L$. Choose a $K$-invariant hermitian inner product $(\ ,\ )$ on $V$. The equivalence class containing $\rho$ will be denoted by the same letter $\rho$. Choose an orthonormal basis \{u_i; 1 \leq i \leq d_\rho\} of $V$ such that $u_i$ is $L$-invariant. Define $\varphi_i \in C^\infty(S)$ (1 \leq i \leq d_\rho) by
$$
\varphi_i(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.
$$
We know that they are linearly independent, in view of orthogonality relations of matrix elements $(u_i, \rho(k)u_j)$. For any $k' \in K$ we have
$$
\varphi_i(k'^{-1}k) = (u_i, \rho(k'^{-1}k)u_i) = (\rho(k')u_i, \rho(k)u_i)
$$
$$
= \sum_j (\rho(k')u_i, u_j)(u_j, \rho(k)u_i)
$$
$$
= \sum_j (\rho(k')u_i, u_j) \varphi_j(k),
$$
i.e.
$$
k'\varphi_i = \sum_j (\rho(k')u_i, u_j) \varphi_j \quad (1 \leq i \leq d_\rho).
$$
In particular
$$
l\varphi_i = \varphi_i \quad \text{for any} \quad l \in L,
$$
and
$$
\varphi_i(e) = 1.
$$
Therefore the system \{\varphi_i; 1 \leq i \leq d_\rho\} forms a basis of $\mathfrak{o}_\rho(K, L)$ and the zonal spherical function $\omega_\rho$ is given by
$$
\omega_\rho(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.
$$
Furthermore orthogonality relations implies that the system
$$
\{\sqrt{d_\rho}\varphi_i; 1 \leq i \leq d_\rho\}
$$
forms an orthonormal basis of $\mathfrak{o}_\rho(K, L)$ and that
$$
\langle \omega_\rho, \omega_{\rho'} \rangle = \delta_{\rho\rho'} \frac{1}{d_\rho}.
$$
Henceforth we assume that the pair $(K, L)$ is a symmetric pair, i.e. there exists an involutive automorphism $\theta$ of $K$ such that if we put
$$
K_\theta = \{k \in K; \theta(k) = k\},
$$
$L$ lies between $K_\theta$ and the connected component $K_\theta^0$ of $K_\theta$. Then the pair $(K, L)$ satisfies the condition (*) (E. Cartan [1]). For example, a compact connected Lie group $S$ admits a symmetric pair $(K, L)$ such that $S = K/L$. In fact,
$$
K = S \times S,
$$
$$
L = \{(x, x); x \in S\}$
In the following we summarize some known facts on a symmetric pair (cf. Helgason \[4\]).

Let \( \mathfrak{f} \) (resp. \( \mathfrak{l} \)) be the Lie algebra of \( K \) (resp. of \( L \)). The involutive automorphism of \( \mathfrak{f} \) obtained by differentiating the automorphism \( \theta \) of \( K \) will be also denoted by the same letter \( \theta \).

Choose and fix once for all a \( \mathbb{C} \)-bilinear symmetric form \( (\cdot, \cdot) \) on the complexification \( \mathfrak{f}^\mathbb{C} \) of \( \mathfrak{f} \), which is invariant under both the \( \mathbb{C} \)-linear extension to \( \mathfrak{f}^\mathbb{C} \) of \( \theta \) and the adjoint action of \( \mathfrak{f}^\mathbb{C} \) and furthermore is negative definite on \( \mathfrak{f} \times \mathfrak{f} \). Then \( S \) is a Riemannian symmetric space with respect to the \( K \)-invariant Riemannian metric on \( S \) defined by \(-\langle \cdot, \cdot \rangle\). We put

\[
\mathfrak{s} = \{ X \in \mathfrak{f}; \theta X = -X \} = \{ X \in \mathfrak{f}; (X, I) = \{0\} \}.
\]

Then we have orthogonal decompositions

\[\mathfrak{f} = \mathfrak{b} \oplus \mathfrak{a},\]

where \( \mathfrak{c} \) is the center of \( \mathfrak{f} \) and \( \mathfrak{f}' \) is the derived algebra \([\mathfrak{f}, \mathfrak{f}]\) of \( \mathfrak{f} \). We choose a maximal abelian subalgebra \( \mathfrak{a} \) in \( \mathfrak{s} \). Such \( \mathfrak{a} \) are mutually conjugate under the adjoint action of \( L \). \( \dim \mathfrak{a} \) is the rank of the symmetric pair \( (K, L) \). Extend \( \mathfrak{a} \) to a maximal abelian subalgebra \( \mathfrak{t} \) of \( \mathfrak{f} \) containing \( \mathfrak{a} \). Then we have the decomposition

\[\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}, \quad \text{where} \quad \mathfrak{b} = \mathfrak{b} \cap \mathfrak{i}.
\]

Let \( \mathfrak{t}' = \mathfrak{t} \cap \mathfrak{f}' \) and \( \mathfrak{a}' = \mathfrak{a} \cap \mathfrak{f}' \). The real vector space \( \sqrt{-1} \mathfrak{t} \) has the natural inner product \( (\cdot, \cdot) \) induced from the bilinear form \( (\cdot, \cdot) \) on \( \mathfrak{f}^\mathbb{C} \). We shall identify \( \sqrt{-1} \mathfrak{t} \) with the dual space of \( \sqrt{-1} \mathfrak{t} \) by means of the inner product \( (\cdot, \cdot) \). We have the orthogonal decomposition

\[\sqrt{-1} \mathfrak{t} = \sqrt{-1} \mathfrak{b} \oplus \sqrt{-1} \mathfrak{a}.
\]

Let \( \sigma \) be the orthogonal transformation on \( \sqrt{-1} \mathfrak{t} \) defined by

\[\sigma|\sqrt{-1} \mathfrak{b} = -1 \quad \text{and} \quad \sigma|\sqrt{-1} \mathfrak{a} = 1 \]

and

\[\varpi = \frac{1}{2}(1+\sigma); \quad \sqrt{-1} \mathfrak{t} \rightarrow \sqrt{-1} \mathfrak{a}\]

be the orthogonal projection of \( \sqrt{-1} \mathfrak{t} \) onto \( \sqrt{-1} \mathfrak{a} \). Let \( \Sigma_\mathfrak{t} \) denote the set of roots of \( \mathfrak{f}^\mathbb{C} \) with respect to the complexification \( \mathfrak{f}^\mathbb{C} \) of \( \mathfrak{t} \). Let \( W_\mathfrak{l} = N_K(T)/T \) be the Weyl group of \( \mathfrak{f} \), where \( T \) is the connected subgroup of \( K \) generated by \( \mathfrak{t} \) and \( N_K(T) \) is the normalizer of \( T \) in \( K \). \( \Sigma_\mathfrak{t} \) is a \( \sigma \)-invariant reduced root system in
\[ \sqrt{-1}t'. \] As a group of orthogonal transformations of \( \sqrt{-1}t \), \( W_t \) is generated by reflections with respect to roots in \( \Sigma_t \). Put
\[
\Sigma_t^0 = \Sigma_t \cap \sqrt{-1}b = \{ \alpha \in \Sigma_t; \sigma \alpha = 0 \},
\Sigma_s = \{ \sigma \alpha; \alpha \in \Sigma_t - \Sigma_t^0 \} = \sigma \Sigma_t - \{0\},
W_s = N_L(A)/Z_L(A),
\]
where \( A \) is the connected subgroup of \( K \) generated by \( a \) and \( N_L(A) \) (resp. \( Z_L(A) \)) the normalizer (resp. the centralizer) of \( A \) in \( L \). An element of \( \Sigma_s \) is a restricted root of the symmetric space \( S \) and \( W_s \) is the Weyl group of \( S \). \( \Sigma_s \) is a (not necessarily reduced) root system in \( \sqrt{-1}a' \). As a group of orthogonal transformations of \( \sqrt{-1}a \), \( W_s \) is generated by reflections with respect to roots in \( \Sigma_s \). A linear order \( > \) on \( \sqrt{-1}t \) is said to be compatible for \( \Sigma_t \) with respect to \( \sigma \) (or with respect to the orthogonal decomposition \( \sqrt{-1}t = \sqrt{-1}b \oplus \sqrt{-1}a \)) if \( \alpha \in \Sigma_t, \alpha > 0 \) and \( \sigma \alpha \neq -\alpha \) imply \( \sigma \alpha > 0 \). Take a compatible order \( > \) on \( \sqrt{-1}t \) and fix it once and for all. Let
\[
\Pi_t = \{ \alpha_1, \ldots, \alpha_t \}
\]
be the fundamental root system of \( \Sigma_t \) with respect to the order \( > \) and put
\[
\Pi_t^0 = \Pi_t \cap \Sigma_t^0.
\]
\( W_t \) is also generated by reflections with respect to roots in \( \Pi_t \). We have the decomposition
\[
\sigma = sp \quad \text{where } s \in W_t, \quad p \Pi_t = \Pi_t
\]
of \( \sigma \) in such a way that \( p^2 = 1 \), \( p(\Pi_t - \Pi_t^0) = \Pi_t - \Pi_t^0 \) and \( \sigma \alpha_i \equiv p \alpha_i \mod \{ \Pi_t^0 \} \) for any \( \alpha_i \in \Pi_t - \Pi_t^0 \) (Satake [10]). We put
\[
\Pi_s = \{ \sigma \alpha_i; \alpha_i \in \Pi_t - \Pi_t^0 \} = \sigma \Pi_t - \{0\}.
\]
We may assume that \( \Pi_s = \{ \gamma_1, \ldots, \gamma_p \} \) with \( \sigma \alpha_i = \gamma_i (1 \leq i \leq p) \), changing indices of the \( \alpha_i \)'s if necessary. \( \Pi_s \) is the fundamental root system of \( \Sigma_s \) with respect to the order \( > \). We put
\[
\Sigma_s^* = \{ \gamma \in \Sigma_s; 2 \gamma \in \Sigma_s \}.
\]
Then \( \Sigma_s^* \) is a reduced root system in \( \sqrt{-1}a' \). The fundamental root system \( \Pi_s^* \) of \( \Sigma_s^* \) with respect to the order \( > \) is given by
\[
\Pi_s^* = \{ \beta_1, \ldots, \beta_p \}
\]
where
\[
\beta_i = \begin{cases} 
\gamma_i & \text{if } 2\gamma_i \in \Sigma_s \\
2\gamma_i & \text{if } 2\gamma_i \in \Sigma_s.
\end{cases}
\]
\( W_s \) is also generated by reflections with respect to roots of \( \Pi_s \) or of \( \Pi_s^* \). Let
\[ \Sigma^+_t \text{ (resp. } \Sigma^+_s, (\Sigma^*_s)^+) \text{ denote the set of positive roots in } \Sigma_t \text{ (resp. } \Sigma_s, \Sigma^*_s). \]

Then
\[ \Sigma^+_s = \sigma (\Sigma^+_t - \Sigma^0) = \sigma \Sigma^+_t - \{0\}. \]

For \( \lambda \in \sqrt{-1} t, \lambda \neq 0 \), we define

\[ \lambda^* = \frac{2}{(\lambda, \lambda)} \lambda. \]

**Theorem 2.1.** (E. Cartan) Assume that \( K \) is simply connected. Then

1) \( K_\alpha \) is connected.

2) The kernel of \( \exp: \alpha \to K \) is the subgroup of \( \alpha \) generated by \{2\pi \sqrt{-1} \gamma^*; \gamma \in \Sigma_s\}.

**Theorem 2.2.** (Harish-Chandra) Let \( S^*_\ell(\mathfrak{g}) \) (resp. \( S^*_W(\alpha) \)) be the space of polynomial functions on \( \mathfrak{g} \) (resp. on \( \alpha \)), which are invariant under the adjoint actions of \( L \) (resp. of \( W_s \)). Then the restriction map
\[ S^*_\ell(\mathfrak{g}) \to S^*_W(\alpha) \]

is an isomorphism.

Now we shall consider \( W_S \)-invariant characters of a maximal torus of \( S \). Put
\[ \Gamma = \Gamma(K, L) = \{ H \in \alpha; \exp H \in L \} \]
and
\[ \Gamma_c = \Gamma \cap \mathfrak{c}_\alpha \text{ where } \mathfrak{c}_\alpha = \mathfrak{c} \cap \alpha. \]

Then \( \Gamma \) is a \( W_S \)-invariant lattice in \( \alpha \) and \( \Gamma_c \) is a lattice in \( \mathfrak{c}_\alpha \). Let \( \mathcal{C}_a \) be the connected subgroup of \( K \) generated by \( \mathfrak{c}_a \). Then the \( A \)-orbit \( \hat{A} \) in \( S \) through the origin \( x_0 \) of \( S \) and the \( \mathcal{C}_a \)-orbit \( \hat{\mathcal{C}}_a \) in \( S \) through the origin have identifications
\[ \hat{A} = \alpha/\Gamma \]
and
\[ \hat{\mathcal{C}}_a = \mathfrak{c}_a/\Gamma_c. \]

Hence both \( \hat{A} \) and \( \hat{\mathcal{C}}_a \) have structures of toral groups. The toral group \( \hat{A} \) is said to be a maximal torus of the symmetric space \( S \). The adjoint action of \( W_S \) on \( A \) induces the action of \( W_S \) on \( A \). This action is compatible with the natural action of \( W_S \) on \( \alpha/\Gamma \) relative to the identification: \( \hat{A} = \alpha/\Gamma \). Put
\[ Z = Z(K, L) = \{ \lambda \in \sqrt{-1} \alpha; (\lambda, H) \in 2\pi \sqrt{-1} Z \text{ for any } H \in \Gamma \} \]

\( Z \) is isomorphic with the group \( \mathcal{D}(\hat{A}) \) of characters of \( \hat{A} \) by the correspondence \( \lambda \mapsto e^\lambda \), where \( e^\lambda \in \mathcal{D}(\hat{A}) \) is defined by \( e^\lambda((\exp H)x_0) = \exp (\lambda, H) \) for \( H \in \alpha \). Put
\begin{align*}
D = D(K, L) &= \{\lambda \in \mathbb{Z}; (\lambda, \gamma_i) \geq 0 \text{ for any } \gamma_i \in \Pi_{s_i}\} \\
&= \{\lambda \in \mathbb{Z}; (\lambda, \gamma) \geq 0 \text{ for any } \gamma \in \sum_s^+\}.
\end{align*}

Then we have
\[D = \{\lambda \in \mathbb{Z}; s\lambda \leq \lambda \text{ for any } s \in W_s\}.
\]

An element of $D$ is called a dominant integral form on $\alpha$. We define a lattice $\Gamma_0'$ in $\alpha'$ to be the subgroup of $\alpha'$ generated by \(\{2\pi \sqrt{-1} (\frac{1}{2} \gamma^*); \gamma \in \sum_s\}\). We define a lattice $\Gamma_0$ in $\alpha$ and a toral group $\hat{A}_0$ by
\[\Gamma_0 = \Gamma_0 \oplus \Gamma_0',\]
and
\[\hat{A}_0 = \alpha/\Gamma_0.\]

Put
\[Z_0 = \{\lambda \in \sqrt{-1} \alpha; (\lambda, H) \in 2\pi \sqrt{-1} \mathbb{Z} \text{ for any } H \in \Gamma_0\}
\]
and
\[D_0 = D \cap Z_0.
\]
$Z_0$ is isomorphic with the group $\mathcal{D}(\hat{A}_0)$ of characters of $\hat{A}_0$. Put furthermore
\[Z_0' = Z_0 \cap \sqrt{-1} \alpha' = \left\{\lambda \in \sqrt{-1} \alpha'; \frac{2(\lambda, \gamma)}{(\gamma, \gamma)} \in 2\mathbb{Z} \text{ for any } \gamma \in \sum_s\right\}
\]
and
\[D_0' = D_0 \cap \sqrt{-1} \alpha' = D \cap Z_0'.
\]

Lemma 1. If $L=K_0$, then
\[\Gamma = \{\frac{1}{2} H; H \in \alpha, \exp H = e\}.
\]

Proof. For $H \in \alpha$, $\exp H = e \iff \exp \frac{H}{2} \exp \frac{H}{2} = e \iff \exp \frac{H}{2} = (\exp \frac{H}{2})^{-1} \iff \exp \frac{H}{2} = \theta\left(\exp \frac{H}{2}\right) \iff \exp \frac{H}{2} \in K_0$, which yields Lemma 1. q.e.d.

Lemma 2. 1) $\Gamma_0' = 2\pi \sqrt{-1} \sum_{i=1}^k Z(\frac{1}{2} \beta_i^*)$

and it is $W_s$-invariant. Therefore $\Gamma_0$ is $W_s$-invariant.

2) $\Gamma_0 \subset \Gamma$. Therefore $Z_0 \supset Z$ and $D_0 \supset D$.

3) If $S$ is simply connected, then $\Gamma = \Gamma_0 = \Gamma_0'$ (thus $Z = Z_0 = Z_0'$, $D = D_0 = D_0'$) and $\hat{A}_0$ can be identified with $\hat{A}$.

Proof. 1) Denoting the reflection of $\sqrt{-1} \alpha$ with respect to $\beta_i \in \Pi_s^*$ by $s_i \in W_s$, we have
\[ s_i\gamma^* = (s_i\gamma)^* = \gamma^* - \frac{2(\beta_i, \gamma)}{(\gamma, \gamma)} \beta_i^* \quad \text{for} \quad \gamma \in \sum_s. \]

It follows that \( \Gamma_0' \) is \( W_s \)-invariant. Since we have
\[ (2\lambda)^* = \frac{2 \cdot 2\lambda}{4(\lambda, \lambda)} = \frac{\lambda}{(\lambda, \lambda)} = \frac{1}{2} \lambda^* \quad \text{for} \quad \lambda \in \sqrt{-1} a, \lambda \neq 0, \]
\( \Gamma_0' \) is the subgroup of \( \alpha' \) generated by \( 2\pi\sqrt{-1}(\frac{1}{2} \gamma^*) \) for \( \gamma \in \sum_s^* \). Thus it suffices to show that
\[ \gamma^* \in \sum_{\sum_s^*} \mathbb{Z} \beta_i^* \quad \text{for any} \quad \gamma \in \sum_s^*. \]

But this follows from the first equality since there exist \( \beta_{i_1}, \ldots, \beta_{i_r} \in \Pi_s^* \) such that
\[ s_1 \cdots s_r \gamma \in \Pi_s^*. \]

2) Since \( \Gamma \subset \Gamma' \), it suffices to show that \( \Gamma_0' \subset \Gamma' \) for \( \Gamma' = \Gamma \cap \alpha' \). Let \( K' \) be the connected subgroup of \( K \) generated by \( \nu' \) and \( L' = K' \cap L \). Then \( (K', L') \) is also a symmetric pair with respect to \( \theta \) and \( S' = K'/L' \) can be identified with the \( K' \)-orbit in \( S \) through the origin \( x_0 \) of \( S \). Let
\[ \pi': K_0' \to K' \]
be the covering homomorphism of the universal covering group \( K_0' \) of \( K' \) and put
\[ L_0' = \{ k \in K_0' ; \theta_0(k) = k \}, \]
where \( \theta_0 \) is the involutive automorphism of \( K_0' \) covering the involutive automorphism \( \theta \) of \( K' \). \( K_0' \) is compact since \( K' \) is semi-simple. \( S' \) can be identified with \( K_0' \backslash \pi'^{-1}(L') \). It follows from Theorem 2.1 and Lemma 1 that \( L_0' \) is connected and
\[ \Gamma_0' = \{ H \in \alpha'; \exp_{K_0'} H \in L_0' \}. \]

Let \( \tilde{\alpha}' \) (resp. \( \tilde{A}' \)) be the connected subgroup of \( K'(\text{resp. of } K_0') \) generated by \( \alpha' \) and \( \tilde{\alpha}' \) (resp. \( \tilde{A}' \)) be the \( \alpha' \)-orbit in \( S' \) (resp. the \( \alpha_0' \)-orbit in \( S_0' = K_0'/L_0' \)) through the origin. Then we have identifications
\[ \tilde{\alpha}' = \alpha'/\Gamma' \]
and
\[ \tilde{A}_0' = \alpha'/\Gamma_0'. \]

On the other hand, since \( \pi'^{-1}(L') \supseteq L_0' \), the covering homomorphism \( \pi' \) induces the commutative diagram
\[ S_0' \xrightarrow{\pi'} S' \]
\[ \bigcup \bigcup \]
\[ \tilde{A}_0' \xrightarrow{\pi'} \tilde{A}'. \]
It follows that
\[ \Gamma'_o \subseteq \Gamma'. \]

3) Under the notation in 2), we have a covering map
\[ \hat{\mathcal{C}} \times S' \to S. \]
It follows from the assumption that \( \hat{\mathcal{C}} = \{ e \} \) and \( S' \) is simply connected. Thus the covering map \( \pi' \) is trivial and \( \Gamma = \Gamma'_o \). Moreover \( c_0 = \{ 0 \} \) implies that \( \Gamma = \Gamma' \) and \( \Gamma_o = \Gamma'_o \). q.e.d.

REMARK. Define \( \Lambda_i = \sqrt{-1} \alpha' \) \( (1 \leq i \leq l) \) by
\[ (\Lambda_i, \alpha'_j) = \delta_{ij} \quad (1 \leq i, j \leq l). \]
Then define \( M_i \) \( (1 \leq i \leq p) \) by
\[ M_i = \begin{cases} 2\Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_i') = \{ 0 \} \\ \Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_i') \neq \{ 0 \} \\ \Lambda_i + \Lambda_i & \text{if } p\alpha_i = \alpha_i \mp \alpha_i. \end{cases} \]
Then it can be verified (cf. Sugiura [12]) that \( M_i \in \sqrt{-1} \alpha' \) \( (1 \leq i \leq p) \) and
\[ (M_i, \frac{1}{2} \beta'_j) = \delta_{ij} \quad (1 \leq i, j \leq p). \]
It follows that
\[ Z'_o = \sum_{i=1}^{l} ZM_i \]
and
\[ D'_o = \{ \sum_{i=1}^{l} m_i M_i; \ m_i \in \mathbb{Z}, m_i \geq 0 \ (1 \leq i \leq p) \}. \]
It follows from Lemma 2,1) that \( W_s \) acts on \( \hat{\mathfrak{A}}_s = \alpha/\Gamma_o \) and from Lemma 2,2) that we have a \( W_s \)-equivariant homomorphism
\[ \pi_o: \hat{\mathfrak{A}}_o \to \hat{\mathfrak{A}}. \]
Let \( \mathcal{R}(\hat{\mathfrak{A}}) \) denote the character ring of \( \hat{\mathfrak{A}} \). Then \( W_s \) acts on \( \mathcal{R}(\hat{\mathfrak{A}}) \) (or more generally on the space \( C^\infty(\hat{\mathfrak{A}}) \) of \( C \)-valued \( C^\infty \)-functions on \( \hat{\mathfrak{A}} \)) by
\[ (s\chi)(\hat{a}) = \chi(s^{-1}\hat{a}) \quad \text{for } s \in W_s, \hat{a} \in \hat{\mathfrak{A}}. \]
This action coincides on \( Z = \mathcal{D}(\hat{\mathfrak{A}}) \subseteq \mathcal{R}(\hat{\mathfrak{A}}) \) with the adjoint action of \( W_s \) on \( Z \). Let \( \mathcal{R}_{W_s}(\hat{\mathfrak{A}}) \) be the subring of \( W_s \)-invariant characters of \( \hat{\mathfrak{A}} \) and \( \mathcal{R}_{W_s}(\hat{\mathfrak{A}})^C \) the \( C \)-span of \( \mathcal{R}_{W_s}(\hat{\mathfrak{A}}) \) in \( C^\infty(\hat{\mathfrak{A}}) \). Let \( \mathcal{R}(\hat{\mathfrak{A}}), \mathcal{R}_{W_s}(\hat{\mathfrak{A}}), \mathcal{R}_{W_s}(\hat{\mathfrak{A}})^C \) denote the same objects for \( \hat{\mathfrak{A}}_o \). Then \( \pi_o \) induces a \( W_s \)-equivariant monomorphism
\[ \pi_o^*: \mathcal{R}(\hat{\mathfrak{A}}) \to \mathcal{R}(\hat{\mathfrak{A}}_o) \]
and monomorphisms
\[ \pi^* : R_{W_S}(\hat{A}) \rightarrow R_{W_S}(\hat{A}_0) , \]
\[ \pi^* : R_{W_S}(\hat{A})^c \rightarrow R_{W_S}(\hat{A}_0)^c . \]

Henceforth we shall identify \( R_{W_S}(\hat{A}) \) with a subring of \( R_{W_S}(\hat{A}_0) \) and \( R_{W_S}(\hat{A})^c \) with a subalgebra of \( R_{W_S}(\hat{A}_0)^c \) by means of these monomorphisms \( \pi^* \).

For \( \lambda \in \sqrt{-1} e \), we shall denote by \( \lambda_e \) the \( \sqrt{-1} e \)-component of \( \lambda \) with respect to the orthogonal decomposition
\[ \sqrt{-1} e = \sqrt{-1} e_1 \oplus \sqrt{-1} e.' \]

The following facts can be proved in the same way as the classical results for a compact connected Lie group \( S \), so the proofs are omitted.

We define an element \( \delta \) in \( Z_\circ \) by
\[ \delta = \sum_{\gamma \in (\Sigma_R)^*} \gamma . \]

For \( \lambda \in Z_\circ \), we define \( \xi_\lambda \in R(\hat{A}_0) \) by
\[ \xi_\lambda = \sum_{\gamma \in W_S} (\det \gamma) e^{\gamma \lambda} . \]

For \( \lambda \in Z \), \( \xi_\lambda \) is divisible by \( \xi_\delta \) in the ring \( R(\hat{A}_0) \) and
\[ \chi_\lambda = \frac{\xi_{\lambda + \delta}}{\xi_\delta} \]
is in \( R_{W_S}(\hat{A}) \). If \( \chi_\lambda \) has the expression
\[ \chi_\lambda = \sum m_\mu e^\mu \quad \text{with} \quad \mu \in Z, m_\mu \in Z, m_\mu \neq 0 , \]
then \( m_\mu \) are the same for any \( \mu \). In particular, if \( \lambda \in D \), then the highest component in the above expression of \( \chi_\lambda \) is \( e^\lambda \) with \( m_\lambda = 1 \). Any \( W_S \)-invariant character \( \chi \in R_{W_S}(\hat{A}) \) of \( \hat{A} \) has an expression
\[ \chi = \sum m_\lambda \chi_\lambda \quad \text{with} \quad \lambda \in D, m_\lambda \in Z . \]

The expression is unique for \( \chi \). In particular, the system \{\( \chi_\lambda ; \lambda \in D \)\} forms a basis of the space \( R_{W_S}(\hat{A})^c \).

Now we come back to spherical representations of a symmetric pair \( (K, L) \).

**Theorem 2.3.** (E. Cartan [1]) \( \lambda \in \mathfrak{d}(K, L) \) have the highest weight \( \lambda \in \sqrt{-1} t \) and \( \omega_\lambda \) be the zonal spherical function for \( (K, L) \) belonging to \( \rho \). Then
1) \( \lambda \in D \),
2) \( \omega_\lambda \) restricted to \( \hat{A} \) is in \( R_{W_S}(\hat{A})^c \) and has an expression
\[ \omega_\lambda = \sum a_\mu e^{-\mu} \quad \text{with} \quad \mu \in Z, a_\mu \in R, a_\mu > 0, \sum a_\mu = 1 , \]
with the lowest component $a_x e^{-\lambda}$.

Proof. Proof of E. Cartan [1] was done in the case where $K$ is semi-simple and $L=K_a$. His proof can be applied for our case without difficulties. But his proof of $\lambda \in \sqrt{-1} a$ is not complete. A correct proof is seen, for example, in Schmid [11]. q.e.d.

**Lemma 3.** For any $\lambda \in \mathcal{D}$, there exists an irreducible representation $\rho$ of $K$ such that the highest weight of $\rho$ on $\mathfrak{fr}^c$ is $\lambda$.

Proof. Let $H \in \mathfrak{t}$ with $\exp H=e$. Decompose $H$ as

$$H = H' + H'' \quad \text{with} \quad H' \in \mathfrak{b}, \ H'' \in \mathfrak{a}.$$  

Then $\exp H'' = (\exp H')^{-1} \in L$, i.e. $H'' \in \mathfrak{g}$. It follows from $\lambda \in \mathbb{Z} \subset \sqrt{-1} a$ that $(\lambda, H) = (\lambda, H') + (\lambda, H'') = 2\pi \sqrt{-1} Z$. Moreover $(\lambda, \alpha_i) = (\lambda, \varepsilon \alpha_i) > 0$ for any $\alpha_i \in \mathfrak{t}$ since $\lambda \in \mathcal{D}$. Thus $e^{\lambda}$ is a dominant character of the maximal torus $T$ of $K$. Then the classical representation theory of compact connected Lie groups assures the existence of $\rho$. q.e.d.

**Lemma 4.** Let $Z_L(A)$ be the centralizer in $L$ of $A$ and $Z_L(A)^0$ the connected component of $Z_L(A)$. Then

$$Z_L(A)^0 = Z_L(A)^0 \exp \Gamma.$$  

Proof. The centralizer $Z_L(A)$ in $L$ of $A$ has the decomposition

$$Z_L(A) = Z(L)^0 A,$$  

where $Z(L)$ is the centralizer in $L$ of $A$. Since the centralizer $Z_K(A)$ in the compact connected Lie group $K$ of the torus $A$ is connected, we have the decomposition

$$Z_K(A) = Z_L(A)^0 A.$$  

It follows that any element $m \in Z_L(A)$ can be written as

$$m = m'a \quad \text{with} \quad m' \in Z_L(A)^0, \ a \in A.$$  

Then $a = m'^{-1} m \in L$ so that $a \in \exp \Gamma$. Thus $m \in Z_L(A)^0 \exp \Gamma$, which proves Lemma 4. q.e.d.

**Lemma 5.** Let $K^c$ denote the Chevalley complexification of $K$. Put

$$K^* = L \exp \sqrt{-1} \mathfrak{g}$$  

and

$$(K^*^0) = L^0 \exp \sqrt{-1} \mathfrak{g},$$  

where $L^0$ denotes the connected component of $L$. Then $(K^*)^0$ is a closed subgroup of
$K^c$ normalized by $K^*$ and

$$K^* = (K^*)^\circ \exp \Gamma.$$  

Therefore $K^*$ is a closed subgroup of $K^c$ with the connected component $(K^*)^\circ$.

Proof. The first statement is clear. Take any element $l \in L$. From the conjugateness of maximal abelian subalgebras in § under the adjoint action of $L^\circ$, there exists $l_i \in L^\circ$ such that $l_i l \in N_L(A)$. Since

$$N_L(A)/Z_L(A) = N_{L^\circ}(A)/Z_{L^\circ}(A) = W_s,$$

we can choose $l_z \in L^\circ$ such that $l_z l_i l \in Z_L(A)$. It follows from Lemma 4 that there exist $l_z \in Z_L(A)^\circ$ and $a \in \exp \Gamma$ such that $l_z l_i l = l_i a$. Therefore $l = l_i^{-1} l_z^{-1} l_i a$ with $l_i^{-1} l_z^{-1} l_i \in L^\circ \subset (K^*)^\circ$, i.e. $l \in (K^*)^\circ \exp \Gamma$. This completes the proof of Lemma 5. q.e.d.

Now we can prove the following

**Theorem 2.4.** (E. Cartan [1], Sugiura [12], Helgason [5]) For any $\lambda \in D$, there exists an irreducible spherical representation $\rho$ of $K$ relative to $L$ such that the highest weight of $\rho$ on $t^c$ is $\lambda$.

Together with Theorem 2.3 we have the following

**Corollary.** For $\rho \in \mathcal{D}(K, L)$, let $\lambda(\rho)$ denote the highest weight of $\rho$ on $t^c$. Then the correspondence $\rho \mapsto \lambda(\rho)$ gives a bijection:

$$\mathcal{D}(K, L) \rightarrow D(K, L).$$

Proof of Theorem 2.4. This theorem for the case where $K$ is semi-simple and $L = K_\circ$ was stated in E. Cartan [1] but its proof is not complete. It was stated for simply connected $K$ without proof in Sugiura [12]. It was proved in Helgason [5] for the case where $K$ is semi-simple and $L$ is connected. Helgason's proof can be applied for our case without difficulties, so we shall confine ourselves to point out necessary modifications.

Let $\rho: K \rightarrow GL(V)$ be the irreducible representation of $K$ with the highest weight $\lambda$ (Lemma 3). By extending $\rho$ to the Chevalley complexification $K^c$ of $K$ and restricting it to the closed subgroup $K^*$ of $K^c$ (Lemma 5), we have an irreducible representation of $K^*$, which will be denoted by the same letter $\rho$. It suffices to show that $\rho$ has a non-zero $L$-invariant. Let $N$ be the connected subgroup of $K^*$ generated by the subalgebra

$$n = \mathfrak{t}^* \cap \bigcup_{\sigma \in \Sigma^+} \mathfrak{t}^\sigma.$$
where \( \mathfrak{t}^* \) is the Lie algebra of \( K^* \) and \( \mathfrak{t}_0^* \) is the root space of \( \mathfrak{t}^* \) for \( \alpha \). We shall first prove that the representation \( \rho \) of \( K^* \) is a conical representation of \( K^* \) in the sense of Helgason [5], i.e. if \( v_\lambda \in V \), \( v_\lambda \neq 0 \), is a highest weight vector for \( \rho \) with respect to \( \mathfrak{t}^* \), we have

\[
\rho(mn)v_\lambda = v_\lambda \quad \text{for any} \quad m \in Z_L(A), \ n \in N .
\]

Denoting the infinitesimal action of \( \mathfrak{t}^* \) on \( V \) by the same letter \( \rho \), we have

\[
\rho(n)v_\lambda = \rho(\delta_l(a))v_\lambda = \{0\} .
\]

In fact, \( \rho(n)v_\lambda = \{0\} \) since \( n \subset \sum_{\alpha \in \Sigma^0} \mathfrak{t}_\alpha \). \( \rho(\mathfrak{t}^*)v_\lambda = \{0\} \) for the complexification \( \mathfrak{b}^\mathbb{C} \) of \( \mathfrak{b} \) since \( (\sqrt{-1} \mathfrak{b}, \lambda) = \{0\} . \). \( \rho(\mathfrak{t}^*)v_\lambda = \{0\} \) for \( \alpha \in \Sigma^0, \alpha > 0 \). It follows from \( (\alpha, \lambda) \in (\sqrt{-1} \mathfrak{b}, \lambda) = \{0\} \) for \( \alpha \in \Sigma^0 \) that \( \lambda - \alpha \) is not a weight of \( \rho \) for \( \alpha \in \Sigma^0, \alpha > 0 \). Since the complexification of \( \delta_l(a) \) is spanned by \( \mathfrak{b}^\mathbb{C} \) and the \( \mathfrak{t}_\alpha^\mathbb{C} \)'s for \( \alpha \in \Sigma^0 \), we have \( \rho(\delta_l(a))v_\lambda = \{0\} \). Therefore it suffices from Lemma 4 to show that

\[
\rho(\exp H)v_\lambda = v_\lambda \quad \text{for any} \quad H \in \Gamma .
\]

But it is clear since \( \lambda \in \mathbb{Z} \), i.e. \( (\lambda, H) \in 2\pi \sqrt{-1} \mathbb{Z} \) for any \( H \in \Gamma \).

Thus we can prove in the same way as Helgason [5] that \( V \) has a non-zero \( L \)-invariant vector, by constructing a \( K^* \)-submodule \( V' \) of the \( K^* \)-module \( C^\infty(K^*) \) of \( C^\infty \)-functions on \( K^* \), having a non-zero \( L \)-invariant, and by constructing a \( K^* \)-equivariant isomorphism of \( V \) onto \( V' \).

Next we shall describe zonal spherical functions in terms of the basis \( \{ X_\lambda; \lambda \in D \} \) of \( \mathcal{B}_{W_S}(\hat{A})^c \).

For \( \hat{a} = (\exp H)x_\alpha \in \hat{A}, H \in \mathfrak{a} \), we put

\[
D(\hat{a}) = \left| \prod_{\alpha \in \Sigma^0 - \Sigma^0} 2 \sin(\alpha, \sqrt{-1} H) \right| .
\]

Let \( d\hat{a} \) denote the normalized Haar measure of \( \hat{A} \) and \( |W_S| \) the order of the Weyl group \( W_S \). For \( W_S \)-invariant functions \( \chi, \chi' \) on \( \hat{A} \), we define

\[
\langle \chi, \chi' \rangle = \frac{c}{|W_S|} \int_{\hat{A}} \chi(\hat{a})\chi'(\hat{a})D(\hat{a})d\hat{a} ,
\]

where

\[
c = \left( \frac{1}{|W_S|} \int_{\hat{A}} D(\hat{a})d\hat{a} \right)^{-1} .
\]

\( c = 1 \) in the case where \( S \) is a compact connected Lie group. In particular, if \( \chi \) and \( \chi' \) can be extended to \( L \)-invariant functions \( f \) and \( f' \) on \( S \), then \( \langle \chi, \chi' \rangle \) coincides with the inner product \( \langle f, f' \rangle \) in \( L^2(S) \) (cf. Helgason [4]).

Fix a dominant integral form \( \lambda \in D \). We define a finite subset \( D_\lambda \) of \( D \) by
D_\lambda = \{\mu \in D; \mu_c = \lambda_c, \mu \leq \lambda\}.

Since the system \{X_\mu; \mu \in D\} forms a basis of \mathcal{R}_W(\hat{A})^c, the matrix

\langle \langle X_\mu, X_\nu \rangle \rangle_{\mu, \nu \in D_\lambda}

is a positive definite hermitian matrix. Let

\left(b^{\mu \nu}\right)_{\mu, \nu \in D_\lambda}

be the inverse matrix of the above matrix. In particular $b^{\lambda \mu} > 0$. For any $\mu \in D_\lambda$, we put

\[ c^\mu_\lambda = \frac{b^{\lambda \mu}}{\sqrt{d_\lambda b^{\lambda \lambda}}}, \]

where $d_\lambda$ is the degree of an irreducible representation of $K$ with the highest weight $\lambda$. Then we have

**Theorem 2.5.** Let $\lambda \in D$ and $\omega_\lambda$ be the zonal spherical function belonging to the class of an irreducible representation of $K$ with the highest weight $\lambda$. Then $\omega_\lambda$ restricted to $\hat{A}$ is given by

\[ \omega_\lambda = \sum_{\mu \in D_\lambda} c^\mu_\lambda X_\mu. \]

Proof. The idea of the following proof owes to Hua [6]. Let $\mu \in D_\lambda$. Then $\omega_\mu$ restricted to $\hat{A}$ is in $\mathcal{R}_W(\hat{A})^c$ by Theorem 2.3. It follows by Theorem 2.3 and Corollary of Theorem 2.4 that $\omega_\mu$ has an expression

\[ \omega_\mu = \sum_{\nu \in D_\lambda} c^{\nu \mu}_\mu X_\nu \quad \text{with} \quad c^{\nu \mu}_\mu \in \mathbb{R}, c^{\nu \mu}_\mu > 0, c^{\nu \mu}_\mu = 0 \quad \text{if} \quad \nu > \mu. \]

We define an upper triangular matrix $C'$ by

\[ C' = (c^{\nu \mu}_\mu)_{\mu, \nu \in D_\lambda}. \]

Then we have

\[ \langle \langle \omega_\mu, \omega_\nu \rangle \rangle_{\mu, \nu \in D_\lambda} = 'C'\langle \langle X_\mu, X_\nu \rangle \rangle_{\mu, \nu \in D_\lambda} C'. \]

Since $\langle \langle \omega_\mu, \omega_\nu \rangle \rangle = d_\mu^{\nu - 1} \delta_{\mu \nu}$, we have

\[ (d_\mu \delta_{\mu \nu})_{\mu, \nu \in D_\lambda} = C'^{-1} B'^{-1} C'^{-1}, \]

where

\[ B' = (b^{\mu \nu})_{\mu, \nu \in D_\lambda} = \langle \langle X_\mu, X_\nu \rangle \rangle_{\mu, \nu \in D_\lambda}^{-1}. \]

It follows that

\[ C'(d_\mu \delta_{\mu \nu})_{\mu, \nu \in D_\lambda} 'C' = B'. \]

Comparing $(\mu, \lambda)$-components of both sides, we have
In particular

\[(c^\lambda_\mu d_\lambda c^\lambda_\mu = b^\lambda \mu \lambda .\]

hence

\[c^\mu_\lambda = \frac{b^\mu_\lambda}{\sqrt{d_\lambda b^\lambda \mu}} .\]

Since \(b^\mu_\nu = b^\nu_\mu\), we have

\[c^\mu_\lambda = \frac{b^\mu_\lambda}{\sqrt{d_\lambda b^\lambda \mu}} = c^\mu_\lambda .\]

q.e.d.

**EXAMPLE.** If \(S\) is a compact connected Lie group and \((K, L)\) the symmetric pair with \(K/L = S\) as mentioned before, then the set \(\mathcal{D}(S)\) of equivalence classes of irreducible representations of \(S\) is in the bijective correspondence with \(\mathcal{D}(K, L)\) by the assignment \(\rho \mapsto \rho \boxtimes \rho^*\), where \(\rho^*\) denotes the contragredient representation of \(\rho\). \(\hat{A}\) is a maximal torus of the compact Lie group \(S\). Let \(\chi_\rho\) be the invariant character of \(\hat{A}\) for the dominant integral form in \(D(K, L)\) corresponding to \(\rho \boxtimes \rho^*\) by the bijection in Corollary of Theorem 2.4. Then it is nothing but the character of \(\rho\). It follows from orthogonality relations of irreducible characters that the matrix \((b^\mu_\nu)\) is the identity matrix. Thus the zonal spherical function \(\omega_{\rho \boxtimes \rho^*}\) belonging to \(\rho \boxtimes \rho^*\) is given by

\[\omega_{\rho \boxtimes \rho^*} = \frac{1}{d_\rho} \chi_\rho ,\]

where \(d_\rho\) is the degree of \(\rho\).

3. **Polynomial representations associated with symmetric bounded domains**

Let \(D\) be an irreducible symmetric bounded domain with rank \(p\) realized in \((p^c)^-\) as in Introduction. We shall use the same notation as in Introduction.

Let

\[\Pi = \{\alpha_1, \cdots, \alpha_l\}\]

be the fundamental root system of \(\Sigma\) with respect to the order \(>\) and let \(\Pi_0 = \Pi \cap \Sigma_0\). It is known that \(\Pi_0\) is the fundamental root system of \(\Sigma_0\), \(\Pi - \Pi_0\) consists of one element, say \(\alpha_0\), which is the lowest root in \(\Sigma_0^+\), and for any \(\alpha = \sum_{i=1}^l m_i \alpha_i \in \Sigma_0^+\), \(m_i = 1\). Let \(\Sigma_0^+\) denote the set of positive compact roots.

Put

\[b = \{H \in a; (\sqrt{-1}H, \Delta) = \{0\}\} .\]
Then we have the orthogonal decomposition
\[ \sqrt{-1}t = \sqrt{-1}b \oplus \sqrt{-1}a \]
with respect to \((\ , \ )\). We define an orthogonal transformation \(\sigma\) on \(\sqrt{-1}t\) by
\[ \sigma|b = -1 \quad \text{and} \quad \sigma|\sqrt{-1}a = 1. \]
Let
\[ \varpi = \frac{1}{2}(1+\sigma); \quad \sqrt{-1}t \to \sqrt{-1}a \]
be the orthogonal projection of \(\sqrt{-1}t\) onto \(\sqrt{-1}a\). Let \(\kappa\) be the unique involutive element of the Weyl group \(\mathcal{W}\) of \(K\) such that \(\kappa \Pi_1 = -\Pi_1\). Since \(\Sigma^n_0\) is the set of weights on \(t^c\) of the irreducible \(K\)-module \((p^c)^*\), we have \(\kappa \Sigma^n_0 = \Sigma^n_0\) and \(\kappa \gamma_i = \alpha_i\). Put
\[ \Delta' = \kappa \Delta = \{\gamma'_1, \ldots, \gamma'_p\}, \quad \gamma'_i = \kappa \gamma_i \quad (1 \leq i \leq p), \quad \gamma'_i = \alpha_i. \]
It is the original maximal strongly orthogonal subsystem of \(\Sigma^n_0\) of Harish-Chandra [3]. For the system \(\Delta'\), the orthogonal projection
\[ \varpi' : \sqrt{-1}t \to \sqrt{-1}a' \]
on to the \(R\)-span \(\sqrt{-1}a'\) of \(\Delta'\) is defined in the same way as for \(\Delta\). Put
\[ P'_1 = \{\alpha \in \Sigma^n_0; \quad \varpi' (\alpha) = \frac{1}{2}(\gamma'_i + \gamma'_j) \quad \text{for some} \quad 1 \leq i < j \leq p\}, \]
\[ P'_i = \{\alpha \in \Sigma^n_0; \quad \varpi' (\alpha) = \frac{1}{2} \gamma'_i \quad \text{for some} \quad 1 \leq i \leq p\}, \]
\[ K'_0 = \{\alpha \in \Sigma^n_0; \quad \varpi' (\alpha) = \frac{1}{2}(\gamma'_i - \gamma'_j) \quad \text{for some} \quad 1 \leq i, j \leq p\}, \]
\[ K'_i = \{\alpha \in \Sigma^n_0; \quad \varpi' (\alpha) = \frac{1}{2} \gamma'_i \quad \text{for some} \quad 1 \leq i \leq p\}. \]
Then (Harish-Chandra [3]) \(\Sigma\) is the disjoint union of \(P'_1, -P'_1, P'_i, -P'_i, K'_0, K'_i\) and we have
\[ \varpi' P'_1 = \left\{ \frac{1}{2}(\gamma'_i + \gamma'_j); \quad 1 \leq i \leq p \right\}, \]
\[ \varpi' P'_i = \left\{ \frac{1}{2} \gamma'_i; \quad 1 \leq i \leq p \right\} \quad \text{if} \quad P'_i \neq \phi, \]
\[ \varpi' K'_0 - \{0\} = \left\{ \pm \frac{1}{2}(\gamma'_i - \gamma'_j); \quad 1 \leq i < j \leq p \right\}, \]
\[ \varpi' K'_i = \left\{ \frac{1}{2} \gamma'_i; \quad 1 \leq i \leq p \right\} \quad \text{if} \quad P'_i \neq \phi. \]
Furthermore the multiplicity (with respect to \(\varpi'\)) of any \(\gamma'_i\) is 1 and that of any \(\frac{1}{2} \gamma'_i\) is even. It follows that
\[ \varpi' \Sigma - \{0\} = \left\{ \begin{array} { l } { \pm \frac{1}{2}(\gamma'_i \pm \gamma'_j); \quad 1 \leq i < j \leq p, \quad \pm \gamma'_i; \quad 1 \leq i \leq p } \end{array} \right\} \quad \text{if} \quad P'_i = \phi \]
\[ \left\{ \begin{array} { l } { \pm \frac{1}{2}(\gamma'_i \pm \gamma'_j); \quad 1 \leq i < j \leq p, \quad \pm \gamma'_i, \quad \pm \frac{1}{2} \gamma'_j; \quad 1 \leq i \leq p } \end{array} \right\} \quad \text{if} \quad P'_i = \phi. \]
Moreover we have (Moore [8])
\[ \varpi' \Pi - \{0\} = \left\{ \begin{array} { l } { \gamma'_1, \frac{1}{2}(\gamma'_2 - \gamma'_1), \ldots, \frac{1}{2}(\gamma'_p - \gamma'_{p-1}) } \end{array} \right\} \quad \text{if} \quad P'_1 = \phi \]
\[ \left\{ \begin{array} { l } { \gamma'_1, \frac{1}{2}(\gamma'_2 - \gamma'_1), \ldots, \frac{1}{2}(\gamma'_p - \gamma'_{p-1}), \quad -\frac{1}{2} \gamma'_p } \end{array} \right\} \quad \text{if} \quad P'_1 = \phi, \]
and

\[ \varpi' \Pi_t - \{0\} = \begin{cases} \{ \frac{1}{2} (\gamma_i' - \gamma_{i-1}'), \cdots, \frac{1}{2} (\gamma_p' - \gamma_{p-1}') \} & \text{if } P_i' = \phi \\ \{ \frac{1}{2} (\gamma_i' - \gamma_{i-1}'), \cdots, \frac{1}{2} (\gamma_p' - \gamma_{p-1}'), -\frac{1}{2} \gamma_p' \} & \text{if } P_i' \neq \phi . \end{cases} \]

**Lemma 1.**

1) \[ \varpi \alpha_i = \varpi \alpha_i = \begin{cases} \gamma_p' & \text{if } P_i' = \phi \\ \frac{1}{2} \gamma_p' & \text{if } P_i' \neq \phi . \end{cases} \]

2) (Schmid [11]) If \( P_i' \neq \phi \) and

\[ \sum_{\beta \in \mathcal{P}_i} m_\beta \beta \quad \text{with } m_\beta \geq 0 \]

is in the \( R \)-span \( \{ P_i' \}_R \) of \( P_i' \), then \( m_\beta = 0 \) for any \( \beta \).

**Proof.** For any \( \alpha \in \sum_{\beta \geq 0} = \mathcal{P}_i \cup P_i' \), \( \varpi' \alpha \) can be written as

\[ \varpi' \alpha = \frac{1}{2} m_1 (\gamma_2' - \gamma_1') + \frac{1}{2} m_2 (\gamma_3' - \gamma_2') + \cdots + \frac{1}{2} m_{p-1} (\gamma_p' - \gamma_{p-1}') \\
- \frac{1}{2} m_p \gamma_p' + m_{p+1} \gamma_{i+1}' \\
= \frac{1}{2} (2m_{p+1} - m_i) \gamma_1' + \frac{1}{2} (m_i - m_1) \gamma_2' + \cdots + \frac{1}{2} (m_{p+2} - m_{p+1}) \gamma_{p-1}' \\
+ \frac{1}{2} (m_{p+1} - m_p) \gamma_p' \]

where \( m_i \in \mathbb{Z}, m_i \geq 0, m_{p+1} = 1 \). Since \( \varpi' \alpha = \frac{1}{2} (\gamma_i' + \gamma_j') \) or \( \frac{1}{2} \gamma_i' \) for some \( i, j \), we have

\[ 2 \geq m_1 \geq m_2 \geq \cdots \geq m_{p+1} \geq m_p \geq 0 . \]

Furthermore \( \alpha \in \mathcal{P}_i' \) (resp. \( \alpha \in \mathcal{P}_i \)) if and only if \( m_p = 0 \) (resp. \( m_p = 1 \)).

1) If \( P_i' = \phi \), then \( \gamma_1' = r_i' \). For \( \alpha = r_i' \), the coefficients in the above expression are \( m_i = \cdots = m_{p+1} = 2, m_p = 0 \) and \( \varpi' r_i' = \gamma_p' \). If \( P_i' = \phi \), then for \( \alpha = r_i' \), the coefficients are \( m_i = \cdots = m_{p+1} = 2, m_p = 1 \) and \( \varpi' r_i' = \frac{1}{2} \gamma_p' \). Now the assertion 1) follows from \( \varpi \alpha_i = \kappa^{-1} \varpi' \kappa \alpha_i = \kappa^{-1} \varpi' r_i' \).

2) Let

\[ \alpha = \sum_{i=1}^{l} n_i \alpha_i \quad \text{with } n_i \in \mathbb{Z}, n_i \geq 0 \]

be in \( \sum_{\beta \geq 0} \). It follows from the first argument that

(a) if \( \alpha \in \mathcal{P}_i' \), \( \varpi' \alpha_i = -\frac{1}{2} \gamma_p' \), then \( n_i = 0 \),

(b) if \( \alpha \in \mathcal{P}_i \), then there exists \( \alpha_i \in \Pi_t \) such that \( n_i > 0 \) and \( \varpi' \alpha_i = -\frac{1}{2} \gamma_p' \).

This implies the assertion 2).

q.e.d.
Now $P_1, P_3, K_o$ and $K_i$ are defined for $\Delta$ in the same way as for $\Delta'$. Then $\kappa$ transforms $P_1$ (resp. $P_3, K_o, K_i$) onto $P_1'$ (resp. $P_3', K_o', K_i'$). It follows that the above mentioned properties due to Harish-Chandra are also satisfied by our objects for $\Delta$. But Moore's results should be modified as follows.

They follows from Lemma 1, 1) and

$$\varpi \Pi = \kappa^{-1} \varpi' \kappa \Pi = -\kappa^{-1} \varpi' \Pi.$$

Note that $K_1 \subset \Sigma^+_t$ while $K_4' \subset -\Sigma^+_t$.

Lemma 2. 1) The order $>^t$ is a compatible order for $\Sigma$ with respect to $\sigma$ in the sense of 2.

2) $\varpi K_o - \{0\}$ is a root system with the fundamental root system

$$\{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p) \}$$

with respect to the order $>^t$.

3) If $P_1 \neq \phi$ and

$$\sum_{\beta \in P_1} m_\beta \beta$$

with $m_\beta \geq 0$

is in the $R$-span $\{P_i\}_R$ of $P_i$, then $m_\beta = 0$ for any $\beta$.

Proof. 1) is clear from the form of $\varpi \Pi - \{0\}$ above.

2) is clear since

$$\varpi K_o - \{0\} = \{ \pm \frac{1}{2} (\gamma_i - \gamma_j); 1 \leq i < j \leq p \}.$$

3) follows from Lemma 1, 2) and $\kappa P_1 = P_1', \kappa P_3 = P_3'$. q.e.d.

For $\lambda \in \sqrt{-1}t$, $\lambda \neq 0$, we define as in 2

$$\lambda^* = \frac{2}{(\lambda, \lambda)} \lambda.$$
and put
\[ Z_0 = \frac{1}{2} \sum_{\beta \in \Delta} \gamma^*. \]

Since \((\frac{1}{2} \gamma_i, \gamma_j^*) = \delta_{ij}\) for \(1 \leq i, j \leq p\), we have
\[ P_i = \{ \alpha \in \Sigma_p; (\alpha, Z_0) = 1 \}, \]
\[ P_i = \{ \alpha \in \Sigma_p; (\alpha, Z_0) = \frac{1}{2} \}, \]
\[ K_0 = \{ \alpha \in \Sigma_p; (\alpha, Z_0) = 0 \}, \]
\[ K_i = \{ \alpha \in \Sigma_p; (\alpha, Z_0) = \frac{1}{2} \}. \]

Hence eigenvalues of \(\text{ad} Z_0\) are \(\pm 1, \pm \frac{1}{2}\) on \(\mathfrak{p}_c\), \(0, \pm \frac{1}{2}\) on \(\mathfrak{k}_c\). Let \(\mathfrak{p}_{c+}, \mathfrak{p}_{c-}, \mathfrak{k}_c\), \(\mathfrak{r}_{c+}\) denote the corresponding eigenspaces. Note that the origin \(X_0\) of the Shilov boundary \(S\) is in \(\mathfrak{p}_c\).

The following results are due to Korányi-Wolf [7]. We define an element \(c\) of \(G^c\), which is called the Cayley transform, by
\[ c = \exp \left( -\frac{\pi}{4} \sum_{\gamma \in \Delta} (X_\gamma + X_{-\gamma}) \right) \]
and define an automorphism of \(G^c\) by
\[ \theta(x) = c^2 x c^{-2} \quad \text{for} \quad x \in G^c. \]

The automorphism \(\text{Ad} c^2\) of \(g^c\) obtained by differentiating \(\theta\) will be also denoted by the same letter \(\theta\). Then \(\theta^* = 1\) and on \(\sqrt{-1} \mathfrak{t}\) it coincides with \(-\sigma\). Put
\[ g_0 = \{ X \in \mathfrak{g}; \theta^a X = X \}, \]
\[ \mathfrak{f}_0 = g_0 \cap \mathfrak{f}, \]
and
\[ \mathfrak{p}_0 = g_0 \cap \mathfrak{p}. \]

Then \(\mathfrak{f}_0\) is \(\theta\)-invariant and
\[ \mathfrak{f}_0 = \{ X \in \mathfrak{f}; [Z_0, X] = 0 \}. \]

Hence \(\mathfrak{f}_0\) is a real form of \(\mathfrak{f}_0^c\) containing \(\mathfrak{t}\) as a maximal abelian subalgebra. \(K_0\) is nothing but the set of roots of \(\mathfrak{p}_0^c\) with respect to \(\mathfrak{t}_c^0\). The complexification \(\mathfrak{p}_c^0\) of \(\mathfrak{p}_0\) is the direct sum of \(\mathfrak{p}_{c+}^c\) and \(\mathfrak{p}_{c-}^c\). \(g_0\) is a reductive subalgebra of \(g\) with a Cartan decomposition
\[ g_0 = \mathfrak{f}_0 + \mathfrak{p}_0. \]
Let $G_0$ (resp. $K_0$) be the connected subgroup of $G$ generated by $g_0$ (resp. by $l_0$) and let

$$L_0 = \{ k \in K_0; \text{Ad}kX_0 = X_0 \} = K_0 \cap L.$$ 

Put

$$D_0 = D \cap p^c_1$$

and

$$S_0 = S \cap p^c_1.$$ 

Then $G_0$ acts on $D_0$ transitively and $K \cap G_0$ coincides with $K_0$, so that $D_0$ is identified with the quotient space $G_0/K_0$. Furthermore $K_0$ acts on $S_0$ transitively so that $S_0$ is identified with $K_0/L_0$. $D_0$ is totally geodesic in $D$ with respect to Bergmann metric of $D$ and it is also an irreducible symmetric bounded domain with the same rank as $D$. $S_0$ is the Shilov boundary of $D_0$. The complex structure of $D_0$ is given at the origin by $\text{ad}H_0$ with $\sqrt{-1}H_0 = Z_0$. We have

$$\omega Z = Z_0.$$ 

The inclusion $D_0 \subset p^c_1$ is nothing but the Harish-Chandra's imbedding of $D_0 = G_0/K_0$. $(K_0, L_0)$ is a symmetric pair with respect to $\theta$, having the same rank as $D$. Hence

$$l_0 = \{ X \in l_0; \theta X = X \}$$ 

is the Lie algebra of $L_0$ and $\alpha$ is a maximal abelian subalgebra of

$$\mathfrak{g}_0 = \{ X \in l_0; \theta X = -X \}.$$ 

We can define a semi-linear transformation $X \mapsto \bar{X}$ of $p^c_1$ by

$$\bar{X} = \tau \theta X = \theta \tau X \quad \text{for} \quad X \in p^c_1.$$ 

Put

$$p_{-1} = \{ X \in p^c_1; \bar{X} = X \}.$$ 

It is a real form of $p^c_1$ and is invariant under the adjoint action of $L_0$ on $p^c_1$. The correspondence $X \mapsto [X, \bar{X}]$ gives an isomorphism

$$\psi: \sqrt{-1} \mathfrak{g}_0 \rightarrow p_{-1},$$

which is equivariant with respect to the adjoint actions of $L_0$.

Now we shall consider the polynomial representation $S^*((p^c)^{-})$ of $K$. Let $S_0((p^c)^{+})$ be the symmetric algebra over $(p^c)^{+}$. $K$ acts on $S_0((p^c)^{+})$ by the natural extension Ad of the adjoint action of $K$ on $(p^c)^{+}$. On the other hand, the non-degenerate pairing

$$(p^c)^{+} \times (p^c)^{-} \rightarrow \mathbb{C}$$
by means of the Killing form \((\cdot,\cdot)\) induces the identification
\[ S_\#((\mathfrak{p}^c)^+) = S^*((\mathfrak{p}^c)^-) .\]
This identification is compatible with the actions of \(K\), since the Killing form is invariant under the adjoint action of \(K\). In the same way we have a \(K_0\)-equivariant identification
\[ S_\#(\mathfrak{p}_1^c) = S^*(\mathfrak{p}_1^c) .\]
\(S_\#(\mathfrak{p}_1^c)\) can be considered as a \(K_0\)-submodule of \(S_\#((\mathfrak{p}^c)^+)\) by means of the natural monomorphism \(S_\#(\mathfrak{p}_1^c) \to S_\#((\mathfrak{p}^c)^+)\) induced from the inclusion \(\mathfrak{p}_1^c \subset (\mathfrak{p}^c)^+\).

**Theorem 3.1.** (i) Any irreducible \(K\)-submodule of \(S_\#((\mathfrak{p}^c)^+)\) (resp. \(K_0\)-submodule of \(S_\#(\mathfrak{p}_1^c)\)) is contained exactly once in \(S_\#((\mathfrak{p}^c)^+)\) (resp. in \(S_\#(\mathfrak{p}_1^c)\)).

(ii) For an irreducible \(K\)-submodule \(V\) of \(S_\#((\mathfrak{p}^c)^+)\), we put
\[ V_0 = V \cap S_\#(\mathfrak{p}_1^c) .\]
Then \(V \mapsto V_0\) is the one to one correspondence between the set of irreducible \(K\)-submodules of \(S_\#((\mathfrak{p}^c)^+)\) and the set of irreducible \(K_0\)-submodules of \(S^*(\mathfrak{p}_1^c)\) in such a way that
1) The highest weights on \(\mathfrak{t}^c\) of \(V\) and \(V_0\) are the same.
2) The subspace of \(L\)-invariants in \(V\) is 1-dimensional and contained in \(V_0\).

(iii) The highest weight \(\lambda \in \sqrt{-1} \mathfrak{t}\) of an irreducible \(K\)-submodule \(V\) of \(S_\#((\mathfrak{p}^c)^+)\) is of the form
\[ \lambda = \sum_i n_i \gamma_i , \quad n_i \in \mathbb{Z}, n_1 \geq n_2 \geq \cdots \geq n_p \geq 0 .\]
If \(\sum_i n_i = \nu\), then \(V\) is contained in \(S_\#((\mathfrak{p}^c)^+)\). i.e. \(S^*(D) \subset S^*(K, L)\) under the notation in Introduction.

For the proof of the theorem, we need the following

**Lemma 3.** (Murakami [9]) Let \(\mathfrak{k}\) be a Lie algebra over \(\mathbb{R}\) and \(\mathfrak{k}^c\) the complexification of \(\mathfrak{k}\). Assume that there exists \(Y \in \sqrt{-1} \mathfrak{k} \subset \mathfrak{k}^c\) such that \(\mathfrak{k}^c\) is the direct sum of 0-eigenspace \(\mathfrak{k}_0^c\), \((+1)\)-eigenspace \(\mathfrak{k}_1^c\) and \((-1)\)-eigenspace \(\mathfrak{k}_2^c\) of \(\text{ad} Y\), respectively. Let \((\rho, V)\) be a complex irreducible \(\mathfrak{k}\)-module with \(\mathfrak{t}\)-invariant hermitian inner product. Denoting the extension to \(\mathfrak{k}^c\) of \(\rho\) by the same letter \(\rho\), let \(a_1, a_2, \ldots, a_m\) \((a_i \in \mathbb{R})\) be eigenvalues of \(\rho(Y)\), and \(S_t\) be \(a_t\)-eigenspace of \(\rho(Y)\) \((1 \leq t \leq m)\). Put \(\mathfrak{k}_0^c = \mathfrak{k}_0^c \cap \mathfrak{k}\) (which is a real form of \(\mathfrak{k}_0^c\)). Then
1) \(a_t = a_{t-1} - t + 1\) \((1 \leq t \leq m)\).
2) Each \(S_t\) is a \(\mathfrak{k}_0^c\)-submodule of \(V\) and
\[ V = S_1 + \cdots + S_m \]
is the orthogonal direct sum.
3) \( S_i \) and \( S_m \) are irreducible \( \mathfrak{t}_e \)-submodules of \( V \) and characterized by
\[
S_i = \{ v \in V; \rho(X)v = 0 \text{ for any } X \in \mathfrak{t}_e \}, \\
S_m = \{ v \in V; \rho(X)v = 0 \text{ for any } X \in \mathfrak{t}_e \}.
\]

Proof of Theorem 3.1. The infinitesimal action of \( \mathfrak{t}_e \) on \( S_{*}(\mathfrak{p}_C)^{+} \) induced from the adjoint action \( \text{Ad} \) of \( K \) will be denoted by \( \text{ad} \).

Let \( V \) be an irreducible \( K \)-submodule of \( S_{*}(\mathfrak{p}_C)^{+} \). Since \( Z \) is in the center of \( \mathfrak{t}_e \), it follows from Schur's lemma that \( V \) is contained in an eigenspace of \( \text{ad} Z \) in \( S_{*}(\mathfrak{p}_C)^{+} \). But since \( \text{ad} Z \) is the scalar operator \( \nu \) on \( S_{*}(\mathfrak{p}_C)^{+} \), \( V \) is contained in \( S_{*}(\mathfrak{p}_C)^{+} \) for some \( \nu \). Let \( \lambda \in \sqrt{-1} t \) be the highest weight of \( V \). Put \( Y = 2Z_{0} \in \sqrt{-1} t \subset \mathfrak{t}_C \). Then the decomposition
\[
\mathfrak{t}_C = \mathfrak{t}_0^{+} + \mathfrak{t}_i + \mathfrak{t}_0^{-}
\]
satisfies the assumption in Lemma 3. So we have a decomposition
\[ V = S_{1} + \cdots + S_{m} \]
into \( K_{e} \)-submodules, where \( S_{i} \) is an irreducible \( K_{e} \)-submodule and is the eigenspace for the maximum eigenvalue of \( \text{ad} Y \) in \( V \). It is characterized by
\[
S_{i} = \{ v \in V; \text{ad} (X)v = 0 \text{ for any } X \in \mathfrak{t}_e \}.
\]
Thus a highest weight vector \( v_{\lambda} \) of the \( K \)-module \( V \) is contained in \( S_{i} \) because of \( K_{i} \subset \mathfrak{t}_e^{+} \). It follows that putting \( V_{0} = S_{1} \), \( V_{0} \) is an irreducible \( K_{e} \)-submodule of \( S_{*}(\mathfrak{p}_C)^{+} \) with the highest weight \( \lambda \).

We shall show that \( V_{0} = V \cap S_{*}(\mathfrak{p}_C) \). We have the decomposition
\[
S_{*}(\mathfrak{p}_C)^{+} = \sum_{r, s} S_{r}(\mathfrak{p}_C^r) \otimes S_{s}(\mathfrak{p}_C^s)
\]
as \( K_{e} \)-modules. \( \text{ad} Z_{0} \) is the scalar operator \( r + \frac{1}{2} s = \frac{1}{2}(r + \nu) \) on \( S_{r}(\mathfrak{p}_C^r) \otimes S_{s}(\mathfrak{p}_C^s) \). In the same way as the first argument, we can get the decomposition
\[ V = V_{1} + \cdots + V_{h} \]
into irreducible \( K_{e} \)-submodules such that any \( V_{i} \) is contained in \( S_{r}(\mathfrak{p}_C^r) \otimes S_{s}(\mathfrak{p}_C^s) \) for some \( r, s \). Since \( S_{*}(\mathfrak{p}_C)^{-} \) is \( K \)-isomorphic with \( S_{*}(S) \subset C_{\infty}(S) \), \( V \) has an \( L \)-invariant \( w \neq 0 \). Decompose \( w \) as
\[ w = w_{1} + \cdots + w_{h}, \quad w_{i} \in V_{i} \quad (1 \leq i \leq k). \]
At least one of the \( w_{i} \)'s, say \( w_{1} \), is not zero. Let \( \lambda_{1} \in \sqrt{-1} t \) be the highest weight of the irreducible \( K_{e} \)-module \( V_{1} \). Since \( w_{1} \) is a non-zero \( L_{0} \)-invariant of \( V_{1} \), \( V_{1} \) is a spherical \( K_{e} \)-module relative to \( L_{0} \). \( (K_{0}, L_{0}) \) is a symmetric pair, \( a \) is a maximal abelian subalgebra of \( S_{0} \) and the order \( > \) on \( \sqrt{-1} t \) is a compatible order for \( K_{0} \) with respect to \( \sigma \) by Lemma 1, 1), so we shall use the notations
\( \Gamma(K_o, L_o), Z(K_o, L_o), D(K_o, L_o) \) in 2. Then it follows from Theorem 2.3 that \( \lambda, \in D(K_o, L_o) \). On the other hand, if \( V_1 \subset S_\gamma(p^c) \otimes S_\lambda(p^c) \), \( \lambda, \) is of the form

\[
\lambda = \sum_{\alpha \in P_1} m_\alpha \alpha + \sum_{\beta \in P_2} m_\beta \beta, \quad m_\alpha, m_\beta \in \mathbb{Z}, \quad m_\alpha \geq 0, \quad m_\beta \geq 0
\]

with \( \sum m_\alpha = r, \sum m_\beta = s \). Since \( D(K_o, L_o) \subset \sqrt{-1} \Delta = \{ \Delta \}_R \subset \{ P_1 \}_R \), we have

\[
\sum_{\beta \in P_2} m_\beta \beta \in \{ P_1 \}_R.
\]

It follows from Lemma 2,3) that \( r = u, s = 0 \), i.e. \( V_1 \subset V \cap S_\gamma(p^c) \). On the other hand, \( V \cap S_\gamma(p^c) \subset V \) since the possible maximum eigenvalue of \( \text{ad} F \) on \( F \) is 2\( v \). Thus we have that \( V_o = V_1 = V \cap S_\gamma(p^c) \).

The above argument shows also that any \( L \)-invariant in \( V \) is contained in \( V_o \). It is unique up to scalar since \( (K_o, L_o) \) is a symmetric pair.

Conversely, let \( V_o \) be an irreducible \( K_o \)-submodule of \( S_\gamma(p^c) \) with the highest weight \( \lambda \in \sqrt{-1} \mathfrak{t} \). In the same way as the first argument, we know that \( V_o \) is contained in \( S_\lambda(p^c) \) for some \( \nu \). Let \( \psi_\lambda \in V_o \) be a highest weight vector. Then \( \text{ad} f_i \psi_\lambda = \{ 0 \} \) because of \( [f_i, \psi_\lambda] = \{ 0 \} \). Hence \( \text{ad} X_\alpha \psi_\lambda = 0 \) for any \( \alpha \in \sum^+ \). We define \( V \) to be the \( C \)-span of \( \{ \text{ad} k \psi_\lambda; k \in K \} \) in \( S_\lambda((p^c)^+) \). Then \( V \) is an irreducible \( K \)-submodule of \( S_\lambda((p^c)^+) \) with the highest weight \( \lambda \in \sqrt{-1} \mathfrak{t} \).

It is easy to see that each of the above correspondences \( V \mapsto V_o \) and \( V_o \mapsto V \) is the inverse of the other. This proves assertions (i) and (ii).

(iii) We have \( [\frac{1}{2} \gamma_i, X_{-\gamma_i}] = -\delta_{ij} X_{-\gamma_j} \) (\( 1 \leq i, j \leq p \)) because of \( (\frac{1}{2} \gamma_i, \gamma_i) = \delta_{ij} \) (\( 1 \leq i, j \leq p \)). It follows that for \( H = 2\pi \sqrt{-1} \sum_{i=1}^p \gamma_i, \gamma_i \) we have

\[
\text{Ad}(\exp H)X_{\gamma_i} = -\sum_{i=1}^p \exp(-2\pi \sqrt{-1} x_i) X_{-\gamma_i}.
\]

Thus we have

\[
\Gamma(K_o, L_o) = 2\pi \sqrt{-1} \sum_{i=1}^p Z(\frac{1}{2} \gamma_i)
\]

and

\[
Z(K_o, L_o) = \sum_{i=1}^p Z \gamma_i.
\]

It follows from Lemma 2,2) that

\[
D(K_o, L_o) = \{ \sum_{i=1}^p n_i \gamma_i; n_i \in \mathbb{Z}, n_1 \geq n_2 \geq \cdots \geq n_p \}.
\]

Therefore \( \lambda \) is of the form

\[
\lambda = \sum_{i=1}^p n_i \gamma_i \quad \text{with} \quad n_i \in \mathbb{Z}, \quad n_1 \geq n_2 \geq \cdots \geq n_p.
\]

On the other hand, \( \lambda \) is of the form
\[ \lambda = \sum_{a \in P_1} m_a \alpha \] with \( m_a \in \mathbb{Z}, m_a \geq 0 \),

which implies that \( n_1 \geq \cdots \geq n_p \geq 0 \). If \( V \subset S_n((\mathfrak{p}^\circ)^+) \), then \( V_0 \subset S_0(\mathfrak{p}^0_k) \) and \( \text{ad} \, Z_0 \) is the scalar operator \( \nu \) on \( V_0 \), which equals \( (\lambda, Z_0) = \sum_{i=1}^p n_i \).

**Remark.** In terms of polynomial functions \( S^*((\mathfrak{p}^\circ)^-) \), for an irreducible \( K \)-submodule \( V \) of \( S^*((\mathfrak{p}^\circ)^-) \), \( V_0 \) is obtained by restriction to \( \mathfrak{p}^0_k \) of functions in \( V \).

**Proof of Theorem A.** Orthogonality relations for the \( S^*(D)^* \)'s (resp. for the \( S^*(S)^* \)'s) and the assertion that the restriction \( S^*(D)^* \to S^*(S)^* \) is a similitude follow from Schur's lemma. So it suffices to show that the cardinalities of \( S^*(D) \) and \( S^*(K, L) \) are the same.

From the first argument in the proof of Theorem 3.1 (iii), we see that \( \psi((\frac{1}{2} \gamma^*)) = X_\gamma (\gamma \in \Delta) \) for the \( L_0 \)-equivariant isomorphism \( \psi: \sqrt{\cdot - 1} \xi_0 \to \mathfrak{p} \). We put

\[ \alpha^- = \psi(\sqrt{\cdot - 1} \alpha) = \{ X_\gamma; \gamma \in \Delta \} \subset \mathfrak{p} \text{.} \]

Since the Weyl group \( W_{S_0} \) of \( S_0 \) is isomorphic with the group of permutations of \( \Delta \) by Lemma 2.2, the "Weyl group" \( W_{S_0}^- = N_{L_0}(\alpha^-)/Z_{L_0}(\alpha^-) \), where \( N_{L_0}(\alpha^-) \) (resp. \( Z_{L_0}(\alpha^-) \)) is the normalizer (resp. centralizer) of \( \alpha^- \) in \( L_0 \), is isomorphic with the group of permutations of \( \{ X_\gamma; \gamma \in \Delta \} \). On the other hand, since \( S^*_{L_0}(\mathfrak{p}) \) is isomorphic with \( S^*_{W_{S_0}}(\alpha) \) by Theorem 2.2, \( S^*_{L_0}(\mathfrak{p}) \) is isomorphic with \( S^*_{W_{S_0}^-}(\alpha^-) \). Hence \( S^*_{L_0}(\mathfrak{p}) \) is isomorphic with \( S^*_{W_{S_0}^-}(\alpha^-) \). It follows from Theorem 3.1, (ii), 2) that the cardinality of \( S^*(D) \) is equal to \( \dim S^*_{L_0}(\mathfrak{p}) = \dim S^*_{W_{S_0}^-}(\alpha^-) \) = the number of linearly independent symmetric polynomials in \( p \)-variables with degree \( n \), which is known to be the cardinality of \( S^*(K, L) \).

**q.e.d.**

### 4. Normalizing factor \( h_\chi \)

Let \( \hat{A} = \text{Ad}A(X_0) \), denoting by \( A \) the connected subgroup of \( K_0 \) generated by \( \alpha \). \( \hat{A} \) has a natural group structure induced from that of \( \alpha \). Let

\[ T = \{ t \in C^*; |t| = 1 \} \]

be the 1-dimensional torus. Under the identification in Introduction of \( (\alpha^-)^C \) with \( C^p \), \( \alpha^- \) is identified with \( R^p \) and \( \hat{A} \) with \( T^p \). We see that the latter identification is compatible with group structures and complex conjugations, in view of the expression of \( \text{Ad} \, (\exp H)X_0 \) in the proof of Theorem 3.1, (iii). Moreover, under the same identification we have (Moore [8])

\[ D \cap \alpha^- = \{ x \in R^p; |x_i| < 1 \} \]

denoting by \( z_i \) (1 ≤ \( i \) ≤ \( p \)) the \( i \)-th component of \( z \in C^p \). By means of this
identification we define a measure on $\alpha^-$ by
\[ dH = dx_1 \cdots dx_p \]
and a function $D(H)$ on $\alpha^-$ by
\[ D(H) = \prod_{i=1}^p (2x_i)x_i^{2s} \prod_{1 \leq i < j \leq p} ((x_i + x_j)(x_i - x_j))^r \quad \text{for} \quad H \in \alpha^-, \]
where $r$, $2s$ are multiplicities defined in Introduction. Then we have the following

**Lemma 1.** There exists a constant $c' > 0$ such that
\[ \int_{\alpha^-} f(X)d\mu(X) = c' \int_{\alpha^-} f(H)|D(H)|dH \]
for any integrable $K$-invariant function $f$ on $D$.

Proof. It is easy to see that $\text{Ad} \ cH = H$ for any $H \in \mathfrak{b}$ and $\text{Ad} \ c\gamma^* = X_{\gamma} - X_{-\gamma} \in \mathfrak{p}$ for any $\gamma \in \Delta$. Put
\[ \alpha^o = \text{Ad} \ c(\sqrt{-1}\alpha) = \{X_{\gamma} - X_{-\gamma}; \gamma \in \Delta\}_R, \]
\[ \mathfrak{h} = \text{Ad} \ c(\mathfrak{b} \oplus \sqrt{-1}\alpha) = \mathfrak{b} \oplus \alpha^o \]
and
\[ \mathfrak{h}_R = \sqrt{-1}\mathfrak{b} \oplus \alpha^o. \]

Then $\alpha^o$ is a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\alpha^o$ and $\mathfrak{h}_R$ is the real part of the complexification $\mathfrak{h}^c$ of $\mathfrak{h}$. We define linear forms $h_i$ ($1 \leq i \leq p$) on $\alpha^o$ by
\[ h_i(X_{\gamma} - X_{-\gamma}) = \delta_{ij} \quad (1 \leq i, j \leq p). \]
If $h_i$ is identified with an element of $\alpha^o$ by means of the Killing form, we have $\text{Ad} \ c(\frac{1}{2}\gamma_i) = h_i$ ($1 \leq i \leq p$). The linear order on $\mathfrak{h}_R$ induced by $\text{Ad} \ c$ from the order $>$ on $\sqrt{-1} \mathfrak{t}$ is a compatible order for $\text{Ad} \ c \sum$ with respect to the decomposition $\mathfrak{h}_R = \sqrt{-1}\mathfrak{b} \oplus \alpha^o$. This follows from 3, Lemma 2,1). Thus positive restricted roots on $\alpha^o$ of the symmetric space $D = G/K$ are
\[ \{h_i \pm h_j; 1 \leq i < j \leq p, 2h_i; 1 \leq i \leq p\} \quad \text{if} \quad P_i = \phi, \]
\[ \{h_i \pm h_j; 1 \leq i < j \leq p, 2h_i, h_i; 1 \leq i \leq p\} \quad \text{if} \quad P_i \neq \phi. \]
The multiplicity of $h_i \pm h_j$ ($1 \leq i < j \leq p$), i.e. the number of roots in $\text{Ad} \ c \sum$ projecting to $h_i \pm h_j$, is the same as that of $\frac{1}{2}(\gamma_i \pm \gamma_j)$. Since the Weyl group $W_D$ on $\alpha^o$ of $D = G/K$ is generated by reflections with respect to $h_i - h_{p-i}, \ldots, h_{p-1} - h_p, h_p$, hence transitive on the set $\{\pm h_i \pm h_j; 1 \leq i < j \leq p\}$, it follows that
multiplicities of these roots are the same \( r \). By the same reason, multiplicities of \( h_i \) (\( 1 \leq i \leq p \)) are the same \( 2s \), which is even from the results of Harish-Chandra mentioned in 3. In the same way we know that multiplicities of \( 2h_i \) (\( 1 \leq i \leq p \)) are 1. Thus the product \( D^0 \) of positive restricted roots (multiplicity counted) is given by

\[
D^0(H^\circ) = \prod_{i=1}^{p} 2h_i(H^\circ)h_i(H^\circ)^{2r} \prod_{1 \leq i < j \leq p} ((h_i + h_j)(H^\circ)(h_i - h_j)(H^\circ))^{r} \quad \text{for } H^\circ \in \alpha^0.
\]

Let \( dX \) (resp. \( dH^\circ \)) denote the Euclidean measure of \( \mathfrak{p} \) (resp. of \( \alpha^0 \)) induced from the Killing form \( (\,,) \), and \( dk \) the normalized Haar measure of \( K \). Then (cf. Helgason [4]) under the surjective map \( K \times \alpha^0 \rightarrow \mathfrak{p} \) defined by \( (k, H^\circ) \mapsto \text{Ad } kH^\circ \), these measures are related as follows:

\[
dX = c'' |D^0(H^\circ)| dH^\circ \quad \text{with some constant } c'' > 0.
\]

Now we define a \( K \)-equivariant \( \mathcal{R} \)-isomorphism \( j : \mathfrak{p} \rightarrow (\mathfrak{p}^c)^\circ \) by

\[
j(X) = \frac{1}{2} (X - [Z, X]) \quad \text{for } X \in \mathfrak{p}.
\]

It is easy to see that \( j(X_\gamma - X_\gamma) = -X_\gamma \) for any \( \gamma \in \Delta \), hence \( j\alpha^0 = \alpha^- \). Since \( K \) acts irreducibly on \( \mathfrak{p} \), the map \( j \) is a similitude with respect to inner products \( (\,,) \) and the real part of \( (\,,) \). Therefore under the surjective map \( K \times \alpha^- \rightarrow (\mathfrak{p}^c)^\circ \) defined by \( (k, H) \mapsto \text{Ad } kH \), we have

\[
d\mu(X) = c' |D(H)| dH \quad \text{with some constant } c' > 0.
\]

Seeing \( \text{Ad } K(D \cap \alpha^-) = D \), we get the proof of Lemma 1.

q.e.d.

Take a form \( \lambda \in S^*_*(K, L) \). Choose an orthonormal basis \( \{u_i ; 1 \leq i \leq d_\lambda\} \) of \( S^*_\mathfrak{p}((\mathfrak{p}^c)^\circ) \) with respect to \( (\,,) \), such that \( \{u_i ; 1 \leq i \leq d_{\lambda, 0}\} \) spans \( S^*_\mathfrak{p}(\mathfrak{p}^c_0) \) and \( u_i \) is \( L \)-invariant. Put

\[
\rho_j'(k) = (\text{Ad } ku_j, u_i), \quad \text{for } k \in K \quad (1 \leq i, j \leq d_\lambda),
\]

\[
\varphi_j'(k) = \rho_j'(k) \quad \text{for } k \in K \quad (1 \leq i \leq d_\lambda),
\]

\[
f_i' = \sqrt{d_\lambda} \varphi_i' \quad (1 \leq i \leq d_\lambda).
\]

The arguments in 2 show that \( \{f_i' ; 1 \leq i \leq d_\lambda\} \) form an orthonormal basis of \( S_\mathfrak{p}(S) \) with respect to \( (\,,) \) and \( \varphi_i \) is the zonal spherical function \( \omega \) for \( (K, L) \) belonging to \( \lambda \), identifying \( C^\omega(S) \) with the space of right \( L \)-invariant \( C^\omega \)-functions on \( K \). The zonal spherical polynomial \( \Omega_\lambda \) for \( \mathcal{D} \) belonging to \( \lambda \) defined in Introduction is characterized by that its restriction to \( S_\mathfrak{p} \) coincides with \( \omega_\lambda \). \( \Omega_\lambda \) restricted to \( \mathfrak{p}^c_0 \) is the zonal spherical polynomial for \( D_0 \) belonging to \( \lambda \) and \( \omega_\lambda \) restricted to \( S_0 \) is the zonal spherical function for \( (K_0, L_0) \) belonging to \( \lambda \). \( \Omega_\lambda \)
restricted to \((\alpha^-)^c\) is a symmetric polynomial since it is \(W_{\tilde{g}_0}\)-invariant. Let \(f_i \in S^\pm(\mathfrak{p}^-)\) \((1 \leq i \leq d_\lambda)\) be the unique polynomial such that its restriction to \(S\) is \(f_i\). Then \(\{f_i; 1 \leq i \leq d_\lambda\}\) form an orthogonal basis of \(S^\pm(\mathfrak{p}^-)\) with respect to \((,\; ,)\), such that \(\{f_i; 1 \leq i \leq d_{\lambda, 0}\}\) form an orthogonal basis of \(S_{\mathfrak{p}_-}^\pm(\mathfrak{p}^-) \cap S^*(\mathfrak{p}_-^c)\). They satisfy relations

\[
f_i(\text{Ad} k^{-1} X) = \frac{d_\lambda}{d_\lambda} \sum_{j=1}^{d_\lambda} \rho^j(k) f_j(X) \quad \text{for} \quad k \in K, \quad X \in (\mathfrak{p}^-)^- \quad (1 \leq i \leq d_\lambda).
\]

We put

\[
\Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \quad \text{for} \quad X \in (\mathfrak{p}^-)^-.
\]

Then for any \(k \in K\) we have

\[
\Phi_\lambda(\text{Ad} k^{-1} X) = \frac{1}{d_\lambda} \sum_i \left( \sum_j \rho^j(k) f_j(X) \right) \left( \sum_k \overline{\rho^j(k)} f_k(X) \right)
\]

\[
= \frac{1}{d_\lambda} \sum_i \left( \sum_j \rho^j(k) \overline{\rho^j(k)} f_j(X) f_k(X) \right)
\]

\[
= \frac{1}{d_\lambda} \sum_i \delta_{jk} f_j(X) \overline{f_k(X)} = \Phi_\lambda(X) \quad \text{for} \quad X \in (\mathfrak{p}^-)^-,
\]

i.e. \(\Phi_\lambda\) is a \(K\)-invariant \(C^\infty\)-function on \((\mathfrak{p}^-)^-\). Note that

\[
\Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{a=1}^{d_{\lambda, 0}} |f_a(X)|^2 \quad \text{for} \quad X \in \mathfrak{p}^-^c.
\]

**Lemma 2.**

\[
h_\lambda = c' \int_{\partial \mathfrak{p}^-} \Phi_\lambda(H) |D(H)| dH
\]

Proof.

\[
\int_\partial \Phi_\lambda(X) d\mu(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \left\langle f_i, f_i \right\rangle = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} h_\lambda \left\langle f_i, f_i \right\rangle = h_\lambda.
\]

On the other hand, by Lemma 1 we have

\[
\int_\partial \Phi_\lambda(X) d\mu(X) = c' \int_{\partial \mathfrak{p}^-} \Phi_\lambda(H) |D(H)| dH.
\]

Proof of Theorem B. Making use of the complex conjugation \(X \mapsto \overline{X}\) of \(\mathfrak{p}_-^c\) defined in 3, we define \(\Phi_\lambda \in S^*(\mathfrak{p}_-^c)\) by

\[
\Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{a=1}^{d_{\lambda, 0}} f_a(X) \overline{f_a(X)} \quad \text{for} \quad X \in \mathfrak{p}_-^c.
\]

Then \(\Phi_\lambda = \Phi_\lambda\) on \(\mathfrak{p}_-\) and we have for any \(k \in K_0\)
\[ \Phi_\lambda(\mathrm{Ad} k X_\alpha) = \frac{1}{d_\lambda} \sum f_\alpha(\mathrm{Ad} k X_\alpha) f_\alpha(\mathrm{Ad} \theta(k) X_\alpha) \]
\[ = \frac{1}{d_\lambda} \sum f_\alpha(\mathrm{Ad} k X_\alpha) f_\alpha(\mathrm{Ad} \theta(k) X_\alpha) \]
\[ = \frac{1}{d_\lambda} \sum f_\alpha'(k) f_\alpha'(\theta(k)) = \sum f_\alpha'(k) f_\alpha'(\theta(k)) \]
\[ = \sum \rho_\alpha^2(k) \rho_\alpha^2(\theta(k)) = \sum \rho_\alpha^2(k) \rho_\alpha^2(\theta(k)^{-1}) \]
\[ = \rho_\alpha^2(\theta(k)^{-1}) = \omega_\lambda(\theta(k)^{-1} k) . \]

In particular for any \( a \in A \)
\[ \Phi_\lambda(\mathrm{Ad} a X_\alpha) = \omega_\lambda(a^2) , \]
i.e. for any \( \tilde{a} \in \tilde{A} \)
\[ \tilde{\Phi}_\lambda(\tilde{a}) = \omega_\lambda(\tilde{a}^2) = \Omega_\lambda(\tilde{a}^2) . \]

Since \( \tilde{A} = T^p \) is a compact real form of \( C^*p \) and \( C^*p \) is open in \( C^*(\alpha^*) \), we have
\[ \Phi_\lambda(x_1, \ldots, x_p) = \Omega_\lambda(x_1^2, \ldots, x_p^2) \quad \text{for any } z \in C^p = (\alpha^*)^c . \]

By Lemma 2 we have
\[ h_\lambda = c' \int_{\partial \mathcal{A}^-} \Phi_\lambda(H) |D(H)| dH \]
\[ = c' \int_{\partial \mathcal{A}^-} \Omega_\lambda(x_1^2, \ldots, x_p^2) | \prod_{i=1}^p \bigl( (x_i + x_j)(x_i - x_j) \bigr) | \prod_{1 \leq i < j \leq p} | \prod (y_i - y_j) | \prod y_i^r dy_1 \cdots dy_p \]
for some constant \( c(D) > 0 \), which does not depend on \( \lambda \). In particular, for \( \lambda = 0 \)
\[ \mu(D) = h_0 = c(D) \int_{\partial \mathcal{A}^-} \prod_{1 \leq i < j \leq p} (y_i - y_j) | \prod y_i^r dy_1 \cdots dy_p , \]
since \( \Omega_0 \equiv 1 \). This completes the proof of Theorem B. q.e.d.

**Remark.** It can be proved that \( \Phi_\lambda \) is an \( L_\lambda \)-invariant polynomial on \( \mathcal{P}_z \).

The multiplicities \( r, s \) are given as follows.

<table>
<thead>
<tr>
<th>( D )</th>
<th>( \text{rank } D )</th>
<th>( r )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)_{p,s} (p \leq q)</td>
<td>( p )</td>
<td>( 2 )</td>
<td>( q-p )</td>
</tr>
<tr>
<td>(II)_{n}</td>
<td>( [n/2] )</td>
<td>( 4 )</td>
<td>( { \begin{array}{ll} 2 &amp; \text{n odd} \ 0 &amp; \text{n even} \end{array} )</td>
</tr>
<tr>
<td>(III)_{n}</td>
<td>( n )</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>(IV)_{n} (n \geq 3)</td>
<td>( 2 )</td>
<td>( n-2 )</td>
<td>( 0 )</td>
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<tr>
<td>(EIII)</td>
<td>( 2 )</td>
<td>( 6 )</td>
<td>( 4 )</td>
</tr>
<tr>
<td>(EVII)</td>
<td>( 3 )</td>
<td>( 8 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
The zonal spherical polynomial $\Omega_\lambda$ is given as follows.

For integers $n_1, \ldots, n_p$ we define the Schur function $\{n_1, \ldots, n_p\}$ on the $p$-dimensional torus $T^p$ by

$$\{n_1, \ldots, n_p\}(t) = \frac{\det(t_1^{i+j+p-f})_{i,j \leq p}}{\det(t_1^{i+j})_{i,j \leq p}} \quad \text{for} \quad t = \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \in T^p \subset C^p.$$

$\{n_1, \ldots, n_p\}$ is symmetric in variables $t_1, \ldots, t_p$ and it is a polynomial in $t_1, \ldots, t_p$ if and only if $n_i \geq 0$ ($1 \leq i \leq p$). For an element $\lambda = \sum_{i=1}^p n_i \gamma_i \in \sum_{i=1}^p Z \gamma_i = Z(K_0, L_0)$, the $i$-th coefficient $n_i$ will be denoted by $n_i(\lambda)$.

Then we have

**Theorem 4.1.** The zonal spherical polynomial $\Omega_\lambda$ for $D$ belonging to $\lambda \in S^*(K, L)$ is determined on $(\alpha^-)^c$ by the relation

$$\Omega_\lambda(t) = \sum_{\mu \in D^\lambda} c^\alpha_\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for any} \quad t \in T^p = \hat{A} \subset (\alpha^-)^c,$$

where the $c^\alpha_\mu$'s are coefficients in Theorem 2.5 for the symmetric pair $(K_0, L_0)$.

**Proof.** As we have seen in the proof of Theorem B, $\Omega_\lambda$ is determined on $(\alpha^-)^c$ by

$$\Omega_\lambda(t) = \omega_\lambda(t) \quad \text{for any} \quad t \in T^p = \hat{A}.$$

By Theorem 2.5, $\omega_\lambda$ has an expression

$$\omega_\lambda(t) = \sum_{\mu \in D^\lambda} c^\alpha_\mu \chi_\mu(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$

Since the Weyl group $W_{S_0}$ acts on $Z(K_0, L_0)$ by the group of permutations of $\gamma_1, \ldots, \gamma_p$, $W_{S_0}$-invariant characters $\chi_\lambda$ of $\hat{A}$ are nothing but Schur functions. As we have seen in the proof of Theorem 3.1, (iii), the $i$-th component of $\text{Ad}(\exp H)X_\gamma \in T^p = \hat{A}$ is $\exp(-\gamma_i^\vee H)$ for any $H \in \alpha$. It follows that

$$\chi_\mu(t) = \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$

Hence we have

$$\Omega_\lambda(t) = \sum_{\mu \in D^\lambda} c^\alpha_\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t)$$

$$= \sum_{\mu \in D^\lambda} c^\alpha_\mu \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A}. \quad \text{q.e.d.}$$

In the case of the domain $D$ of type $(I)_{p,q}$ ($p \leq q$), $S_0$ is the unitary group $U(p)$ of degree $p$. We have in view of Example in 2 that

$$\Omega_\lambda(t) = \frac{1}{d_\lambda} \{n_1(\lambda), \ldots, n_p(\lambda)\}(t) \quad \text{for} \quad t \in T^p = \hat{A},$$
where $d_\lambda$ is the degree of the irreducible representation of $U(p)$ with the signature $(n_1(\lambda), \cdots, n_p(\lambda))$. In the case of the domain $D$ of type (IV)$_n$, $S$ is the Lie sphere and $\Omega_\lambda$ can be described in terms of Gegenbauer polynomials, which are zonal spherical functions for the sphere. So our integral formula in Theorem B clarifies the meaning of integrals of Hua [6].

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**References**


