Polynomial representations associated with symmetric bounded domains

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Osaka University
POLYNOMIAL REPRESENTATIONS ASSOCIATED WITH
SYMMETRIC BOUNDED DOMAINS

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Introduction. In this note we want to construct a complete orthonormal
system of the Hilbert space $H(D)$ of square integrable holomorphic functions on
an irreducible symmetric bounded domain $D$. A symmetric bounded domain $D$
is canonically realizable as a circular starlike bounded domain with the center
0 in a complex cartesian space by means of Harish-Chandra's imbedding (Harish-
Chandra [3]), which is constructed as follows. The largest connected group $G$
of holomorphic automorphisms of $D$ is a connected semi-simple Lie group without
center, which is transitive on $D$. Thus denoting the stablizer in $G$ of a point
0 $\in D$ by $K$, $D$ is identified with the quotient space $G/K$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the
Lie algebra of $G$ (resp. $K$) and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ with
respect to $\mathfrak{k}$. Then there exists uniquely an element $H$ of the center of $\mathfrak{k}$ such
that $\text{ad} H$ restricted to $\mathfrak{p}$ coincides with the complex structure tensor on the
tangent space $T_0(D)$ of at the origin $0$, identifying as usual $\mathfrak{p}$ with $T_0(D)$. Let $\mathfrak{g}^c$
be the Lie algebra of the complexification $G^c$ of $G$ and put $Z=\sqrt{-1}H \in \mathfrak{g}^c$. Let $(\mathfrak{p}^c)^\pm$
be the $(\pm 1)$-eigenspace in $\mathfrak{g}^c$ of $\text{ad} Z$. Then they are invariant under
the adjoint action of $K$ and the complexification $\mathfrak{p}^c$ of $\mathfrak{p}$ is the direct sum of $(\mathfrak{p}^c)^+$
and $(\mathfrak{p}^c)^-$. Let $U^c$ denote the normalizer of $(\mathfrak{p}^c)^+$ in $G^c$. Then $D=G/K$ is
holomorphically imbedded as an open submanifold into the quotient space $G^c/U^c$
in the natural way. For any point $z \in D$, there exists uniquely a vector $X \in (\mathfrak{p}^c)^-$
such that

$$\exp X \mod U^c = z.$$ 

The map $z \mapsto X$ of $D$ into $(\mathfrak{p}^c)^-$ is the desired imbedding. Note that the natural
action of $K$ on $D$ can be extended to the adjoint action of $K$ on the ambient space
$(\mathfrak{p}^c)^-$. Henceforth we assume that $D$ is a bounded domain in $(\mathfrak{p}^c)^-$ realized in the
above manner. Let $( , )$ denote the Killing form of $\mathfrak{g}^c$ and $\tau$ the complex conjugation
of $\mathfrak{g}^c$ with respect to the compact real form $\mathfrak{k}+\sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}^c$. We define
a $K$-invariant hermitian inner product $( , )$, on $\mathfrak{g}^c$ by

$$(X, Y) = -(X, \tau Y) \quad \text{for } X, Y \in \mathfrak{g}^c.$$
This defines a $K$-invariant Euclidean measure $d\mu(X)$ on $(\mathfrak{g}^-)^*$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(X)$. The inner product of $H^2(D)$ will be denoted by $\langle , \rangle$. $K$ acts on $H^2(D)$ as unitary operators by
\[ (kf)(X) = f(k^{-1}X) \quad \text{for} \quad k \in K, X \in D. \]

Let $S^*((\mathfrak{g}^-)^*)$ denote the graded space of polynomial functions on $(\mathfrak{g}^-)^*$. It has the natural hermitian inner product $( , )$, induced from the inner product $( , )$, on $(\mathfrak{g}^-)^*$. $K$ acts on $S^*((\mathfrak{g}^-)^*)$ as unitary operators by
\[ (kf)(X) = f(Ad k^{-1}X) \quad \text{for} \quad k \in K, X \in (\mathfrak{g}^-)^*. \]

Now let $S$ denote the Shilov boundary of $D$. It is known (Korányi-Wolf [7]) that $K$ acts transitively on $S$. Thus denoting by $L$ the stabilizer in $K$ of a point $X_0 \in S$, $S$ is identified with the quotient space $K/L$. Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure of $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle , \rangle$. $K$ acts on $L^2(S)$ as unitary operators by
\[ (kf)(X) = f(Ad k^{-1}X) \quad \text{for} \quad k \in K, X \in S. \]

The space $C^\infty(S)$ of $C$-valued $C^\infty$-functions on $S$ is a $K$-submodule of $L^2(S)$. The restrictions $S^*((\mathfrak{g}^-)^*) \to H^2(D)$ and $S^*((\mathfrak{g}^-)^*) \to L^2(S)$ are both $K$-equivariant monomorphisms. Their images will be denoted by $S^*(D)$ and $S^*(S)$, respectively. They have natural gradings induced from that of $S^*((\mathfrak{g}^-)^*)$. Then the "restriction" $S^*(D) \to S^*(S)$ is defined in the natural manner and it is a $K$-equivariant isomorphism. Since $D$ is a circular starlike bounded domain, a theorem of H. Cartan [2] yields that the subspace $S^*(D)$ of $H^2(D)$ is dense in $H^2(D)$ (cf. 1).

We decompose first the $K$-module $S^*(D)$ into irreducible components. We take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and identify the real part $\sqrt{-1} \mathfrak{t}$ of the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$ with its dual space by means of Killing form of $\mathfrak{g}^c$. Let $\Sigma \subset \sqrt{-1} \mathfrak{t}$ denote the set of roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$. We choose root vectors $X_\alpha \in \mathfrak{g}^c$ for $\alpha \in \Sigma$ such that
\[
[X_\alpha, X_{-\alpha}] = -\frac{2}{(\alpha, \alpha)} \alpha,
\]
\[ \tau X_\alpha = X_{-\alpha}. \]

A root is called compact if it is also a root of the complexification $\mathfrak{t}^c$ of $\mathfrak{t}$, otherwise it is called non-compact. $\Sigma_\mathfrak{t}$ (resp. $\Sigma_\mathfrak{p}$) denotes the set of compact roots (resp. of non-compact roots). We choose and fix once for all a linear order $\succ$ on $\sqrt{-1} \mathfrak{t}$ such that $(\mathfrak{g}^c)^+$ is spanned by the root spaces for non-compact positive
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roots $\sum_0^+$. Two roots $\alpha, \beta \in \sum$ are called **strongly orthogonal** if $\alpha \pm \beta$ is not a root. We define a maximal strongly orthogonal subsystem

$$\Delta = \{\gamma_1, \ldots, \gamma_p\}, \quad \gamma_1 > \gamma_2 > \cdots > \gamma_p > 0, \quad p = \text{rank } D$$

of $\sum_0^+$ as follows (cf. Harish-Chandra [3]). Let $\gamma_i$ be the highest root of $\sum$ and for each $j$, $\gamma_{j+1}$ be the highest positive non-compact root that is strongly orthogonal to $\gamma_1, \ldots, \gamma_j$. We put

$$X_0 = -\sum_{\gamma \in \Delta} X_{-\gamma}.$$ 

Then it is known (Korányi-Wolf [7]) that $X_0$ is on the Shilov boundary $S$ of $D$. Henceforth we shall take the above point $X_0$ as the origin of $S$. We put for $\nu \in \mathbb{Z}, \nu \geq 0$

$$S^*(K, L) = \left\{ \sum_{i=1}^{p} n_i \gamma_i ; n_i \in \mathbb{Z}, n_i \geq n_{i+1} \geq \cdots \geq n_p \geq 0, \sum_{i=1}^{p} n_i = \nu \right\},$$

and

$$S^*(K, L) = \sum_{\nu \geq 0} S^*(K, L).$$

We shall prove the following

**Theorem A.** Any irreducible $K$-submodule of $S^*(D)$ is contained exactly once in $S^*(D)$. The set $S^*(D)$ of highest weights (with respect to $\langle , \rangle$) of irreducible $K$-submodules contained in $S^*(D)$ coincides with $S^*(K, L)$. Denoting by $S^*_\lambda(D)$ (resp. $S^*_\lambda(S)$) the irreducible $K$-submodule of $S^*(D)$ (resp. of $S^*(S)$) with the highest weight $\lambda \in S^*(K, L)$,

$$S^*(D) = \sum_{\lambda \in S^*(K, L)} \oplus S^*_\lambda(D)$$

and

$$S^*(S) = \sum_{\lambda \in S^*(K, L)} \oplus S^*_\lambda(S)$$

are the orthogonal sum relative to the inner product $\langle , \rangle$ and $\langle , \rangle$, respectively. The restriction $f \mapsto f^*$ of $S^*_\lambda(D) \to S^*_\lambda(S)$ is a similitude for each $\lambda \in S^*(K, L)$, i.e., there exists a constant $h_\lambda > 0$ such that

$$\langle f, g \rangle = h_\lambda \langle f^*, g^* \rangle \quad \text{for any } f, g \in S^*_\lambda(D).$$

Thus, if

$$\{f^*_\lambda, i ; 1 \leq i \leq d_\lambda \}, \quad \lambda \in S^*(K, L)$$

is an orthonormal basis of $S^*_\lambda(S)$, then

$$\{\sqrt{h_\lambda^{-1}} f^*_\lambda, i ; \lambda \in S^*(K, L), 1 \leq i \leq d_\lambda \}$$

is a complete orthonormal system of $H^2(D)$.
A basis $\{f_{\lambda,i}; 1 \leq i \leq d_\lambda\}$ is, for instance, constructed as follows. Take an irreducible $K$-module $(\rho, V)$ with the highest weight $\lambda$, carrying a $K$-invariant hermitian inner product $(\cdot, \cdot)$. Choose an orthonormal basis $\{u_i; 1 \leq i \leq d_\lambda\}$ of $V$ such that the first vector $u_i$ is $L$-invariant. This can be done in view of Frobenius' reciprocity since the $K$-module $V$ is $K$-isomorphic with a $K$-submodule of $C^\infty(S)$. Then the functions $f_{\lambda,i}(1 \leq i \leq d_\lambda)$ defined by

$$f_{\lambda,i}(kX_\lambda) = \sqrt{d_\lambda}(u_i, \rho(k)u_i) \quad \text{for} \quad k \in K$$

form an orthonormal basis of $S^*_\lambda(S)$ (cf. 2).

We compute next the normalizing factor $h_\lambda$. Let

$$a = \{\sqrt{-1}\Delta\}_R$$

be the $R$-span of $\sqrt{-1}\Delta$ in $t$ and

$$\sigma: \sqrt{-1}t \rightarrow \sqrt{-1}a$$

denote the orthogonal projection of $\sqrt{-1}t$ onto $\sqrt{-1}a$. For $\gamma \in \sigma\mathbb{Z} - \{0\}$, the number of roots $\alpha \in \Sigma$ such that $\sigma\alpha = \gamma$ is called the multiplicity of $\gamma$. Let $r$ (resp. $2s$) be the multiplicity of $\frac{1}{2}(\gamma_i - \gamma_i)$ (resp. of $\frac{1}{2}\gamma_i$). If follows from Theorem A and Frobenius' reciprocity that for each $\lambda \in S^*(K, L)$ there exists uniquely an $L$-invariant polynomial $\Omega_\lambda$ in $S^*_\lambda((\mathfrak{p}^C)^{-})$ such that $\Omega_\lambda(X_\lambda) = 1$, where $S^*_\lambda((\mathfrak{p}^C)^{-})$ denotes the irreducible $K$-submodule of $S^*((\mathfrak{p}^C)^{-})$ with the highest weight $\lambda$. The polynomial $\Omega_\lambda$ is called the zonal spherical polynomial for $D$ belonging to $\lambda$. Let

$$(a^{-})^c = \{X_{-\gamma}; \gamma \in \Delta\}_C$$

be the $C$-span of $\{X_{-\gamma}; \gamma \in \Delta\}$ in $\mathfrak{p}^C$. It is identified with the complex cartesian space $C^p$ by the map

$$-\sum_{i=1}^p x_iX_{-\gamma_i} \mapsto \left(\begin{array}{c} x_1 \\ \vdots \\ x_p \end{array}\right).$$

Thus the zonal spherical polynomial $\Omega_\lambda$ restricted to $(a^-)^c$ is a polynomial $\Omega_\lambda(Y_1, \ldots, Y_p)$ in $\mathfrak{p}$-variables. Let $\mu(D)$ denote the volume of $D$ with respect to the measure $d\mu(X)$. We shall prove the following

**Theorem B.** For $\lambda \in S^*(K, L)$, the normalizing factor $h_\lambda$ is given by

$$h_\lambda = c(D)\int_{0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_p} \Omega_\lambda(y_1, \ldots, y_p) \prod_{i \leq j \leq p} (y_i - y_j)^r \prod_{i=1}^p y_i^s dy_1 \cdots dy_p$$

where

$$c(D) = \mu(D)\left(\int_{0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_p} \prod_{i \leq j \leq p} (y_i - y_j)^r \prod_{i=1}^p y_i^s dy_1 \cdots dy_p\right)^{-1}.$$
Hua [6] proved Theorem A for classical domains by decomposing the character of the $K$-module $\mathcal{S}^{*}((\mathfrak{g} c)^{-})$ into the sum of irreducible characters of $K$, while Schmid [11] proved it for general domain $D$. Schmid proved

(a) \[ \mathcal{S}^{*}(D) \subset \mathcal{S}^{*}(K, L) \]

by seeing the character of the $K$-module $\mathcal{S}^{*}((\mathfrak{g} c)^{-})$ and by making use of E. Cartan's theory on spherical representations of a compact symmetric pair. But his proof of

(b) \[ \mathcal{S}^{*}(K, L) \subset \mathcal{S}^{*}(D) \]

is complicated and was done after nine successive lemmas. In this note we give another proof of (a) by means of a lemma of Murakami and Cartan's theory, and give a relatively short proof of (b) by means of a theorem of Harish-Chandra on invariant polynomials for a symmetric pair.

Hua [6] computed the factors $h_\lambda$ for certain classical domains by integrating certain polynomials. Our integral formula in Theorem B will clarify the meaning of integrals of Hua.

1. Circular domains

A domain $D \subset \mathbb{C}^n$ containing the origin 0 is said to be a circular domain with the center 0 if together with any point $z \in D$ the point $e^{\sqrt{-1} \theta} z$ is in $D$ for any real $\theta \in \mathbb{R}$. $D$ is said to be a starlike domain with the center 0 if together with any point $z \in D$ the point $r z$ is in $D$ for any real $r \in \mathbb{R}$ with $0 \leq r < 1$.

**Theorem 1.1.** (H. Cartan [2]) Let $D \subset \mathbb{C}^n$ be a circular domain with the center 0. Then any holomorphic function $f$ on $D$ can be developed in the sum of homogeneous polynomials $P_\nu$, in $n$-variables with degree $\nu$ ($\nu=0, 1, 2, \cdots$):

\[
  f(z) = \sum_{\nu=0}^\infty P_\nu(z) \quad \text{for } z \in D.
\]

The sum converges uniformly on any compact subset of $D$. The homogeneous polynomials $P_\nu$ are uniquely determined for $f$.

Let $D$ be a bounded domain in $\mathbb{C}^n$, $d\mu(z)$ the Euclidean measure on $\mathbb{C}^n$, induced from the standard hermitian inner product of $\mathbb{C}^n$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(z)$. The inner product of $H^2(D)$ will be denoted by $\langle \ , \rangle$. Let $\mathcal{S}^{*}(\mathbb{C}^n)$ be the graded space of polynomials in $n$-variables and $\mathcal{S}^{*}(D)$ the subspace of $H^2(D)$ consisting of all functions on $D$ obtained by the restriction of polynomials in $\mathcal{S}^{*}(\mathbb{C}^n)$. Then Theorem 1.1 yields the following

**Corollary.** Let $D \subset \mathbb{C}^n$ be a circular starlike bounded domain with the center 0. Then the subspace $\mathcal{S}^{*}(D)$ of $H^2(D)$ is dense in $H^2(D)$. 
Proof. If suffices to show that if \( f \in H^r(D) \) with \( \langle f, S^r(D) \rangle = \{0\} \), then \( f = 0 \). Theorem 1.1 implies that \( f \) can be developed as

\[
f = \sum_{\nu} P_{\nu, i} \quad P_{\nu, i} \in S^r(D),
\]
uniformly convergent on any compact subset of \( D \). Choose an orthonormal basis \( \{P_{\nu, i}\} \) of \( S^r(D) \) with respect to \( \langle \, , \rangle \) for each \( \nu \). Then we have

\[
\langle P_{\nu, j}, P_{\mu, i} \rangle = \delta_{\nu, \mu} \delta_{j, i}.
\]
In fact, since \( d\mu(e^{\sqrt{-1} \theta z}) = d\mu(z) \) for any \( \theta \in \mathbb{R} \), we have \( \langle P_{\nu, j}, P_{\mu, i} \rangle = e^{\sqrt{-1} \theta \mu} \delta_{j, i} \langle P_{\nu, j}, P_{\mu, i} \rangle \) for any \( \theta \in \mathbb{R} \). Then \( f \) can be developed as

\[
f = \sum_{\nu, i} a_{\nu, j} P_{\nu, j} \quad \text{with} \quad a_{\nu, j} \in C,
\]
uniformly convergent on any compact subset of \( D \). Since \( D \) is a starlike domain, the closure \( \overline{rD} \) of \( rD \) is a compact subset of \( D \) for any \( r \in R \) with \( 0 < r < 1 \), so that the above series converges uniformly on \( rD \). Therefore for any \( P_{\mu, i} \) we have

\[
\int_{rD} f(z) \overline{P_{\mu, i}(z)} d\mu(z) = \sum_{\nu, j} a_{\nu, j} \int_{rD} P_{\nu, j}(z) \overline{P_{\mu, i}(z)} d\mu(z).
\]
If we put

\[
z' = \frac{1}{r} z \quad \text{for} \quad z \in rD,
\]
then \( z = rz' \), \( d\mu(z) = r^n d\mu(z') \) so that

\[
\int_{rD} P_{\nu, j}(z) \overline{P_{\mu, i}(z)} d\mu(z) = r^{n+\mu} \int_{D} P_{\nu, j}(z') \overline{P_{\mu, i}(z')} d\mu(z')
\]
\[
= r^{n+\mu} \langle P_{\nu, j}, P_{\mu, i} \rangle = r^{n+\mu} \delta_{\nu, \mu} \delta_{j, i}.
\]
Hence we have

\[
\int_{rD} f(z) \overline{P_{\mu, i}(z)} d\mu(z) = a_{\mu, i} r^{n+\mu}
\]
and

\[
a_{\mu, i} = \lim_{r \to 1} a_{\mu, i} r^{n+\mu} = \lim_{r \to 1} \int_{rD} f(z) \overline{P_{\mu, i}(z)} d\mu(z)
\]
\[
= \langle f, P_{\mu, i} \rangle = 0 \quad \text{(from the assumption)}.
\]
This implies that \( f = 0 \). q.e.d.
2. Spherical representations of a compact symmetric pair

Let $K$ be a compact connected Lie group, $L$ a closed subgroup of $K$ and $S$ be the quotient space $K/L$. The space of $C$-valued $C^\infty$-functions on $S$ will be denoted by $C^\infty(S)$. We shall often identify $C^\infty(S)$ with the space of $C^\infty$-functions $f$ on $K$ such that

$$f(kl) = f(k) \quad \text{for any} \quad k \in K, \ l \in L.$$ 

Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure on $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle , \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$(kf)(x) = f(k^{-1}x) \quad \text{for} \quad k \in K, \ x \in S.$$ 

Then $C^\infty(S)$ is a $K$-submodule of $L^2(S)$. A (continuous finite dimensional complex) representation

$$\rho: K \to GL(V)$$

of $K$ is said to be spherical relative to $L$ if the $K$-module $V$ is equivalent to a $K$-submodule of $C^\infty(S)$, which amounts to the same from Frobenius’ reciprocity that the $K$-module $V$ has a non-zero $L$-invariant vector. We denote by $\mathcal{D}(K, L)$ the set of equivalence classes of irreducible spherical representations of $K$ relative to $L$. The totality of $f \in C^\infty(S)$ contained in a finite dimensional $K$-submodule of $C^\infty(S)$, which will be denoted by $\mathfrak{o}(K, L)$, is a $K$-submodule of $C^\infty(S)$. A function in $\mathfrak{o}(K, L)$ is called a spherical function for the pair $(K, L)$. For $\rho \in \mathcal{D}(K, L)$, the totality of $f \in \mathfrak{o}(K, L)$ that transforms according to $\rho$, which will be denoted by $\mathfrak{o}_\rho(K, L)$, is a finite dimensional $K$-submodule of $\mathfrak{o}(K, L)$. Then

$$\mathfrak{o}(K, L) = \bigoplus_{\rho \in \mathcal{D}(K, L)} \mathfrak{o}_\rho(K, L)$$

is the orthogonal sum with respect to the inner product $\langle , \rangle$. Peter-Weyl approximation theorem implies that the subspace $\mathfrak{o}(K, L)$ of $L^2(S)$ is dense in $L^2(S)$. We assume furthermore that the pair $(K, L)$ satisfies the condition

(*) \quad \text{any} \ \rho \in \mathcal{D}(K, L) \text{ is contained exactly once in} \ \mathfrak{o}(K, L),

which is by Frobenius’ reciprocity equivalent to that for any spherical representation

$$\rho: K \to GL(V)$$

of $K$ relative to $L$, an $L$-invariant vector of $V$ is unique up to scalar multiplication. Then for each $\rho \in \mathcal{D}(K, L)$, there exists uniquely an $L$-invariant function $\omega_\rho \in \mathfrak{o}_\rho(K, L)$ such that $\omega_\rho(e) = 1$. $\omega_\rho$ is called the zonal spherical function for $(K, L)$ belonging to $\rho$. Let

$$\rho: K \to GL(V)$$
be a spherical representation of \( K \) relative to \( L \). Choose a \( K \)-invariant hermitian inner product \((\cdot,\cdot)\) on \( V \). The equivalence class containing \( \rho \) will be denoted by the same letter \( \rho \). Choose an orthonormal basis \( \{ u_i; 1 \leq i \leq d_\rho \} \) of \( V \) such that \( u_i \) is \( L \)-invariant. Define \( \varphi_i \in C^\infty(S) \) \((1 \leq i \leq d_\rho)\) by
\[
\varphi_i(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.
\]
We know that they are linearly independent, in view of orthogonality relations of matrix elements \((u_i, \rho(k)u_j)\). For any \( k' \in K \) we have
\[
\varphi_i(k'^{-1}k) = (u_i, \rho(k'^{-1}k)u_i) = (\rho(k')u_i, \rho(k)u_i)
\]
\[
= \sum_j (\rho(k')u_i, \rho(k)u_i) = \sum_j \rho(k')u_i, \rho(k)u_j \varphi_j(k),
\]
i.e.
\[
k' \varphi_i = \sum_j (\rho(k')u_i, \rho(k)u_j) \varphi_j \quad (1 \leq i \leq d_\rho).
\]
In particular
\[
l \varphi_i = \varphi_i \quad \text{for any} \quad l \in L,
\]
and
\[
\varphi_i(e) = 1.
\]
Therefore the system \( \{ \varphi_i; 1 \leq i \leq d_\rho \} \) forms a basis of \( o_\rho(K,L) \) and the zonal spherical function \( \omega_\rho \) is given by
\[
\omega_\rho(k) = (u_i, \rho(k)u_i) \quad \text{for} \quad k \in K.
\]
Furthermore orthogonality relations implies that the system
\[
\{ \sqrt{d_\rho} \varphi_i; 1 \leq i \leq d_\rho \}
\]
forms an orthonormal basis of \( o_\rho(K,L) \) and that
\[
\langle \omega_\rho, \omega_{\rho'} \rangle = \delta_{\rho\rho'} \frac{1}{d_\rho}.
\]
Henceforth we assume that the pair \((K,L)\) is a symmetric pair, i.e. there exists an involutive automorphism \( \theta \) of \( K \) such that if we put
\[
K_\theta = \{ k \in K; \theta(k) = k \},
\]
\( L \) lies between \( K_\theta \) and the connected component \( K_\theta^0 \) of \( K_\theta \). Then the pair \((K,L)\) satisfies the condition \((\ast)\) (E. Cartan [1]). For example, a compact connected Lie group \( S \) admits a symmetric pair \((K,L)\) such that \( S = K/L \). In fact,
\[
K = S \times S,
\]
\[
L = \{(x, x); x \in S\}.
θ: (x, y) ↦ (y, x) for x, y ∈ S

have desired properties.

In the following we summarize some known facts on a symmetric pair (cf. Helgason [4]).

Let ℱ (resp. ℱ') be the Lie algebra of K (resp. of L). The involutive automorphism of ℱ obtained by differentiating the automorphism θ of K will be also denoted by the same letter θ.

Choose and fix once for all a C-bilinear symmetric form ( , ) on the complexification ℱ of ℱ, which is invariant under both the C-linear extension to ℱ of θ and the adjoint action of ℱ and furthermore is negative definite on ℱ × ℱ. Then S is a Riemannian symmetric space with respect to the K-invariant Riemannian metric on S defined by −( , ). We put

\[ S = \{ X ∈ ℱ; θX = −X \} = \{ X ∈ ℱ; (X, I) = \{0\} \} . \]

Then we have orthogonal decompositions

\[ ℱ = ℱ ⊕ S = c ⊕ ℱ' , \]

where c is the center of ℱ and ℱ' is the derived algebra [ℱ, ℱ] of ℱ. We choose a maximal abelian subalgebra a in S. Such a are mutually conjugate under the adjoint action of L. dim a is the rank of the symmetric pair (K, L). Extend a to a maximal abelian subalgebra t of ℱ containing a. Then we have the decomposition

\[ ℱ = b ⊕ a \]

where \( b = ℱ \cap ℱ' \). Let t' = t ∩ ℱ' and \( a' = a ∩ ℱ' \). The real vector space \( √{-1} t \) has the natural inner product ( , ) induced from the bilinear form ( , ) on ℱ. We shall identify \( √{-1} t \) with the dual space of \( √{-1} t \) by means of the inner product ( , ). We have the orthogonal decomposition

\[ √{-1} t = √{-1} b ⊕ √{-1} a . \]

Let σ be the orthogonal transformation on \( √{-1} t \) defined by

\[ σ|√{-1} b = -1 \quad \text{and} \quad σ|√{-1} a = 1 \]

and

\[ ω = \frac{1}{2}(1 + σ): √{-1} t → √{-1} a \]

be the orthogonal projection of \( √{-1} t \) onto \( √{-1} a \). Let \( Σ_τ \) denote the set of roots of ℱ with respect to the complexification ℱ of ℱ. Let \( W_τ = N_K(T)/T \) be the Weyl group of ℱ, where T is the connected subgroup of K generated by t and \( N_K(T) \) is the normalizer of T in K. \( Σ_τ \) is a σ-invariant reduced root system in
As a group of orthogonal transformations of $\sqrt{-1}t$, $W_t$ is generated by reflections with respect to roots in $\Sigma_t$. Put

$$\Sigma_t^0 = \Sigma_t \cap \sqrt{-1}b = \{ \alpha \in \Sigma_t; \omega \alpha = 0 \},$$
$$\Sigma_s = \{ \omega \alpha; \alpha \in \Sigma_t - \Sigma_t^0 \} = \omega \Sigma_t - \{0\},$$
$$W_s = N_L(A)/Z_L(A),$$

where $A$ is the connected subgroup of $K$ generated by $\alpha$ and $N_L(A)$ (resp. $Z_L(A)$) the normalizer (resp. the centralizer) of $A$ in $L$. An element of $\Sigma_s$ is a root of the symmetric space $S$ and $W_s$ is the Weyl group of $S$. $\Sigma_s$ is a (not necessarily reduced) root system in $\sqrt{-1}a'$. As a group of orthogonal transformations of $\sqrt{-1}a$, $W_s$ is generated by reflections with respect to roots in $\Sigma_s$. A linear order $>$ on $\sqrt{-1}t$ is said to be compatible for $\Sigma_t$ with respect to $\sigma$ (or with respect to the orthogonal decomposition $\sqrt{-1}t = \sqrt{-1}b + \sqrt{-1}a$) if $\alpha \in \Sigma_t$, $\alpha > 0$ and $\sigma \alpha = -\alpha$ imply $\sigma \alpha > 0$. Take a compatible order $>$ on $\sqrt{-1}t$ and fix it once and for all. Let

$$\Pi_t = \{ \alpha_1, \ldots, \alpha_r \}$$

be the fundamental root system of $\Sigma_t$ with respect to the order $>$ and put

$$\Pi_t^0 = \Pi_t \cap \Sigma_t^0.$$ 

$W_t$ is also generated by reflections with respect to roots in $\Pi_t$. We have the decomposition

$$\sigma = sp \quad \text{where} \quad s \in W_t, \quad p \Pi_t = \Pi_t$$

of $\sigma$ in such a way that $s^2 = 1, \quad p(\Pi_t - \Pi_t^0) = \Pi_t - \Pi_t^0$ and $\sigma \alpha_i = p \alpha_i \mod \{ \Pi_t^0 \}$ for any $\alpha_i \in \Pi_t - \Pi_t^0$ (Satake [10]). We put

$$\Pi_s = \{ \omega \alpha_i; \alpha_i \in \Pi_t - \Pi_t^0 \} = \omega \Pi_t - \{0\}.$$ 

We may assume that $\Pi_s = \{ \gamma_1, \ldots, \gamma_p \}$ with $\omega \alpha_i = \gamma_i (1 \leq i \leq p)$, changing indices of the $\alpha_i$'s if necessary. $\Pi_s$ is the fundamental root system of $\Sigma_s$ with respect to the order $>$. We put

$$\Sigma_s^* = \{ \gamma \in \Sigma_s; 2 \gamma \in \Sigma_s \}.$$ 

Then $\Sigma_s^*$ is a reduced root system in $\sqrt{-1}a'$. The fundamental root system $\Pi_s^*$ of $\Sigma_s^*$ with respect to the order $>$ is given by

$$\Pi_s^* = \{ \beta_1, \ldots, \beta_p \}$$

where

$$\beta_i = \begin{cases} \gamma_i & \text{if} \ 2\gamma_i \in \Sigma_s \\ 2\gamma_i & \text{if} \ 2\gamma_i \in \Sigma_s \end{cases}.$$ 

$W_s$ is also generated by reflections with respect to roots of $\Pi_s$ or of $\Pi_s^*$. Let
\[ \Sigma_t^+ \text{ (resp. } \Sigma_s^+, (\Sigma_s^*)^+) \text{ denote the set of positive roots in } \Sigma_t \text{ (resp. } \Sigma_s, \Sigma_s^*). \]

Then

\[ \Sigma_s = \sigma (\Sigma_t^+ - \Sigma_t^-) = \sigma \Sigma_t^+ \setminus \{0\}. \]

For \( \lambda \in \sqrt{-1} \mathfrak{t}, \lambda \neq 0 \), we define

\[ \lambda^* = \frac{2}{(\lambda, \lambda)} \lambda. \]

**Theorem 2.1.** (E. Cartan) Assume that \( K \) is simply connected. Then

1) \( K_o \) is connected.

2) The kernel of \( \exp: \mathfrak{a} \to K \) is the subgroup of \( \mathfrak{a} \) generated by \( \{2\pi \sqrt{-1} \gamma^*; \gamma \in \Sigma_s\}. \)

**Theorem 2.2.** (Harish-Chandra) Let \( S^*_L(\mathfrak{s}) \) (resp. \( S^*_W(\mathfrak{a}) \)) be the space of polynomial functions on \( \mathfrak{s} \) (resp. on \( \mathfrak{a} \)), which are invariant under the adjoint actions of \( L \) (resp. of \( W_s \)). Then the restriction map

\[ S^*_L(\mathfrak{s}) \to S^*_W(\mathfrak{a}) \]

is an isomorphism.

Now we shall consider \( W_s \)-invariant characters of a maximal torus of \( S \). Put

\[ \Gamma = \Gamma(K, L) = \{H \in \mathfrak{a}; \exp H \in L\} \]

and

\[ \Gamma_c = \Gamma \cap \mathfrak{c}_a \text{ where } \mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}. \]

Then \( \Gamma \) is a \( W_s \)-invariant lattice in \( \mathfrak{a} \) and \( \Gamma_c \) is a lattice in \( \mathfrak{c}_a \). Let \( \mathcal{C}_a \) be the connected subgroup of \( K \) generated by \( \mathfrak{c}_a \). Then the \( \mathfrak{a} \)-orbit \( \mathcal{A} \) in \( S \) through the origin \( x_0 \) of \( S \) and the \( \mathfrak{c}_a \)-orbit \( \mathcal{C}_a \) in \( S \) through the origin have identifications

\[ \mathcal{A} = \mathfrak{a}/\Gamma \]

and

\[ \mathcal{C}_a = \mathfrak{c}_a/\Gamma_c. \]

Hence both \( \mathcal{A} \) and \( \mathcal{C}_a \) have structures of toral groups. The toral group \( \mathcal{A} \) is said to be a **maximal torus** of the symmetric space \( S \). The adjoint action of \( W_s \) on \( \mathcal{A} \) induces the action of \( W_s \) on \( \mathcal{A} \). This action is compatible with the natural action of \( W_s \) on \( \mathfrak{a}/\Gamma \) relative to the identification: \( \mathcal{A} \approx \mathfrak{a}/\Gamma \). Put

\[ Z = Z(K, L) = \{\lambda \in \sqrt{-1} \mathfrak{a}; (\lambda, H) \in 2\pi \sqrt{-1} \mathcal{Z} \text{ for any } H \in \Gamma\}. \]

\( Z \) is isomorphic with the group \( \mathcal{D}(\mathcal{A}) \) of characters of \( \mathcal{A} \) by the correspondence \( \lambda \mapsto e^\lambda \), where \( e^\lambda \in \mathcal{D}(\mathcal{A}) \) is defined by \( e^\lambda((\exp H)x_0) = \exp (\lambda, H) \) for \( H \in \mathfrak{a} \). Put
\[ D = D(K, L) = \{ \lambda \in \mathbb{Z}; (\lambda, \gamma_i) \geq 0 \text{ for any } \gamma_i \in \Pi_s^* \} \]
\[ = \{ \lambda \in \mathbb{Z}; (\lambda, \gamma) \geq 0 \text{ for any } \gamma \in \Sigma_s^* \}. \]

Then we have
\[ D = \{ \lambda \in \mathbb{Z}; s\lambda \leq \lambda \text{ for any } s \in W_s \}. \]

An element of \( D \) is called a **dominant integral form** on \( \alpha \). We define a lattice \( \Gamma_0' \) in \( \alpha' \) to be the subgroup of \( \alpha' \) generated by \( \{ 2\pi \sqrt{-1} (\frac{1}{2} \gamma^*); \gamma \in \Sigma_s \} \). We define a lattice \( \Gamma_0 \) in \( \alpha \) and a toral group \( \hat{A}_0 \) by
\[ \Gamma_0 = \Gamma \oplus \Gamma_0' \]
and
\[ \hat{A}_0 = \alpha/\Gamma_0. \]

Put
\[ Z_0 = \{ \lambda \in \sqrt{-1} \alpha; (\lambda, H) \in 2\pi \sqrt{-1} \mathbb{Z} \text{ for any } H \in \Gamma_0 \} \]
and
\[ D_0 = D \cap Z_0. \]

\( Z_0 \) is isomorphic with the group \( \mathcal{D}(\hat{A}_0) \) of characters of \( \hat{A}_0 \). Put furthermore
\[ Z_0' = Z_0 \cap \sqrt{-1} \alpha' = \left\{ \lambda \in \sqrt{-1} \alpha'; \frac{2(\lambda, \gamma)}{(\gamma, \gamma)} \in 2\mathbb{Z} \text{ for any } \gamma \in \Sigma_s \right\} \]
and
\[ D_0' = D_0 \cap \sqrt{-1} \alpha' = D \cap Z_0'. \]

**Lemma 1.** If \( L = K_0 \), then
\[ \Gamma = \{ \frac{1}{2} H; H \in \alpha, \exp H = e \}. \]

**Proof.** For \( H \in \alpha \), \( \exp H = e \iff \exp \frac{H}{2} \exp \frac{H}{2} = e \iff \exp \frac{H}{2} = (\exp \frac{H}{2})^{-1} \iff \exp \frac{H}{2} = \theta \left( \exp \frac{H}{2} \right) \iff \exp \frac{H}{2} \in K_0 \), which yields Lemma 1. \( \text{q.e.d.} \)

**Lemma 2.**
1) \[ \Gamma_0' = 2\pi \sqrt{-1} \sum_{i=1}^s Z(\frac{1}{2} \beta^*_i) \]
and it is \( W_s \)-invariant. Therefore \( \Gamma_0 \) is \( W_s \)-invariant.
2) \( \Gamma_0 \subset \Gamma \). Therefore \( Z_0 \supset Z \) and \( D_0 \supset D \).
3) If \( S \) is simply connected, then \( \Gamma = \Gamma_0 = \Gamma_0' \) (thus \( Z = Z_0 = Z_0' \), \( D = D_0 = D_0' \)) and \( \hat{A}_0 \) can be identified with \( \hat{A} \).

**Proof.**
1) Denoting the reflection of \( \sqrt{-1} \alpha \) with respect to \( \beta_i \in \Pi_s^* \) by \( s_i \in W_s \), we have
It follows that $\Gamma_o'$ is $W_S$-invariant. Since we have

\[
(2\lambda)^* = \frac{2 \cdot 2\lambda}{4(\lambda, \lambda)} = \frac{\lambda}{(\lambda, \lambda)} = \frac{1}{2}\lambda^* \quad \text{for} \quad \lambda \in \sqrt{-1}a, \lambda \neq 0,
\]

$\Gamma_o'$ is the subgroup of $\alpha'$ generated by $2\pi\sqrt{-1}(\frac{1}{2}\gamma^*)$ for $\gamma \in \Sigma_s^*$. Thus it suffices to show that

\[
\gamma^* \in \sum_{i=1}^f \mathbb{Z}\beta_i^* \quad \text{for any} \quad \gamma \in \Sigma_s^*.
\]

But this follows from the first equality since there exist $\beta_{i_1}, \ldots, \beta_{i_r} \in \Pi_s^*$ such that $s_{i_1} \cdots s_{i_r} \gamma \in \Pi_s^*$.

2) Since $\Gamma \subset \Gamma'$, it suffices to show that $\Gamma_o' \subset \Gamma'$ for $\Gamma' = \Gamma \cap \alpha'$. Let $K'$ be the connected subgroup of $K$ generated by $\Gamma'$ and $L' = K' \cap L$. Then $(K', L')$ is also a symmetric pair with respect to $\theta$ and $S' = K'/L'$ can be identified with the $K'$-orbit in $S$ through the origin $x_0$ of $S$. Let

\[
\pi': K_o' \to K'
\]

be the covering homomorphism of the universal covering group $K_o'$ of $K'$ and put

\[
L_o' = \{k \in K_o'; \theta_0(k) = k\},
\]

where $\theta_0$ is the involutive automorphism of $K_o'$ covering the involutive automorphism $\theta$ of $K'$. $K_o'$ is compact since $K'$ is semi-simple. $S'$ can be identified with $K_o'/\pi'^{-1}(L')$. It follows from Theorem 2.1 and Lemma 1 that $L_o'$ is connected and

\[
\Gamma_o' = \{H \in \alpha'; \exp_{K_o'} H \in L_o'\}.
\]

Let $A'$ (resp. $A_o'$) be the connected subgroup of $K'$ (resp. of $K_o'$) generated by $\alpha'$ and $\hat{A}'$ (resp. $\hat{A}_o'$) be the $A'$-orbit in $S'$ (resp. the $A_o'$-orbit in $S_o' = K_o'/L_o'$) through the origin. Then we have identifications

\[
\hat{A}' = \alpha'/\Gamma'
\]

and

\[
\hat{A}_o' = \alpha'/\Gamma_o'.
\]

On the other hand, since $\pi'^{-1}(L') \supset L_o'$, the covering homomorphism $\pi'$ induces the commutative diagram

\[
\begin{array}{ccc}
S_o' & \stackrel{\pi'}{\longrightarrow} & S' \\
\cup & \cup & \cup \\
\hat{A}_o' & \longrightarrow & \hat{A}'.
\end{array}
\]
It follows that
\[ \Gamma'_0 \subset \Gamma'. \]

3) Under the notation in 2), we have a covering map
\[ \hat{\mathcal{C}}_a \times S' \to S. \]

It follows from the assumption that \( \chi_a = \{ e \} \) and \( S' \) is simply connected. Thus the covering map \( \pi' \) is trivial and \( \Gamma' = \Gamma'_0 \). Moreover \( c_0 = \{ 0 \} \) implies that \( \Gamma = \Gamma' \) and \( \Gamma_0 = \Gamma'_0 \). q.e.d.

**Remark.** Define \( \Lambda_i \in \sqrt{-1} \alpha' (1 \leq i \leq l) \) by
\[
(\Lambda_i, \alpha^j) = \delta_{ij} \quad (1 \leq i, j \leq l).
\]

Then define \( M_i \) (1 \leq i \leq p) by
\[
M_i = \begin{cases}
2\Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_0^+) = \{ 0 \} \\
\Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_0^+) \neq \{ 0 \} \\
\Lambda_i + \lambda_i & \text{if } p\alpha_i = \alpha_i + \alpha_i.
\end{cases}
\]

Then it can be verified (cf. Sugiura [12]) that \( M_i \in \sqrt{-1} \alpha' (1 \leq i \leq p) \) and
\[
(M_i, \frac{1}{2} \beta_0^+) = \delta_{ij} \quad (1 \leq i, j \leq p).
\]

It follows that
\[
\sum_{i=1}^{p} ZM_i
\]
and
\[
D'_0 = \left\{ \sum_{i=1}^{p} m_i M_i; m_i \in \mathbb{Z}, m_i \geq 0 \ (1 \leq i \leq p) \right\}.
\]

It follows from Lemma 2.1) that \( W_0 \) acts on \( \hat{\mathcal{A}}_0 = \mathcal{A}/\Gamma_0 \) and from Lemma 2.2) that we have a \( W_0 \)-equivariant homomorphism
\[
\pi_0: \hat{\mathcal{A}}_0 \to \hat{\mathcal{A}}.
\]

Let \( R(\hat{\mathcal{A}}) \) denote the character ring of \( \hat{\mathcal{A}} \). Then \( W_0 \) acts on \( R(\hat{\mathcal{A}}) \) (or more generally on the space \( C^\infty(\hat{\mathcal{A}}) \) of \( C \)-valued \( C^\infty \)-functions on \( \hat{\mathcal{A}} \)) by
\[
(s\chi)(\hat{a}) = \chi(s^{-1} \hat{a}) \quad \text{for } s \in W_0, \hat{a} \in \hat{\mathcal{A}}.
\]

This action coincides on \( Z = \mathcal{O}(\hat{\mathcal{A}}) \subset R(\hat{\mathcal{A}}) \) with the adjoint action of \( W_0 \) on \( Z \). Let \( R_{W_0}(\hat{\mathcal{A}}) \) be the subring of \( W_0 \)-invariant characters of \( \hat{\mathcal{A}} \) and \( R_{W_0}(\hat{\mathcal{A}}) \) the \( C \)-span of \( R_{W_0}(\hat{\mathcal{A}}) \) in \( C^\infty(\hat{\mathcal{A}}) \). Let \( R(\hat{\mathcal{A}}_0), R_{W_0}(\hat{\mathcal{A}}_0) \) and \( R_{W_0}(\hat{\mathcal{A}}_0) \) denote the same objects for \( \hat{\mathcal{A}}_0 \). Then \( \pi_0 \) induces a \( W_0 \)-equivariant monomorphism
\[
\pi_0^*: R(\hat{\mathcal{A}}) \to R(\hat{\mathcal{A}}_0)
\]
and monomorphisms
\[ \pi^\#: \mathcal{R}_{W_s}(\hat{A}) \to \mathcal{R}_{W_s}(A_0), \]
\[ \pi^\#: \mathcal{R}_{W_s}(\hat{A})^c \to \mathcal{R}_{W_s}(A_0)^c. \]

Henceforth we shall identify \( \mathcal{R}_{W_s}(\hat{A}) \) with a subring of \( \mathcal{R}_{W_s}(A_0) \) and \( \mathcal{R}_{W_s}(\hat{A})^c \) with a subalgebra of \( \mathcal{R}_{W_s}(A_0)^c \) by means of these monomorphisms \( \pi^\# \).

For \( \lambda \in \sqrt{-1}a \), we shall denote by \( \lambda_\epsilon \) the \( \sqrt{-1}c_\epsilon \)-component of \( \lambda \) with respect to the orthogonal decomposition
\[ \sqrt{-1}a = \sqrt{-1}c_\epsilon \oplus \sqrt{-1}a'. \]

The following facts can be proved in the same way as the classical results for a compact connected Lie group \( S \), so the proofs are omitted.

We define an element \( \delta \) in \( Z_s \) by
\[ \delta = \sum_{\gamma \in (\Sigma_{G^0})^+} \gamma. \]

For \( \lambda \in Z_s \), we define \( \xi_\lambda \in \mathcal{R}(\hat{A}) \) by
\[ \xi_\lambda = \sum_{\alpha \in W_s} (\det \xi)e^{\alpha}. \]

For \( \lambda \in Z \), \( \xi_\lambda \) is divisible by \( \xi_\delta \) in the ring \( \mathcal{R}(\hat{A}) \) and
\[ \chi_\lambda = \frac{\xi_{\lambda + \delta}}{\xi_\delta} \]
is in \( \mathcal{R}_{W_s}(\hat{A}) \). If \( \chi_\lambda \) has the expression
\[ \chi_\lambda = \sum m_\mu e^{\mu} \quad \text{with} \quad \mu \in Z, m_\mu \in Z, m_\mu \neq 0, \]
then \( \mu_\epsilon \) are the same for any \( \mu \). In particular, if \( \lambda \in D \), then the highest component in the above expression of \( \chi_\lambda \) is \( e^{\lambda} \) with \( m_\lambda = 1 \). Any \( W_s \)-invariant character \( \chi \in \mathcal{R}_{W_s}(\hat{A}) \) of \( \hat{A} \) has an expression
\[ \chi = \sum m_\lambda \chi_\lambda \quad \text{with} \quad \lambda \in D, \ m_\lambda \in Z. \]
The expression is unique for \( \chi \). In particular, the system \( \{\chi_\lambda ; \lambda \in D\} \) forms a basis of the space \( \mathcal{R}_{W_s}(\hat{A})^c \).

Now we come back to spherical representations of a symmetric pair \( (K, L) \).

**Theorem 2.3.** (E. Cartan [1]) Let \( \rho \in \mathcal{D}(K, L) \) have the highest weight \( \lambda \in \sqrt{-1}t \) and \( \omega_\lambda \) be the zonal spherical function for \( (K, L) \) belonging to \( \rho \). Then
1) \( \lambda \in D \),
2) \( \omega_\lambda \) restricted to \( \hat{A} \) is in \( \mathcal{R}_{W_s}(\hat{A})^c \) and has an expression
\[ \omega_\lambda = \sum a_\mu e^{-\mu} \quad \text{with} \quad \mu \in Z, a_\mu \in \mathcal{R}, a_\mu > 0, \sum a_\mu = 1, \]
with the lowest component $a_\lambda e^{-\lambda}$.

Proof. Proof of E. Cartan [1] was done in the case where $K$ is semi-simple and $L=K_\mathbb{R}$. His proof can be applied for our case without difficulties. But his proof of $\lambda \in \sqrt{-1} a$ is not complete. A correct proof is seen, for example, in Schmid [11]. q.e.d.

**Lemma 3.** For any $\lambda \in D$, there exists an irreducible representation $\rho$ of $K$ such that the highest weight of $\rho$ on $\mathfrak{t}^C$ is $\lambda$.

Proof. Let $H \in \mathfrak{t}$ with $\exp H = e$. Decompose $H$ as

$$H = H' + H'' \quad \text{with} \quad H' \in \mathfrak{b}, \ H'' \in \mathfrak{a}.$$  

Then $\exp H'' = (\exp H')^{-1} \in L$, i.e. $H'' \in \mathfrak{g}$. It follows from $\lambda \in Z \subset \sqrt{-1} a$ that $(\lambda, H) = (\lambda, H') + (\lambda, H'') \in 2\pi \sqrt{-1} \mathbb{Z}$. Moreover $(\lambda, \alpha_i) = (\lambda, \sigma_i \alpha_i) > 0$ for any $\alpha_i \in \Pi$ since $\lambda \in D$. Thus $e^\lambda$ is a dominant character of the maximal torus $T$ of $K$. Then the classical representation theory of compact connected Lie groups assures the existence of $\rho$. q.e.d.

**Lemma 4.** Let $Z_L(A)$ be the centralizer in $L$ of $A$ and $Z_L(A)^0$ the connected component of $Z_L(A)$. Then

$$Z_L(A) = Z_L(A)^0 \exp \Gamma.$$  

Proof. The centralizer $\mathfrak{z}_L(A)$ in $L$ of $A$ has the decomposition

$$\mathfrak{z}_L(A) = \mathfrak{z}_L(A)^0 \oplus \mathfrak{a},$$

where $\mathfrak{z}_L(A)$ is the centralizer in $L$ of $A$. Since the centralizer $Z_K(A)$ in the compact connected Lie group $K$ of the torus $A$ is connected, we have the decomposition

$$Z_K(A) = Z_L(A)^0 A.$$  

It follows that any element $m \in Z_L(A)$ can be written as

$$m = m' a \quad \text{with} \quad m' \in Z_L(A)^0, \ a \in A.$$  

Then $a = m'^{-1} m \in L$ so that $a \in \exp \Gamma$. Thus $m \in Z_L(A)^0 \exp \Gamma$, which proves Lemma 4. q.e.d.

**Lemma 5.** Let $K^C$ denote the Chevalley complexification of $K$. Put

$$K^* = L \exp \sqrt{-1} \mathfrak{g}$$

and

$$(K^*)^0 = L^0 \exp \sqrt{-1} \mathfrak{g},$$

where $L^0$ denotes the connected component of $L$. Then $(K^*)^0$ is a closed subgroup of
\( K^c \) normalized by \( K^* \) and

\[ K^* = (K^*)^\circ \exp \Gamma. \]

Therefore \( K^* \) is a closed subgroup of \( K^c \) with the connected component \( (K^*)^\circ \).

Proof. The first statement is clear. Take any element \( l \in L \). From the conjugateness of maximal abelian subalgebras in \( \mathfrak{s} \) under the adjoint action of \( L^0 \), there exists \( l_i \in L^0 \) such that \( l_il_i \in N_L(A) \). Since

\[ N_L(A)/Z_L(A) = N_{L^0}(A)/Z_{L^0}(A) = W_S, \]

we can choose \( l_2 \in L^0 \) such that \( l_1l_2l_1 \in Z_L(A) \). It follows from Lemma 4 that there exist \( l_2 \in Z_L(A)^0 \) and \( a \in \exp \Gamma \) such that \( l_2l_2l_2 = l_2a \). Therefore \( l = l_2l_2^{-1}l_2^{-1}a \) with \( l_2l_2^{-1}l_2^{-1} \in L^0(\mathfrak{k}^*)^\circ \), i.e. \( l \in (K^*)^\circ \exp \Gamma \). This completes the proof of Lemma 5.

q.e.d.

Now we can prove the following

**Theorem 2.4.** (E. Cartan [1], Sugiura [12], Helgason [5]) For any \( \lambda \in D \), there exists an irreducible spherical representation \( \rho \) of \( K \) relative to \( L \) such that the highest weight of \( \rho \) on \( t^c \) is \( \lambda \).

Together with Theorem 2.3 we have the following

**Corollary.** For \( \rho \in D(K, L) \), let \( \lambda(\rho) \) denote the highest weight of \( \rho \) on \( t^c \). Then the correspondence \( \rho \mapsto \lambda(\rho) \) gives a bijection:

\[ D(K, L) \rightarrow D(K, L). \]

Proof of Theorem 2.4. This theorem for the case where \( K \) is semi-simple and \( L = K_\Theta \) was stated in E. Cartan [1] but its proof is not complete. It was stated for simply connected \( K \) without proof in Sugiura [12]. It was proved in Helgason [5] for the case where \( K \) is semi-simple and \( L \) is connected. Helgason's proof can be applied for our case without difficulties, so we shall confine ourselves to point out necessary modifications.

Let

\[ \rho: K \rightarrow GL(V) \]

be the irreducible representation of \( K \) with the highest weight \( \lambda \) (Lemma 3). By extending \( \rho \) to the Chevalley complexification \( K^c \) of \( K \) and restricting it to the closed subgroup \( K^* \) of \( K^c \) (Lemma 5), we have an irreducible representation of \( K^* \), which will be denoted by the same letter \( \rho \). It suffices to show that \( \rho \) has a non-zero \( L \)-invariant. Let \( N \) be the connected subgroup of \( K^* \) generated by the subalgebra

\[ n = \mathfrak{l}^* \cap \sum_{\alpha \in \Sigma^+} \mathfrak{l}^*_\alpha \]
where \( \mathfrak{t}^* \) is the Lie algebra of \( K^* \) and \( \mathfrak{t}_\alpha^c \) is the root space of \( \mathfrak{t}^c \) for \( \alpha \). We shall first prove that the representation \( \rho \) of \( K^* \) is a conical representation of \( K^* \) in the sense of Helgason [5], i.e. if \( v_\lambda \in V, v_\lambda \neq 0 \), is a highest weight vector for \( \rho \) with respect to \( \mathfrak{t}^c \), we have

\[
\rho(mn)v_\lambda = v_\lambda \quad \text{for any } \ m \in Z_L(A), \ n \in N.
\]

Denoting the infinitesimal action of \( \mathfrak{t}^c \) on \( V \) by the same letter \( \rho \), we have

\[
\rho(n)v_\lambda = \rho(\delta_\alpha(a))v_\lambda = \{0\}.
\]

In fact, \( \rho(n)v_\lambda = \{0\} \) since \( n \subset \sum_\alpha \mathfrak{t}_\alpha^c \), \( \rho(\mathfrak{b}^c)v_\lambda = \{0\} \) for the complexification \( \mathfrak{b}^c \) of \( \mathfrak{b} \) since \( (\sqrt{-1} \mathfrak{b}, \lambda) = \{0\} \). \( \rho(\mathfrak{t}_\alpha^c)v_\lambda = \{0\} \) for \( \alpha \in \Sigma^0_t, \alpha > 0 \). It follows from \( (\alpha, \lambda) (\sqrt{-1} \mathfrak{b}, \lambda) = \{0\} \) for \( \alpha \in \Sigma^0_t \) that \( \lambda - \alpha \) is not a weight of \( \rho \) for \( \alpha \in \Sigma^0_t, \alpha > 0 \). Since the complexification of \( \delta_\alpha(a) \) is spanned by \( \mathfrak{b}^c \) and the \( \mathfrak{t}_\alpha^c \)'s for \( \alpha \in \Sigma^0_t \), we have \( \rho(\delta_\alpha(a))v_\lambda = \{0\} \). Therefore it suffices from Lemma 4 to show that

\[
\rho(\exp H)v_\lambda = v_\lambda \quad \text{for any } \ H \in \Gamma.
\]

But it is clear since \( \lambda \in Z, i.e. (\lambda, H) \in 2\pi \sqrt{-1} Z \) for any \( H \in \Gamma \).

Thus we can prove in the same way as Helgason [5] that \( V \) has a non-zero \( L \)-invariant vector, by constructing a \( K^* \)-submodule \( V' \) of the \( K^* \)-module \( C^\infty(K^*) \) of \( C^\infty \)-functions on \( K^* \), having a non-zero \( L \)-invariant, and by constructing a \( K^* \)-equivariant isomorphism of \( V \) onto \( V' \).

Next we shall describe zonal spherical functions in terms of the basis \( \{X_\lambda; \lambda \in D\} \) of \( R(W_s(A))^c \).

For \( \alpha = (\exp H)x_\alpha \in \hat{A}, H \in \mathfrak{a} \), we put

\[
D(\alpha) = \left| \prod_{\alpha \in \Sigma^+_t - \Sigma^0_t} \frac{2 \sin(\alpha, \sqrt{-1} H)}{2 \sin(\alpha, \sqrt{-1} H)} \right|.
\]

Let \( d\hat{\alpha} \) denote the normalized Haar measure of \( \hat{A} \) and \( |W_s| \) the order of the Weyl group \( W_s \). For \( W_s \)-invariant functions \( \chi, \chi' \) on \( \hat{A} \), we define

\[
\langle \chi, \chi' \rangle = \frac{c}{|W_s|} \int_{\hat{A}} \chi(\hat{a})\chi'(\hat{a})D(\hat{a})d\hat{a},
\]

where

\[
c = \left( \frac{1}{|W_s|} \int_{\hat{A}} D(\hat{a})d\hat{a} \right)^{-1}.
\]

\( c=1 \) in the case where \( S \) is a compact connected Lie group. In particular, if \( \chi \) and \( \chi' \) can be extended to \( L \)-invariant functions \( f \) and \( f' \) on \( S \), then \( \langle \chi, \chi' \rangle \) coincides with the inner product \( \langle f, f' \rangle \) in \( L^2(S) \) (cf. Helgason [4]).

Fix a dominant integral form \( \lambda \in D \). We define a finite subset \( D_\lambda \) of \( D \) by
\( D_\lambda = \{ \mu \in D; \mu_c = \lambda_c, \mu \leq \lambda \} \).

Since the system \( \{ \chi_\mu; \mu \in D \} \) forms a basis of \( \mathcal{R}_{W_\lambda}(\hat{A})^c \), the matrix

\[
(\langle \chi_\mu, \chi_\nu \rangle)_{\mu, \nu \in D_\lambda}
\]

is a positive definite hermitian matrix. Let

\[
(b^{\mu \nu})_{\mu, \nu \in D_\lambda}
\]

be the inverse matrix of the above matrix. In particular \( b^{\lambda \lambda} > 0 \). For any \( \mu \in D_\lambda \), we put

\[
c^\mu_\lambda = \frac{b^{\lambda \mu}}{\sqrt{d_\lambda b^{\lambda \lambda}}},
\]

where \( d_\lambda \) is the degree of an irreducible representation of \( K \) with the highest weight \( \lambda \). Then we have

**Theorem 2.5.** Let \( \lambda \in D \) and \( \omega_\lambda \) be the zonal spherical function belonging to the class of an irreducible representation of \( K \) with the highest weight \( \lambda \). Then \( \omega_\lambda \) restricted to \( \hat{A} \) is given by

\[
\omega_\lambda = \sum_{\mu \in D_\lambda} c^\mu_\lambda \chi_\mu.
\]

Proof. The idea of the following proof owes to Hua [6]. Let \( \mu \in D_\lambda \). Then \( \omega_\mu \) restricted to \( \hat{A} \) is in \( \mathcal{R}_{W_\lambda}(\hat{A})^c \) by Theorem 2.3. It follows by Theorem 2.3 and Corollary of Theorem 2.4 that \( \omega_\mu \) has an expression

\[
\omega_\mu = \sum_{\nu \in D_\lambda} c^{\nu}_\mu \chi_\nu \quad \text{with} \quad c^{\nu}_\mu \in \mathbb{R}, \quad c^{\nu}_\mu > 0, \quad c^{\nu}_\mu = 0 \quad \text{if} \quad \nu > \mu.
\]

We define an upper triangular matrix \( C' \) by

\[
C' = (c^{\nu}_\mu)_{\nu, \mu \in D_\lambda}.
\]

Then we have

\[
(\langle \omega_\mu, \omega_\nu \rangle)_{\mu, \nu \in D_\lambda} = 'C'(\langle \chi_\mu, \chi_\nu \rangle)_{\mu, \nu \in D_\lambda} C'.
\]

Since \( \langle \omega_\mu, \omega_\nu \rangle = d_\mu^{-1} \delta_{\mu \nu} \), we have

\[
(d_\mu \delta_{\mu \nu})_{\mu, \nu \in D_\lambda} = C'^{-1} B' t C'^{-1},
\]

where

\[
B' = (b^{\mu \nu})_{\mu, \nu \in D_\lambda} = (\langle \chi_\mu, \chi_\nu \rangle)_{\mu, \nu \in D_\lambda}^{-1}.
\]

It follows that

\[
C' (d_\mu \delta_{\mu \nu})_{\mu, \nu \in D_\lambda} 'C' = B'.
\]

Comparing \( (\mu, \lambda) \)-components of both sides, we have
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\[ c_\lambda^* d_\lambda c_\lambda = b^{\rho \lambda} . \]

In particular

\[ (c_\lambda^*)^2 d_\lambda = b^{\rho \lambda}, \quad \text{i.e.} \quad c_\lambda^* = \sqrt{\frac{b^{\rho \lambda}}{d_\lambda}} , \]

hence

\[ c_\lambda^* = \frac{b^{\rho \lambda}}{d_\lambda c_\lambda} = \frac{b^{\rho \lambda}}{\sqrt{d_\lambda b^{\rho \lambda}}} . \]

Since \( b^{\rho \nu} = b^{\rho \mu} \), we have

\[ c_\lambda^* = \frac{b^{\rho \mu}}{\sqrt{d_\lambda b^{\rho \lambda}}} = c_\lambda^\nu . \quad \text{q.e.d.} \]

**EXAMPLE.** If \( S \) is a compact connected Lie group and \((K, L)\) the symmetric pair with \( K/L = S \) as mentioned before, then the set \( D(S) \) of equivalence classes of irreducible representations of \( S \) is in the bijective correspondence with \( D(K, L) \) by the assignment \( \rho \mapsto \rho \otimes \rho^* \), where \( \rho^* \) denotes the contragredient representation of \( \rho \). \( \hat{A} \) is a maximal torus of the compact Lie group \( S \). Let \( \chi_\rho \) be the invariant character of \( \hat{A} \) for the dominant integral form in \( D(K, L) \) corresponding to \( \rho \otimes \rho^* \) by the bijection in Corollary of Theorem 2.4. Then it is nothing but the character of \( \rho \). It follows from orthogonality relations of irreducible characters that the matrix \((b^{\rho \nu})\) is the identity matrix. Thus the zonal spherical function \( \omega_{p \otimes \rho^*} \) belonging to \( \rho \otimes \rho^* \) is given by

\[ \omega_{p \otimes \rho^*} = \frac{1}{d_\rho} \chi_\rho , \]

where \( d_\rho \) is the degree of \( \rho \).

3. **Polynomial representations associated with symmetric bounded domains**

Let \( D \) be an irreducible symmetric bounded domain with rank \( p \) realized in \((p^c)^c\) as in Introduction. We shall use the same notation as in Introduction.

Let

\[ \Pi = \{ \alpha_1, \ldots, \alpha_t \} \]

be the fundamental root system of \( \Sigma \) with respect to the order \( > \) and let \( \Pi_l = \Pi \cap \Sigma_l \). It is known that \( \Pi_l \) is the fundamental root system of \( \Sigma_l \), \( \Pi - \Pi_l \) consists of one element, say \( \alpha_1 \), which is the lowest root in \( \Sigma_l^+ \), and for any

\[ \alpha = \sum_{t=1}^{t} m_i \alpha_i \in \Sigma_l^+ , m_i \geq 1. \]

Let \( \Sigma_l^+ \) denote the set of positive compact roots. Put

\[ b = \{ H \in \mathfrak{a} ; (\sqrt{-1} H, \Delta) = \{0\} \} . \]
Then we have the orthogonal decomposition
\[ \sqrt{-1} t = \sqrt{-1} b \oplus \sqrt{-1} a \]
with respect to ( , ). We define an orthogonal transformation \( \sigma \) on \( \sqrt{-1} t \) by \( \sigma | b = -1 \) and \( \sigma | \sqrt{-1} a = 1 \). Let
\[ \varpi = \frac{1}{2} (1 + \sigma): \sqrt{-1} t \rightarrow \sqrt{-1} a \]
be the orthogonal projection of \( \sqrt{-1} t \) onto \( \sqrt{-1} a \). Let \( \kappa \) be the unique involutive element of the Weyl group \( W_t \) of \( K \) such that \( \kappa \Pi_t = - \Pi_t \). Since \( \Sigma^+_{\rho} \) is the set of weights on \( t^c \) of the irreducible \( K \)-module \( (p^c)^* \), we have \( \kappa \Sigma^+_{\rho} = \Sigma^+_{\rho} \) and \( \kappa \gamma_i = \alpha_i \). Put
\[ \Delta' = \kappa \Delta = \{ \gamma_1', \ldots, \gamma_p' \}, \quad \gamma_i' = \kappa \gamma_i (1 \leq i \leq p), \gamma_p' = \alpha_1. \]
It is the original maximal strongly orthogonal subsystem of \( \Sigma^+_{\rho} \) of Harish-Chandra [3]. For the system \( \Delta' \), the orthogonal projection
\[ \varpi': \sqrt{-1} t \rightarrow \sqrt{-1} a' \]
on the \( R \)-span \( \sqrt{-1} a' \) of \( \Delta' \) is defined in the same way as for \( \Delta \). Put
\[ P_i' = \{ \alpha \in \Sigma^+_{\rho}; \varpi'(\alpha) = \frac{1}{2} (\gamma_i' + \gamma_j') \} \text{ for some } 1 \leq i < j \leq p, \]
\[ P_i' = \{ \alpha \in \Sigma^+_{\rho}; \varpi'(\alpha) = \frac{1}{2} \gamma_i' \} \text{ for some } 1 \leq i \leq p, \]
\[ K_i' = \{ \alpha \in \Sigma^+_{\rho}; \varpi'(\alpha) = \frac{1}{2} \gamma_i' \} \text{ for some } 1 \leq i \leq p. \]
Then (Harish-Chandra [3]) \( \Sigma \) is the disjoint union of \( P_i', -P_i', P_i', -P_i', K_i', -K_i' \) and we have
\[ \varpi' P_i' = \{ \frac{1}{2} (\gamma_i' + \gamma_j'); 1 \leq i < j \leq p \}, \]
\[ \varpi' P_i' = \{ \frac{1}{2} \gamma_i'; 1 \leq i \leq p \} \text{ if } P_i' \neq \phi, \]
\[ \varpi' K_i' - \{ 0 \} = \{ \pm \frac{1}{2} (\gamma_i' - \gamma_j'); 1 \leq i < j \leq p \}, \]
\[ \varpi' K_i' = \{ \frac{1}{2} \gamma_i'; 1 \leq i \leq p \} \text{ if } P_i' \neq \phi. \]
Furthermore the multiplicity (with respect to \( \varpi' \)) of any \( \gamma_i' \) is 1 and that of any \( \frac{1}{2} \gamma_i' \) is even. It follows that
\[ \varpi' \Sigma - \{ 0 \} = \{ \pm \frac{1}{2} (\gamma_i' \pm \gamma_j'); 1 \leq i < j \leq p, \pm \gamma_i; 1 \leq i \leq p \} \text{ if } P_i' = \phi \]
\[ \{ \pm \frac{1}{2} (\gamma_i' \pm \gamma_j'); 1 \leq i < j \leq p, \pm \gamma_i', \pm \frac{1}{2} \gamma_i'; 1 \leq i \leq p \} \text{ if } P_i' \neq \phi. \]
Moreover we have (Moore [8])
\[ \varpi' \Pi - \{ 0 \} = \{ \gamma_i', \frac{1}{2} (\gamma_j' - \gamma_i'); \ldots, \frac{1}{2} (\gamma_p' - \gamma_p' - 1) \} \text{ if } P_i' = \phi \]
\[ \{ \gamma_i', \frac{1}{2} (\gamma_j' - \gamma_i'); \ldots, \frac{1}{2} (\gamma_p' - \gamma_p' - 1), -\frac{1}{2} \gamma_p' \} \text{ if } P_i' \neq \phi, \]
\[ \omega' \Pi_1 - \{0\} = \begin{cases} \{ \frac{1}{2}(\gamma'_i - \gamma_i), \ldots, \frac{1}{2}(\gamma'_p - \gamma'_p) \} & \text{if } P'_1 = \phi \\ \{ \frac{1}{2}(\gamma'_i - \gamma_i), \ldots, \frac{1}{2}(\gamma'_p - \gamma'_p), -\frac{1}{2} \gamma_p \} & \text{if } P'_1 \neq \phi. \end{cases} \]

**Lemma 1.**

1) \[
\omega' \alpha_i = \begin{cases} \gamma_i & \text{if } P'_1 = \phi \\ \frac{1}{2} \gamma_i & \text{if } P'_1 \neq \phi. \end{cases}
\]

2) (Schmid [11]) If \( P'_1 \neq \phi \) and

\[
\sum_{\beta \in P'_1} m_\beta \beta \quad \text{with } m_\beta \geq 0
\]
is in the \( R \)-span \( \{ P'_1 \}_R \) of \( P'_1 \), then \( m_\beta = 0 \) for any \( \beta \).

**Proof.** For any \( \alpha \in \sum_+ = P'_1 \cup P'_1 \), \( \omega' \alpha \) can be written as

\[
\omega' \alpha = \frac{1}{2} m_1 (\gamma'_2 - \gamma'_1) + \frac{1}{2} m_2 (\gamma'_3 - \gamma'_2) + \cdots + \frac{1}{2} m_{p-1} (\gamma'_p - \gamma'_{p-1})
\]

\[
- \frac{1}{2} m_p \gamma'_p + m_{p+1} \gamma'_i
\]

\[
= \frac{1}{2} (2m_{p+1} - m_i) \gamma'_1 + \frac{1}{2} (m_1 - m_i) \gamma'_2 + \cdots + \frac{1}{2} (m_{p-2} - m_{p-1}) \gamma'_{p-1}
\]

\[
+ \frac{1}{2} (m_{p-1} - m_p) \gamma'_p
\]

where \( m_i \in \mathbb{Z}, m_i \geq 0, m_{p+1} = 1 \). Since \( \omega' \alpha = \frac{1}{2} (\gamma'_i + \gamma'_j) \) or \( \frac{1}{2} \gamma'_i \) for some \( i, j \), we have

\[ 2 \geq m_1 \geq m_2 \geq \cdots \geq m_{p-1} \geq m_p \geq 0. \]

Furthermore \( \alpha \in P'_1 \) (resp. \( \alpha \in P'_1 \)) if and only if \( m_p = 0 \) (resp. \( m_p = 1 \)).

1) If \( P'_1 = \phi \), then \( \gamma'_1 \in P'_1 \). For \( \alpha = \gamma'_1 \), the coefficients in the above expression are \( m_1 = \cdots = m_{p-1} = 2, m_p = 0 \) and \( \omega' \gamma'_1 = \gamma'_p \). If \( P'_1 = \phi \), then for \( \alpha = \gamma'_1 \), the coefficients are \( m_1 = \cdots = m_{p-1} = 2, m_p = 1 \) and \( \omega' \gamma'_1 = \frac{1}{2} \gamma'_p \). Now the assertion 1) follows from \( \omega' \alpha = \kappa^{-1} \omega' \kappa \alpha_i = \kappa^{-1} \omega' \gamma'_1 \).

2) Let

\[ \alpha = \sum_{i=1}^t n_i \alpha_i \quad \text{with } n_i \in \mathbb{Z}, n_i \geq 0 \]

be in \( \sum_+ \). It follows from the first argument that

(a) if \( \alpha \in P'_1 \), \( \omega' \alpha_i = -\frac{1}{2} \gamma'_p \), then \( n_i = 0 \),

(b) if \( \alpha \in P'_1 \), then there exists \( \alpha_i \in \Pi_1 \) such that \( n_i > 0 \) and \( \omega' \alpha_i = -\frac{1}{2} \gamma'_p \).

This implies the assertion 2).

q.e.d.
Now $P_1$, $P_3$, $K_o$ and $K_1$ are defined for $\Delta$ in the same way as for $\Delta'$. Then $\kappa$ transforms $P_1$ (resp. $P_3$, $K_o$, $K_1$) onto $P_1'$ (resp. $P_3'$, $K_o'$, $K_1'$). It follows that the above mentioned properties due to Harish-Chandra are also satisfied by our objects for $\Delta$. But Moore's results should be modified as follows.

$$\omega \Pi - \{0\} = \left\{ \begin{array}{ll} \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p), \gamma_p \} & \text{if } P_1 = \phi \\ \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p), \frac{1}{2} \gamma_p \} & \text{if } P_1 \neq \phi. \end{array} \right.$$

$$\omega \Pi_t - \{0\} = \left\{ \begin{array}{ll} \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p) \} & \text{if } P_1 = \phi \\ \{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p), \frac{1}{2} \gamma_p \} & \text{if } P_1 \neq \phi. \end{array} \right.$$

They follow from Lemma 1, 1) and

$$\omega \Pi_t = \kappa^{-1} \omega' \kappa \Pi_t = -\kappa^{-1} \omega' \Pi_t.$$

Note that $K_1 \subset \sum^+_t$ while $K_1' \subset -\sum^+_t$.

**Lemma 2.** 1) The order $>$ is a compatible order for $\sum$ with respect to $\sigma$ in the sense of 2.

2) $\omega K_0 - \{0\}$ is a root system with the fundamental root system

$$\{ \frac{1}{2} (\gamma_1 - \gamma_2), \cdots, \frac{1}{2} (\gamma_{p-1} - \gamma_p) \}$$

with respect to the order $>$.  

3) If $P_1 \neq \phi$ and

$$\sum_{\beta \in P_1} m_\beta \beta \text{ with } m_\beta \geq 0$$

is in the $R$-span $\{ P_1 \}_R$ of $P_1$, then $m_\beta = 0$ for any $\beta$.

Proof. 1) is clear from the form of $\omega \Pi - \{0\}$ above.

2) is clear since

$$\omega K_0 - \{0\} = \{ \pm \frac{1}{2} (\gamma_i - \gamma_j); 1 \leq i < j \leq p \}.$$

3) follows from Lemma 1, 2) and $\kappa P_1 = P_1'$, $\kappa P_3 = P_3'$. q.e.d.

For $\lambda \in \sqrt{-1} t$, $\lambda \neq 0$, we define as in 2

$$\lambda^* = \frac{2}{(\lambda, \lambda)} \lambda.$$
and put

\[ Z_0 = \frac{1}{2} \sum_{\alpha \in \Delta} \gamma^*. \]

Since \((\frac{1}{2} \gamma_i, \gamma_j^*) = \delta_{ij}\) for \(1 \leq i, j \leq p\), we have

\[ P_i = \{ \alpha \in \Sigma_p; (\alpha, Z_0) = 1 \}, \]
\[ P_{\bar{i}} = \{ \alpha \in \Sigma_p; (\alpha, Z_0) = \frac{1}{2} \}, \]
\[ K_0 = \{ \alpha \in \Sigma_\bar{\iota}; (\alpha, Z_0) = 0 \}, \]
\[ K_{\bar{i}} = \{ \alpha \in \Sigma_{\bar{\iota}}; (\alpha, Z_0) = \frac{1}{2} \}. \]

Hence eigenvalues of \(\text{ad} \ Z_0\) are \(\pm 1, \pm \frac{1}{2}\) on \(\mathfrak{p}^c, 0, \pm \frac{1}{2}\) on \(\mathfrak{h}^c\). Let \(\mathfrak{p}^c_{\pm 1}, \mathfrak{p}^c_{\pm \frac{1}{2}}, \mathfrak{r}^c_0, \mathfrak{r}^c_{\bar{\iota}}\) denote the corresponding eigenspaces. Note that the origin \(X_0\) of the Shilov boundary \(S\) is in \(\mathfrak{p}^c_1\).

The following results are due to Korányi-Wolf [7]. We define an element \(c\) of \(G^c\), which is called Cayley transform, by

\[ c = \exp \left( -\frac{\pi}{4} \sum_{\gamma \in \Delta} (X_\gamma + X_{-\gamma}) \right) \]

and define an automorphism of \(G^c\) by

\[ \theta(x) = c^2 x c^{-2} \quad \text{for} \quad x \in G^c. \]

The automorphism \(\text{Ad} \ c^2\) of \(g^c\) obtained by differentiating \(\theta\) will be also denoted by the same letter \(\theta\). Then \(\theta' = 1\) and on \(\sqrt{-1} \mathfrak{t}\) it coincides with \(-\sigma\). Put

\[ \mathfrak{g}_0 = \{ X \in \mathfrak{g}; \theta^* X = X \}, \]
\[ \mathfrak{f}_0 = \mathfrak{g}_0 \cap \mathfrak{f}, \]

and

\[ \mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{p}. \]

Then \(\mathfrak{f}_0\) is \(\theta\)-invariant and

\[ \mathfrak{f}_0 = \{ X \in \mathfrak{f}; [Z_0, X] = 0 \}. \]

Hence \(\mathfrak{f}_0\) is a real form of \(\mathfrak{f}^c_0\) containing \(\mathfrak{t}\) as a maximal abelian subalgebra. \(K_0\) is nothing but the set of roots of \(\mathfrak{f}^c_0\) with respect to \(\mathfrak{t}^c\). The complexification \(\mathfrak{p}^c_0\) of \(\mathfrak{p}_0\) is the direct sum of \(\mathfrak{p}^c_{\pm 1}\) and \(\mathfrak{p}^c_{\pm \frac{1}{2}}\). \(\mathfrak{g}_0\) is a reductive subalgebra of \(\mathfrak{g}\) with a Cartan decomposition

\[ \mathfrak{g}_0 = \mathfrak{f}_0 + \mathfrak{p}_0. \]
Let \( G_0 \) (resp. \( K_0 \)) be the connected subgroup of \( G \) generated by \( g_0 \) (resp. by \( t_0 \)) and let
\[
L_0 = \{ k \in K_0; \text{Ad} X_0 = X_0 \} = K_0 \cap L.
\]
Put
\[
D_0 = D \cap \mathfrak{p} \mathfrak{c}_1
\]
and
\[
S_0 = S \cap \mathfrak{p} \mathfrak{c}_1.
\]
Then \( G_0 \) acts on \( D_0 \) transitively and \( K \cap G_0 \) coincides with \( K_0 \), so that \( D_0 \) is identified with the quotient space \( G_0 / K_0 \). Furthermore \( K_0 \) acts on \( S_0 \) transitively so that \( S_0 \) is identified with \( K_0 / L_0 \). \( D_0 \) is totally geodesic in \( D \) with respect to Bergmann metric of \( D \) and it is also an irreducible symmetric bounded domain with the same rank as \( D \). \( S_0 \) is the Shilov boundary of \( D_0 \). The complex structure of \( D_0 \) is given at the origin by \( \text{ad} H_0 \) with \( \sqrt{-1} H_0 Z_0 \). We have
\[
\forall Z = Z_0.
\]
The inclusion \( D_0 \subset \mathfrak{p} \mathfrak{c}_1 \) is nothing but the Harish-Chandra's imbedding of \( D_0 = G_0 / K_0 \). \( (K_0, L_0) \) is a symmetric pair with respect to \( \theta \), having the same rank as \( D \). Hence
\[
\mathfrak{l}_0 = \{ X \in \mathfrak{k}_0; \theta X = X \}
\]
is the Lie algebra of \( L_0 \) and \( \mathfrak{a} \) is a maximal abelian subalgebra of
\[
\mathfrak{s}_0 = \{ X \in \mathfrak{k}_0; \theta X = - X \}.
\]
We can define a semi-linear transformation \( X \mapsto \bar{X} \) of \( \mathfrak{p} \mathfrak{c}_1 \) by
\[
X = \tau \theta X = \theta \tau X \quad \text{for} \quad X \in \mathfrak{p} \mathfrak{c}_1.
\]
Put
\[
\mathfrak{p}^{-1} = \{ X \in \mathfrak{p} \mathfrak{c}_1; \bar{X} = X \}.
\]
It is a real form of \( \mathfrak{p} \mathfrak{c}_1 \) and is invariant under the adjoint action of \( L_0 \) on \( \mathfrak{p} \mathfrak{c}_1 \). The correspondence \( X \mapsto [X, X] \) gives an isomorphism
\[
\psi: \sqrt{-1} \mathfrak{s}_0 \rightarrow \mathfrak{p}^{-1},
\]
which is equivariant with respect to the adjoint actions of \( L_0 \).

Now we shall consider the polynomial representation \( S^*((\mathfrak{p}^c)^-) \) of \( K \). Let \( S_\#((\mathfrak{p}^c)^+) \) be the symmetric algebra over \( (\mathfrak{p}^c)^+ \). \( K \) acts on \( S_\#((\mathfrak{p}^c)^+) \) by the natural extension \( \text{Ad} \) of the adjoint action of \( K \) on \( (\mathfrak{p}^c)^+ \). On the other hand, the non-degenerate pairing
\[
(\mathfrak{p}^c)^+ \times (\mathfrak{p}^c)^- \rightarrow \mathbb{C}
\]
by means of the Killing form \((\ , \ )\) induces the identification
\[ S_\mathfrak{k}((\mathfrak{p}^c)^+) = S^*((\mathfrak{p}^c)^-) . \]
This identification is compatible with the actions of \(\mathcal{K}\), since the Killing form is invariant under the adjoint action of \(\mathcal{K}\). In the same way we have a \(\mathcal{K}_0\)-equivariant identification
\[ S_\mathfrak{k}(\mathfrak{p}^c_1) = S^*(\mathfrak{p}^c_1) . \]

\(S_\mathfrak{k}(\mathfrak{p}^c_1)\) can be considered as a \(\mathcal{K}_0\)-submodule of \(S_\mathfrak{k}((\mathfrak{p}^c)^+)\) by means of the natural monomorphism \(S_\mathfrak{k}(\mathfrak{p}^c_1) \to S_\mathfrak{k}((\mathfrak{p}^c)^+)\) induced from the inclusion \(\mathfrak{p}^c_1 \subset (\mathfrak{p}^c)^+\).

**Theorem 3.1.** (i) Any irreducible \(\mathcal{K}\)-submodule of \(S_\mathfrak{k}((\mathfrak{p}^c)^+)\) (resp. \(\mathcal{K}_0\)-submodule of \(S_\mathfrak{k}(\mathfrak{p}^c_1)\)) is contained exactly once in \(S_\mathfrak{k}((\mathfrak{p}^c)^+)\) (resp. in \(S_\mathfrak{k}(\mathfrak{p}^c_1)\)).

(ii) For an irreducible \(\mathcal{K}\)-submodule \(V\) of \(S_\mathfrak{k}((\mathfrak{p}^c)^+)\), we put
\[ V_0 = V \cap S_\mathfrak{k}(\mathfrak{p}^c_1) . \]
Then \(V \mapsto V_0\) is the one to one correspondence between the set of irreducible \(\mathcal{K}\)-submodules of \(S_\mathfrak{k}((\mathfrak{p}^c)^+)\) and the set of irreducible \(\mathcal{K}_0\)-submodules of \(S_\mathfrak{k}(\mathfrak{p}^c_1)\) in such a way that
1) The highest weights on \(\mathfrak{t}^c\) of \(V\) and \(V_0\) are the same.
2) The subspace of \(L\)-invariants in \(V\) is 1-dimensional and contained in \(V_0\).

(iii) The highest weight \(\lambda \in \sqrt{-1}\mathfrak{t}\) of an irreducible \(\mathcal{K}\)-submodule \(V\) of \(S_\mathfrak{k}((\mathfrak{p}^c)^+)\) is of the form
\[ \lambda = \sum n_i \gamma_i , \quad n_i \in \mathbb{Z} , n_1 \geq n_2 \geq \cdots \geq n_p \geq 0 . \]
If \(\sum n_i = \nu\), then \(V\) is contained in \(S_\nu((\mathfrak{p}^c)^+)\). i.e. \(S^\nu(D) \subset S^\nu(K, L)\) under the notation in Introduction.

For the proof of the theorem, we need the following

**Lemma 3.** (Murakami [9]) Let \(\mathfrak{g}\) be a Lie algebra over \(\mathbb{R}\) and \(\mathfrak{g}^c\) the complexification of \(\mathfrak{g}\). Assume that there exists \(Y \in \sqrt{-1}\mathfrak{g} \subset \mathfrak{g}^c\) such that \(\mathfrak{g}^c\) is the direct sum of 0-eigenspace \(\mathfrak{g}_0^c\), \((-1)\)-eigenspace \(\mathfrak{g}_1^c\) and \((-1)\)-eigenspace \(\mathfrak{g}_2^c\) of \(\text{ad} Y\), respectively. Let \((\rho, V)\) be a complex irreducible \(\mathfrak{g}\)-module with \(\mathfrak{g}\)-invariant hermitian inner product. Denoting the extension to \(\mathfrak{g}^c\) of \(\rho\) by the same letter \(\rho\), let \(a_1 > a_2 > \cdots > a_m (a_i \in \mathbb{R})\) be eigenvalues of \(\rho(Y)\), and \(S_t\) be \(a_t\)-eigenspace of \(\rho(Y)\) \((1 \leq t \leq m)\). Put \(\mathfrak{g}_0 = \mathfrak{g}_0^c \cap \mathfrak{g} (\ , \ which\ is\ a\ real\ form\ of\ \mathfrak{g}_0^c)\). Then
1) \(a_t = a_t - t + 1 (1 \leq t \leq m)\).
2) Each \(S_t\) is a \(\mathfrak{g}_0\)-submodule of \(V\) and
\[ V = S_1 + \cdots + S_m \]
is the orthogonal direct sum.
3) $S_i$ and $S_m$ are irreducible $\mathfrak{t}_c$-submodules of $V$ and characterized by

\[ S_i = \{ v \in V ; \rho(X)v = 0 \text{ for any } X \in \mathfrak{t}_c \} , \]
\[ S_m = \{ v \in V ; \rho(X)v = 0 \text{ for any } X \in \mathfrak{t}_c \} . \]

Proof of Theorem 3.1. The infinitesimal action of $\mathfrak{t}_c$ on $S_*(\mathfrak{g}_c^*)$ induced from the adjoint action $\text{Ad}$ of $K$ will be denoted by $\text{ad}$.

Let $V$ be an irreducible $K$-submodule of $S_*(\mathfrak{g}_c^*)$. Since $Z$ is in the center of $\mathfrak{g}_c$, it follows from Schur's lemma that $V$ is contained in an eigenspace of $\text{ad} Z$ in $S_*(\mathfrak{g}_c^*)$. But since $\text{ad} Z$ is the scalar operator on $S_*(\mathfrak{g}_c^*)$, $V$ is contained in $S_*(\mathfrak{h}_c^*)$ for some $\nu$. Let $\lambda \in \mathfrak{h}_c$ be the highest weight of $V$. Put $Y = Z \lambda \in \mathfrak{h}_c$. Then the decomposition

\[ \mathfrak{t}_c = \mathfrak{t}_c^0 + \mathfrak{t}_c^1 + \mathfrak{t}_c^{-1} \]

satisfies the assumption in Lemma 3. So we have a decomposition

\[ V = S_1 + \cdots + S_m \]

into $\mathfrak{t}_c$-submodules, where $S_i$ is an irreducible $\mathfrak{t}_c$-submodule and is the eigenspace for the maximum eigenvalue of $\text{ad} Y$ in $V$. It is characterized by

\[ S_i = \{ v \in V ; \text{ad} (X)v = 0 \text{ for any } X \in \mathfrak{t}_c \} . \]

Thus a highest weight vector $v_\lambda$ of the $K$-module $V$ is contained in $S_i$ because of $\mathfrak{g}_c^0 \subseteq \mathfrak{t}_c^+$. It follows that putting $V_0 = S_1$, $V_0$ is an irreducible $\mathfrak{t}_c$-submodule of $S_*(\mathfrak{g}_c^*)$ with the highest weight $\lambda$.

We shall show that $V_0 = V \cap S_*(\mathfrak{g}_c^*)$. We have the decomposition

\[ S_*(\mathfrak{g}_c^*) = \sum_{r, s} S_r(\mathfrak{p}_c^r) \otimes S_s(\mathfrak{p}_c^s) \]

as $\mathfrak{g}_c$-modules. $\text{ad} Z_0$ is the scalar operator $r + \frac{1}{2}s = \frac{1}{2}(r + \nu)$ on $S_r(\mathfrak{p}_c^r) \otimes S_s(\mathfrak{p}_c^s)$.

In the same way as the first argument, we can get the decomposition

\[ V = V_1 + \cdots + V_h \]

into irreducible $\mathfrak{g}_c$-submodules such that any $V_i$ is contained in $S_r(\mathfrak{p}_c^r) \otimes S_s(\mathfrak{p}_c^s)$ for some $(r, s)$. Since $S^*((\mathfrak{g}_c^-))$ is $K$-isomorphic with $S^*(S) \subseteq C^*(S)$, $V$ has an $L$-invariant $w \neq 0$. Decompose $w$ as

\[ w = w_1 + \cdots + w_h, \quad w_i \in V_i \ (1 \leq i \leq k) . \]

At least one of the $w_i$'s, say $w_1$, is not zero. Let $\lambda_1 \in \mathfrak{h}_c$ be the highest weight of the irreducible $\mathfrak{g}_c$-module $V_1$. Since $w_1$ is a non-zero $L_0$-invariant of $V_1$, $V_1$ is a spherical $\mathfrak{g}_c$-module relative to $L_0$. $(\mathfrak{g}_o, L_0)$ is a symmetric pair, $a$ is a maximal abelian subalgebra of $\mathfrak{g}_o$ and the order $> \in \mathfrak{h}_c$ is a compatible order for $\mathfrak{g}_o$ with respect to $a$ by Lemma 1, 1), so we shall use the notations
\( \Gamma(\mathcal{K}_o, L_o), Z(\mathcal{K}_o, L_o), D(\mathcal{K}_o, L_o) \) in 2. Then it follows from Theorem 2.3 that 
\( \lambda_i \in D(\mathcal{K}_o, L_o) \). On the other hand, if \( V_1 \subset S_\nu(\mathfrak{p}_f^\ast) \otimes S_\nu(\mathfrak{p}_s^\ast) \), \( \lambda_i \) is of the form

\[
\lambda_i = \sum_{a \in P_1} m_a \alpha + \sum_{b \in P_2} m_b \beta, \quad m_a, m_b \in \mathbb{Z}, \quad m_a \geq 0, \quad m_b \geq 0
\]

with \( \sum m_a = r, \sum m_b = s \). Since \( D(\mathcal{K}_o, L_o) \subset \sqrt{-1} \Delta = \{ \Delta \}_R \subset \{ P_1 \}_R \), we have

\[
\sum_{b \in P_2} m_b \beta \in \{ P_1 \}_R.
\]

It follows from Lemma 2.3) that \( r = \nu, s = 0 \), i.e. \( V_1 \subset V \cap S_\nu(\mathfrak{p}_s^\ast) \). On the other hand, \( V \cap S_\nu(\mathfrak{p}_s^\ast) \subset V_o \) since the possible maximum eigenvalue of \( \text{ad} Y \) on \( V \) is \( 2\nu \). Thus we have that \( V_o = V_1 = V \cap S_\nu(\mathfrak{p}_s^\ast) \).

The above argument shows also that any \( L \)-invariant in \( V \) is contained in \( V_o \). It is unique up to scalar since \( (\mathcal{K}_o, L_o) \) is a symmetric pair.

Conversely, let \( V_o \) be an irreducible \( K_0 \)-submodule of \( S_\nu(\mathfrak{p}_f^\ast) \) with the highest weight \( \lambda \in \sqrt{-1} I \). In the same way as the first argument, we know that \( V_o \) is contained in \( S_\nu(\mathfrak{p}_f^\ast) \) for some \( \nu \). Let \( \nu_0 \in V_o \) be a highest weight vector. Then \( \text{ad} \mathfrak{p}_f \nu_0 = \{ 0 \} \) because of \( [\mathfrak{p}_f^\ast, \mathfrak{p}_f^\ast] = \{ 0 \} \). Hence \( \text{ad} X_{\alpha} \nu_0 = 0 \) for any \( \alpha \in \sum^*_+ \). We define \( V \) to be the \( C \)-span of \( \{ \text{Ad} k \nu_0; k \in K \} \) in \( S_\nu((\mathfrak{p}_s^\ast)^+) \). Then \( V \) is an irreducible \( K \)-submodule of \( S_\nu((\mathfrak{p}_s^\ast)^+) \) with the highest weight \( \lambda \in \sqrt{-1} I \).

It is easy to see that each of the above correspondences \( V_1 \mapsto V_o \) and \( V_o \mapsto V \) is the inverse of the other. This proves assertions (i) and (ii).

(iii) We have \( [\frac{1}{2} \gamma^*, X_{-\gamma}] = -\delta_{ij} X_{-\gamma} \) \( 1 \leq i \leq j \leq \text{p} \) because of \( (\frac{1}{2} \gamma^*, \gamma) = \delta_{ij} \) \( 1 \leq i \leq j \leq \text{p} \). It follows that for \( H = 2\pi \sqrt{-1} \sum_{i=1}^{\text{p}} x_i (\frac{1}{2} \gamma_i^*, \gamma) \in \mathfrak{a} \) we have

\[
\text{Ad}(\exp H)X_i = -\sum_{i=1}^{\text{p}} \exp(-2\pi \sqrt{-1} x_i) X_{-\gamma_i}.
\]

Thus we have

\[
\Gamma(\mathcal{K}_o, L_o) = 2\pi \sqrt{-1} \sum_{i=1}^{\text{p}} Z(\frac{1}{2} \gamma_i^*)
\]

and

\[
Z(\mathcal{K}_o, L_o) = \sum_{i=1}^{\text{p}} Z \gamma_i.
\]

It follows from Lemma 2.2) that

\[
D(\mathcal{K}_o, L_o) = \{ \sum_{i=1}^{\text{p}} n_i \gamma_i; n_i \in \mathbb{Z}, n_1 \geq n_2 \geq \cdots \geq n_\text{p} \}.
\]

Therefore \( \lambda \) is of the form

\[
\lambda = \sum_{i=1}^{\text{p}} n_i \gamma_i \quad \text{with} \quad n_i \in \mathbb{Z}, \quad n_1 \geq n_2 \geq \cdots \geq n_\text{p}.
\]

On the other hand, \( \lambda \) is of the form
\[ \lambda = \sum_{a \in \mathbb{P}_1} m_a \alpha \quad \text{with} \quad m_a \in \mathbb{Z}, \ m_a \geq 0, \]

which implies that \( n_1 \geq \cdots \geq n_p \geq 0 \). If \( V \subset S_c((\mathfrak{p}^c)^+) \), then \( V_0 \subset S_c(\mathfrak{p}^c) \) and \( \text{ad } Z_0 \)

is the scalar operator \( \nu \) on \( V_0 \), which equals \( (\lambda, Z_0) = \sum_{i=1}^p n_i \). q.e.d.

**Remark.** In terms of polynomial functions \( S^*((\mathfrak{p}^c)^-)) \), for an irreducible \( K \)-submodule \( V \) of \( S^*((\mathfrak{p}^c)^-) \), \( V_0 \) is obtained by restriction to \( \mathfrak{p} \subset \mathfrak{p}^c_0 \) of functions in \( V \).

Proof of Theorem A. Orthogonality relations for the \( S^c((\mathfrak{p}^c)^-))'s \) (resp. for the \( S^c(\mathfrak{p}^c)'s \) and the assertion that the restriction \( S^c((\mathfrak{p}^c)^-) \to S^c(\mathfrak{p}^c) \) is a similitude follow from Schur's lemma. So it suffices to show that the cardinalities of \( S^c(\mathfrak{p}^c) \) and \( S^c(K, L) \) are the same.

From the first argument in the proof of Theorem 3.1 (iii), we see that
\[ \psi(\frac{1}{2} \gamma^*) = X_\gamma (\gamma \in \Delta) \] for the \( L_0 \)-equivariant isomorphism \( \psi: \sqrt{1-1} \xrightarrow{\text{ad } Z_0} \mathfrak{p} \).

We put
\[ \alpha^- = \psi(\sqrt{1-1} \alpha) = \{ X_\gamma; \gamma \in \Delta \} \subset \mathfrak{p} \cdot \]

Since the Weyl group \( W_0 \) of \( S_0 \) is isomorphic with the group of permutations of \( \Delta \) by Lemma 2.2), the "Weyl group" \( W_0' = N_{L_0}(\alpha^-)/Z_{L_0}(\alpha^-) \), where \( N_{L_0}(\alpha^-) \) (resp. \( Z_{L_0}(\alpha^-) \)) is the normalizer (resp. centralizer) of \( \alpha^- \) in \( L_0 \), is isomorphic with the group of permutations of \( \{ X_\gamma; \gamma \in \Delta \} \). On the other hand, since \( S^c_{L_0}(\mathfrak{p}^c) \) is isomorphic with \( S^c_{W_0}(\alpha) \) by Theorem 2.2, \( S^c_{L_0}(\mathfrak{p}^c) \) is isomorphic with \( S^c_{W_0'}(\alpha^-) \). Hence \( S^c_{L_0}(\mathfrak{p}^c) \) is isomorphic with \( S^c_{W_0'}(\alpha^c) \). It follows from Theorem 3.1, (ii), 2) that the cardinality of \( S^c(\mathfrak{p}^c) \) is equal to \( \dim S^c_{L_0}(\mathfrak{p}^c) = \dim S^c_{W_0}(\alpha^c) \) = the number of linearly independent symmetric polynomials in \( \mathfrak{p} \)-variables with degree \( \nu \), which is known to be the cardinality of \( S^c(K, L) \).

q.e.d.

4. **Normalizing factor** \( h_\lambda \)

Let \( \hat{A} = \text{Ad } A(X_0) \), denoting by \( \hat{A} \) the connected subgroup of \( K_0 \) generated by \( \alpha \). \( \hat{A} \) has a natural group structure induced from that of \( \alpha \). Let
\[ T = \{ t \in C^*; |t| = 1 \} \]
be the 1-dimensional torus. Under the identification in Introduction of \( (\alpha^-)^c \) with \( C^\rho \), \( \alpha^- \) is identified with \( R^\rho \) and \( \hat{A} \) with \( T^\rho \). We see that the latter identification is compatible with group structures and complex conjugations, in view of the expression of \( \text{Ad } (\exp H)X_0 \) in the proof of Theorem 3.1, (iii). Moreover, under the same identification we have (Moore [8])
\[ D \cap \alpha^- = \{ x \in R^\rho; |x_i| < 1 \quad (1 \leq i \leq p) \}, \]

denoting by \( z_i (1 \leq i \leq p) \) the \( i \)-th component of \( z \in C^\rho \). By means of this
identification we define a measure on $\alpha^-$ by

$$ dH = dx_1 \cdots dx_p $$

and a function $D(H)$ on $\alpha^-$ by

$$ D(H) = \prod_{i=1}^p (2x_i)x_i^{rs} \prod_{1 \leq i < j \leq p} ((x_i + x_j)(x_i - x_j))^r \quad \text{for } H \in \alpha^-, $$

where $r, 2s$ are multiplicities defined in Introduction. Then we have the following

**Lemma 1.** There exists a constant $c' > 0$ such that

$$ \int_{\Delta} f(X) d\mu(X) = c' \int_{\Delta \cap \alpha^-} f(H) |D(H)| dH $$

for any integrable $K$-invariant function $f$ on $D$.

**Proof.** It is easy to see that $\text{Ad} cH = H$ for any $H \in \mathfrak{b}$ and $\text{Ad} c\gamma^* = X_\gamma - X_{-\gamma} \in \mathfrak{p}$ for any $\gamma \in \Delta$. Put

$$ \alpha^0 = \text{Ad} c(\sqrt{-1} \alpha) = \{X_\gamma - X_{-\gamma} ; \gamma \in \Delta\}_R, $$

$$ \mathfrak{h} = \text{Ad} c(b \oplus \sqrt{-1} \mathfrak{a}) = b \oplus \alpha^0 $$

and

$$ \mathfrak{h}_R = \sqrt{-1} b \oplus \alpha^0. $$

Then $\alpha^0$ is a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\alpha^0$ and $\mathfrak{h}_R$ is the real part of the complexification $\mathfrak{h}^c$ of $\mathfrak{h}$. We define linear forms $h_i$ $(1 \leq i \leq p)$ on $\alpha^0$ by

$$ h_i(X_{\gamma_j} - X_{-\gamma_j}) = \delta_{ij} \quad (1 \leq i, j \leq p). $$

If $h_i$ is identified with an element of $\alpha^0$ by means of the Killing form, we have $\text{Ad} c(\frac{1}{2} \gamma_i) = h_i$ $(1 \leq i \leq p)$. The linear order on $\mathfrak{h}_R$ induced by $\text{Ad} c$ from the order $>$ on $\sqrt{-1} \mathfrak{t}$ is a compatible order for $\text{Ad} c \Sigma$ with respect to the decomposition $\mathfrak{h}_R = \sqrt{-1} b \oplus \alpha^0$. This follows from 3, Lemma 2.1. Thus positive restricted roots on $\alpha^0$ of the symmetric space $D=G/K$ are

$$ \{h_i \pm h_j ; 1 \leq i < j \leq p, 2h_i ; 1 \leq i \leq p\} \quad \text{if } P_1 = \varnothing, $$

$$ \{h_i \pm h_j ; 1 \leq i < j \leq p, 2h_i, h_i ; 1 \leq i \leq p\} \quad \text{if } P_1 = \neq \varnothing. $$

The multiplicity of $h_i \pm h_j$ $(1 \leq i < j \leq p)$, i.e. the number of roots in $\text{Ad} c \Sigma$ projecting to $h_i \pm h_j$, is the same as that of $\frac{1}{2}(\gamma_i \pm \gamma_j)$. Since the Weyl group $W_D$ on $\alpha^0$ of $D=G/K$ is generated by reflections with respect to $h_1 - h_p, \ldots, h_{p-1} - h_p, h_p$, hence transitive on the set $\{ \pm h_i \pm h_j ; 1 \leq i < j \leq p\}$, it follows that
multiplicities of these roots are the same $r$. By the same reason, multiplicities of $h_i$ ($1 \leq i \leq p$) are the same $2s$, which is even from the results of Harish-Chandra mentioned in 3. In the same way we know that multiplicities of $2h_i$ ($1 \leq i \leq p$) are 1. Thus the product $D^\theta$ of positive restricted roots (multiplicity counted) is given by

$$D^\theta(H^\circ) = \prod_{i=1}^p 2h_i(H^\circ) h_i(H^\circ)^s \prod_{1 \leq i < j \leq p} ((h_i+h_j)(H^\circ)(h_i-h_j)(H^\circ))^r$$

for $H^\circ \in \alpha^\circ$.

Let $dX$ (resp. $dH^\circ$) denote the Euclidean measure of $\mathfrak{p}$ (resp. of $\alpha^\circ$) induced from the Killing form $(\ , \ )$, and $dk$ the normalized Haar measure of $K$. Then (cf. Helgason [4]) under the surjective map $K \times \alpha^\circ \rightarrow \mathfrak{p}$ defined by $(k, H^\circ) \mapsto \text{Ad} kH^\circ$, these measures are related as follows:

$$dX = c'' |D^\theta(H^\circ)| \, dk \, dH^\circ$$

with some constant $c'' > 0$.

Now we define a $K$-equivariant $\mathcal{R}$-isomorphism $j: \mathfrak{p} \rightarrow (\mathfrak{p}^c)^-$ by

$$j(X) = \frac{1}{2} (X - [Z, X])$$

for $X \in \mathfrak{p}$.

It is easy to see that $j(X_\gamma - X_\lambda) = -X_\gamma$ for any $\gamma \in \Delta$, hence $j\alpha^\circ = \alpha^-$. Since $K$ acts irreducibly on $\mathfrak{p}$, the map $j$ is a similitude with respect to inner products $(\ , \ )$ and the real part of $(\ , \ )$. Therefore under the surjective map $K \times \alpha^- \rightarrow (\mathfrak{p}^c)^-$ defined by $(k, H) \mapsto \text{Ad} kH$, we have

$$d\mu(X) = c' |D(H)| \, dk \, dH$$

with some constant $c' > 0$.

Seeing $\text{Ad} K(D \cap \alpha^-) = D$, we get the proof of Lemma 1. q.e.d.

Take a form $\lambda \in S^*(K, L)$. Choose an orthonormal basis $\{u_i; 1 \leq i \leq d_\lambda\}$ of $S^*_\mathfrak{c}_a((\mathfrak{p}^c)^-)$ with respect to $(\ , \ )$, such that $\{u_i; 1 \leq i \leq d_{\lambda, 0}\}$ spans $S^*_\mathfrak{c}_a((\mathfrak{p}^c)^-) \cap S^*_\mathfrak{c}_0$ and $u_i$ is $L$-invariant. Put

$$\rho'_j(k) = (\text{Ad} \, k u_j, u_i)$$

for $k \in K$ ($1 \leq i, j \leq d_\lambda$),

$$\varphi'_i(k) = \rho'_i(k)$$

for $k \in K$ ($1 \leq i \leq d_\lambda$),

$$f'_i = \sqrt{d_\lambda} \varphi'_i$$

($1 \leq i \leq d_\lambda$).

The arguments in 2 show that $\{f'_i; 1 \leq i \leq d_\lambda\}$ form an orthonormal basis of $S^*_\mathfrak{c}_0(S)$ with respect to $(\ , \ )$, and $\varphi'_i$ is the zonal spherical function $\omega_\lambda$ for $(K, L)$ belonging to $\lambda$, identifying $C^\omega(S)$ with the space of right $L$-invariant $C^\omega$-functions on $K$. The zonal spherical polynomial $\Omega_\lambda$ for $D$ belonging to $\lambda$ defined in Introduction is characterized by that its restriction to $S$ coincides with $\omega_\lambda$. $\Omega_\lambda$ restricted to $\mathfrak{p}^c_0$ is the zonal spherical polynomial for $D_0$ belonging to $\lambda$ and $\omega_\lambda$ restricted to $S_0$ is the zonal spherical function for $(K_0, L_0)$ belonging to $\lambda$. $\Omega_\lambda$
restricted to \((\alpha^-)C\) is a symmetric polynomial since it is \(W_\varnothing\)-invariant. Let \(f_i \in S^*_x((\varphi^-)')\ (1 \leq i \leq d_\lambda)\) be the unique polynomial such that its restriction to \(S\) is \(f_i'\). Then \(\{f_i; 1 \leq i \leq d_\lambda\}\) form an orthogonal basis of \(S^*_x((\varphi^-)')\) with respect to \((,\,\)) such that \(\{f_i; 1 \leq i \leq d_\lambda, 0\}\) form an orthogonal basis of \(S^*_x((\varphi^-)') \cap S^*(\varphi^\prime)\). They satisfy relations

\[
   f_i(\text{Ad} \, k^{-1} \, X) = \sum_{j=1}^{d_\lambda} \rho(j) f_j(X) \quad \text{for} \quad k \in K, \ X \in (\varphi^-) \ (1 \leq i \leq d_\lambda).
\]

We put

\[
   \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \quad \text{for} \quad X \in (\varphi^-).
\]

Then for any \(k \in K\) we have

\[
   \Phi_\lambda(\text{Ad} \, k^{-1} \, X) = \frac{1}{d_\lambda} \sum_{i,j} (\sum_j \rho(j) f_j(X))(\sum_i \rho_i(k) f_i(X))
\]

\[
   = \frac{1}{d_\lambda} \sum_{i,j} (\sum_j \rho(j) \rho_i(k)) f_j(X) f_i(X)
\]

\[
   = \frac{1}{d_\lambda} \sum_{i,j} \delta_{jk} f_j(X) f_i(X) = \Phi_\lambda(X) \quad \text{for} \quad X \in (\varphi^-),
\]

i.e. \(\Phi_\lambda\) is a \(K\)-invariant \(C^\infty\)-function on \((\varphi^-)\). Note that

\[
   \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \quad \text{for} \quad X \in \varphi^\prime.
\]

**Lemma 2.**

\[
   h_\lambda = c' \int_{D \, a^-} \Phi_\lambda(H) |D(H)| \, dH
\]

**Proof.**

\[
   \int_D \Phi_\lambda(X) d\mu(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \langle f_i, f_i \rangle = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} h_\lambda \langle f_i', f_i' \rangle = h_\lambda.
\]

On the other hand, by Lemma 1 we have

\[
   \int_D \Phi_\lambda(X) d\mu(X) = c' \int_{D \, a^-} \Phi_\lambda(H) |D(H)| \, dH.
\]

**Proof of Theorem B.** Making use of the complex conjugation \(X \mapsto \overline{X}\) of \(\varphi^\prime\) defined in 3, we define \(\Phi_\lambda \in S^*(\varphi^\prime)\) by

\[
   \Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{a=1}^{d_\lambda} f_a(X) \overline{f_a(X)} \quad \text{for} \quad X \in \varphi^\prime.
\]

Then \(\Phi_\lambda = \Phi_\lambda\) on \(\varphi_{-1}\) and we have for any \(k \in K_0\).
\[ \Phi_\lambda(\text{Ad } k X_\circ) = \frac{1}{d_\lambda} \sum_{a} f_a(\text{Ad } k X_\circ) f_a(\text{Ad } \theta(k) X_\circ) \]
\[ = \frac{1}{d_\lambda} \sum_{a} f_a'(k) f_a'((\theta(k)) = \sum_{a} \varphi_a(k) \varphi_a'(\theta(k)) \]
\[ = \sum_{a} \frac{\rho^2(k)}{\rho^2(k) \rho^2(\theta(k)) = \sum_{a} \rho^2(k) \rho^2(\theta(k)^{-1}) \]
\[ = \rho^2(\theta(k)^{-1} k) = \omega_k(\theta(k)^{-1} k). \]

In particular for any \( a \in A \)
\[ \Phi_\lambda(\text{Ad } a X_\circ) = \omega_k(a^2), \]
i.e. for any \( \bar{a} \in \bar{A} \)
\[ \Phi_\lambda(\bar{a}) = \omega_\lambda(\bar{a}^2) = \Omega_\lambda(\bar{a}^2). \]

Since \( \bar{A} = T^p \) is a compact real form of \( \mathbb{C}^* \) and \( \mathbb{C}^* \) is open in \( \mathbb{C}^p = (\mathbb{C}^*)^c \), we have
\[ \Phi_\lambda(z_1, \ldots, z_p) = \Omega_\lambda(z_1^2, \ldots, z_p^2) \quad \text{for any } z \in \mathbb{C}^p = (\mathbb{C}^*)^c. \]

By Lemma 2 we have
\[ h_\lambda = c' \int_{D \cap \bar{A}} \Phi_\lambda(H) |D(H)| dH \]
\[ = c' \int_{\{0\leq i < 1, 1 \leq i \leq p\}} \Omega_\lambda(x_1^2, \ldots, x_p^2) \prod_{i=1}^p \prod_{1 \leq i < j \leq p} ((x_i-x_j)(x_i-x_j))^{r} \prod_{i=1}^p y_i^{s} dy_1 \cdots dy_p \]
\[ = c(D) \int_{\{0 \leq i < 1, 1 \leq i \leq p\}} \Omega_\lambda(y_1, \ldots, y_p) \prod_{1 \leq i < j \leq p} (y_i-y_j)^{r} \prod_{i=1}^p y_i^{s} dy_1 \cdots dy_p \]
for some constant \( c(D) > 0 \), which does not depend on \( \lambda \). In particular, for \( \lambda = 0 \)
\[ \mu(D) = h_0 = c(D) \int_{\{0 \leq i < 1, 1 \leq i \leq p\}} \prod_{1 \leq i < j \leq p} (y_i-y_j)^{2} \prod_{i=1}^p y_i^{s} dy_1 \cdots dy_p, \]
since \( \Omega_0 \equiv 1 \). This completes the proof of Theorem B. q.e.d.

**Remark.** It can be proved that \( \Phi_\lambda \) is an \( L_\omega \)-invariant polynomial on \( \mathbb{C}^p \).

The multiplicities \( r, s \) are given as follows.

<table>
<thead>
<tr>
<th>( D )</th>
<th>rank ( D )</th>
<th>( r )</th>
<th>( s )</th>
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<tbody>
<tr>
<td>(I)_{p, s} (p \leq q)</td>
<td>( p )</td>
<td>2</td>
<td>( q-p )</td>
</tr>
<tr>
<td>(II)_{s}</td>
<td>([n/2])</td>
<td>4</td>
<td>{ \begin{array}{ll} 2 &amp; n \text{ odd} \ 0 &amp; n \text{ even} \end{array} }</td>
</tr>
<tr>
<td>(III)_{s}</td>
<td>( n )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(IV)_{s} (n \geq 3)</td>
<td>2</td>
<td>( n-2 )</td>
<td>0</td>
</tr>
<tr>
<td>(EIII)</td>
<td>2</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>(EVII)</td>
<td>3</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>
The zonal spherical polynomial $\Omega_\lambda$ is given as follows.

For integers $n_1, \ldots, n_p$ we define the Schur function $\{n_1, \ldots, n_p\}$ on the $p$-dimensional torus $T^p$ by

$$\{n_1, \ldots, n_p\}(t) = \frac{\det(t_1^{n_i+n_{p-j}})_{i,j\leq p}}{\det(t_1^{n_i})_{i,j< p}} \quad \text{for} \quad t = \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \in T^p \subset C^p.$$ 

$\{n_1, \ldots, n_p\}$ is symmetric in variables $t_1, \ldots, t_p$ and it is a polynomial in $t_1, \ldots, t_p$ if and only if $n_i \geq 0$ ($1 \leq i \leq p$). For an element $\lambda = \sum n_i \gamma_i \in \sum \mathbb{Z} \gamma_i = \mathbb{Z}(K_0, L_0)$, the $i$-th coefficient $n_i$ will be denoted by $n_i(\lambda)$.

Then we have

**Theorem 4.1.** The zonal spherical polynomial $\Omega_\lambda$ for $D$ belonging to $\lambda \in S^p(K, L)$ is determined on $(\alpha^-)^c$ by the relation

$$\Omega_\lambda(t) = \sum_{\mu \in \mathcal{B}_\lambda} c^\mu_\lambda \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for any} \quad t \in T^p = \hat{A} \subset (\alpha^-)^c,$$

where the $c^\mu_\lambda$'s are coefficients in Theorem 2.5 for the symmetric pair $(K_0, L_0)$.

**Proof.** As we have seen in the proof of Theorem B, $\Omega_\lambda$ is determined on $(\alpha^-)^c$ by

$$\Omega_\lambda(t) = \omega_\lambda(t) \quad \text{for any} \quad t \in T^p = \hat{A}.$$ 

By Theorem 2.5, $\omega_\lambda$ has an expression

$$\omega_\lambda(t) = \sum_{\mu \in \mathcal{B}_\lambda} c^\mu_\lambda \chi_\mu(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$ 

Since the Weyl group $W_{S_0}$ acts on $Z(K_0, L_0)$ by the group of permutations of $\gamma_1, \ldots, \gamma_p$, $W_{S_0}$-invariant characters $\chi_\lambda$ of $\hat{A}$ are nothing but Schur functions. As we have seen in the proof of Theorem 3.1, (iii), the $i$-th component of $\text{Ad}(\exp H)X_\lambda \in T^p = \hat{A}$ is $\exp(-\langle \gamma_i, H \rangle)$ for any $H \in \alpha$. It follows that

$$\chi_\mu(t) = \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A}.$$ 

Hence we have

$$\Omega_\lambda(t) = \sum_{\mu \in \mathcal{B}_\lambda} c^\mu_\lambda \{n_1(\mu), \ldots, n_p(\mu)\}(t)$$

$$= \sum_{\mu \in \mathcal{B}_\lambda} c^\mu_\lambda \{n_1(\mu), \ldots, n_p(\mu)\}(t) \quad \text{for} \quad t \in T^p = \hat{A}. \quad \text{q.e.d.}$$

In the case of the domain $D$ of type $(I)_{p,q}$ ($p \leq q$), $S_0$ is the unitary group $U(p)$ of degree $p$. We have in view of Example in 2 that

$$\Omega_\lambda(t) = \frac{1}{d_\lambda} \{n_1(\lambda), \ldots, n_p(\lambda)\}(t) \quad \text{for} \quad t \in T^p = \hat{A},$$

where $d_\lambda$ is a constant.
where \( d_\lambda \) is the degree of the irreducible representation of \( U(p) \) with the signature \((n_1(\lambda), \ldots, n_p(\lambda))\). In the case of the domain \( D \) of type (IV), \( S_0 \) is the Lie sphere and \( \Omega_\lambda \) can be described in terms of Gegenbauer polynomials, which are zonal spherical functions for the sphere. So our integral formula in Theorem B clarifies the meaning of integrals of Hua [6].

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**References**


