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# POLYNOMIAL REPRESENTATIONS ASSOCIATED WITH SYMMETRIC BOUNDED DOMAINS

MASARU TAKEUCHI

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**Introduction.** In this note we want to construct a complete orthonormal system of the Hilbert space $H^2(D)$ of square integrable holomorphic functions on an irreducible symmetric bounded domain $D$. A symmetric bounded domain $D$ is canonically realizable as a circular starlike bounded domain with the center 0 in a complex cartesian space by means of Harish-Chandra's imbedding (Harish-Chandra [3]), which is constructed as follows. The largest connected group $G$ of holomorphic automorphisms of $D$ is a connected semi-simple Lie group without center, which is transitive on $D$. Thus denoting the stablizer in $G$ of a point $o\in D$ by $K, D$ is identified with the quotient space $G/K$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $G$ (resp. $K$) and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Then there exists uniquely an element $H$ of the center of $\mathfrak{k}$ such that $\operatorname{ad} H$ restricted to $\mathfrak{p}$ coincides with the complex structure tensor on the tangent space $T_o(D)$ of $D$ at the origin $o$, identifying as usual $\mathfrak{p}$ with $T_o(D)$. Let $\mathfrak{g}^C$ be the Lie algebra of the complexification $G^C$ of $G$ and put $Z=\sqrt{-1}H\in\mathfrak{g}^C$. Let $(\mathfrak{p}^C)^{\pm}$ be the $(\pm 1)$-eigenspace in $\mathfrak{g}^C$ of ad $Z$. Then they are invariant under the adjoint action of $K$ and the complexification $\mathfrak{p}^C$ of $\mathfrak{p}$ is the direct sum of $(\mathfrak{p}^C)^+$ and $(\mathfrak{p}^C)^-$. Let $U^C$ denote the normalizer of $(\mathfrak{p}^C)^+$ in $G^C$. Then $D=G/K$ is holomorphically imbedded as an open submanifold into the quotient space $G^C/U^C$ in the natural way. For any point $z\in D$, there exists uniquely a vector $X\in(\mathfrak{p}^C)^-$ such that

$$\exp X \bmod U^C = z\,.$$

The map $z\mapsto X$ of $D$ into $(\mathfrak{p}^C)^-$ is the desired imbedding. Note that the natural action of $K$ on $D$ can be extended to the adjoint action of $K$ on the ambient space $(\mathfrak{p}^C)^-$.

Henceforth we assume that $D$ is a bounded domain in $(\mathfrak{p}^C)^-$ realized in the above manner. Let $(\ ,\ )$ denote the Killing form of $\mathfrak{g}^C$ and $\tau$ the complex conjugation of $\mathfrak{g}^C$ with respect to the compact real form $\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$ of $\mathfrak{g}^C$. We define a $K$-invariant hermitian inner product $(\ ,\ )_\tau$ on $\mathfrak{g}^C$ by

$$(X,\ Y)_\tau = -(X,\ \tau Y) \qquad \text{for} \quad X,\ Y\in\mathfrak{g}^C\,.$$

This defines a $K$-invariant Euclidean measure $d\mu(X)$ on $(\mathfrak{p}^C)^-$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(X)$. The inner product of $H^2(D)$ will be denoted by $\langle\!\langle\ ,\ \rangle\!\rangle$. $K$ acts on $H^2(D)$ as unitary operators by

$$(kf)(X) = f(k^{-1}X) \qquad \text{for} \quad k \in K,\ X \in D\,.$$

Let $S^*((\mathfrak{p}^C)^-)$ denote the graded space of polynomial functions on $(\mathfrak{p}^C)^-$. It has the natural hermitian inner product $(\ ,\ )_\tau$ induced from the inner product $(\ ,\ )_\tau$ on $(\mathfrak{p}^C)^-$. $K$ acts on $S^*((\mathfrak{p}^C)^-)$ as unitary operators by

$$(kf)(X) = f(\mathrm{Ad}\, k^{-1}X) \qquad \text{for} \quad k \in K,\ X \in (\mathfrak{p}^C)^-\,.$$

Now let $S$ denote the Shilov boundary of $D$. It is known (Korányi-Wolf [7]) that $K$ acts transitively on $S$. Thus denoting by $L$ the stabilizer in $K$ of a point $X_0 \in S$, $S$ is identified with the quotient space $K/L$. Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure of $K$ and $L^2(S)$ the Hilbert space of square integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle\ ,\ \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$(kf)(X) = f(\mathrm{Ad}\, k^{-1}X) \qquad \text{for} \quad k \in K,\ X \in S\,.$$

The space $C^\infty(S)$ of $\boldsymbol{C}$-valued $C^\infty$-functions on $S$ is a $K$-submodule of $L^2(S)$. The restrictions $S^*((\mathfrak{p}^C)^-) \to H^2(D)$ and $S^*((\mathfrak{p}^C)^-) \to L^2(S)$ are both $K$-equivariant monomorphisms. Their images will be denoted by $S^*(D)$ and $S^*(S)$, respectively. They have natural gradings induced from that of $S^*((\mathfrak{p}^C)^-)$. Then the "restriction" $S^*(D) \to S^*(S)$ is defined in the natural manner and it is a $K$-equivariant isomorphism. Since $D$ is a circular starlike bounded domain, a theorem of H. Cartan [2] yields that the subspace $S^*(D)$ of $H^2(D)$ is dense in $H^2(D)$ (cf. **1**).

We decompose first the $K$-module $S^*(D)$ into irreducible components. We take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and idenitfy the real part $\sqrt{-1}\,\mathfrak{t}$ of the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ with its dual space by means of Killing form of $\mathfrak{g}^C$. Let $\Sigma \subset \sqrt{-1}\,\mathfrak{t}$ denote the set of roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$. We choose root vectors $X_\alpha \in \mathfrak{g}^C$ for $\alpha \in \Sigma$ such that

$$[X_\alpha, X_{-\alpha}] = -\frac{2}{(\alpha, \alpha)}\alpha\,,$$
$$\tau X_\alpha = X_{-\alpha}\,.$$

A root is called *compact* if it is also a root of the complexification $\mathfrak{k}^C$ of $\mathfrak{k}$, otherwise it is called *non-compact*. $\Sigma_\mathfrak{k}$ (resp. $\Sigma_\mathfrak{p}$) denotes the set of compact roots (resp. of non-compact roots). We choose and fix once for all a linear order $>$ on $\sqrt{-1}\,\mathfrak{t}$ such that $(\mathfrak{p}^C)^+$ is spanned by the root spaces for non-compact positive roots $\sum_{\mathfrak{p}}^{+}$. Two roots $\alpha$, $\beta\in\sum$ are called *strongly orthogonal* if $\alpha\pm\beta$ is not a root. We define a maximal strongly orthogonal subsystem

$$\Delta=\{\gamma_1\cdots,\gamma_p\},\quad \gamma_1>\gamma_2>\cdots>\gamma_p>0,\quad p=\operatorname{rank} D$$

of $\sum_{\mathfrak{p}}^{+}$ as follows (cf. Harish-Chandra [3]). Let $\gamma_1$ be the highest root of $\sum$ and for each $j$, $\gamma_{j+1}$ be the highest positive non-compact root that is strongly orthogonal to $\gamma_1,\cdots,\gamma_j$. We put

$$X_0=-\sum_{\gamma\in\Delta}X_{-\gamma}\,.$$

Then it is known (Korányi-Wolf [7]) that $X_0$ is on the Shilov boundary $S$ of $D$. Henceforth we shall take the above point $X_0$ as the origin of $S$. We put for $\nu\in\boldsymbol{Z}$, $\nu\geqslant 0$

$$\mathcal{S}^{\nu}(K,L)=\{\sum_{i=1}^{p}n_i\gamma_i;\ n_i\in\boldsymbol{Z},\ n_1\geqslant n_2\geqslant\cdots\geqslant n_p\geqslant 0,\ \sum_{i=1}^{p}n_i=\nu\}\ ,$$

and

$$\mathcal{S}^{*}(K,L)=\sum_{\nu\geqslant 0}\mathcal{S}^{\nu}(K,L)\,.$$

We shall prove the following

**Theorem A.** *Any irreducible $K$-submodule of $S^*(D)$ is contained exactly once in $S^*(D)$. The set $\mathcal{S}^{\nu}(D)$ of highest weights (with respect to $\mathfrak{t}^C$) of irrducible $K$-submodules contained in $S^{\nu}(D)$ coincides with $\mathcal{S}^{\nu}(K,L)$. Denoting by $S^*_{\lambda}(D)$ (resp. $S^*_{\lambda}(S)$) the irreducible $K$-submodule of $S^*(D)$ (resp. of $S^*(S)$) with the highest weight $\lambda\in\mathcal{S}^*(K,L)$,*

$$S^*(D)=\sum_{\lambda\in\mathcal{S}^*(K,L)}\oplus S^*_{\lambda}(D)$$

*and*

$$S^*(S)=\sum_{\lambda\in\mathcal{S}^*(K,L)}\oplus S^*_{\lambda}(S)$$

*are the orthogonal sum relative to the inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ and $\langle\ ,\ \rangle$, respectively. The restriction $f\mapsto f'$ of $S^*_{\lambda}(D)\to S^*_{\lambda}(S)$ is a similitude for each $\lambda\in\mathcal{S}^*(K,L)$, i.e. there exists a constant $h_{\lambda}>0$ such that*

$$\langle\!\langle f,g\rangle\!\rangle=h_{\lambda}\langle f',g'\rangle\qquad\text{for any}\quad f,g\in S^*_{\lambda}(D)\,.$$

*Thus, if*

$$\{f'_{\lambda,i};\ 1\leqslant i\leqslant d_{\lambda}\}\ ,\quad \lambda\in\mathcal{S}^*(K,L)$$

*is an orthonormal basis of $S^*_{\lambda}(S)$, then*

$$\{\sqrt{h_{\lambda}}^{-1}f_{\lambda,i};\ \lambda\in\mathcal{S}^*(K,L),\ 1\leqslant i\leqslant d_{\lambda}\}$$

*is a complete orthonormal system of $H^2(D)$.*

A basis $\{f'_{\lambda,i};\ 1\leqslant i\leqslant d_\lambda\}$ is, for instance, constructed as follows. Take an irreducible $K$-module $(\rho, V)$ with the highest weight $\lambda$, carrying a $K$-invariant hermitian inner product $(\ ,\ )$. Choose an orthonormal basis $\{u_i;\ 1\leqslant i\leqslant d_\lambda\}$ of $V$ such that the first vector $u_1$ is $L$-invariant. This can be done in view of Frobenius' reciprocity since the $K$-module $V$ is $K$-isomorphic with a $K$-submodule of $C^\infty(S)$. Then the functions $f'_{\lambda,i}$ $(1\leqslant i\leqslant d_\lambda)$ defined by

$$f'_{\lambda,i}(kX_0)=\sqrt{d_\lambda}(u_i,\ \rho(k)u_1) \qquad \text{for} \quad k\in K$$

form an orthonormal basis of $S^*_\lambda(S)$ (cf. 2).

We compute next the normalizing factor $h_\lambda$. Let

$$\mathfrak{a}=\{\sqrt{-1}\Delta\}_{\boldsymbol{R}}$$

be the $\boldsymbol{R}$-span of $\sqrt{-1}\Delta$ in $\mathfrak{t}$ and

$$\varpi:\ \sqrt{-1}\mathfrak{t}\rightarrow\sqrt{-1}\mathfrak{a}$$

denote the orthogonal projection of $\sqrt{-1}\mathfrak{t}$ onto $\sqrt{-1}\mathfrak{a}$. For $\gamma\in\varpi\sum-\{0\}$, the number of roots $\alpha\in\sum$ such that $\varpi\alpha=\gamma$ is called the *multplicity* of $\gamma$. Let $r$ (resp. 2s) be the multiplicity of $\frac{1}{2}(\gamma_1-\gamma_2)$ (resp. of $\frac{1}{2}\gamma_1$). If follows from Theorem A and Frobenius' reciprocity that for each $\lambda\in\mathcal{S}^*(K, L)$ there exists uniquely an $L$-invariant polynomial $\Omega_\lambda$ in $S^*_\lambda((\mathfrak{p}^C)^-)$ such that $\Omega_\lambda(X_0)=1$, where $S^*_\lambda((\mathfrak{p}^C)^-)$ denotes the irreducible $K$-submodule of $S^*((\mathfrak{p}^C)^-)$ with the highest weight $\lambda$. The polynomial $\Omega_\lambda$ is called the *zonal spherical polynomial* for $D$ belonging to $\lambda$. Let

$$(\mathfrak{a}^-)^C=\{X_{-\gamma};\ \gamma\in\Delta\}_C$$

be the $\boldsymbol{C}$-span of $\{X_{-\gamma};\ \gamma\in\Delta\}$ in $(\mathfrak{p}^C)^-$. It is identified with the complex cartesian space $\boldsymbol{C}^p$ by the map

$$-\sum_{i=1}^{p} z_iX_{-\gamma i}\mapsto\begin{pmatrix} z_1\\ \vdots\\ z_p\end{pmatrix}.$$

Thus the zonal spherical polynomial $\Omega_\lambda$ restricted to $(\mathfrak{a}^-)^C$ is a polynomial $\Omega_\lambda(Y_1, \cdots, Y_p)$ in $p$-variables. Let $\mu(D)$ denote the volume of $D$ with respect to the measure $d\mu(X)$. We shall prove the following

**Theorem B.** *For $\lambda\in\mathcal{S}^*(K, L)$, the normalizing factor $h_\lambda$ is given by*

$$h_\lambda=c(D)\int_{0\leqslant y_i<1\ (1\leqslant i\leqslant p)}\Omega_\lambda(y_1, \cdots, y_p)\,|\prod_{1\leqslant i<j\leqslant p}(y_i-y_j)^r|\prod_{i=1}^{p}y_i^s\,dy_1\cdots dy_p$$

*where*

$$c(D)=\mu(D)\Big(\int_{0\leqslant y_i<1\ (1\leqslant i\leqslant p)}|\prod_{1\leqslant i<j\leqslant p}(y_i-y_j)^r|\prod_{i=1}^{p}y_i^s\,dy_1\cdots dy_p\Big)^{-1}.$$

Hua [6] proved Theorem A for classical domains by decomposing the character of the $K$-module $S^*((\mathfrak{p}^C)^-)$ into the sum of irreducible characters of $K$, while Schmid [11] proved it for general domain $D$. Schmid proved

(a) $$\mathcal{S}^\nu(D) \subset \mathcal{S}^\nu(K, L)$$

by seeing the character of the $K$-module $S^*((\mathfrak{p}^C)^-)$ and by making use of $E$. Cartan's theory on spherical representations of a compact symmetric pair. But his proof of

(b) $$\mathcal{S}^\nu(K, L) \subset \mathcal{S}^\nu(D)$$

is complicated and was done after nine successive lemmas. In this note we give another proof of (a) by means of a lemma of Murakami and Cartan's theory, and give a relatively short proof of (b) by means of a theorm of Harish-Chandra on invariant polynomials for a symmetric pair.

Hua [6] computed the factors $h_\lambda$ for certain classical domains by integrating certain polynomials. Our integral formula in Theorem B will clarify the meaning of integrals of Hua.

## 1. Circular domains

A domain $D \subset \boldsymbol{C}^n$ containing the origin 0 is said to be a *circular domain* with the center 0 if together with any point $z \in D$ the point $e^{\sqrt{-1}\theta} z$ is in $D$ for any real $\theta \in \boldsymbol{R}$. $D$ is said to be a *starlike domain* with the center 0 if together with any point $z \in D$ the point $rz$ is in $D$ for any real $r \in \boldsymbol{R}$ with $0 \leqslant r < 1$.

**Theorem 1.1.** (H. Cartan [2]) *Let $D \subset \boldsymbol{C}^n$ be a circular domain with the center* 0. *Then any holomorphic function $f$ on $D$ can be developed in the sum of homogeneous polynomials $P_\nu$ in $n$-variables with degree $\nu$ ($\nu = 0, 1, 2, \cdots$):*

$$f(z) = \sum_{\nu=0}^{\infty} P_\nu(z) \qquad \text{for} \quad z \in D\,.$$

*The sum converges uniformly on any compact subset of $D$. The homogeneous polynomials $P_\nu$ are uniquely determined for $f$.*

Let $D$ be a bounded domain in $\boldsymbol{C}^n$, $d\mu(z)$ the Euclidean measure on $\boldsymbol{C}^n$, induced from the standard hermitian inner product of $\boldsymbol{C}^n$. Let $H^2(D)$ denote the Hilbert space of holomorphic functions on $D$, which are square integrable with respect to the measure $d\mu(z)$. The inner product of $H^2(D)$ will be denoted by $\langle\!\langle\ ,\ \rangle\!\rangle$. Let $S^*(\boldsymbol{C}^n)$ be the graded space of polynomials in $n$-variables and $S^*(D)$ the subspace of $H^2(D)$ consisting of all functions on $D$ obtained by the restriction of polynomials in $S^*(\boldsymbol{C}^n)$. Then Theorem 1.1 yields the following

**Corollary.** *Let $D \subset \boldsymbol{C}^n$ be a circular starlike bounded domain with the center* 0. *Then the subspace $S^*(D)$ of $H^2(D)$ is dense in $H^2(D)$.*

Proof. If suffices to show that if $f \in H^2(D)$ with $\langle\langle f, S^*(D)\rangle\rangle = \{0\}$, then $f=0$. Theorem 1.1 implis that $f$ can be developed as

$$f = \sum_{\nu=0}^{\infty} P_\nu, \qquad P_\nu \in S^\nu(D),$$

uniformly convergent on any compact subset of $D$. Choose an orthonormal basis $\{P_{\nu,j}\}$ of $S^\nu(D)$ with respect to $\langle\langle\ ,\ \rangle\rangle$ for each $\nu$. Then we have

$$\langle\langle P_{\nu,j}, P_{\mu,i}\rangle\rangle = \delta_{\nu\mu}\delta_{ji}.$$

In fact, since $d\mu(e^{\sqrt{-1}\theta}z) = d\mu(z)$ for any $\theta \in \boldsymbol{R}$, we have $\langle\langle P_{\nu,j}\ P_{\mu,i}\rangle\rangle = e^{\sqrt{-1}(\nu-\mu)\theta}\langle\langle P_{\nu,j}, P_{\mu,i}\rangle\rangle$ for any $\theta \in \boldsymbol{R}$. Then $f$ can be developed as

$$f = \sum_{\nu,j} a_{\nu,j} P_{\nu,j} \qquad \text{with} \quad a_{\nu,j} \in \boldsymbol{C},$$

uniformly convergent on any compact subset of $D$. Since $D$ is a starlike domain, the closure $\overline{rD}$ of $rD$ is a compact subset of $D$ for any $r \in R$ with $0 < r < 1$, so that the above series converges uniformly on $rD$. Therefore for any $P_{\mu,i}$ we have

$$\int_{rD} f(z)\overline{P_{\mu,i}(z)}\, d\mu(z) = \sum_{\nu,j} a_{\nu,j} \int_{rD} P_{\nu,j}(z)\overline{P_{\mu,i}(z)}\, d\mu(z).$$

If we put

$$z' = \frac{1}{r} z \qquad \text{for} \quad z \in rD,$$

then $z = rz'$, $d\mu(z) = r^{2n} d\mu(z')$ so that

$$\begin{aligned}\int_{rD} P_{\nu,j}(z)\overline{P_{\mu,i}(z)}\, d\mu(z) &= r^{2n+\nu+\mu}\int_{D} P_{\nu,j}(z')\overline{P_{\mu,i}(z')}\, d\mu(z') \\ &= r^{2n+\nu+\mu}\langle\langle P_{\nu,j}, P_{\mu,i}\rangle\rangle = r^{2n+2\mu}\delta_{\nu\mu}\delta_{ji}.\end{aligned}$$

Hence we have

$$\int_{rD} f(z)\overline{P_{\mu,i}(z)}\, d\mu(z) = a_{\mu,i} r^{2n+2\mu}$$

and

$$\begin{aligned}a_{\mu,i} &= \lim_{r\uparrow 1} a_{\mu,i} r^{2n+2\mu} = \lim_{r\uparrow 1} \int_{rD} f(z)\overline{P_{\mu,i}(z)}\, d\mu(z) \\ &= \langle\langle f, P_{\mu,i}\rangle\rangle = 0 \qquad \text{(from the assumption)}.\end{aligned}$$

This implies that $f=0$. q.e.d.

## 2. Spherical representations of a compact symmetric pair

Let $K$ be a compact connected Lie group, $L$ a closed subgroup of $K$ and $S$ be the quotient space $K/L$. The space of $\boldsymbol{C}$-valued $C^\infty$-functions on $S$ will be denoted by $C^\infty(S)$. We shall often identify $C^\infty(S)$ with the space of $C^\infty$-functions $f$ on $K$ such that

$$f(kl) = f(k) \qquad \text{for any} \quad k \in K,\ l \in L\,.$$

Let $dx$ denote the $K$-invariant measure on $S$ induced from the normalized Haar measure on $K$ and $L^2(S)$ the Hilbert space of sequare integrable functions on $S$ with respect to the measure $dx$. The inner product of $L^2(S)$ will be denoted by $\langle\ ,\ \rangle$. $K$ acts on $L^2(S)$ as unitary operators by

$$(kf)(x) = f(k^{-1}x) \quad \text{for} \quad k \in K,\ x \in S\,.$$

Then $C^\infty(S)$ is a $K$-submodule of $L^2(S)$. A (continuous finite dimensional complex) representation

$$\rho\colon K \to GL(V)$$

of $K$ is said to be *spherical* relative to $L$ if the $K$-module $V$ is equivalent to a $K$-submodule of $C^\infty(S)$, which amounts to the same from Frobenius' reciprocity that the $K$-module $V$ has a non-zero $L$-invariant vector. We denote by $\mathscr{D}(K, L)$ the set of equivalence classes of irreducible spherical representations of $K$ relative to $L$. The totality of $f \in C^\infty(S)$ contained in a finite dimensional $K$-submodule of $C^\infty(S)$, which will be denoted by $\mathfrak{o}(K, L)$, is a $K$-submodule of $C^\infty(S)$. A function in $\mathfrak{o}(K, L)$ is called a *spherical function* for the pair $(K, L)$. For $\rho \in \mathscr{D}(K, L)$, the totality of $f \in \mathfrak{o}(K, L)$ that transforms according to $\rho$, which will be denoted by $\mathfrak{o}_\rho(K, L)$, is a finite dimensional $K$-submodule of $\mathfrak{o}(K, L)$. Then

$$\mathfrak{o}(K, L) = \sum_{\rho \in \mathscr{D}(K, L)} \oplus \mathfrak{o}_\rho(K, L)$$

is the orthogonal sum with respect to the inner product $\langle\ ,\ \rangle$. Peter-Weyl approximation theorem implies that the subspace $\mathfrak{o}(K, L)$ of $L^2(S)$ is dense in $L^2(S)$. We assume furthermore that the pair $(K, L)$ satisfies the condition

(*) any $\rho \in \mathscr{D}(K, L)$ is contained exactly once in $\mathfrak{o}(K, L)$,

which is by Frobenius' reciprocity equivalent to that for any spherical representation

$$\rho\colon K \to GL(V)$$

of $K$ relative to $L$, an $L$-invariant vector of $V$ is unique up to scalor multiplication. Then for each $\rho \in \mathscr{D}(K, L)$, there exists uniquely an $L$-invariant function $\omega_\rho \in \mathfrak{o}_\rho(K, L)$ such that $\omega_\rho(e) = 1$. $\omega_\rho$ is called the *zonal spherical function* for $(K, L)$ belonging to $\rho$. Let

$$\rho\colon K \to GL(V)$$

be a spherical representation of $K$ relative to $L$. Choose a $K$-invariant hermitian inner product $(\ ,\ )$ on $V$. The equivalence class containing $\rho$ will be denoted by the same letter $\rho$. Choose an orthonormal basis $\{u_i;\ 1\leqslant i\leqslant d_\rho\}$ of $V$ such that $u_1$ is $L$-invariant. Define $\varphi_i\in C^\infty(S)$ $(1\leqslant i\leqslant d_\rho)$ by

$$\varphi_i(k)=(u_i,\ \rho(k)u_1)\qquad\text{for}\quad k\in K\,.$$

We know that they are linearly independent, in view of orthogonality relations of matrix elements $(u_i,\ \rho(k)u_j)$. For any $k'\in K$ we have

$$\begin{aligned}\varphi_i(k'^{-1}k)&=(u_i,\ \rho(k'^{-1}k)u_1)=(\rho(k')u_i,\ \rho(k)u_1)\\&=\sum_j(\rho(k')u_i,\ u_j)(u_j,\ \rho(k)u_1)\\&=\sum_j(\rho(k')u_i,\ u_j)\,\varphi_j(k)\,,\end{aligned}$$

i.e.

$$k'\varphi_i=\sum_j(\rho(k')u_i,\ u_j)\varphi_j\qquad(1\leqslant i\leqslant d_\rho)\,.$$

In particular

$$l\varphi_1=\varphi_1\qquad\text{for any}\quad l\in L\,,$$

and

$$\varphi_1(e)=1\,.$$

Therefore the system $\{\varphi_i;\ 1\leqslant i\leqslant d_\rho\}$ forms a basis of $\mathfrak{o}_\rho(K,L)$ and the zonal spherical function $\omega_\rho$ is given by

$$\omega_\rho(k)=(u_1,\ \rho(k)u_1)\qquad\text{for}\quad k\in K\,.$$

Furthermore orthogonality relations implies that the system

$$\{\sqrt{d_\rho}\,\varphi_i;\ 1\leqslant i\leqslant d_\rho\}$$

forms an orthonormal basis of $\mathfrak{o}_\rho(K,L)$ and that

$$\langle\omega_\rho,\ \omega_{\rho'}\rangle=\delta_{\rho\rho'}\frac{1}{d_\rho}\,.$$

Henceforth we assume that the pair $(K,L)$ is a *symmetric pair*, i.e. there exists an involutive automorphism $\theta$ of $K$ such that if we put

$$K_\theta=\{k\in K;\ \theta(k)=k\}\ ,$$

$L$ lies between $K_\theta$ and the connected component $K_\theta^0$ of $K_\theta$. Then the pair $(K,L)$ satisfies the condition $(*)$ (E. Cartan [1]). For example, a compact connected Lie group $S$ admits a symmetric pair $(K,L)$ such that $S=K/L$. In fact,

$$\begin{aligned}K&=S\times S\,,\\L&=\{(x,\ x);\ x\in S\}\end{aligned}$$

and

$$\theta\colon (x, y) \mapsto (y, x) \qquad \text{for} \quad x, y \in S$$

have desired properties.

In the following we summarize some known facts on a symmetric pair (cf. Helgason [4]).

Let $\mathfrak{k}$ (resp. $\mathfrak{l}$) be the Lie algebra of $K$ (resp. of $L$). The involutive automorphism of $\mathfrak{k}$ obtained by differentiating the automorphism $\theta$ of $K$ will be also denoted by the same letter $\theta$.

Choose and fix once for all a $\boldsymbol{C}$-bilinear symmetric form $(\ ,\ )$ on the complexification $\mathfrak{k}^C$ of $\mathfrak{k}$, which is invariant under both the $\boldsymbol{C}$-linear extension to $\mathfrak{k}^C$ of $\theta$ and the adjoint action of $\mathfrak{k}^C$ and furthermore is negative definite on $\mathfrak{k}\times\mathfrak{k}$. Then $S$ is a Riemannian symmetric space with respect to the $K$-invariant Riemannian metric on $S$ defined by $-(\ ,\ )$. We put

$$\mathfrak{s} = \{X \in \mathfrak{k};\, \theta X = -X\} = \{X \in \mathfrak{k};\, (X, \mathfrak{l}) = \{0\}\}\,.$$

Then we have orthogonal decompositions

$$\mathfrak{k} = \mathfrak{l} + \mathfrak{s} = \mathfrak{c} \oplus \mathfrak{k}'\,,$$

where $\mathfrak{c}$ is the center of $\mathfrak{k}$ and $\mathfrak{k}'$ is the derived algebra $[\mathfrak{k}, \mathfrak{k}]$ of $\mathfrak{k}$. We choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{s}$. Such $\mathfrak{a}$ are mutually conjugate under the adjoint action of $L$. $\dim \mathfrak{a}$ is the rank of the symmetric pair $(K, L)$. Extend $\mathfrak{a}$ to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ containing $\mathfrak{a}$. Then we have the decomposition

$$\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a} \qquad \text{where} \quad \mathfrak{b} = \mathfrak{t} \cap \mathfrak{l}\,.$$

Let $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{k}'$ and $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{k}'$. The real vector space $\sqrt{-1}\,\mathfrak{t}$ has the natural inner product $(\ ,\ )$ induced from the bilinear form $(\ ,\ )$ on $\mathfrak{k}^C$. We shall identify $\sqrt{-1}\,\mathfrak{t}$ with the dual space of $\sqrt{-1}\,\mathfrak{t}$ by means of the inner product $(\ ,\ )$. We have the orthogonal decomposition

$$\sqrt{-1}\,\mathfrak{t} = \sqrt{-1}\,\mathfrak{b} \oplus \sqrt{-1}\,\mathfrak{a}\,.$$

Let $\sigma$ be the orthogonal transformation on $\sqrt{-1}\,\mathfrak{t}$ defined by

$$\sigma | \sqrt{-1}\,\mathfrak{b} = -1 \quad \text{and} \quad \sigma | \sqrt{-1}\,\mathfrak{a} = 1$$

and

$$\varpi = \tfrac{1}{2}(1+\sigma)\colon \sqrt{-1}\,\mathfrak{t} \to \sqrt{-1}\,\mathfrak{a}$$

be the orthogonal projection of $\sqrt{-1}\,\mathfrak{t}$ onto $\sqrt{-1}\,\mathfrak{a}$. Let $\sum_{\mathfrak{k}}$ denote the set of roots of $\mathfrak{k}^C$ with repsect to the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$. Let $W_{\mathfrak{k}} = N_K(T)/T$ be the Weyl group of $\mathfrak{k}$, where $T$ is the connected subgroup of $K$ generated by $\mathfrak{t}$ and $N_K(T)$ is the normalizer of $T$ in $K$. $\sum_{\mathfrak{k}}$ is a $\sigma$-invariant reduced root system in

$\sqrt{-1}\mathfrak{t}'$. As a group of orthogonal transformations of $\sqrt{-1}\mathfrak{t}$, $W_{\mathfrak{t}}$ is generated by reflections with respect to roots in $\Sigma_{\mathfrak{t}}$. Put

$$\Sigma_{\mathfrak{t}}^0 = \Sigma_{\mathfrak{t}} \cap \sqrt{-1}\mathfrak{b} = \{\alpha \in \Sigma_{\mathfrak{t}};\ \varpi\alpha = 0\}\ ,$$
$$\Sigma_S = \{\varpi\alpha;\ \alpha \in \Sigma_{\mathfrak{t}} - \Sigma_{\mathfrak{t}}^0\} = \varpi\,\Sigma_{\mathfrak{t}} - \{0\}\ ,$$
$$W_S = N_L(A)/Z_L(A)\ ,$$

where $A$ is the connected subgroup of $K$ generated by $\mathfrak{a}$ and $N_L(A)$ (resp. $Z_L(A)$) the normalizer (resp. the centralizer) of $A$ in $L$. An element of $\Sigma_S$ is a restricted root of the symmetric space $S$ and $W_S$ is the Weyl group of $S$. $\Sigma_S$ is a (not necessarily reduced) root system in $\sqrt{-1}\mathfrak{a}'$. As a group of orthogonal transformations of $\sqrt{-1}\mathfrak{a}$, $W_S$ is generated by reflections with respect to roots in $\Sigma_S$. A linear order $>$ on $\sqrt{-1}\mathfrak{t}$ is said to be *compatible* for $\Sigma_{\mathfrak{t}}$ with respect to $\sigma$ (or with respect to the orthogonal decomposition $\sqrt{-1}\mathfrak{t} = \sqrt{-1}\mathfrak{b} \oplus \sqrt{-1}\mathfrak{a}$) if $\alpha \in \Sigma_{\mathfrak{t}}$, $\alpha > 0$ and $\sigma\alpha \neq -\alpha$ imply $\sigma\alpha > 0$. Take a compatible order $>$ on $\sqrt{-1}\mathfrak{t}$ and fix it once and for all. Let

$$\Pi_{\mathfrak{t}} = \{\alpha_1, \cdots, \alpha_l\}$$

be the fundamental root system of $\Sigma_{\mathfrak{t}}$ with respect to the order $>$ and put

$$\Pi_{\mathfrak{t}}^0 = \Pi_{\mathfrak{t}} \cap \Sigma_{\mathfrak{t}}^0\ .$$

$W_{\mathfrak{t}}$ is also generated by reflections with respect to roots in $\Pi_{\mathfrak{t}}$. We have the decomposition

$$\sigma = sp \qquad \text{where} \quad s \in W_{\mathfrak{t}},\quad p\Pi_{\mathfrak{t}} = \Pi_{\mathfrak{t}}$$

of $\sigma$ in such a way that $p^2 = 1$, $p(\Pi_{\mathfrak{t}} - \Pi_{\mathfrak{t}}^0) = \Pi_{\mathfrak{t}} - \Pi_{\mathfrak{t}}^0$ and $\sigma\alpha_i \equiv p\alpha_i \bmod \{\Pi_{\mathfrak{t}}^0\}_Z$ for any $\alpha_i \in \Pi_{\mathfrak{t}} - \Pi_{\mathfrak{t}}^0$ (Satake [10]). We put

$$\Pi_S = \{\varpi\alpha_i;\ \alpha_i \in \Pi_{\mathfrak{t}} - \Pi_{\mathfrak{t}}^0\} = \varpi\Pi_{\mathfrak{t}} - \{0\}\ .$$

We may assume that $\Pi_S = \{\gamma_1, \cdots, \gamma_p\}$ with $\varpi\alpha_i = \gamma_i$ $(1 \leqslant i \leqslant p)$, changing indices of the $\alpha_i$'s if necessary. $\Pi_S$ is the fundamental root system of $\Sigma_S$ with respect to the order $>$. We put

$$\Sigma_S^* = \{\gamma \in \Sigma_S;\ 2\gamma \notin \Sigma_S\}\ .$$

Then $\Sigma_S^*$ is a reduced root system in $\sqrt{-1}\mathfrak{a}'$. The fundamental root system $\Pi_S^*$ of $\Sigma_S^*$ with respect to the order $>$ is given by

$$\Pi_S^* = \{\beta_1, \cdots, \beta_p\}$$

where
$$\beta_i = \begin{cases} \gamma_i & \text{if } 2\gamma_i \notin \Sigma_S \\ 2\gamma_i & \text{if } 2\gamma_i \in \Sigma_S\ . \end{cases}$$

$W_S$ is also generated by reflections with respect to roots of $\Pi_S$ or of $\Pi_S^*$. Let

$\Sigma_{\mathfrak{k}}^{+}$ (resp. $\Sigma_S^{+}$, $(\Sigma_S^{*})^{+}$) denote the set of positive roots in $\Sigma_{\mathfrak{k}}$ (resp. $\Sigma_S$, $\Sigma_S^{*}$). Then

$$\Sigma_S^{+} = \varpi(\Sigma_{\mathfrak{k}}^{+} - \Sigma_{\mathfrak{k}}^{0}) = \varpi\Sigma_{\mathfrak{k}}^{+} - \{0\} .$$

For $\lambda \in \sqrt{-1}\mathfrak{t}$, $\lambda \neq 0$, we define

$$\lambda^{*} = \frac{2}{(\lambda, \lambda)}\lambda .$$

**Theorem 2.1.** (E. Cartan) *Assume that $K$ is simply connected. Then*

1) *$K_\theta$ is connected.*

2) *The kernel of* $\exp: \mathfrak{a} \to K$ *is the subgroup of* $\mathfrak{a}$ *generated by* $\{2\pi\sqrt{-1}\gamma^{*};\ \gamma \in \Sigma_S\}$.

**Theorem 2.2.** (Harish-Chandra) *Let $S_L^{*}(\mathfrak{s})$ (resp. $S_{W_S}^{*}(\mathfrak{a})$) be the space of polynomial functions on $\mathfrak{s}$ (resp. on $\mathfrak{a}$), which are invariant under the adjoint actions of $L$ (resp. of $W_S$). Then the restriction map*

$$S_L^{*}(\mathfrak{s}) \to S_{W_S}^{*}(\mathfrak{a})$$

*is an isomorphism.*

Now we shall consider $W_S$-invariant characters of a maximal torus of $S$. Put

$$\Gamma = \Gamma(K, L) = \{H \in \mathfrak{a};\ \exp H \in L\}$$

and

$$\Gamma_{\mathfrak{c}} = \Gamma \cap \mathfrak{c}_{\mathfrak{a}} \quad \text{where} \quad \mathfrak{c}_{\mathfrak{a}} = \mathfrak{c} \cap \mathfrak{a} .$$

Then $\Gamma$ is a $W_S$-invariant lattice in $\mathfrak{a}$ and $\Gamma_{\mathfrak{c}}$ is a lattice in $\mathfrak{c}_{\mathfrak{a}}$. Let $C_{\mathfrak{a}}$ be the connected subgroup of $K$ generated by $\mathfrak{c}_{\mathfrak{a}}$. Then the $A$-orbit $\hat{A}$ in $S$ through the origin $x_0$ of $S$ and the $C_{\mathfrak{a}}$-orbit $\hat{C}_{\mathfrak{a}}$ in $S$ through the origin have identifications

$$\hat{A} = \mathfrak{a}/\Gamma$$

and

$$\hat{C}_{\mathfrak{a}} = \mathfrak{c}_{\mathfrak{a}}/\Gamma_{\mathfrak{c}} .$$

Hence both $\hat{A}$ and $\hat{C}_{\mathfrak{a}}$ have structures of toral groups. The toral group $\hat{A}$ is said to be a *maximal torus* of the symmetric space $S$. The adjoint action of $W_S$ on $A$ induces the action of $W_S$ on $A$. This action is compatible with the natural action of $W_S$ on $\mathfrak{a}/\Gamma$ relative to the identification: $\hat{A} = \mathfrak{a}/\Gamma$. Put

$$Z = Z(K, L) = \{\lambda \in \sqrt{-1}\mathfrak{a};\ (\lambda, H) \in 2\pi\sqrt{-1}\boldsymbol{Z} \text{ for any } H \in \Gamma\} .$$

$Z$ is isomorphic with the group $\mathcal{D}(\hat{A})$ of characters of $\hat{A}$ by the correspondence $\lambda \mapsto e^{\lambda}$, where $e^{\lambda} \in \mathcal{D}(\hat{A})$ is defined by $e^{\lambda}((\exp H)x_0) = \exp(\lambda, H)$ for $H \in \mathfrak{a}$. Put

$$D = D(K, L) = \{\lambda \in Z;\ (\lambda, \gamma_i) \geqslant 0 \text{ for any } \gamma_i \in \Pi_S\}$$
$$= \{\lambda \in Z;\ (\lambda, \gamma) \geqslant 0 \text{ for any } \gamma \in \textstyle\sum_S^+\}\ .$$

Then we have

$$D = \{\lambda \in Z;\ s\lambda \leqslant \lambda \text{ for any } s \in W_S\}\ .$$

An element of $D$ is called a *dominant integral form* on $\mathfrak{a}$. We define a lattice $\Gamma_0'$ in $\mathfrak{a}'$ to be the subgroup of $\mathfrak{a}'$ generated by $\{2\pi\sqrt{-1}(\frac{1}{2}\gamma^*);\ \gamma \in \sum_S\}$. We define a lattice $\Gamma_0$ in $\mathfrak{a}$ and a toral group $\hat{A}_0$ by

$$\Gamma_0 = \Gamma_{\mathfrak{c}} \oplus \Gamma_0'$$

and

$$\hat{A}_0 = \mathfrak{a}/\Gamma_0\,.$$

Put

$$Z_0 = \{\lambda \in \sqrt{-1}\,\mathfrak{a};\ (\lambda, H) \in 2\pi\sqrt{-1}\,\boldsymbol{Z} \text{ for any } H \in \Gamma_0\}$$

and

$$D_0 = D \cap Z_0\,.$$

$Z_0$ is isomorphic with the group $\mathcal{D}(\hat{A}_0)$ of characters of $\hat{A}_0$. Put furthermore

$$Z_0' = Z_0 \cap \sqrt{-1}\,\mathfrak{a}' = \left\{\lambda \in \sqrt{-1}\,\mathfrak{a}';\ \frac{2(\lambda, \gamma)}{(\gamma, \gamma)} \in 2\boldsymbol{Z} \text{ for any } \gamma \in \textstyle\sum_S\right\}$$

and

$$D_0' = D_0 \cap \sqrt{-1}\,\mathfrak{a}' = D \cap Z_0'\,.$$

**Lemma 1.** *If $L = K_\theta$, then*

$$\Gamma = \{\tfrac{1}{2}H;\ H \in \mathfrak{a},\ \exp H = e\}\ .$$

Proof. For $H \in \mathfrak{a}$, $\exp H = e \Leftrightarrow \exp\frac{H}{2}\exp\frac{H}{2} = e \Leftrightarrow \exp\frac{H}{2} = \left(\exp\frac{H}{2}\right)^{-1} \Leftrightarrow \exp\frac{H}{2} = \theta\left(\exp\frac{H}{2}\right) \Leftrightarrow \exp\frac{H}{2} \in K_\theta$, which yields Lemma 1. q.e.d.

**Lemma 2.** 1) $$\Gamma_0' = 2\pi\sqrt{-1}\sum_{i=1}^{p}\boldsymbol{Z}(\tfrac{1}{2}\beta_i^*)$$
*and it is $W_S$-invariant. Therefore $\Gamma_0$ is $W_S$-invariant.*

2) $\Gamma_0 \subset \Gamma$. *Therefore $Z_0 \supset Z$ and $D_0 \supset D$.*

3) *If $S$ is simply connected, then $\Gamma = \Gamma_0 = \Gamma_0'$ (thus $Z = Z_0 = Z_0'$, $D = D_0 = D_0'$) and $\hat{A}_0$ can be identified with $\hat{A}$.*

Proof. 1) Denoting the reflection of $\sqrt{-1}\,\mathfrak{a}$ with respect to $\beta_i \in \Pi_S^*$ by $s_i \in W_S$, we have

$$s_i\gamma^* = (s_i\gamma)^* = \gamma^* - \frac{2(\beta_i, \gamma)}{(\gamma, \gamma)}\beta_i^* \qquad \text{for} \quad \gamma \in \Sigma_S .$$

It follows that $\Gamma_0'$ is $W_S$-invariant. Since we have

$$(2\lambda)^* = \frac{2\cdot 2\lambda}{4(\lambda, \lambda)} = \frac{\lambda}{(\lambda, \lambda)} = \tfrac{1}{2}\lambda^* \qquad \text{for} \quad \lambda \in \sqrt{-1}\,\mathfrak{a},\ \lambda \neq 0 ,$$

$\Gamma_0'$ is the subgroup of $\mathfrak{a}'$ generated by $2\pi\sqrt{-1}(\frac{1}{2}\gamma^*)$ for $\gamma \in \Sigma_S^*$. Thus it suffices to show that

$$\gamma^* \in \sum_{i=1}^{p} \boldsymbol{Z}\beta_i^* \qquad \text{for any} \quad \gamma \in \Sigma_S^* .$$

But this follows from the first equality since there exist $\beta_{i_1}, \cdots, \beta_{i_r} \in \Pi_S^*$ such that $s_{i_1}\cdots s_{i_r}\gamma \in \Pi_S^*$.

2) Since $\Gamma_c \subset \Gamma$, it suffices to show that $\Gamma_0' \subset \Gamma'$ for $\Gamma' = \Gamma \cap \mathfrak{a}'$. Let $K'$ be the connected subgroup of $K$ generated by $\mathfrak{k}'$ and $L' = K' \cap L$. Then $(K', L')$ is also a symmetric pair with respect to $\theta$ and $S' = K'/L'$ can be identified with the $K'$-orbit in $S$ through the origin $x_0$ of $S$. Let

$$\pi' : K_0' \to K'$$

be the covering homomorphism of the universal covering group $K_0'$ of $K'$ and put

$$L_0' = \{k \in K_0' ;\ \theta_0(k) = k\} ,$$

where $\theta_0$ is the involutive automorphism of $K_0'$ covering the involutive automorphism $\theta$ of $K'$. $K_0'$ is compact since $K'$ is semi-simple. $S'$ can be identified with $K_0'/\pi'^{-1}(L')$. It follows from Theorem 2.1 and Lemma 1 that $L_0'$ is connected and

$$\Gamma_0' = \{H \in \mathfrak{a}' ;\ \exp_{K_0'} H \in L_0'\} .$$

Let $A'$ (resp. $A_0'$) be the connected subgroup of $K'$ (resp. of $K_0'$) generated by $\mathfrak{a}'$ and $\hat{A}'$ (resp. $\hat{A}_0'$) be the $A'$-orbit in $S'$ (resp. the $A_0'$-orbit in $S_0' = K_0'/L_0'$) through the origin. Then we have identifictions

$$\hat{A}' = \mathfrak{a}'/\Gamma'$$

and

$$\hat{A}_0' = \mathfrak{a}'/\Gamma_0' .$$

On the other hand, since $\pi'^{-1}(L') \supset L_0'$, the covering homomorphism $\pi'$ induces the commutative diagram

$$\begin{array}{ccc} S_0' & \xrightarrow{\pi'} & S' \\ \cup & & \cup \\ \hat{A}_0' & \xrightarrow[\pi']{} & \hat{A}' . \end{array}$$

It follows that

$$\Gamma_0' \subset \Gamma' .$$

3) Under the notation in 2), we have a covering map

$$\hat{C}_{\mathfrak{a}} \times S' \to S .$$

It follows from the assumption that $\hat{C}_{\mathfrak{a}} = \{e\}$ and $S'$ is simply connected. Thus the covering map $\pi'$ is trivial and $\Gamma' = \Gamma_0'$. Moreover $\mathfrak{c}_{\mathfrak{a}} = \{0\}$ implies that $\Gamma = \Gamma'$ and $\Gamma_0 = \Gamma_0'$. q.e.d.

REMARK. Define $\Lambda_i \in \sqrt{-1}\,\mathfrak{t}'$ $(1 \leqslant i \leqslant l)$ by

$$(\Lambda_i, \alpha_j^*) = \delta_{ij} \quad (1 \leqslant i, j \leqslant l) .$$

Then define $M_i$ $(1 \leqslant i \leqslant p)$ by

$$M_i = \begin{cases} 2\Lambda_i & \text{if } \ p\alpha_i = \alpha_i \ \text{ and } \ (\alpha_i, \Pi_{\mathfrak{k}}^0) = \{0\} \\ \Lambda_i & \text{if } \ p\alpha_i = \alpha_i \ \text{ and } \ (\alpha_i, \Pi_{\mathfrak{k}}^0) \neq \{0\} \\ \Lambda_i + \Lambda_{i'} & \text{if } \ p\alpha_i = \alpha_{i'} \neq \alpha_i . \end{cases}$$

Then it can be verified (cf. Sugiura [12]) that $M_i \in \sqrt{-1}\,\mathfrak{a}'$ $(1 \leqslant i \leqslant p)$ and

$$(M_i, \tfrac{1}{2}\beta_j^*) = \delta_{ij} \quad (1 \leqslant i, j \leqslant p) .$$

It follows that

$$Z_0' = \sum_{i=1}^{p} \boldsymbol{Z} M_i$$

and

$$D_0' = \{ \sum_{i=1}^{p} m_i M_i ;\ m_i \in \boldsymbol{Z},\ m_i \geqslant 0 \ (1 \leqslant i \leqslant p) \} .$$

It follows from Lemma 2,1) that $W_S$ acts on $\hat{A}_0 = \mathfrak{a}/\Gamma_0$ and from Lemma 2,2) that we have a $W_S$-equivariant homomorphism

$$\pi_0 \colon \hat{A}_0 \to \hat{A} .$$

Let $\mathcal{R}(\hat{A})$ denote the character ring of $\hat{A}$. Then $W_S$ acts on $\mathcal{R}(\hat{A})$ (or more generally on the space $C^\infty(\hat{A})$ of $\boldsymbol{C}$-valued $C^\infty$-functions on $\hat{A}$) by

$$(s\chi)(\hat{a}) = \chi(s^{-1}\hat{a}) \qquad \text{for } \ s \in W_S,\ \hat{a} \in \hat{A} .$$

This action coincides on $Z = \mathcal{D}(\hat{A}) \subset \mathcal{R}(\hat{A})$ with the adjoint action of $W_S$ on $Z$. Let $\mathcal{R}_{W_S}(\hat{A})$ be the subring of $W_S$-invariant characters of $\hat{A}$ and $\mathcal{R}_{W_S}(\hat{A})^C$ the $\boldsymbol{C}$-span of $\mathcal{R}_{W_S}(\hat{A})$ in $C^\infty(\hat{A})$. Let $\mathcal{R}(\hat{A}_0)$, $\mathcal{R}_{W_S}(\hat{A}_0)$ and $\mathcal{R}_{W_S}(\hat{A}_0)^C$ denote the same objects for $\hat{A}_0$. Then $\pi_0$ induces a $W_S$-equivariant monomorphism

$$\pi_0^* \colon \mathcal{R}(\hat{A}) \to \mathcal{R}(\hat{A}_0)$$

and monomorphisms

$$\pi_0^*: \mathcal{R}_{W_S}(\hat{A}) \to \mathcal{R}_{W_S}(\hat{A}_0),$$
$$\pi_0^*: \mathcal{R}_{W_S}(\hat{A})^C \to \mathcal{R}_{W_S}(\hat{A}_0)^C.$$

Henceforth we shall identify $\mathcal{R}_{W_S}(\hat{A})$ with a subring of $\mathcal{R}_{W_S}(\hat{A}_0)$ and $\mathcal{R}_{W_S}(\hat{A})^C$ with a subalgebra of $\mathcal{R}_{W_S}(\hat{A}_0)^C$ by means of these monomorphisms $\pi_0^*$.

For $\lambda \in \sqrt{-1}\,\mathfrak{a}$, we shall denote by $\lambda_{\mathfrak{c}}$ the $\sqrt{-1}\,\mathfrak{c}_{\mathfrak{a}}$-component of $\lambda$ with respect to the orthogonal decomposition

$$\sqrt{-1}\,\mathfrak{a} = \sqrt{-1}\,\mathfrak{c}_{\mathfrak{a}} \oplus \sqrt{-1}\,\mathfrak{a}'.$$

The following facts can be proved in the same way as the classical results for a compact connected Lie group $S$, so the proofs are omitted.

We define an element $\delta$ in $Z_0$ by

$$\delta = \sum_{\gamma \in (\Sigma_S^*)^+} \gamma.$$

For $\lambda \in Z_0$, we define $\xi_\lambda \in \mathcal{R}(\hat{A}_0)$ by

$$\xi_\lambda = \sum_{s \in W_S} (\det s) e^{s\lambda}.$$

For $\lambda \in Z$, $\xi_\lambda$ is divisible by $\xi_\delta$ in the ring $\mathcal{R}(\hat{A}_0)$ and

$$\chi_\lambda = \frac{\xi_{\lambda+\delta}}{\xi_\delta}$$

is in $\mathcal{R}_{W_S}(\hat{A})$. If $\chi_\lambda$ has the expression

$$\chi_\lambda = \sum m_\mu e^\mu \quad \text{with} \quad \mu \in Z,\ m_\mu \in \boldsymbol{Z},\ m_\mu \neq 0,$$

then $\mu_{\mathfrak{c}}$ are the same for any $\mu$. In particular, if $\lambda \in D$, then the highest component in the above expression of $\chi_\lambda$ is $e^\lambda$ with $m_\lambda = 1$. Any $W_S$-invariant character $\chi \in \mathcal{R}_{W_S}(\hat{A})$ of $\hat{A}$ has an expression

$$\chi = \sum m_\lambda \chi_\lambda \quad \text{with} \quad \lambda \in D,\ m_\lambda \in \boldsymbol{Z}.$$

The expression is unique for $\chi$. In particular, the system $\{\chi_\lambda;\ \lambda \in D\}$ forms a basis of the space $\mathcal{R}_{W_S}(\hat{A})^C$.

Now we come back to spherical representations of a symmetric pair $(K, L)$.

**Theorem 2.3.** (E. Cartan [1]) *Let $\rho \in \mathcal{D}(K, L)$ have the highest weight $\lambda \in \sqrt{-1}\,\mathfrak{t}$ and $\omega_\lambda$ be the zonal spherical function for $(K, L)$ belonging to $\rho$. Then*

1) $\lambda \in D$,

2) *$\omega_\lambda$ restricted to $\hat{A}$ is in $\mathcal{R}_{W_S}(\hat{A})^C$ and has an expression*

$$\omega_\lambda = \sum a_\mu e^{-\mu} \quad \text{with} \quad \mu \in Z,\ a_\mu \in \boldsymbol{R},\ a_\mu > 0,\ \sum a_\mu = 1,$$

*with the lowst component* $a_\lambda e^{-\lambda}$.

Proof. Proof of E. Cartan [1] was done in the case where $K$ is semi-simple and $L=K_\theta$. His proof can be applied for our case without difficulties. But his proof of $\lambda\in\sqrt{-1}\mathfrak{a}$ is not complete. A correct proof is seen, for example, in Schmid [11]. q.e.d.

**Lemma 3.** *For any* $\lambda\in D$, *there exists an irreducible representation* $\rho$ *of* $K$ *such that the highest weight of* $\rho$ *on* $\mathfrak{t}^C$ *is* $\lambda$.

Proof. Let $H\in\mathfrak{t}$ with $\exp H=e$. Decompose $H$ as

$$H = H'+H'' \quad \text{with} \quad H'\in\mathfrak{b},\ H''\in\mathfrak{a}\,.$$

Then $\exp H''=(\exp H')^{-1}\in L$, i.e. $H''\in\Gamma$. It follows from $\lambda\in Z\subset\sqrt{-1}\mathfrak{a}$ that $(\lambda, H)=(\lambda, H')+(\lambda, H'')=(\lambda, H'')\in 2\pi\sqrt{-1}\boldsymbol{Z}$. Moreover $(\lambda, \alpha_i)=(\lambda, \varpi\alpha_i)\geqslant 0$ for any $\alpha_i\in\Pi_{\mathfrak{k}}$ since $\lambda\in D$. Thus $e^\lambda$ is a dominant character of the maximal torus $T$ of $K$. Then the classical representation theory of compact connected Lie groups assures the existence of $\rho$. q.e.d.

**Lemma 4.** *Let* $Z_L(A)$ *be the centralizer in* $L$ *of* $A$ *and* $Z_L(A)^0$ *the connected component of* $Z_L(A)$. *Then*

$$Z_L(A) = Z_L(A)^0 \exp\Gamma\,.$$

Proof. The centralizer $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ in $\mathfrak{k}$ of $\mathfrak{a}$ has the decomposition

$$\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{l}}(\mathfrak{a})\oplus\mathfrak{a}\,,$$

where $\mathfrak{z}_{\mathfrak{l}}(\mathfrak{a})$ is the centralizer in $\mathfrak{l}$ of $\mathfrak{a}$. Since the centralizer $Z_K(A)$ in the compact connected Lie group $K$ of the torus $A$ is connected, we have the decomposition

$$Z_K(A) = Z_L(A)^0 A\,.$$

It follows that any element $m\in Z_L(A)$ can be written as

$$m = m'a \quad \text{with} \quad m'\in Z_L(A)^0,\ a\in A\,.$$

Then $a=m'^{-1}m\in L$ so that $a\in\exp\Gamma$. Thus $m\in Z_L(A)^0\exp\Gamma$, which proves Lemma 4. q.e.d.

**Lemma 5.** *Let* $K^C$ *denote the Chevalley complexification of* $K$. *Put*

$$K^* = L\exp\sqrt{-1}\mathfrak{s}$$

*and*

$$(K^*)^0 = L^0\exp\sqrt{-1}\mathfrak{s}\,,$$

*where* $L^0$ *denotes the connected component of* $L$. *Then* $(K^*)^0$ *is a closed subgroup of*

*$K^C$ normalized by $K^*$ and*

$$K^* = (K^*)^0 \exp \Gamma .$$

*Therefore $K^*$ is a closed subgroup of $K^C$ with the connected compoenet $(K^*)^0$.*

Proof. The first statement is clear. Take any element $l \in L$. From the conjugteness of maximal abelian subalgebras in $\mathfrak{s}$ under the adjoint action of $L^0$, there exists $l_1 \in L^0$ such that $l_1 l \in N_L(A)$. Since

$$N_L(A)/Z_L(A) = N_{L^0}(A)/Z_{L^0}(A) = W_S ,$$

we can choose $l_2 \in L^0$ such that $l_2 l_1 l \in Z_L(A)$. It follows from Lemma 4 that there exist $l_3 \in Z_L(A)^0$ and $a \in \exp \Gamma$ such that $l_2 l_1 l = l_3 a$. Therefore $l = l_1^{-1} l_2^{-1} l_3 a$ with $l_1^{-1} l_2^{-1} l_3 \in L^0 \subset (K^*)^0$, i.e. $l \in (K^*)^0 \exp \Gamma$. This completes the proof of Lemma 5. q.e.d.

Now we can prove the following

**Theorem 2.4.** (E. Cartan [1], Sugiura [12], Helgason [5]) *For any $\lambda \in D$, there exists an irreducible spherical representation $\rho$ of $K$ relative to $L$ such that the highest weight of $\rho$ on $\mathfrak{t}^C$ is $\lambda$.*

Together with Theorem 2.3 we have the following

**Corollary**. *For $\rho \in \mathcal{D}(K, L)$, let $\lambda(\rho)$ denote the highest weight of $\rho$ on $\mathfrak{t}^C$. Then the correspondence $\rho \mapsto \lambda(\rho)$ gives a bijection*:

$$\mathcal{D}(K, L) \to D(K, L) .$$

Proof of Theorem 2.4. This theorem for the case where $K$ is semi-simple and $L = K_\theta$ was stated in E. Cartan [1] but its proof is not complete. It was stated for simply connected $K$ without proof in Sugiura [12]. It was proved in Helgason [5] for the case where $K$ is semi-simple and $L$ is connected. Helgason's proof can be applied for our case without difficulties, so we shall confine ourselves to point out necessary modifications.

Let

$$\rho \colon K \to GL(V)$$

be the irreducible representation of $K$ with the highest weight $\lambda$ (Lemma 3). By extending $\rho$ to the Chevalley complexification $K^C$ of $K$ and restricting it to the closed subgroup $K^*$ of $K^C$ (Lemma 5), we have an irreducible representation of $K^*$, which will be denoted by the same letter $\rho$. It suffices to show that $\rho$ has a non-zero $L$-invariant. Let $N$ be the connected subgroup of $K^*$ generated by the subalgebra

$$\mathfrak{n} = \mathfrak{k}^* \cap \sum_{\alpha \in \Sigma_{\mathfrak{t}}^+ - \Sigma_{\mathfrak{t}}^0} \mathfrak{k}_\alpha^C ,$$

where $\mathfrak{k}^*$ is the Lie algebra of $K^*$ and $\mathfrak{k}^C_\alpha$ is the root space of $\mathfrak{k}^C$ for $\alpha$. We shall first prove that the representation $\rho$ of $K^*$ is a conical representation of $K^*$ in the sense of Helgason [5], i.e. if $v_\lambda \in V$, $v_\lambda \neq 0$, is a highest weight vector for $\rho$ with repsect to $\mathfrak{t}^C$, we have

$$\rho(mn)v_\lambda = v_\lambda \qquad \text{for any} \quad m \in Z_L(A),\, n \in N\,.$$

Denoting the infinitesimal action of $\mathfrak{k}^C$ on $V$ by the same letter $\rho$, we have

$$\rho(\mathfrak{n})v_\lambda = \rho(\mathfrak{z}_{\mathfrak{l}}(\mathfrak{a}))v_\lambda = \{0\}\ .$$

In fact, $\rho(\mathfrak{n})v_\lambda = \{0\}$ since $\mathfrak{n} \subset \sum_{\alpha \in \Sigma_{\mathfrak{k}}^+} \mathfrak{k}^C_\alpha$. $\rho(\mathfrak{b}^C)v_\lambda = \{0\}$ for the complexification $\mathfrak{b}^C$ of $\mathfrak{b}$ since $(\sqrt{-1}\,\mathfrak{b}, \lambda) = \{0\}$. $\rho(\mathfrak{k}^C_\alpha)v_\lambda = \{0\}$ for $\alpha \in \sum_{\mathfrak{k}}^0$, $\alpha > 0$. It follows from $(\alpha, \lambda) \in (\sqrt{-1}\,\mathfrak{b}, \lambda) = \{0\}$ for $\alpha \in \sum_{\mathfrak{k}}^0$ that $\lambda - \alpha$ is not a weight of $\rho$ for $\alpha \in \sum_{\mathfrak{k}}^0$, $\alpha > 0$. Since the complexification of $\mathfrak{z}_{\mathfrak{l}}(\mathfrak{a})$ is spanned by $\mathfrak{b}^C$ and the $\mathfrak{k}^C_\alpha$'s for $\alpha \in \sum_{\mathfrak{k}}^0$, we have $\rho(\mathfrak{z}_{\mathfrak{l}}(\mathfrak{a}))v_\lambda = \{0\}$. Therefore it suffices from Lemma 4 to show that

$$\rho(\exp H)v_\lambda = v_\lambda \qquad \text{for any} \quad H \in \Gamma\,.$$

But it is clear since $\lambda \in Z$, i.e. $(\lambda, H) \in 2\pi\sqrt{-1}\,\boldsymbol{Z}$ for any $H \in \Gamma$.

Thus we can prove in the same way as Helgason [5] that $V$ has a non-zero $L$-invariant vector, by constructing a $K^*$-submodule $V'$ of the $K^*$-module $C^\infty(K^*)$ of $C^\infty$-functions on $K^*$, having a non-zero $L$-invariant, and by constructing a $K^*$-equivariant isomorphism of $V$ onto $V'$. q.e.d.

Next we shall describe zonal spherical functions in terms of the basis $\{\chi_\lambda;\, \lambda \in D\}$ of $\mathcal{R}_{W_S}(\hat{A})^C$.

For $\hat{a} = (\exp H)x_0 \in \hat{A}$, $H \in \mathfrak{a}$, we put

$$D(\hat{a}) = \left| \prod_{\alpha \in \Sigma_{\mathfrak{k}}^+ - \Sigma_{\mathfrak{k}}^0} 2\sin(\alpha, \sqrt{-1}H) \right|.$$

Let $d\hat{a}$ denote the normalized Haar measure of $\hat{A}$ and $|W_S|$ the order of the Weyl group $W_S$. For $W_S$-invariant functions $\chi$, $\chi'$ on $\hat{A}$, we define

$$\langle \chi, \chi' \rangle = \frac{c}{|W_S|} \int_{\hat{A}} \chi(\hat{a})\overline{\chi'(\hat{a})}D(\hat{a})d\hat{a}\,,$$

where

$$c = \left( \frac{1}{|W_S|} \int_{\hat{A}} D(\hat{a})d\hat{a} \right)^{-1}.$$

$c=1$ in the case where $S$ is a compact connected Lie group. In particular, if $\chi$ and $\chi'$ can be extended to $L$-invariant functions $f$ and $f'$ on $S$, then $\langle \chi, \chi' \rangle$ coincides with the inner product $\langle f, f' \rangle$ in $L^2(S)$ (cf. Helgason [4]).

Fix a dominant integral form $\lambda \in D$. We define a finite subset $D_\lambda$ of $D$ by

$$D_\lambda = \{\mu \in D;\ \mu_c = \lambda_c,\ \mu \leqslant \lambda\}\ .$$

Since the system $\{\chi_\mu;\ \mu \in D\}$ forms a basis of $\mathcal{R}_{W_S}(\hat{A})^C$, the matrix

$$(\langle \chi_\mu,\ \chi_\nu \rangle)_{\mu,\nu \in D_\lambda}$$

is a positive definite hermitian matrix. Let

$$(b^{\mu\nu})_{\mu,\nu \in D_\lambda}$$

be the inverse matrix of the above matrix. In particular $b^{\lambda\lambda} > 0$. For any $\mu \in D_\lambda$, we put

$$c_\lambda^\mu = \frac{b^{\lambda\mu}}{\sqrt{d_\lambda b^{\lambda\lambda}}}\ ,$$

where $d_\lambda$ is the degree of an irrducible representation of $K$ with the highest weight $\lambda$. Then we have

**Theorem 2.5.** *Let $\lambda \in D$ and $\omega_\lambda$ be the zonal spherical function belonging to the class of an irreducible representation of $K$ with the highest weight $\lambda$. Then $\omega_\lambda$ restricted to $\hat{A}$ is given by*

$$\omega_\lambda = \sum_{\mu \in D_\lambda} c_\lambda^\mu \bar{\chi}_\mu\ .$$

Proof. The idea of the following proof owes to Hua [6]. Let $\mu \in D_\lambda$. Then $\omega_\mu$ restricted to $\hat{A}$ is in $\mathcal{R}_{W_S}(\hat{A})^C$ by Theorem 2.3. It follows by Theorem 2.3 and Corollary of Theorem 2.4 that $\omega_\mu$ has an expression

$$\omega_\mu = \sum_{\nu \in D_\lambda} c'^{\nu}_{\mu} \bar{\chi}_\nu \qquad \text{with } c'^{\nu}_{\mu} \in \boldsymbol{R},\ c'^{\mu}_{\mu} > 0,\ c'^{\nu}_{\mu} = 0 \text{ if } \nu > \mu\ .$$

We define an upper triangular matrix $C'$ by

$$C' = (c'^{\nu}_{\mu})_{\nu,\mu \in D_\lambda}\ .$$

Then we have

$$(\langle \omega_\mu,\ \omega_\nu \rangle)_{\mu,\nu \in D_\lambda} = {}^tC'(\langle \chi_\mu,\ \chi_\nu \rangle)_{\mu,\nu \in D_\lambda} C'\ .$$

Since $\langle \omega_\mu,\ \omega_\nu \rangle = d_\mu^{-1} \delta_{\mu\nu}$, we have

$$(d_\mu \delta_{\mu\nu})_{\mu,\nu \in D_\lambda} = C'^{-1} B' {}^tC'^{-1}\ ,$$

where

$$B' = (b'^{\mu\nu})_{\mu,\nu \in D_\lambda} = (\langle \bar{\chi}_\mu,\ \bar{\chi}_\nu \rangle)^{-1}_{\mu,\nu \in D_\lambda}\ .$$

It follows that

$$C'(d_\mu \delta_{\mu\nu})_{\mu,\nu \in D_\lambda} {}^tC' = B'\ .$$

Comparing $(\mu,\ \lambda)$-components of both sides, we have

$$c'^{\mu}_{\lambda} d_{\lambda} c'^{\lambda}_{\lambda} = b'^{\mu\lambda} .$$

In particular

$$(c'^{\lambda}_{\lambda})^2 d_{\lambda} = b'^{\lambda\lambda}, \quad \text{i.e.} \quad c^{\lambda}_{\lambda} = \sqrt{\frac{b'^{\lambda\lambda}}{d_{\lambda}}} ,$$

hence

$$c'^{\mu}_{\lambda} = \frac{b'^{\mu\lambda}}{d_{\lambda} c'^{\lambda}_{\lambda}} = \frac{b'^{\mu\lambda}}{\sqrt{d_{\lambda} b'^{\lambda\lambda}}} .$$

Since $b'^{\mu\nu} = b^{\nu\mu}$, we have

$$c'^{\mu}_{\lambda} = \frac{b^{\lambda\mu}}{\sqrt{d_{\lambda} b^{\lambda\lambda}}} = c^{\mu}_{\lambda} . \qquad \text{q.e.d.}$$

EXAMPLE. If $S$ is a compact connected Lie group and $(K, L)$ the symmetric pair with $K/L = S$ as mentioned before, then the set $\mathcal{D}(S)$ of equivalence classes of irreducible representations of $S$ is in the bijective correspondence with $\mathcal{D}(K, L)$ by the assignment $\rho \mapsto \rho \boxtimes \rho^*$, where $\rho^*$ denotes the contragredient representation of $\rho$. $\hat{A}$ is a maximal torus of the compact Lie group $S$. Let $\chi_{\rho}$ be the invariant character of $\hat{A}$ for the dominant integral form in $D(K, L)$ corresponding to $\rho \boxtimes \rho^*$ by the bijection in Cororally of Theorem 2.4. Then it is nothing but the character of $\rho$. It follows from orthogonality relations of irreducible characters that the matrix $(b^{\lambda\mu})$ is the identity matrix. Thus the zonal spherical function $\omega_{\rho\boxtimes\rho^*}$ belonging to $\rho \boxtimes \rho^*$ is given by

$$\omega_{\rho\boxtimes\rho^*} = \frac{1}{d_{\rho}} \bar{\chi}_{\rho} ,$$

where $d_{\rho}$ is the degree of $\rho$.

## 3. Polynomial representations associated with symmetric bounded domains

Let $D$ be an irreducible symmetric bounded domain with rank $p$ realized in $(\mathfrak{p}^C)^-$ as in Introduction. We shall use the same notation as in Introduction.

Let

$$\Pi = \{\alpha_1, \cdots, \alpha_l\}$$

be the fundamental root system of $\sum$ with respect to the order $>$ and let $\Pi_{\mathfrak{k}} = \Pi \cap \sum_{\mathfrak{k}}$. It is known that $\Pi_{\mathfrak{k}}$ is the fundamental root system of $\sum_{\mathfrak{k}}$, $\Pi - \Pi_{\mathfrak{k}}$ consists of one element, say $\alpha_1$, which is the lowest root in $\sum_{\mathfrak{p}}^+$, and for any $\alpha = \sum_{i=1}^{l} m_i \alpha_i \in \sum_{\mathfrak{p}}^+$, $m_1 = 1$. Let $\sum_{\mathfrak{k}}^+$ denote the set of positive compact roots. Put

$$\mathfrak{b} = \{H \in \mathfrak{a}; (\sqrt{-1} H, \Delta) = \{0\}\} .$$

Then we have the orthogonal decomposition

$$\sqrt{-1}\mathfrak{t}=\sqrt{-1}\mathfrak{b}\oplus\sqrt{-1}\mathfrak{a}$$

with respect to ( , ). We define an orthogonal transformation $\sigma$ on $\sqrt{-1}\mathfrak{t}$ by $\sigma|\mathfrak{b}=-1$ and $\sigma|\sqrt{-1}\mathfrak{a}=1$. Let

$$\varpi=\tfrac{1}{2}(1+\sigma)\colon \sqrt{-1}\mathfrak{t}\to\sqrt{-1}\mathfrak{a}$$

be the orthogonal projection of $\sqrt{-1}\mathfrak{t}$ onto $\sqrt{-1}\mathfrak{a}$. Let $\kappa$ be the unique involutive element of the Weyl group $W_{\mathfrak{k}}$ of $K$ such that $\kappa\,\Pi_{\mathfrak{k}}=-\Pi_{\mathfrak{k}}$. Since $\sum_{\mathfrak{p}}^{+}$ is the set of weights on $\mathfrak{t}^{C}$ of the irreducible $K$-module $(\mathfrak{p}^{C})^{+}$, we have $\kappa\sum_{\mathfrak{p}}^{+}=\sum_{\mathfrak{p}}^{+}$ and $\kappa\gamma_1=\alpha_1$. Put

$$\Delta'=\kappa\Delta=\{\gamma_1',\cdots,\gamma_p'\},\qquad \gamma_i'=\kappa\gamma_i\ (1\leqslant i\leqslant p),\ \gamma_1'=\alpha_1\,.$$

It is the original maxiaml strongly orthogonal subsystem of $\sum_{\mathfrak{p}}^{+}$ of Harish-Chandra [3]. For the system $\Delta'$, the orthogonal projection

$$\varpi'\colon \sqrt{-1}\mathfrak{t}\to\sqrt{-1}\mathfrak{a}'$$

onto the $\boldsymbol{R}$-span $\sqrt{-1}\mathfrak{a}'$ of $\Delta'$ is defined in the same way as for $\Delta$. Put

$$\begin{aligned}
\boldsymbol{P}_1'&=\{\alpha\in\textstyle\sum_{\mathfrak{p}}^{+};\ \varpi'(\alpha)=\tfrac{1}{2}(\gamma_i'+\gamma_j')\text{ for some }1\leqslant i\leqslant j\leqslant p\}\,,\\
\boldsymbol{P}_{\frac{1}{2}}'&=\{\alpha\in\textstyle\sum_{\mathfrak{p}}^{+};\ \varpi'(\alpha)=\tfrac{1}{2}\gamma_i'\text{ for some }1\leqslant i\leqslant p\}\,,\\
\boldsymbol{K}_0'&=\{\alpha\in\textstyle\sum_{\mathfrak{k}};\ \varpi'(\alpha)=\tfrac{1}{2}(\gamma_i'-\gamma_j')\text{ for some }1\leqslant i,j\leqslant p\}\,,\\
\boldsymbol{K}_{\frac{1}{2}}'&=\{\alpha\in\textstyle\sum_{\mathfrak{k}};\ \varpi'(\alpha)=\tfrac{1}{2}\gamma_i'\text{ for some }1\leqslant i\leqslant p\}\,.
\end{aligned}$$

Then (Harish-Chandra [3]) $\sum$ is the disjoint union of $\boldsymbol{P}_1'$, $-\boldsymbol{P}_1'$, $\boldsymbol{P}_{\frac{1}{2}}'$, $-\boldsymbol{P}_{\frac{1}{2}}'$, $\boldsymbol{K}_0'$, $\boldsymbol{K}_{\frac{1}{2}}'$, $-\boldsymbol{K}_{\frac{1}{2}}'$ and we have

$$\begin{aligned}
\varpi'\boldsymbol{P}_1'&=\{\tfrac{1}{2}(\gamma_i'+\gamma_j');\ 1\leqslant i\leqslant i\leqslant p\}\,,\\
\varpi'\boldsymbol{P}_{\frac{1}{2}}'&=\{\tfrac{1}{2}\gamma_i';\ 1\leqslant i\leqslant p\}\qquad\text{if}\quad \boldsymbol{P}_{\frac{1}{2}}'\neq\phi\,,\\
\varpi'\boldsymbol{K}_0'-\{0\}&=\{\pm\tfrac{1}{2}(\gamma_i'-\gamma_j');\ 1\leqslant i<j\leqslant p\}\,,\\
\varpi'\boldsymbol{K}_{\frac{1}{2}}'&=\{\tfrac{1}{2}\gamma_i';\ 1\leqslant i\leqslant p\}\quad\text{if}\quad \boldsymbol{P}_{\frac{1}{2}}'\neq\phi\,.
\end{aligned}$$

Furthermore the multiplicity (with respect to $\varpi'$) of any $\gamma_i'$ is 1 and that of any $\frac{1}{2}\gamma_i'$ is even. It follows that

$$\varpi'\textstyle\sum-\{0\}=\begin{cases}\{\pm\tfrac{1}{2}(\gamma_i'\pm\gamma_j');\ 1\leqslant i<i\leqslant p,\ \pm\gamma_i;\ 1\leqslant i\leqslant p\}\ \text{if}\ \boldsymbol{P}_{\frac{1}{2}}'=\phi\\ \{\pm\tfrac{1}{2}(\gamma_i'\pm\gamma_j');\ 1\leqslant i<i\leqslant p,\ \pm\gamma_i',\ \pm\tfrac{1}{2}\gamma_i';\ 1\leqslant i\leqslant p\}\ \text{if}\ \boldsymbol{P}_{\frac{1}{2}}'\neq\phi.\end{cases}$$

Moreover we have (Moore [8])

$$\varpi'\,\Pi-\{0\}=\begin{cases}\{\gamma_1',\ \tfrac{1}{2}(\gamma_2'-\gamma_1'),\cdots,\ \tfrac{1}{2}(\gamma_p'-\gamma_{p-1}')\}\qquad\text{if}\quad \boldsymbol{P}_{\frac{1}{2}}'=\phi\\ \{\gamma_1',\ \tfrac{1}{2}(\gamma_2'-\gamma_1'),\cdots,\ \tfrac{1}{2}(\gamma_p'-\gamma_{p-1}'),\ -\tfrac{1}{2}\gamma_p'\}\qquad\text{if}\quad \boldsymbol{P}_{\frac{1}{2}}'\neq\phi\,,\end{cases}$$

and

$$\varpi'\Pi_{\mathfrak{k}}-\{0\}=\begin{cases}\{\frac{1}{2}(\gamma_2'-\gamma_1'),\cdots,\frac{1}{2}(\gamma_p'-\gamma_{p-1}')\} & \text{if } \boldsymbol{P}_{\frac{1}{2}}'=\phi\\ \{\frac{1}{2}(\gamma_2'-\gamma_1'),\cdots,\frac{1}{2}(\gamma_p'-\gamma_{p-1}'),\ -\frac{1}{2}\gamma_p'\} & \text{if } \boldsymbol{P}_{\frac{1}{2}}'\neq\phi\,.\end{cases}$$

**Lemma 1.** 1)

$$\varpi\alpha_1=\begin{cases}\gamma_p & \text{if } \boldsymbol{P}_{\frac{1}{2}}'=\phi\\ \frac{1}{2}\gamma_p & \text{if } \boldsymbol{P}_{\frac{1}{2}}'\neq\phi\,.\end{cases}$$

2) (Schmid [11]) *If* $\boldsymbol{P}_{\frac{1}{2}}'\neq\phi$ *and*

$$\sum_{\beta\in\boldsymbol{P}_{\frac{1}{2}}'} m_\beta\beta \qquad \text{with} \quad m_\beta\geqslant 0$$

*is in the* $\boldsymbol{R}$*-span* $\{\boldsymbol{P}_1'\}_{\boldsymbol{R}}$ *of* $\boldsymbol{P}_1'$, *then* $m_\beta=0$ *for any* $\beta$.

Proof. For any $\alpha\in\sum_{\mathfrak{p}}^+=\boldsymbol{P}_1'\cup\boldsymbol{P}_{\frac{1}{2}}'$, $\varpi'\alpha$ can be writen as

$$\begin{aligned}\varpi'\alpha &= \tfrac{1}{2}m_1(\gamma_2'-\gamma_1')+\tfrac{1}{2}m_2(\gamma_3'-\gamma_2')+\cdots+\tfrac{1}{2}m_{p-1}(\gamma_p'-\gamma_{p-1}')\\ &\quad -\tfrac{1}{2}m_p\gamma_p'+m_{p+1}\gamma_1'\\ &= \tfrac{1}{2}(2m_{p+1}-m_1)\gamma_1'+\tfrac{1}{2}(m_1-m_2)\gamma_2'+\cdots+\tfrac{1}{2}(m_{p-2}-m_{p-1})\gamma_{p-1}'\\ &\quad +\tfrac{1}{2}(m_{p-1}-m_p)\gamma_p'\end{aligned}$$

where $m_i\in\boldsymbol{Z}$, $m_i\geqslant 0$, $m_{p+1}=1$. Since $\varpi'\alpha=\frac{1}{2}(\gamma_i'+\gamma_j')$ or $\frac{1}{2}\gamma_i'$ for some $i$, $j$, we have

$$2\geqslant m_1\geqslant m_2\geqslant\cdots\geqslant m_{p-1}\geqslant m_p\geqslant 0\,.$$

Furthermore $\alpha\in\boldsymbol{P}_1'$ (resp. $\alpha\in\boldsymbol{P}_{\frac{1}{2}}'$) if and only if $m_p=0$ (resp. $m_p=1$).

1) If $\boldsymbol{P}_{\frac{1}{2}}'=\phi$, then $\gamma_1\in\boldsymbol{P}_1'$. For $\alpha=\gamma_1$, the coefficients in the above expression are $m_1=\cdots=m_{p-1}=2$, $m_p=0$ and $\varpi'\gamma_1=\gamma_p'$. If $\boldsymbol{P}_{\frac{1}{2}}'\neq\phi$, then for $\alpha=\gamma_1$, the coefficients are $m_1=\cdots=m_{p-1}=2$, $m_p=1$ and $\varpi'\gamma_1=\frac{1}{2}\gamma_p'$. Now the assertion 1) follows from $\varpi\alpha_1=\kappa^{-1}\varpi'\kappa\alpha_1=\kappa^{-1}\varpi'\gamma_1$.

2) Let

$$\alpha=\sum_{i=1}^{l}n_i\alpha_i \quad \text{with} \quad n_i\in\boldsymbol{Z},\ n_i\geqslant 0$$

be in $\sum_{\mathfrak{p}}^+$. It follows from the first argument that

(a) if $\alpha\in\boldsymbol{P}_1'$, $\varpi'\alpha_i=-\frac{1}{2}\gamma_p'$, then $n_i=0$,

(b) if $\alpha\in\boldsymbol{P}_{\frac{1}{2}}'$, then there exists $\alpha_i\in\Pi_{\mathfrak{k}}$ such that $n_i>0$ and $\varpi'\alpha_i=-\frac{1}{2}\gamma_p'$.
This implies the assertion 2). q.e.d.

Now $\boldsymbol{P}_1$, $\boldsymbol{P}_{\frac{1}{2}}$, $\boldsymbol{K}_0$ and $\boldsymbol{K}_{\frac{1}{2}}$ are defined for $\Delta$ in the same way as for $\Delta'$. Then $\kappa$ transforms $\boldsymbol{P}_1$ (resp. $\boldsymbol{P}_{\frac{1}{2}}$, $\boldsymbol{K}_0$, $\boldsymbol{K}_{\frac{1}{2}}$) onto $\boldsymbol{P}_1'$ (resp. $\boldsymbol{P}_{\frac{1}{2}}'$, $\boldsymbol{K}_0'$, $\boldsymbol{K}_{\frac{1}{2}}'$). It follows that the above mentioned properties due to Harish-Chandra are also satisfied by our objects for $\Delta$. But Moore's results should be modified as follows.

$$\varpi\Pi-\{0\}=\begin{cases}\{\frac{1}{2}(\gamma_1-\gamma_2),\cdots,\frac{1}{2}(\gamma_{p-1}-\gamma_p),\gamma_p\} & \text{if } \boldsymbol{P}_{\frac{1}{2}}=\phi\\ \{\frac{1}{2}(\gamma_1-\gamma_2),\cdots,\frac{1}{2}(\gamma_{p-1}-\gamma_p),\frac{1}{2}\gamma_p\} & \text{if } \boldsymbol{P}_{\frac{1}{2}}\neq\phi.\end{cases}$$

$$\varpi\Pi_{\mathfrak{k}}-\{0\}=\begin{cases}\{\frac{1}{2}(\gamma_1-\gamma_2),\cdots,\frac{1}{2}(\gamma_{p-1}-\gamma_p)\} & \text{if } \boldsymbol{P}_{\frac{1}{2}}=\phi\\ \{\frac{1}{2}(\gamma_1-\gamma_2),\cdots,\frac{1}{2}(\gamma_{p-1}-\gamma_p),\frac{1}{2}\gamma_p\} & \text{if } \boldsymbol{P}_{\frac{1}{2}}\neq\phi.\end{cases}$$

They follows from Lemma 1, 1) and

$$\varpi\Pi_{\mathfrak{k}}=\kappa^{-1}\varpi'\kappa\Pi_{\mathfrak{k}}=-\kappa^{-1}\varpi'\Pi_{\mathfrak{k}}\,.$$

Note that $\boldsymbol{K}_{\frac{1}{2}}\subset\sum_{\mathfrak{k}}^{+}$ while $\boldsymbol{K}_{\frac{1}{2}}'\subset-\sum_{\mathfrak{k}}^{+}$.

**Lemma 2.** 1) *The order $>$ is a compatible order for $\sum$ with respect to $\sigma$ in the sense of* **2**.

2) $\varpi\boldsymbol{K}_0-\{0\}$ *is a root system with the fundamental root system*

$$\{\tfrac{1}{2}(\gamma_1-\gamma_2),\cdots,\tfrac{1}{2}(\gamma_{p-1}-\gamma_p)\}$$

*with respect to the order* $>$.

3) *If* $\boldsymbol{P}_{\frac{1}{2}}\neq\phi$ *and*

$$\sum_{\beta\in\boldsymbol{P}_{\frac{1}{2}}}m_\beta\beta\quad\text{with}\quad m_\beta\geqslant 0$$

*is in the* $\boldsymbol{R}$*-span* $\{\boldsymbol{P}_1\}_{\boldsymbol{R}}$ *of* $\boldsymbol{P}_1$, *then* $m_\beta=0$ *for any* $\beta$.

Proof. 1) is clear from the form of $\varpi\Pi-\{0\}$ above.

2) is clear since

$$\varpi\boldsymbol{K}_0-\{0\}=\{\pm\tfrac{1}{2}(\gamma_i-\gamma_j);\,1\leqslant i<j\leqslant p\}\,.$$

3) follows from Lemma 1, 2) and $\kappa\boldsymbol{P}_{\frac{1}{2}}=\boldsymbol{P}_{\frac{1}{2}}'$, $\kappa\boldsymbol{P}_1=\boldsymbol{P}_1'$. q.e.d.

For $\lambda\in\sqrt{-1}\mathfrak{t}$, $\lambda\neq 0$,we define as in **2**

$$\lambda^*=\frac{2}{(\lambda,\lambda)}\lambda$$

and put

$$Z_0 = \tfrac{1}{2}\sum_{\beta\in\Delta} \gamma^* .$$

Since $(\frac{1}{2}\gamma_i, \gamma_j^*)=\delta_{ij}$ for $1\leqslant i, j\leqslant p$, we have

$$\begin{aligned} \boldsymbol{P}_1 &= \{\alpha\in\textstyle\sum_{\mathfrak{p}}; (\alpha, Z_0) = 1\} , \\ \boldsymbol{P}_{\frac{1}{2}} &= \{\alpha\in\textstyle\sum_{\mathfrak{p}}; (\alpha, Z_0) = \tfrac{1}{2}\} , \\ \boldsymbol{K}_0 &= \{\alpha\in\textstyle\sum_{\mathfrak{k}}; (\alpha, Z_0) = 0\} , \\ \boldsymbol{K}_{\frac{1}{2}} &= \{\alpha\in\textstyle\sum_{\mathfrak{k}}; (\alpha, Z_0) = \tfrac{1}{2}\} . \end{aligned}$$

Hence eigenvalues of ad $Z_0$ are $\pm 1$, $\pm\frac{1}{2}$ on $\mathfrak{p}^C$, 0, $\pm\frac{1}{2}$ on $\mathfrak{k}^C$. Let $\mathfrak{p}^C_{\pm 1}$, $\mathfrak{p}^C_{\pm\frac{1}{2}}$, $\mathfrak{k}^C_0$, $\mathfrak{k}^C_{\pm\frac{1}{2}}$ denote the corresponding eigenspaces. Note that the origin $X_0$ of the Shilov boundary $S$ is in $\mathfrak{p}^C_{-1}$.

The following results are due to Korányi-Wolf [7]. We define an element $c$ of $G^C$, which is called *Cayley transform*, by

$$c = \exp\left(-\frac{\pi}{4}\sum_{\gamma\in\Delta}(X_\gamma+X_{-\gamma})\right)$$

and define an automorphism of $G^C$ by

$$\theta(x) = c^2 x c^{-2} \qquad \text{for} \quad x\in G^C .$$

The automorphism $\mathrm{Ad}\, c^2$ of $\mathfrak{g}^C$ obtained by differentiating $\theta$ will be also denoted by the same letter $\theta$. Then $\theta^4=1$ and on $\sqrt{-1}\,\mathfrak{t}$ it coincides with $-\sigma$. Put

$$\begin{aligned} \mathfrak{g}_0 &= \{X\in\mathfrak{g};\, \theta^2 X = X\} , \\ \mathfrak{k}_0 &= \mathfrak{g}_0\cap\mathfrak{k} , \end{aligned}$$

and

$$\mathfrak{p}_0 = \mathfrak{g}_0\cap\mathfrak{p} .$$

Then $\mathfrak{k}_0$ is $\theta$-invariant and

$$\mathfrak{k}_0 = \{X\in\mathfrak{k};\, [Z_0, X] = 0\} .$$

Hence $\mathfrak{k}_0$ is a real form of $\mathfrak{k}^C_0$ containing $\mathfrak{t}$ as a maximal abelian subalgebra. $\boldsymbol{K}_0$ is nothing but the set of roots of $\mathfrak{k}^C_0$ with respect to $\mathfrak{t}^C$. The complexification $\mathfrak{p}^C_0$ of $\mathfrak{p}_0$ is the direct sum of $\mathfrak{p}^C_{+1}$ and $\mathfrak{p}^C_{-1}$. $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$ with a Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0+\mathfrak{p}_0 .$$

Let $G_0$ (resp. $K_0$) be the connected subgroup of $G$ generated by $\mathfrak{g}_0$ (resp. by $\mathfrak{k}_0$) and let

$$L_0 = \{k \in K_0;\ \mathrm{Ad}k\, X_0 = X_0\} = K_0 \cap L .$$

Put

$$D_0 = D \cap \mathfrak{p}^C_{-1}$$

and

$$S_0 = S \cap \mathfrak{p}^C_{-1} .$$

Then $G_0$ acts on $D_0$ transitively and $K \cap G_0$ coincides with $K_0$, so that $D_0$ is identified with the quotient space $G_0/K_0$. Furthermore $K_0$ acts on $S_0$ transitively so that $S_0$ is identified with $K_0/L_0$. $D_0$ is totally geodesic in $D$ with respect to Bergmann metric of $D$ and it is also an irreducible symmeric bounded domain with the same rank as $D$. $S_0$ is the Shilov boundary of $D_0$. The complex structure of $D_0$ is given at the origin by $\mathrm{ad}\, H_0$ with $\sqrt{-1}\, H_0 = Z_0$. We have

$$\varpi Z = Z_0 .$$

The inclusion $D_0 \subset \mathfrak{p}^C_{-1}$ is nothing but the Harish-Chandra's imbedding of $D_0 = G_0/K_0$. $(K_0, L_0)$ is a symmetric pair with respect to $\theta$, having the same rank as $D$. Hence

$$\mathfrak{l}_0 = \{X \in \mathfrak{k}_0;\ \theta X = X\}$$

is the Lie algebra of $L_0$ and $\mathfrak{a}$ is a maximal abelian subalgebra of

$$\mathfrak{s}_0 = \{X \in \mathfrak{k}_0;\ \theta X = -X\} .$$

We can define a semi-linear transformation $X \mapsto \bar{X}$ of $\mathfrak{p}^C_{-1}$ by

$$\bar{X} = \tau\theta X = \theta\tau X \qquad \text{for} \quad X \in \mathfrak{p}^C_{-1} .$$

Put

$$\mathfrak{p}_{-1} = \{X \in \mathfrak{p}^C_{-1};\ \bar{X} = X\} .$$

It is a real form of $\mathfrak{p}^C_{-1}$ and is invariant under the adjoint action of $L_0$ on $\mathfrak{p}^C_{-1}$. The correspondence $X \mapsto [X, X_0]$ gives an isomorphism

$$\psi\colon \sqrt{-1}\,\mathfrak{s}_0 \to \mathfrak{p}_{-1} ,$$

which is equivariant with respect to the adjoint actions of $L_0$.

Now we shall consider the polynomial representation $S^*((\mathfrak{p}^C)^-)$ of $K$. Let $S_*((\mathfrak{p}^C)^+)$ be the symmetric algebra over $(\mathfrak{p}^C)^+$. $K$ acts on $S_*((\mathfrak{p}^C)^+)$ by the natural extension Ad of the adjoint action of $K$ on $(\mathfrak{p}^C)^+$. On the other hand, the non-degenerate pairing

$$(\mathfrak{p}^C)^+ \times (\mathfrak{p}^C)^- \to \boldsymbol{C}$$

by means of the Killing form ( , ) induces the identification

$$S_*((\mathfrak{p}^C)^+) = S^*((\mathfrak{p}^C)^-)\,.$$

This identification is compatible with the actions of $K$, since the Killing form is invariant under the adjoint action of $K$. In the same way we have a $K_0$-equivariant identification

$$S_*(\mathfrak{p}^C_{+1}) = S^*(\mathfrak{p}^C_{-1})\,.$$

$S_*(\mathfrak{p}^C_{+1})$ can be considered as a $K_0$-submodule of $S_*((\mathfrak{p}^C)^+)$ by means of the natural monomorphism $S_*(\mathfrak{p}^C_{+1}) \to S_*((\mathfrak{p}^C)^+)$ induced from the inclusion $\mathfrak{p}^C_{+1} \subset (\mathfrak{p}^C)^+$.

**Theorem 3.1.** (i) *Any irreducible K-submodule of $S_*((\mathfrak{p}^C)^+)$ (resp. $K_0$-submodule of $S_*(\mathfrak{p}^C_{+1})$) is contained exactly once in $S_*((\mathfrak{p}^C)^+)$ (resp. in $S_*(\mathfrak{p}^C_{+1})$).*

(ii) *For an irreducible K-submodule V of $S_*((\mathfrak{p}^C)^+)$, we put*

$$V_0 = V \cap S_*(\mathfrak{p}^C_{+1})\,.$$

*Then $V \mapsto V_0$ is the one to one correspondence between the set of irreducible K-submodules of $S_*((\mathfrak{p}^C)^+)$ and the set of irreducible $K_0$-submodules of $S_*(\mathfrak{p}^C_{+1})$ in such a way that*

1) *The highest weights on $\mathfrak{t}^C$ of V and $V_0$ are the same.*
2) *The subspace of L-invariants in V is 1-dimensional and contained in $V_0$.*

(iii) *The highest weight $\lambda \in \sqrt{-1}\,\mathfrak{t}$ of an irreducible K-submodule V of $S_*((\mathfrak{p}^C)^+)$ is of the form*

$$\lambda = \sum_{i=1}^{p} n_i \gamma_i\,, \qquad n_i \in \boldsymbol{Z},\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_p \geqslant 0\,.$$

*If $\sum_i n_i = \nu$, then V is contained in $S_\nu((\mathfrak{p}^C)^+)$. i.e. $\mathcal{S}^\nu(D) \subset \mathcal{S}^\nu(K, L)$ under the notation in Introduction.*

For the proof of the theorem, we need the following

**Lemma 3.** (Murakami [9]) *Let $\mathfrak{k}$ be a Lie algebra over $\boldsymbol{R}$ and $\mathfrak{k}^C$ the complexification of $\mathfrak{k}$. Assume that there exists $Y \in \sqrt{-1}\,\mathfrak{k} \subset \mathfrak{k}^C$ such that $\mathfrak{k}^C$ is the direct sum of 0-eigenspace $\mathfrak{k}^C_0$, (+1)-eigenspace $\mathfrak{k}^C_+$ and (−1)-eigenspace $\mathfrak{k}^C_-$ of $\mathrm{ad}\,Y$, respectively. Let $(\rho, V)$ be a complex irreducible $\mathfrak{k}$-module with $\mathfrak{k}$-invariant hermitian inner product. Denoting the extension to $\mathfrak{k}^C$ of $\rho$ by the same letter $\rho$, let $a_1 > a_2 > \cdots > a_m$ ($a_i \in \boldsymbol{R}$) be eigenvalues of $\rho(Y)$, and $S_t$ be $a_t$-eigenspace of $\rho(Y)$ ($1 \leqslant t \leqslant m$). Put $\mathfrak{k}_0 = \mathfrak{k}^C_0 \cap \mathfrak{k}$ ( , which is a real form of $\mathfrak{k}^C_0$). Then*

1) *$a_t = a_1 - t + 1$ ($1 \leqslant t \leqslant m$).*
2) *Each $S_t$ is a $\mathfrak{k}_0$-submodule of V and*

$$V = S_1 + \cdots + S_m$$

*is the orthogonal direct sum.*

3) $S_1$ *and* $S_m$ *are irreducible* $\mathfrak{k}_0$*-submodules of* $V$ *and characterized by*

$$S_1 = \{v \in V;\ \rho(X)v = 0 \quad \text{for any } X \in \mathfrak{k}_+^C\}\ ,$$
$$S_m = \{v \in V;\ \rho(X)v = 0 \quad \text{for any } X \in \mathfrak{k}_-^C\}\ .$$

Proof of Theorem 3.1. The infinitesimal action of $\mathfrak{k}^C$ on $S_*((\mathfrak{p}^C)^+)$ induced from the adjoint action Ad of $K$ will be denoted by ad.

Let $V$ be an irreducible $K$-submodule of $S_*((\mathfrak{p}^C)^+)$. Since $Z$ is in the center of $\mathfrak{k}^C$, it follows from Schur's lemma that $V$ is contained in an eigenspace of $\mathrm{ad}\, Z$ in $S_*((\mathfrak{p}^C)^+)$. But since $\mathrm{ad}\, Z$ is the scalor operator $\nu$ on $S_\nu((\mathfrak{p}^C)^+)$, $V$ is contained in $S_\nu((\mathfrak{p}^C)^+)$ for some $\nu$. Let $\lambda \in \sqrt{-1}\,\mathfrak{t}$ be the highest weight of $V$. Put $Y = 2Z_0 \in \sqrt{-1}\,\mathfrak{k} \subset \mathfrak{k}^C$. Then the decomposition

$$\mathfrak{k}^C = \mathfrak{k}_0^C + \mathfrak{k}_{\frac{1}{2}}^C + \mathfrak{k}_{-\frac{1}{2}}^C$$

satisfies the assumption in Lemma 3. So we have a decomposition

$$V = S_1 + \cdots + S_m$$

into $K_0$-submodules, where $S_1$ is an irrducible $K_0$-submodule and is the eigenspace for the maximum eigenvalue of ad $Y$ in $V$. It is characterized by

$$S_1 = \{v \in V;\ \mathrm{ad}\,(X)v = 0 \quad \text{for any } X \in \mathfrak{k}_{\frac{1}{2}}^C\}\ .$$

Thus a highest weight vector $v_\lambda$ of the $K$-module $V$ is contained in $S_1$ because of $\boldsymbol{K}_{\frac{1}{2}} \subset \sum_{\mathfrak{k}}{}^{+}$. It follows that putting $V_0 = S_1$, $V_0$ is an irreducible $K_0$-submodule of $S_\nu((\mathfrak{p}^C)^+)$ with the highest weight $\lambda$.

We shall show that $V_0 = V \cap S_*(\mathfrak{p}_{+1}^C)$. We have the decomposition

$$S_\nu((\mathfrak{p}^C)^+) = \sum_{r+s=\nu} S_r(\mathfrak{p}_1^C) \otimes S_s(\mathfrak{p}_{\frac{1}{2}}^C)$$

as $K_0$-modules. ad $Z_0$ is the scalor operator $r + \frac{1}{2}s = \frac{1}{2}(r+\nu)$ on $S_r(\mathfrak{p}_1^C) \otimes S_s(\mathfrak{p}_{\frac{1}{2}}^C)$. In the same way as the first argument, we can get the decompsotion

$$V = V_1 + \cdots + V_k$$

into irreducible $K_0$-submodules such that any $V_i$ is contained in $S_r(\mathfrak{p}_1^C) \otimes S_s(\mathfrak{p}_{\frac{1}{2}}^C)$ for some $(r, s)$. Since $S^*((\mathfrak{p}^C)^-)$ is $K$-isomorphic with $S^*(S) \subset C^\infty(S)$, $V$ has an $L$-invariant $w \neq 0$. Decompose $w$ as

$$w = w_1 + \cdots + w_k, \qquad w_i \in V_i\ (1 \leqslant i \leqslant k)\ .$$

At least one of the $w_i$'s, say $w_1$, is not zero. Let $\lambda_1 \in \sqrt{-1}\,\mathfrak{t}$ be the highest weight of the irreducible $K_0$-module $V_1$. Since $w_1$ is a non-zero $L_0$-invariant of $V_1$, $V_1$ is a spherical $K_0$-module relative to $L_0$. $(K_0, L_0)$ is a symmetric pair, $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{s}_0$ and the order $>$ on $\sqrt{-1}\,\mathfrak{t}$ is a compatible order for $\boldsymbol{K}_0$ with respect to $\sigma$ by Lemma 1, 1), so we shall use the notations

$\Gamma(K_0, L_0)$, $Z(K_0, L_0)$, $D(K_0, L_0)$ in 2. Then it follows from Theorem 2.3 that $\lambda_1 \in D(K_0, L_0)$. On the other hand, if $V_1 \subset S_r(\mathfrak{p}_1^C) \otimes S_s(\mathfrak{p}_{\frac{1}{2}}^C)$, $\lambda_1$ is of the form

$$\lambda_1 = \sum_{\alpha \in P_1} m_\alpha \alpha + \sum_{\beta \in P_{\frac{1}{2}}} m_\beta \beta, \qquad m_\alpha, m_\beta \in \boldsymbol{Z}, m_\alpha \geqslant 0, m_\beta \geqslant 0$$

with $\sum m_\alpha = r$, $\sum m_\beta = s$. Since $D(K_0, L_0) \subset \sqrt{-1}\,\mathfrak{a} = \{\Delta\}_R \subset \{P_1\}_R$, we have

$$\sum_{\beta \in P_{\frac{1}{2}}} m_\beta \beta \in \{P_1\}_R .$$

It follows from Lemma 2,3) that $r = \nu$, $s = 0$, i.e. $V_1 \subset V \cap S_\nu(\mathfrak{p}_1^C)$. On the other hand, $V \cap S_\nu(\mathfrak{p}_1^C) \subset V_0$ since the possible maximum eigenvalue of $\operatorname{ad} Y$ on $V$ is $2\nu$. Thus we have that $V_0 = V_1 = V \cap S_\nu(\mathfrak{p}^C)$.

The above argument shows also that any $L$-invariant in $V$ is contained in $V_0$. It is unique up to scalor since $(K_0, L_0)$ is a symmetric pair.

Conversely, let $V_0$ be an irreducible $K_0$-submodule of $S_*(\mathfrak{p}_{+1}^C)$ with the highest weight $\lambda \in \sqrt{-1}\,\mathfrak{t}$. In the same way as the first argument, we know that $V_0$ is contained in $S_\nu(\mathfrak{p}_{+1}^C)$ for some $\nu$. Let $v_\lambda \in V_0$ be a highest weight vector. Then $\operatorname{ad} \mathfrak{k}_{\frac{1}{2}}^C v_\lambda = \{0\}$ because of $[\mathfrak{k}_{\frac{1}{2}}^C, \mathfrak{p}_{+1}^C] = \{0\}$. Hence $\operatorname{ad} X_\alpha v_\lambda = 0$ for any $\alpha \in \sum_{\mathfrak{k}}^+$. We define $V$ to be the $\boldsymbol{C}$-span of $\{\operatorname{Ad} k\, v_\lambda; k \in K\}$ in $S_\nu((\mathfrak{p}^C)^+)$. Then $V$ is an irreducible $K$-submodule of $S_*((\mathfrak{p}^C)^+)$ with the highest weight $\lambda \in \sqrt{-1}\,\mathfrak{t}$.

It is easy to see that each of the above correspondences $V \mapsto V_0$ and $V_0 \mapsto V$ is the inverse of the other. This proves assertions (i) and (ii).

(iii) We have $[\frac{1}{2}\gamma_i^*, X_{-\gamma_j}] = -\delta_{ij} X_{-\gamma_j}$ $(1 \leqslant i, j \leqslant p)$ because of $(\frac{1}{2}\gamma_i^*, \gamma_i) = \delta_{ij}$ $(1 \leqslant i, j \leqslant p)$. It follows that for $H = 2\pi\sqrt{-1} \sum_{i=1}^{p} x_i (\frac{1}{2}\gamma_i^*) \in \mathfrak{a}$ we have

$$\operatorname{Ad}(\exp H) X_0 = -\sum_{i=1}^{p} \exp(-2\pi\sqrt{-1}\, x_i) X_{-\gamma_i} .$$

Thus we have

$$\Gamma(K_0, L_0) = 2\pi\sqrt{-1} \sum_{i=1}^{p} \boldsymbol{Z}(\tfrac{1}{2}\gamma_i^*)$$

and

$$Z(K_0, L_0) = \sum_{i=1}^{p} \boldsymbol{Z}\gamma_i .$$

It follows from Lemma 2,2) that

$$D(K_0, L_0) = \{\sum_{i=1}^{p} n_i \gamma_i;\ n_i \in \boldsymbol{Z},\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_p\} .$$

Therefore $\lambda$ is of the form

$$\lambda = \sum_{i=1}^{p} n_i \gamma_i \qquad \text{with} \quad n_i \in \boldsymbol{Z},\ n_1 \geqslant \cdots \geqslant n_p .$$

On the other hand, $\lambda$ is of the form

$$\lambda = \sum_{\alpha \in P_1} m_\alpha \alpha \qquad \text{with} \quad m_\alpha \in \boldsymbol{Z},\ m_\alpha \geqslant 0\,,$$

which implies that $n_1 \geqslant \cdots \geqslant n_p \geqslant 0$. If $V \subset S_\nu((\mathfrak{p}^C)^+)$, then $V_0 \subset S_\nu(\mathfrak{p}^C_{+1})$ and $\mathrm{ad}\, Z_0$ is the scalor operator $\nu$ on $V_0$, which equals $(\lambda, Z_0) = \sum_{i=1}^{p} n_i$. q.e.d.

REMARK. In terms of polynomial functions $S^*((\mathfrak{p}^C)^-)$, for an irreducible $K$-submodule $V$ of $S^*((\mathfrak{p}^C)^-)$, $V_0$ is obtained by restriction to $\mathfrak{p}^C_{-1}$ of functions in $V$.

Proof of Theorem A. Orthogonality relations for the $S^*_\lambda(D)$'s (resp. for the $S^*_\lambda(S)$'s) and the assertion that the restriction $S^*_\lambda(D) \to S^*_\lambda(S)$ is a similitude follow from Schur's lemma. So it suffices to show that the cardinalities of $\mathcal{S}^\nu(D)$ and $\mathcal{S}^\nu(K, L)$ are the same.

From the first argument in the proof of Theorem 3.1 (iii), we see that $\psi(\frac{1}{2}\gamma^*) = X_{-\gamma}\,(\gamma \in \Delta)$ for the $L_0$-equivariant isomorphism $\psi\colon \sqrt{-1}\,\mathfrak{s}_0 \to \mathfrak{p}_{-1}$. We put

$$\mathfrak{a}^- = \psi(\sqrt{-1}\,\mathfrak{a}) = \{X_{-\gamma};\ \gamma \in \Delta\}_{\boldsymbol{R}} \subset \mathfrak{p}_{-1}\,.$$

Since the Weyl group $W_{S_0}$ of $S_0$ is isomorphic with the group of permutations of $\Delta$ by Lemma 2,2), the "Weyl group" $W_{S_0^-} = N_{L_0}(\mathfrak{a}^-)/Z_{L_0}(\mathfrak{a}^-)$, where $N_{L_0}(\mathfrak{a}^-)$ (resp. $Z_{L_0}(\mathfrak{a}^-)$) is the normalizer (resp. centralizer) of $\mathfrak{a}^-$ in $L_0$, is isomorphic with the group of permutations of $\{X_{-\gamma};\ \gamma \in \Delta\}$. On the other hand, since $S^*_{L_0}(\mathfrak{s}_0)$ is isomorphic with $S^*_{W_{S_0}}(\mathfrak{a})$ by Theorem 2.2, $S^*_{L_0}(\mathfrak{p}_{-1})$ is isomorphic with $S^*_{W_{S_0}^-}(\mathfrak{a}^-)$. Hence $S^*_{L_0}(\mathfrak{p}^C_{-1})$ is isomorphic with $S^*_{W_{S_0}^-}((\mathfrak{a}^-)^C)$. It follows from Theorem 3.1, (ii), 2) that the cardinality of $\mathcal{S}^\nu(D)$ is equal to $\dim S^\nu_{L_0}(\mathfrak{p}^C_{-1})$ $= \dim S^\nu_{W_{S_0}^-}((\mathfrak{a}^-)^C) =$ the number of linearly independent symmetric polynomials in $p$-variables with degree $\nu$, which is known to be the cardianlity of $\mathcal{S}^\nu(K, L)$.
q.e.d.

## 4. Normalizing factor $h_\lambda$

Let $\hat{A} = \mathrm{Ad} A(X_0)$, denoting by $A$ the connected subgroup of $K_0$ generated by $\mathfrak{a}$. $\hat{A}$ has a natural group structure induced from that of $\mathfrak{a}$. Let

$$T = \{t \in \boldsymbol{C}^*;\ |t| = 1\}$$

be the 1-dimensional torus. Under the identification in Introduction of $(\mathfrak{a}^-)^C$ with $\boldsymbol{C}^p$, $\mathfrak{a}^-$ is identified with $\boldsymbol{R}^p$ and $\hat{A}$ with $T^p$. We see that the latter identification is compatible with group structures and complex conjugations, in view of the expression of $\mathrm{Ad}(\exp H)X_0$ in the proof of Theorem 3.1, (iii). Moreover, under the same identification we have (Moore [8])

$$D \cap \mathfrak{a}^- = \{x \in \boldsymbol{R}^p;\ |x_i| < 1\ \ (1 \leqslant i \leqslant p)\}\,,$$

denoting by $z_i$ $(1 \leqslant i \leqslant p)$ the $i$-th component of $z \in \boldsymbol{C}^p$. By means of this identification we define a measure on $\mathfrak{a}^-$ by

$$dH = dx_1 \cdots dx_p$$

and a function $D(H)$ on $\mathfrak{a}^-$ by

$$D(H) = \prod_{i=1}^{p} (2x_i)x_i^{2s} \prod_{1 \leqslant i < j \leqslant p} ((x_i + x_j)(x_i - x_j))^r \qquad \text{for} \quad H \in \mathfrak{a}^-,$$

where $r$, $2s$ are multiplicities defined in Introduction. Then we have the following

**Lemma 1.** *There exists a constant $c'>0$ such that*

$$\int_D f(X) d\mu(X) = c' \int_{D \cap \mathfrak{a}^-} f(H) |D(H)| dH$$

*for any integrable K-invariant function f on D.*

Proof. It is easy to see that $\mathrm{Ad}\, cH = H$ for any $H \in \mathfrak{b}$ and $\mathrm{Ad}\, c\gamma^* = X_\gamma - X_{-\gamma} \in \mathfrak{p}$ for any $\gamma \in \Delta$. Put

$$\mathfrak{a}^0 = \mathrm{Ad}\, c(\sqrt{-1}\,\mathfrak{a}) = \{X_\gamma - X_{-\gamma};\ \gamma \in \Delta\}_{\boldsymbol{R}},$$
$$\mathfrak{h} = \mathrm{Ad}\, c(\mathfrak{b} \oplus \sqrt{-1}\,\mathfrak{a}) = \mathfrak{b} \oplus \mathfrak{a}^0$$

and

$$\mathfrak{h}_{\boldsymbol{R}} = \sqrt{-1}\,\mathfrak{b} \oplus \mathfrak{a}^0.$$

Then $\mathfrak{a}^0$ is a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}^0$ and $\mathfrak{h}_{\boldsymbol{R}}$ is the real part of the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$. We define lienear forms $h_i$ $(1 \leqslant i \leqslant p)$ on $\mathfrak{a}^0$ by

$$h_i(X_{\gamma_j} - X_{-\gamma_j}) = \delta_{ij} \qquad (1 \leqslant i, j \leqslant p).$$

If $h_i$ is identified with an element of $\mathfrak{a}^0$ by means of the Killing form, we have $\mathrm{Ad}\, c(\frac{1}{2}\gamma_i) = h_i$ $(1 \leqslant i \leqslant p)$. The linear order on $\mathfrak{h}_{\boldsymbol{R}}$ induced by $\mathrm{Ad}\, c$ from the order $>$ on $\sqrt{-1}\,\mathfrak{t}$ is a compatible order for $\mathrm{Ad}\, c\sum$ with respect to the decomposition $\mathfrak{h}_{\boldsymbol{R}} = \sqrt{-1}\,\mathfrak{b} \oplus \mathfrak{a}^0$. This follows from **3**, Lemma 2,1). Thus positive restricted roots on $\mathfrak{a}^0$ of the symmetric space $D = G/K$ are

$$\{h_i \pm h_j;\ 1 \leqslant i < j \leqslant p,\ 2h_i;\ 1 \leqslant i \leqslant p\} \qquad \text{if} \quad \boldsymbol{P}_{\frac{1}{2}} = \phi,$$
$$\{h_i \pm h_j;\ 1 \leqslant i < j \leqslant p,\ 2h_i,\ h_i;\ 1 \leqslant i \leqslant p\} \qquad \text{if} \quad \boldsymbol{P}_{\frac{1}{2}} \neq \phi.$$

The multiplicity of $h_i \pm h_j$ $(1 \leqslant i < j \leqslant p)$, i.e. the number of roots in $\mathrm{Ad}\, c\sum$ projecting to $h_i \pm h_j$, is the same as that of $\frac{1}{2}(\gamma_i \pm \gamma_j)$. Since the Weyl group $W_D$ on $\mathfrak{a}^0$ of $D = G/K$ is generated by reflections with respect to $h_1 - h_2, \cdots, h_{p-1} - h_p, h_p$, hence transitive on the set $\{\pm h_i \pm h_j;\ 1 \leqslant i < j \leqslant p\}$, it follows that multiplicities of these roots are the same $r$. By the same reason, multiplicities of $h_i$ ($1 \leqslant i \leqslant p$) are the same $2s$, which is even from the results of Harish-Chandra mentioned in **3**. In the same way we know that multiplicities of $2h_i$ ($1 \leqslant i \leqslant p$) are 1. Thus the product $D^0$ of positive restricted roots (multiplicity counted) is given by

$$D^0(H^0) = \prod_{i=1}^{p} 2h_i(H^0)h_i(H^0)^{2s} \prod_{1 \leqslant i < j \leqslant p} ((h_i+h_j)(H^0)(h_i-h_j)(H^0))^r \qquad \text{for} \quad H^0 \in \mathfrak{a}^0 .$$

Let $dX$ (resp. $dH^0$) denote the Euclidean measure of $\mathfrak{p}$ (resp. of $\mathfrak{a}^0$) induced from the Killing form $(\ ,\ )$, and $dk$ the normalied Haar measure of $K$. Then (cf. Helgason [4]) under the surjective map $K \times \mathfrak{a}^0 \to \mathfrak{p}$ defined by $(k, H^0) \mapsto \operatorname{Ad} kH^0$, these measures are related as follows:

$$dX = c'' |D^0(H^0)| \, dk \, dH^0 \quad \text{with some constant} \quad c'' > 0 .$$

Now we define a $K$-equivariant $\boldsymbol{R}$-isomorphism $j$: $\mathfrak{p} \to (\mathfrak{p}^C)^-$ by

$$j(X) = \tfrac{1}{2}(X - [Z, X]) \qquad \text{for} \quad X \in \mathfrak{p} .$$

It is easy to see that $j(X_\gamma - X_{-\gamma}) = -X_{-\gamma}$ for any $\gamma \in \Delta$, hence $j\mathfrak{a}^0 = \mathfrak{a}^-$. Since $K$ acts irreducibly on $\mathfrak{p}$, the map $j$ is a similitude with respect to inner products $(\ ,\ )$ and the real part of $(\ ,\ )_\tau$. Therefore under the surjective map $K \times \mathfrak{a}^- \to (\mathfrak{p}^C)^-$ defined by $(k, H) \mapsto \operatorname{Ad} kH$, we have

$$d\mu(X) = c' |D(H)| \, dk \, dH \quad \text{with some constant} \quad c' > 0 .$$

Seeing $\operatorname{Ad} K(D \cap \mathfrak{a}^-) = D$, we get the proof of Lemma 1. q.e.d.

Take a form $\lambda \in \mathcal{S}^*(K, L)$. Choose an orthonormal basis $\{u_i;\ 1 \leqslant i \leqslant d_\lambda\}$ of $S^*_\lambda((\mathfrak{p}^C)^-)$ with respect to $(\ ,\ )_\tau$ such that $\{u_i;\ 1 \leqslant i \leqslant d_{\lambda,0}\}$ spans $S^*_\lambda((\mathfrak{p}^C)^-) \cap S^*(\mathfrak{p}^C_{-1})$ and $u_1$ is $L$-invariant. Put

$$\begin{aligned} \rho^i_j(k) &= (\operatorname{Ad} k u_j, u_i)_\tau && \text{for } k \in K \quad (1 \leqslant i, j \leqslant d_\lambda) , \\ \varphi'_i(k) &= \overline{\rho^i_1(k)} && \text{for } k \in K \quad (1 \leqslant i \leqslant d_\lambda) , \\ f'_i &= \sqrt{d_\lambda}\, \varphi_i' && (1 \leqslant i \leqslant d_\lambda) . \end{aligned}$$

The arguments in **2** show that $\{f_i';\ 1 \leqslant i \leqslant d_\lambda\}$ form an orthonormal basis of $S^*_\lambda(S)$ with respect to $\langle\ ,\ \rangle$ and $\varphi_1'$ is the zonal spherical function $\omega_\lambda$ for $(K, L)$ belonging to $\lambda$, identifying $C^\infty(S)$ with the space of right $L$-invariant $C^\infty$-functions on $K$. The zonal spherical polynomial $\Omega_\lambda$ for $D$ belonging to $\lambda$ defined in Introduction is characterized by that its restriction to $S$ coincides with $\omega_\lambda$. $\Omega_\lambda$ restricted to $\mathfrak{p}^C_{-1}$ is the zonal spherical polynomial for $D_0$ belonging to $\lambda$ and $\omega_\lambda$ restricted to $S_0$ is the zonal spherical function for $(K_0, L_0)$ belonging to $\lambda$. $\Omega_\lambda$

restricted to $(\mathfrak{a}^-)^C$ is a symmetric polynomial since it is $W_{S_0}^-$-invariant. Let $f_i \in S_\lambda^*((\mathfrak{p}^C)^-)$ $(1 \leqslant i \leqslant d_\lambda)$ be the unique polynomial such that its restriction to $S$ is $f_i'$. Then $\{f_i;\ 1 \leqslant i \leqslant d_\lambda\}$ form an orthogonal basis of $S_\lambda^*((\mathfrak{p}^C)^-)$ with respect to $(\ ,\ )_\tau$ such that $\{f_i;\ 1 \leqslant i \leqslant d_{\lambda,0}\}$ form an orthogonal basis of $S_\lambda^*((\mathfrak{p}^C)^-) \cap S^*(\mathfrak{p}_{-1}^C)$. They satisfy relations

$$f_i(\mathrm{Ad}\, k^{-1} X) = \sum_{j=1}^{d_\lambda} \rho_i^j(k) f_j(X) \qquad \text{for} \quad k \in K,\ X \in (\mathfrak{p}^C)^- \ (1 \leqslant i \leqslant d_\lambda)\,.$$

We put

$$\Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} |f_i(X)|^2 \qquad \text{for} \quad X \in (\mathfrak{p}^C)^-\,.$$

Then for any $k \in K$ we have

$$\begin{aligned} \Phi_\lambda(\mathrm{Ad}\, k^{-1} X) &= \frac{1}{d_\lambda} \sum_i \Big(\sum_j \rho_i^j(k) f_j(X)\Big)\Big(\sum_k \overline{\rho_i^k(k)}\, \overline{f_k(X)}\Big) \\ &= \frac{1}{d_\lambda} \sum_{j,k} \Big(\sum_i \rho_i^j(k)\, \overline{\rho_i^k(k)}\Big) f_j(X) \overline{f_k(X)} \\ &= \frac{1}{d_\lambda} \sum_{j,k} \delta_{jk} f_j(X) \overline{f_k(X)} = \Phi_\lambda(X) \qquad \text{for} \quad X \in (\mathfrak{p}^C)^-\,, \end{aligned}$$

i.e. $\Phi_\lambda$ is a $K$-invariant $C^\infty$-function on $(\mathfrak{p}^C)^-$. Note that

$$\Phi_\lambda(X) = \frac{1}{d_\lambda} \sum_{\alpha=1}^{d_{\lambda,0}} |f_\alpha(X)|^2 \qquad \text{for} \quad X \in \mathfrak{p}_{-1}^C\,.$$

**Lemma 2.**

$$h_\lambda = c' \int_{D \cap \mathfrak{a}^-} \Phi_\lambda(H) |D(H)| \, dH$$

Proof.

$$\int_D \Phi_\lambda(X) d\mu(X) = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} \langle\!\langle f_i, f_i \rangle\!\rangle = \frac{1}{d_\lambda} \sum_{i=1}^{d_\lambda} h_\lambda \langle f_i', f_i' \rangle = h_\lambda\,.$$

On the other hand, by Lemma 1 we have

$$\int_D \Phi_\lambda(X) d\mu(X) = c' \int_{D \cap \mathfrak{a}^-} \Phi_\lambda(H) |D(H)| \, dH\,. \qquad \text{q.e.d.}$$

Proof of Theorem B. Making use of the complex conjugation $X \mapsto \bar{X}$ of $\mathfrak{p}_{-1}^C$ defined in **3**, we define $\tilde{\Phi}_\lambda \in S^*(\mathfrak{p}_{-1}^C)$ by

$$\tilde{\Phi}_\lambda(X) = \frac{1}{d_\lambda} \sum_{\alpha=1}^{d_{\lambda,0}} f_\alpha(X) \overline{f_\alpha(\bar{X})} \qquad \text{for} \quad X \in \mathfrak{p}_{-1}^C\,.$$

Then $\tilde{\Phi}_\lambda = \Phi_\lambda$ on $\mathfrak{p}_{-1}$ and we have for any $k \in K_0$

$$\begin{aligned}
\Phi_\lambda(\mathrm{Ad}\,kX_0) &= \frac{1}{d_\lambda}\sum_\alpha f_\alpha(\mathrm{Ad}\,kX_0)\overline{f_\alpha(\overline{\mathrm{Ad}\,kX_0})} \\
&= \frac{1}{d_\lambda}\sum_\alpha f_\alpha(\mathrm{Ad}\,kX_0)\overline{f_\alpha(\mathrm{Ad}\,\theta(k)X_0)} \\
&= \frac{1}{d_\lambda}\sum_\alpha f_\alpha'(k)\overline{f_\alpha'(\theta(k))} = \sum_\alpha \varphi_\alpha'(k)\overline{\varphi_\alpha'(\theta(k))} \\
&= \sum_\alpha \overline{\rho_1^\alpha(k)}\,\rho_1^\alpha(\theta(k)) = \sum_\alpha \overline{\rho_1^\alpha(k)\,\rho_\alpha^1(\theta(k)^{-1})} \\
&= \overline{\rho_1^1(\theta(k)^{-1}k)} = \omega_\lambda(\theta(k)^{-1}k)\,.
\end{aligned}$$

In particular for any $a\in A$

$$\Phi_\lambda(\mathrm{Ad}\,aX_0) = \omega_\lambda(a^2)\,,$$

i.e. for any $\hat{a}\in\hat{A}$

$$\tilde{\Phi}_\lambda(\hat{a}) = \omega_\lambda(\hat{a}^2) = \Omega_\lambda(\hat{a}^2)\,.$$

Since $\hat{A}=T^p$ is a compact real form of $C^{*p}$ and $C^{*p}$ is open in $C^p=(\mathfrak{a}^-)^C$, we have

$$\tilde{\Phi}_\lambda(z_1, \cdots, z_p) = \Omega_\lambda(z_1^2, \cdots, z_p^2) \qquad \text{for any } z\in C^p = (\mathfrak{a}^-)^C\,.$$

By Lemma 2 we have

$$\begin{aligned}
h_\lambda &= c'\int_{D\cap\mathfrak{a}^-} \tilde{\Phi}_\lambda(H)\,|D(H)|\,dH \\
&= c'\int_{|x_i|<1(1\leqslant i\leqslant p)} \Omega_\lambda(x_1^2, \cdots, x_p^2)\,|\prod_{i=1}^{p}(2x_i)x_i^{2s}\prod_{1\leqslant i<j\leqslant p}((x_i+x_j)(x_i-x_j))^r\,|\,dx_1\cdots dx_p \\
&= c(D)\int_{0\leqslant y_i<1\,(1\leqslant i\leqslant p)} \Omega_\lambda(y_1, \cdots, y_p)\,|\prod_{1\leqslant i<j\leqslant p}(y_i-y_j)^r\,|\prod_{i=1}^{p} y_i^s\,dy_1\cdots dy_p
\end{aligned}$$

for some constant $c(D)>0$, which does not depend on $\lambda$. Inparticular, for $\lambda=0$

$$\mu(D) = h_0 = c(D)\int_{0\leqslant y_i<1\,(1\leqslant i\leqslant p)} |\prod_{1\leqslant i<j\leqslant p}(y_i-y_j)^r\,|\prod_{i=1}^{p} y_i^s\,dy_1\cdots dy_p,$$

since $\Omega_0\equiv 1$. This completes the proof of Theorem B. q.e.d.

REMARK. It can be proved that $\tilde{\Phi}_\lambda$ is an $L_0$-invariant polymomial on $\mathfrak{p}_{-1}^C$.

The multiplicities $r$, $s$ are given as follows.

| $D$ | rank $D$ | $r$ | $s$ |
|---|---|---|---|
| $(\mathrm{I})_{p,q}$ $(p\leqslant q)$ | $p$ | 2 | $q-p$ |
| $(\mathrm{II})_n$ | $[n/2]$ | 4 | $\begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$ |
| $(\mathrm{III})_n$ | $n$ | 1 | 0 |
| $(\mathrm{IV})_n$ $(n\geqslant 3)$ | 2 | $n-2$ | 0 |
| (EIII) | 2 | 6 | 4 |
| (EVII) | 3 | 8 | 0 |

The zonal spherical polynomial $\Omega_\lambda$ is given as follows.

For integers $n_1, \cdots, n_p$ we define the *Schur function* $\{n_1, \cdots, n_p\}$ on the $p$-dimensional torus $T^p$ by

$$\{n_1, \cdots, n_p\}(t) = \frac{\det(t_i^{n_j+p-j})_{1\leq i,j\leq p}}{\det(t_i^{p-j})_{1\leq i,j\leq p}} \qquad \text{for} \quad t = \begin{bmatrix} t_1 \\ \vdots \\ t_p \end{bmatrix} \in T^p \subset \boldsymbol{C}^p .$$

$\{n_1, \cdots, n_p\}$ is symmetric in variables $t_1, \cdots, t_p$ and it is a polynomial in $t_1, \cdots, t_p$ if and only if $n_i \geqslant 0$ $(1 \leqslant i \leqslant p)$. For an element $\lambda = \sum_{i=1}^{p} n_i \gamma_i \in \sum_{i=1}^{p} \boldsymbol{Z}\gamma_i = Z(K_0, L_0)$, the $i$-th coefficient $n_i$ will be denoted by $n_i(\lambda)$.
Then we have

**Theorem 4.1.** *The zonal spherical polynomial* $\Omega_\lambda$ *for* $D$ *belonging to* $\lambda \in \mathcal{S}^*(K, L)$ *is determined on* $(\mathfrak{a}^-)^C$ *by the relation*

$$\Omega_\lambda(t) = \sum_{\mu \in D_\lambda} c_\lambda^\mu \{n_1(\mu), \cdots, n_p(\mu)\}(t) \qquad \text{for any } t \in T^p = \hat{A} \subset (\mathfrak{a}^-)^C ,$$

*where the* $c_\lambda^\mu$*'s are coefficients in Theorem* 2.5 *for the symmetric pair* $(K_0, L_0)$.

Proof. As we have seen in the proof of Theorem B, $\Omega_\lambda$ is determined on $(\mathfrak{a}^-)^C$ by

$$\Omega_\lambda(t) = \omega_\lambda(t) \qquad \text{for any} \quad t \in T^p = \hat{A} .$$

By Theorem 2.5, $\omega_\lambda$ has an expression

$$\omega_\lambda(t) = \sum_{\mu \in D_\lambda} c_\lambda^\mu \overline{\chi_\mu(t)} \qquad \text{for} \quad t \in T^p = \hat{A} .$$

Since the Weyl group $W_{S_0}$ acts on $Z(K_0, L_0)$ by the group of permutations of $\gamma_1, \cdots, \gamma_p$, $W_{S_0}$-invariant characters $\chi_\lambda$ of $\hat{A}$ are nothing but Schur functions. As we have seen in the proof of Theorem 3.1, (iii), the $i$-th component of $\mathrm{Ad}(\exp H)X_0 \in T^p = \hat{A}$ is $\exp(-(\gamma_i, H))$ for any $H \in \mathfrak{a}$. It follows that

$$\chi_\mu(t) = \{n_1(\mu), \cdots, n_p(\mu)\}(\bar{t}) \qquad \text{for} \quad t \in T^p = \hat{A} .$$

Hence we have

$$\begin{aligned} \Omega_\lambda(t) &= \sum_{\mu \in D_\lambda} c_\lambda^\mu \overline{\{n_1(\mu), \cdots, n_p(\mu)\}(\bar{t})} \\ &= \sum_{\mu \in D_\lambda} c_\lambda^\mu \{n_1(\mu), \cdots, n_p(\mu)\}(t) \qquad \text{for} \quad t \in T^p = \hat{A} . \end{aligned}$$

q.e.d.

In the case of the domain $D$ of type $(\mathrm{I})_{p,q}$ $(p \leqslant q)$, $S_0$ is the unitary group $U(p)$ of degree $p$. We have in view of Example in **2** that

$$\Omega_\lambda(t) = \frac{1}{d_\lambda} \{n_1(\lambda), \cdots, n_p(\lambda)\}(t) \qquad \text{for} \quad t \in T^p = \hat{A} ,$$

where $d_\lambda$ is the degree of the irreducible representation of $U(p)$ with the signature $(n_1(\lambda), \cdots, n_p(\lambda))$. In the case of the domain $D$ of type $(\mathrm{IV})_n$, $S_0$ is the Lie sphere and $\Omega_\lambda$ can be described in terms of Gegenbauer polynomials, which are zonal spherical functions for the sphere. So our integral formula in Theorem B clarifies the meaning of integrals of Hua [6].

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