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LINEAR $SU(n)$ -ACTIONS ON COMPLEX PROJECTIVE SPACES

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0. Introduction

Let U_* be the bordism ring of weakly complex manifolds and let G be a compact Lie group. Denote by $SF(G)$, an ideal in U_* of those bordism classes represented by a weakly complex manifold on which the group G acts smoothly without stationary points and the action preserves a weakly complex structure.

For a compact abelian Lie group G the ideal $SF(G)$ was computed by tom Dieck [8]. Such ideals are similarly defined in the bordism ring Ω_* of oriented manifolds and those were computed for certain abelian groups by Floyd [3] and Stong [7]. But it seems that there is no useful method to compute the ideal $SF(G)$ for a non-abelian Lie group G .

First we give an upper bound and a lower bound of $SF(G)$ for any compact Lie group G . To state our result precisely we introduce some notations as follows. Denote by $I(G)$, a set of positive integers such that $n \in I(G)$ if and only if there is an n -dimensional complex G -vector space without G -invariant one-dimensional subspaces, by $m(G)$ the maximum dimension of proper closed subgroups of G , and put

$$n(G) = \dim G - m(G).$$

It is known that the bordism ring $U_* = \sum_{k \geq 0} U_{2k}$ is generated by a set of bordism classes

$$\{[P_n(\mathbf{C})], [H_{p,q}(\mathbf{C})]; n \geq 0, p \geq q > 0\}$$

as a ring. Now we define ideals $L(G)$, $M(G)$ in U_* as follows. Let $L(G)$ be an ideal in U_* generated by a set

$$\{[P_n(\mathbf{C})], [H_{m+n,n}(\mathbf{C})]; n+1 \in I(G), m \geq 0\}$$

and let

$$M(G) = \sum_{2k \geq n(G)} U_{2k}.$$

Then we have following results,

Theorem 0.1. *For any compact Lie group G ,*

$$L(G) \subset SF(G) \subset M(G).$$

Corollary. $SF(SU(2)) = SF(U(2)) = \sum_{n>0} U_{2n}$.

For each positive integer n , $P_n(\mathbf{C})$ admits a linear $SU(2)$ -action without stationary points, but for example $P_3(\mathbf{C})$ does not admit a linear $SU(3)$ -action without stationary points. Thus we next consider $SU(3)$ -actions on $P_3(\mathbf{C})$ and we have a following result. Denote by $hP_3(\mathbf{C})$, a compact smooth 6-dimensional manifold homotopy equivalent to $P_3(\mathbf{C})$.

Theorem 0.2. (a) *Any smooth $SU(3)$ -action on $hP_3(\mathbf{C})$ has at least one stationary point.* (b) *Any non-trivial smooth $SU(3)$ -action on $hP_3(\mathbf{C})$ is equivariantly diffeomorphic to a linear $SU(3)$ -action on $P_3(\mathbf{C})$.*

1. Weakly complex G -manifolds without stationary points

Let G be a Lie group and V be an n -dimensional complex G -vector space. Denote by $P(V)$ the complex projective space $P_{n-1}(\mathbf{C})$ with an induced G -action. We call such a G -action on $P_{n-1}(\mathbf{C})$ a linear G -action. Then $P(V)$ is a weakly complex G -manifold in the sense of Conner-Floyd [1]. Denote by $[v]$ a point of $P(V)$ represented by a non-zero vector v of V . Then

Lemma 1.1. *A point $[v]$ of G -manifold $P(V)$ is a stationary point if and only if the vector v spans a G -invariant one-dimensional subspace of V .*

Lemma 1.2. *Any smooth G -action on a manifold M is trivial, if $\dim M < n(G) = \dim G - m(G)$. Here $m(G)$ is the maximum dimension of proper closed subgroups of G .*

Proof. If $x \in M$ is not a stationary point, then the isotropy subgroup G_x at x is a proper closed subgroup of G , the orbit $G \cdot x$ is a submanifold of M , and $G \cdot x$ is diffeomorphic to the homogeneous space G/G_x . Then

$$\dim M \geq \dim(G \cdot x) = \dim G - \dim G_x \geq \dim G - m(G).$$

REMARK. The integer $m(G)$ was calculated by Mann [5] for compact connected simple Lie groups G , by making use of Dynkin's work [2].

Proof of Theorem 0.1. Let V be an n -dimensional complex G -vector space and W be an m -dimensional complex G -vector space. The canonical G -action on the dual space $V^* = \text{Hom}_{\mathbf{C}}(V, \mathbf{C})$ is defined by

$$(g \cdot u)(v) = u(g^{-1} \cdot v); g \in G, u \in V^*, v \in V.$$

Define

$$H(V \oplus W, V^*) = \{([v+w], [u]) \in P(V \oplus W) \times P(V^*): u(v)=0\},$$

then $H(V \oplus W, V^*)$ is a manifold $H_{m+n-1, n-1}(\mathbf{C})$ with a weakly complex G -action. If V has no G -invariant one-dimensional subspaces, then the G -action on $H(V \oplus W, V^*)$ has no stationary points by Lemma 1.1. Therefore the inclusion $L(G) \subset SF(G)$ is proved. Next the inclusion $SF(G) \subset M(G)$ follows from Lemma 1.2. This completes the proof of Theorem 0.1.

Next we consider the case for $G = SU(n)$, the special unitary group. Let $I(G)$ be the set of positive integers defined in the introduction. Then by definition

$$(1.3) \quad n_1, n_2 \in I(G) \text{ implies } n_1 + n_2 \in I(G).$$

Lemma 1.4. *Any binomial coefficient $\binom{n+k-1}{k}$ is contained in $I(SU(n))$ for $n \geq 2$ and $k \geq 1$.*

Proof. Denote by V_n the complex vector space \mathbf{C}^n with the standard $SU(n)$ -action. Then the k -th symmetric product $S_k(V_n)$ is irreducible as a complex $SU(n)$ -vector space for each $k \geq 1$ and

$$\dim_{\mathbf{C}} S_k(V_n) = \binom{n+k-1}{k}.$$

Corollary 1.5. $SF(SU(2)) = SF(U(2)) = \sum_{n>0} U_{2n}$.

Proof. Since $I(SU(2)) = I(U(2))$ consists all positive integers $n \geq 2$ by Lemma 1.4,

$$L(SU(2)) = L(U(2)) = \sum_{n>0} U_{2n}.$$

On the other hand,

$$M(SU(2)) = M(U(2)) = \sum_{n>0} U_{2n}$$

by the connectivity of $SU(2)$ and $U(2)$.

2. $SU(3)$ -actions on $P_3(\mathbf{C})$

Let us first recall some basic facts in differentiable transformation groups.

(i) Let G be a compact Lie group acting on a manifold M . Then by averaging an arbitrary given Riemannian metric on M , we may have a G -invariant Riemannian metric on M .

(ii) Let $x \in M$, then the isotropy subgroup G_x acts on a normal vector space N_x of the orbit $G \cdot x$ at x orthogonally; we call it the normal representation of G_x at x and denote by ρ_x .

(iii) (The differentiable slice theorem) Let $E(\nu)$ be the normal bundle of the orbit $G \cdot x = G/G_x$. Then

$$E(\nu) = G \times_{G_x} N_x$$

where G_x acts on N_x via ρ_x . We note that G acts naturally on $E(\nu)$ as bundle mappings and we may choose small positive real number ε such that the exponential mapping gives an equivariant diffeomorphism of the ε -disk bundle of $E(\nu)$ onto an invariant tubular neighborhood of $G \cdot x$. ([6], Lemma 3.1)

(iv) Let $H \subset G$ be a closed subgroup. Denote by (H) , the set of all subgroups of G which are conjugate to H in G . We introduce the following partial ordering relation " $<$ " by defining $(H_1) < (H_2)$ if and only if there exist $H_1 \in (H_1)$ and $H_2 \in (H_2)$ such that $H_1 \subset H_2$. If M is connected, then there exists an absolute minimal (H) among the conjugate classes in $\{G_x | x \in M\}$, moreover the set

$$M_{(H)} = \{x \in M | G_x \in (H)\}$$

is a dense open submanifold. The conjugate class (H) is called the type of principal isotropy subgroups. ([6], (2.2) and (2.4))

Combining (iii) and (iv), we have a following lemma.

Lemma 2.1. *If M is connected, then the normal representation of G_x at $x \in M$ is trivial if and only if G_x is a principal isotropy subgroup.*

Now we consider $SU(3)$ -actions. Let H be a closed subgroup of $SU(3)$. Denote by $N(H)$ the normalizer of H in $SU(3)$.

Lemma 2.2. (a) *Let H be a closed connected proper subgroup of $SU(3)$ with $\dim H \geq 3$, then H is conjugate to $SU(2)$, $SO(3)$ or $N(SU(2))$. (b) There are isomorphisms, $N(SU(2))/SU(2) \cong S^1$, the circle group; $N(SO(3))/SO(3) \cong Z_3$, the cyclic group of order 3; $N(N(SU(2))) = N(SU(2))$, as the subgroups of $SU(3)$. (c) $N(SU(2))$ does not contain any subgroup which is conjugate to $SO(3)$.*

Proof. (a) is proved by considering the structure of Lie algebra of $SU(3)$ and the 3-dimensional unitary representations of $SU(2)$. (b) is proved by direct calculation. (c) is true since $N(SU(2)) \subset SU(3)$ is not irreducible but $SO(3) \subset SU(3)$ is irreducible.

REMARK. $\dim SU(3) = 8$ and $\dim SU(2) = \dim SO(3) = 3$.

Lemma 2.3. *Let M be an orientable connected 6-dimensional manifold with smooth $SU(3)$ -action. If an isotropy subgroup $SU(3)_x$ is of 3-dimensional, then $SU(3)_x$ is a principal isotropy subgroup.*

Proof. First we may prove that the homogeneous space $SU(3)/SU(3)_x$ is an orientable 5-manifold by Lemma 2.2. Thus the normal bundle $E(\nu)$ of

$SU(3)/SU(3)_x$ is a trivial line bundle, since M and $SU(3)/SU(3)_x$ are orientable. But if the normal representation of $SU(3)_x$ at $x \in M$ is non-trivial, then the normal bundle $E(\nu)$ is non-orientable. This is a contradiction. Therefore the result follows from Lemma 2.1.

Now we consider non-trivial smooth $SU(3)$ -actions on $hP_3(\mathbf{C})$, a compact 6-dimensional manifold with the homotopy type of $P_3(\mathbf{C})$.

Lemma 2.4. (a) *Any isotropy subgroup is of dimension ≥ 3 .* (b) *$hP_3(\mathbf{C})$ does not admit only one type (H) of isotropy subgroups for any proper subgroup H of $SU(3)$.*

Proof. If $\dim SU(3)_x \leq 1$, then the 6-dimensional manifold $hP_3(\mathbf{C})$ contains a submanifold $SU(3)/SU(3)_x$ of dimension ≥ 7 . This is a contradiction. Next if $\dim SU(3)_x = 2$, then $SU(3)/SU(3)_x$ is an open and closed submanifold of $hP_3(\mathbf{C})$. Therefore

$$hP_3(\mathbf{C}) = SU(3)/SU(3)_x.$$

By an exact sequence of homotopy groups

$$\pi_2(SU(3)) \rightarrow \pi_2(SU(3)/SU(3)_x) \rightarrow \pi_1(SU(3)_x) \rightarrow \pi_1(SU(3)),$$

we obtain $\pi_1(SU(3)_x) = Z$, an infinite cyclic group, since $SU(3)$ is 2-connected. On the other hand, since $\dim SU(3)_x = 2$, the identity component of $SU(3)_x$ is isomorphic to a 2-dimensional toral group, and hence $\pi_1(SU(3)_x) = Z \oplus Z$. This is a contradiction. Next we prove (b). It is sufficient to consider the case

$$\dim H = 3 \text{ or } 4,$$

by (a) and Lemma 2.2. If $hP_3(\mathbf{C})$ admits only one type (H) of isotropy subgroups, then there is a differentiable fibering

$$SU(3)/H \rightarrow h_3P(\mathbf{C}) \xrightarrow{p} h_3P(\mathbf{C})/SU(3),$$

and the orbit space $hP_3(\mathbf{C})/SU(3)$ is a compact manifold without boundary, by the differentiable slice theorem (iii). First if $\dim H = 3$, then the orbit space is of one-dimensional and hence

$$hP_3(\mathbf{C})/SU(3) = S^1.$$

By exact sequences

$$\begin{aligned} \pi_3(S^1) &\rightarrow \pi_2(SU(3)/H) \rightarrow \pi_2(hP_3(\mathbf{C})) \xrightarrow{p_*} \pi_2(S^1), \\ \pi_2(SU(3)) &\rightarrow \pi_2(SU(3)/H) \rightarrow \pi_1(H) \rightarrow \pi_1(SU(3)), \end{aligned}$$

we obtain $\pi_1(H) = Z$. On the other hand $\pi_1(H) = 0$ or Z_2 , since $\pi_1(SU(2)) = 0$

and $\pi_1(SO(3))=Z_2$. This is a contradiction. Next if $\dim H=4$, then H is conjugate to $N(SU(2))$ and the orbit space $hP_3(\mathbf{C})/SU(3)$ is of 2-dimensional. Since

$$SU(3)/N(SU(2))=P_2(\mathbf{C}),$$

there is an exact sequence

$$\pi_1(hP_3(\mathbf{C})) \rightarrow \pi_1(hP_3(\mathbf{C})/SU(3)) \rightarrow \pi_0(P_2(\mathbf{C})).$$

Thus the orbit space is a simply connected 2-dimensional compact manifold without boundary. Therefore

$$hP_3(\mathbf{C})/SU(3) = S^2.$$

Then there is a contradiction in the following exact sequence

$$\pi_4(hP_3(\mathbf{C})) \rightarrow \pi_4(hP_3(\mathbf{C})/SU(3)) \rightarrow \pi_3(P_2(\mathbf{C})),$$

since $\pi_4(S^2)=Z_2$.

REMARK 2.5. By the above consideration, if there is a smooth $SU(3)$ -action on $hP_3(\mathbf{C})$ without stationary points, then $hP_3(\mathbf{C})$ admits just two types (H) and ($N(SU(2))$) of isotropy subgroups, where the identity component of H is $SU(2)$.

Proof of Theorem 0.2 (a). If there is a smooth $SU(3)$ -action on $hP_3(\mathbf{C})$ with just two types (H) and ($N(SU(2))$) of isotropy subgroups, where the identity component of H is $SU(2)$, then $hP_3(\mathbf{C})$ is a special $SU(3)$ -manifold in the sense of Hirzebruch-Mayer [4]. Therefore the orbit space $hP_3(\mathbf{C})/SU(3)$ is a compact smooth manifold with boundary, and hence

$$hP_3(\mathbf{C})/SU(3) = [0,1].$$

Let $p:hP_3(\mathbf{C})\rightarrow[0,1]$ be a projection and

$$X_0 = p^{-1}\left(\left[0, \frac{1}{2}\right]\right), X_1 = p^{-1}\left(\left[\frac{1}{2}, 1\right]\right).$$

Then X_0 and X_1 are diffeomorphic to the disk bundle of n -fold tensor product of the canonical complex line bundle over $P_2(\mathbf{C})$ for certain positive integer n , by the differentiable slice theorem (iii). Therefore $X_0 \cap X_1$ is a 5-dimensional rational homology sphere. Then there is a contradiction in the following exact sequence of cohomology groups with rational coefficients,

$$H^1(X_0 \cap X_1) \rightarrow H^2(hP_3(\mathbf{C})) \rightarrow H^2(X_0) \oplus H^2(X_1) \rightarrow H^2(X_0 \cap X_1).$$

Therefore any smooth $SU(3)$ -action on $hP_3(\mathbf{C})$ has at least one stationary point, by Remark 2.5.

Lemma 2.6. *Consider a non-trivial smooth $SU(3)$ -action on a connected 6-dimensional manifold M . Let $x \in M$ be a stationary point. Then the normal representation $SU(3) \rightarrow O(6)$ is equivalent to the standard inclusion $SU(3) \subset O(6)$, and $SU(2)$ is a principal isotropy subgroup.*

Proof. This follows from the fact that non-trivial 6-dimensional real representation of $SU(3)$ is isomorphic to the real restriction of the standard 3-dimensional complex representation.

REMARK 2.7. Denote by V_3 , the 3-dimensional complex vector space \mathbb{C}^3 with the standard $SU(3)$ -action. Then $P(\mathbb{C}^1 \oplus V_3)$ is the complex projective space $P_3(\mathbb{C})$ with a non-trivial linear $SU(3)$ -action, where the $SU(3)$ -action on \mathbb{C}^1 is trivial. Denote by D^6 the unit disk in V_3 . Then there is an equivariant decomposition

$$P(\mathbb{C}^1 \oplus V_3) = (SU(3) \times_{N(SU(2))} D^2) \cup_h D^6,$$

where the $N(SU(2))$ -action on D^2 is induced from the standard action of $N(SU(2))/SU(2) = S^1$ on D^2 and h is an equivariant diffeomorphism on boundaries.

Lemma 2.8. *Any equivariant diffeomorphism on ∂D^6 is extendable to an equivariant diffeomorphism on D^6 .*

Proof. Since the $SU(3)$ -action on ∂D^6 is transitive, it is easy to prove that any equivariant diffeomorphism on ∂D^6 is given by a scalar multiplication

$$(z_1, z_2, z_3) \rightarrow (uz_1, uz_2, uz_3),$$

where $(z_1, z_2, z_3) \in \partial D^6$, $u \in \mathbb{C}$ and $|u| = 1$. Such a diffeomorphism is canonically extended to an equivariant diffeomorphism on D^6 .

Proof of Theorem 0.2 (b). Let $hP_3(\mathbb{C})$ admit a non-trivial smooth $SU(3)$ -action. Then we can use Lemma 2.6, via Theorem 0.2 (a). Thus $SU(2)$ is a principal isotropy subgroup, and hence the possible types of isotropy subgroups are

$$(SU(2)), (N(SU(2))) \text{ and } (SU(3)),$$

by Lemma 2.2 and Lemma 2.3. In any case, $hP_3(\mathbb{C})$ becomes a special $SU(3)$ -manifold with the orbit space $[0,1]$. If the type $(N(SU(2)))$ does not appear, then $hP_3(\mathbb{C})$ is diffeomorphic to $D^6 \cup D^6$. This is a contradiction. Therefore $hP_3(\mathbb{C})$ has isotropy subgroups of type $(N(SU(2)))$ and of type $(SU(3))$. Hence, by the differentiable slice theorem (iii), there is an equivariant decomposition

$$hP_3(\mathbb{C}) = (SU(3) \times_{N(SU(2))} D^2) \cup_h D^6,$$

where k is an equivariant diffeomorphism on boundaries. Moreover there is an equivariant diffeomorphism from $hP_3(\mathbf{C})$ to $P(\mathbf{C}^1 \oplus V_3)$, by making use of Lemma 2.8 and Remark 2.7.

3. Concluding remarks

3.1. If $G=T^n$, the n -dimensional toral group, then it is known that for any smooth G -action on an oriented compact manifold M without boundary, each connected component of the stationary point set M^G is canonically oriented and the index formula

$$I(M) = I(M^G)$$

holds. Thus we ask whether the above is true or not when G is a compact connected Lie group. The answer is no as follows. Denote by $S_k(V_n)$ the k -th symmetric product of V_n which is \mathbf{C}^n with the standard $SU(n)$ -action. If $n \geq 2$ and $n-1 < 2^a$, then

$$t = \dim_{\mathbf{C}} S_{2^a}(V_n)$$

is odd, and there is a linear $SU(n)$ -action on $P_{s+t}(\mathbf{C})$ with $P_s(\mathbf{C})$ as the stationary point set for each integer s . This example shows that the index formula is false for $SU(n)$ -actions in general. Similarly we can construct linear $SO(n)$ -actions on $P_{s+t}(\mathbf{R})$ with $P_s(\mathbf{R})$ as the stationary point set. This example shows that there are smooth $SO(n)$ -actions for which the stationary point sets are not orientable.

3.2. Let V_n be as above, then $SU(n)$ -manifold $P(\mathbf{C}^1 \oplus V_n)$ has only one stationary point for each $n \geq 2$. Such a phenomenon does not appear for compact G -manifold without boundary when G is an abelian group such as a toral group or a finite cyclic group of prime order.

3.3. Let G be a compact Lie group. Denote by F_A the family of all closed subgroups of G , and by F_P the family of all closed proper subgroups of G . Then there is an exact sequence of bordism modules of weakly complex G -manifolds,

$$\cdots \rightarrow U_*(G; F_P) \xrightarrow{i_*} U_*(G; F_A) \xrightarrow{j_*} U_*(G; F_A, F_P) \xrightarrow{\partial_*} U_*(G; F_P) \rightarrow \cdots$$

It is known that i_* is trivial for $G=T^n$ and almost trivial for G a finite cyclic group of prime order. On the other hand, we can prove that i_* is injective when G is a compact connected semi-simple Lie group, by making use of projective space bundles associated to complex G -vector bundles.

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