ACTIONS OF LINEAR ALGEBRAIC GROUPS
ON PROJECTIVE MANIFOLDS
AND MINIMAL MODEL PROGRAM

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0. Introduction.

Let G be a connected linear algebraic complex group which acts regularly and non trivially on a smooth connected projective complex variety X of dimension n.

In this paper we consider the following question: how does the G-action influence or even determine the structure of X?

As it is stand this is a too general question, thus we will soon add some suitable assumptions; however even in this generality we notice that X is not minimal in the sense of Minimal Model Program (MMP). In particular X admits an extremal ray and an associated extremal (or Fano-Mori) contraction, $\varphi : X \longrightarrow Z$, which turns out to be G-equivariant. It is thus natural to use the tools developed by the MMP, and the good properties of the map φ, to get a classification of such varieties X. This idea was first developed in an important paper by Mukai and Umemura (see [20]), where they studied smooth projective 3-folds on which $G = SL(2)$ acts with a dense orbit. (A complete classification of such quasi-homogeneous 3-folds is contained in a paper of Nakano (see [22]); we refer the reader also to a more recent work by S. Kebekus where the case of singular 3-folds is considered (see [13])).

Note that if X is actually homogeneous with respect to G-action, then X is a Fano manifold and X can be classified in terms of Dynkin diagrams. Fano manifolds are basic blocks of the MMP and moreover in this case there is a beautiful interplay between the representation theory of G and the projective or differential geometry of X.

We want to propose a way to attack the general problem; however, to our knowledge, this way works effectively only in the case when G is a simple group, i.e. the simply connected Lie group associated to a simple Lie algebra. In this case one can in fact perform many computations which seems hard or meaningless otherwise (for instance find the minimal non trivial irreducible representation).

Thus we will also assume that G is a (simply connected) simple Lie group and we will define r_G to be the minimum of the dimension of the homogeneous variety of the group G. That is, r_G is the minimum codimension of the maximal parabolic

subgroup of G (i.e. parabolic subgroup corresponding to one node of the Dynkin diagram). It is easy to calculate r_G if G is simple and this is done in Section 1 (for instance $r_{SL(m)} = m - 1$).

Then we first prove that if $r_G > n$ then there is no such an X, that is, the only possible regular action is the trivial one, while if $r_G = n$ then X is homogeneous. For instance if $n = 3$ this says in particular that the only classical group acting non trivially on a smooth 3-fold are $SL(2)$, $SL(3)$, $SL(4)$, $Sp(4) \simeq Spin(5)$, $SO(4)$ and in the last 3 cases X is homogeneous; this special case was first proved in a paper of T. Nakano (see [21]) which influenced the setup of this paper.

Then we classify all X in the case $r_G = n-1$ (see Theorem 4.1) via the MMP. The special case in which $G = SL(n)$ was obtained first by T. Mabuchi but in a complete different way. Namely he started with the classification of n-codimensional closed subgroups of $SL(n)$, which follows from Dynkin's work, and consequently he discussed the possible completions of their quotient.

Finally we begin to consider the case $r_G = n - 2$; this is much more difficult and it seems reasonable to make an additional general assumption. Namely to assume that X has an open dense orbit; such an X is called a quasi-homogeneous manifold. As remarked above the case with $n = 3$ and $G = SL(2)$ was studied in [20] and [22] while the case with $n = 4$ and $G = SL(3)$ was recently settled by Nakano [23] with the method of computing the closed subgroups of codimension 4 in $SL(3)$. In the present paper, as a test for the MMP, we try to recover this classification; it turns out that the program works easily until the last step, namely the case of Fano manifolds with Picard number one. This requires further investigations; however we believe that once this case is solved, also for the other classical groups and in all dimensions, it will be possible to find a complete classification also for $r_G = n - 2$.

At the beginning I was very much inspired by the papers of Mukai-Umemura, Mabuchi and Nakano which are quoted in the references; after writing a first draft of the paper I came across a beautiful paper of D.N. Ahiezer ([1]) which contains technical tools which simplify many of my original arguments in Section 2.

This note was initiated during my visit at the University of Utah in the fall of 1997. J. Kollár suggested me to investigate in this direction and provided some very useful hints; I like to thank him for all this. I also thank E. Ballico, P. Moeseneder and J. A. Wiśniewski for helpful discussions on this topic.

1. Definitions and preliminaries.

In this paper X will always denote a smooth connected projective variety of dimension n. We use the standard notation from algebraic geometry; more precisely for the Minimal Model Program our notation is compatible with that of [12] while for the Group Action and Representation Theory it is compatible with that of [9].

Definition 1.1.   A smooth projective variety X will be said minimal (in the sense of the MMP) if K_X is nef.

Theorem 1.2 (Mori-Kawamata-Shokurov). *Let X be a smooth variety which is not minimal. Then there exists a map $\varphi : X \longrightarrow Z$ into a normal projective variety Z with connected fibers such that $-K_X$ is φ-ample and φ contracts the set of curves numerically equivalent to a (rational) curve in a non trivial fiber.*

Definition 1.3.   The map $\varphi : X \longrightarrow Z$ given in the above theorem is called an extremal contraction or a Fano-Mori contraction.

Lemma 1.4.   *Let X be a smooth projective manifold on which a connected linear algebraic complex group G acts regularly and non trivially. Then X is uniruled and in particular it is not minimal.*

Proof.   On the generic point the action is not trivial, hence it is contained in an orbit which is unirational since G is rational. Thus the generic point is contained in a rational curve of X. Therefore X is uniruled and not minimal (for this last statement see for instance [14], chapter IV, more precisely 1.3 and 1.9). □

Lemma 1.5.   *Let X and G be as in the previous* Lemma 1.4. *Then there exists a Fano-Mori contraction $\varphi : X \longrightarrow Z$ which is G-equivariant and G acts regularly on Z.*

Proof.   The existence of φ follows from the Lemma 1.4 and the above Mori-Kawamata-Shokurov Theorem 1.2. The equivariance of φ follows from the following two facts: on one end two curves which are carried one to another by the action of G are numerically equivalent, on the other end φ contracts all and only the set of curves in a ray, i.e. a set of curves all numerically equivalent to a (rational) curve in a non trivial fiber. Therefore take two points in a fiber and a curve passing through these two points; this curve will be carried into another curve by the action of G which is numerically equivalent to the first one and therefore it is contained in a fiber.

Let L be an ample line bundle on Z. Then some positive power ϕ^*L^n can be G-linearized, that is, the action of G on X extends to an action on the total space of ϕ^*L^n which is linear on fibers. Since $Z = \text{Proj}(\oplus_{m=0}^{\infty}H^0(X, \phi^*L^{mn}))$, G acts regularly on Z through its actions on the L^{mn}'s. □

Definition 1.6.   If the action of G is transitive on X then X is called a *homogeneous manifold*. If X has a dense open orbit then it is called a *quasi-homogeneous manifold*.

Remark 1.7. If X is homogeneous then T_X is generated by global sections and $-K_X$ is ample (see for instance [14], (v.1.4)); in particular X is a rational Fano manifold. If X is quasi-homogeneous then $-K_X$ is effective; this follows easily taking n elements of the Lie algebra $\mathrm{Lie}(G)$ such that their associated vector fields are linearly independent at a generic point of X. The wedge product of these vector fields gives a non trivial holomorphic section of $-K_X$.

Definition 1.8. Let us fix a simple (or even semisimple), simply connected and connected Lie group G and consider the set of all homogeneous manifolds (of dimension > 0) with respect to this group. They are in a direct correspondence with the parabolic subgroups of G (the isotropy subgroup in one point) which are in turn in direct correspondence with the subsets of the nodes of the Dynkin diagram associated to the group G. We define $r = r_G$ to be the minimal of the dimensions of the manifolds in this set, or equivalently, the *minimal codimension of parabolic subgroups of G*. A homogeneous variety which attains this minimum will be called a *minimal homogeneous variety* for the action of G. The minimal codimension will be attained at a maximal parabolic subgroup, i.e. one corresponding to a single node of the Dynkin diagram.

Example 1.0.1. It is easy to check that if $G = SL(m)$ or $\mathrm{Sp}(2s) = \mathrm{Sp}(m)$ and $s \geq 3$ then $r_G = m - 1$. If $G = SL(m)$ the parabolic subgroup P is the one corresponding to the first (or the last) node of the Dynkin diagram A_m; if $G = \mathrm{Sp}(2s)$ then P is the one corresponding to the first node of the Dynkin diagram C_s. In both cases $G/P = \mathbf{P}^{m-1}$ where G acts through a linear action on $\mathbf{C}^m$, the standard irreducible representation or its dual in the $SL(m)$ case (these are called the standard homogeneous actions).

Also if $G = \mathrm{Sp}(4)$ then $r_G = 3$ but in this case we have two different parabolic groups of codimension 3 which are the subgroup P_1 corresponding to the first node and P_2 corresponding to the second one in the Dynkin diagram; in this case $\mathrm{Sp}(4)/P_1 = \mathbf{Q}^3$ and $\mathrm{Sp}(4)/P_2 = \mathbf{P}^3$.
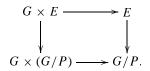
Note that $\mathrm{Spin}(5) \simeq \mathrm{Sp}(4)$ and $\mathrm{Spin}(6) \simeq SL(4)$; thus when we consider the group $G = \mathrm{Spin}(m)$ we will always assume that $m \geq 7$.

If $G = \mathrm{Spin}(m)$ and $m \geq 7$ then $r_G = m - 2$. If $m \neq 8$ the parabolic subgroup P is the one corresponding to the first node of the Dynkin diagrams $B_{(m-1/2)}$ or $D_{(m/2)}$, depending on the cases where m is odd or even and $G/P \simeq \mathbf{Q}^{(m-2)} \subset \mathbf{P}^{(m-1)}$. If $G = \mathrm{Spin}(8)$ in principle we will have two minimal homogeneous varieties (spinor varieties) of dimension 6 (corresponding to each of the two last nodes) but they are both isomorphic to $\mathbf{Q}^6$.

If G is an exceptional group we have the following values for r_G: $r_{G_2} = 5$, $r_{F_4} = 15$, $r_{E_6} = 16$, $r_{E_7} = 27$, $r_{E_8} = 57$. The corresponding minimal homogeneous varieties are not always easy to describe as above. In particular if $G = G_2$ we have two of them,

one being a quadric hypersurface in $\mathbf{P}^6$, the other being described for instance at p. 392 of [9]. If $G = E_6$ then the minimal homogeneous manifold is the fourth Severi variety in the theorem of Zak (see [15] for more details). If $G = F_4$ then we have two of them, one being an hyperplane sections of the above Severi variety (see p. 47 of [15]). If $G = E_7$ or E_8 the parabolic subgroups correspond to the last node of the Dynkin diagrams.

DEFINITION 1.9.   Let $X = G/P$ be an homogeneous variety where G is a simply connected simple group and P is a parabolic subgroup. A vector bundle $E \longrightarrow X = G/P$ is called G-homogeneous or simply homogeneous if there exists an action of G on E such that the following diagram commutes:

$$
\begin{array}{ccc}
G \times E & \longrightarrow & E \\
\downarrow & & \downarrow \\
G \times (G/P) & \longrightarrow & G/P.
\end{array}
$$

REMARK 1.10.   It is evident from the definition that the tangent bundle of X is homogeneous.

One can prove that a vector bundle E on $X = G/P$ is homogeneous if and only if one of the following conditions holds:

i) $\theta_g^* E \simeq E$ for every $g \in G$; θ_g is the automorphism of X given by g.

ii)   There exists a representation $\rho : P \longrightarrow GL(r)$ such that $E \simeq E_\rho$, where E_ρ is the vector bundle with fiber $\mathbf{C}^r$ coming from the principal bundle $G \longrightarrow G/P$ via ρ.

REMARK 1.11.   Let G be a semisimple complex Lie group acting regularly and non trivially on X. If $\pi : \tilde{G} \longrightarrow G$ is the universal covering map of G then it is a finite morphism and hence $\tilde{G}$ acts regularly and non trivially on X through π. Hence we may and shall assume that the acting semisimple group is simply connected without loosing generality.

2.   Points which are fixed by the action of G.

In this section we enlarge slightly our setup: namely we will have an action of a connected and reductive linear algebraic group G on a variety Z with normal singularities. The following result shows how the existence of a fixed point by the action of G determines the structure of Z; the main step, namely that b) implies c), was proved by Ahiezer (see [1] Theorem 3; see also [10] for the analytic case).

In this paper we need only the equivalence between a) and c); for this we could also give a direct proof which doesn't make use of the Ahiezer's result.

Proposition 2.1.   *Suppose that a connected reductive linear algebraic group G acts effectively on a complete normal variety Z. Then the followings are equivalent*:
a)   *There exists a fixed point z such that its projectivized tangent cone, that is the variety $P_z = \mathrm{Proj}(\bigoplus_k m_z^k/m_z^{k+1})$, is a G-homogeneous variety.*
b)   *Z has an open orbit Ω and $A := Z \setminus \Omega$ contains an isolated point z.*
c)   *Z is a projective quasi-homegeneous cone over a homogeneous variety P with respect to G.*

Proof.   By the result of Ahiezer we just need to prove that a) implies b).

In the assumption of a), since z is a fixed point under the action of a reductive group G, there exists a G-stable open affine neighborhood U of z in Z. Let R be the algebra of regular functions on U. Then R has a decreasing filtration by the powers of the ideal of z, and the associated graded ring $\mathrm{gr}(R)$ is the homogeneous coordinate ring of P_z. By assumption, $\mathrm{gr}(R)$ contains no non-constant G-invariant; because G is reductive, R contains no non-constant G-invariant as well. It follows that z is the unique closed G-orbit in U (because invariants separate closed G-invariants subsets in an affine G-variety). In particular, z is contained in the closure of a non-trivial G-orbit. The tangent cone of this orbit and of its closure is G invariant. But, since by assumption P_z is G-homogeneous, this implies that the orbit has dimension equal to the dimension of Z. $\square$

A first application of the above proposition will give the next result.

Lemma 2.2.   *Let X be a smooth projective variety and G a simple, simply connected, connected linear group acting non trivially on X; let r_G be the integer defined in 1.8 and $n = \dim X$. If $n \leq r_G$ there are no fixed points on X. If $r_G = (n-1)$, then X has no fixed points unless $G = SL(n)$ or $\mathrm{Sp}(n = 2s)$, $X = \mathbf{P}^n$ and the action is the one which extends the standard $SL(n)$ or $\mathrm{Sp}(n)$ action on $\mathbf{C}^n$ via the inclusion $\mathbf{C}^n \longrightarrow \mathbf{P}^n, (z_1, \ldots, z_n) \longrightarrow (1, z_1, \ldots, z_n)$ (equivalently the action is induced from the homomorphism $g \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ from $SL(n)$ or $\mathrm{Sp}(n)$ to $PGL(n+1)$).*

Proof.   Note first that the dimensions of the irreducible representations of G are strictly bigger then r_G: in fact for every irreducible representation V there is a unique closed orbit in $\mathbf{P}(V)$ which is the homogeneous variety corresponding to the parabolic subgroup perpendicular to the weight of the representation. Moreover if the minimal of such dimension is equal to $r_G + 1$, then $G = SL(m)$ or $\mathrm{Sp}(m)$ and V is the standard representation; in this case the action on $\mathbf{P}(V) = \mathbf{P}^{m-1}$ is homogeneous.

Assume that $r_G \geq n$ and that $x \in X$ is a fixed point; then G acts on the tangent space $T = T_{X,x}$ and by the above observation this has to be the trivial representation. Let m_x be the maximal ideal of $\mathcal{O} = \mathcal{O}_x$, the local ring of germs of regular functions near x; then G acts trivially on $m/m^2 = T^*$ and on $m^k/m^{(k+1)} = S^k(m/m^2)$. Using

inductively the exact sequences

$$0 \longrightarrow m^k/m^{(k+1)} \longrightarrow \mathcal{O}/m^{(k+1)} \longrightarrow \mathcal{O}/m^k \longrightarrow 0$$

and the fact that G is a reductive group we have that G acts on $\mathcal{O}/m^k$ trivially for all $k > 0$. Thus G acts trivially on the completion $\hat{\mathcal{O}}$, hence trivially on $\mathcal{O}$. This implies that G acts trivially on X itself.

After noticing that G acts trivially on T, one can conclude alternatively via the Luna's étale slice theorem as in the next Lemma 2.3.

Assume now that $r = n - 1$ and let $x \in X$ be a fixed point. If $G = \mathrm{Spin}(m)$ ($m \geq 7$) or an exceptional group then the above proof applies; i.e. the action of G on the tangent space at x must be trivial. In the other cases we can apply the Proposition 2.1 (or the Proposition 2.2) since the action of G on $P_x := \mathrm{Proj}[\mathrm{gr}(\mathcal{O}_{X,x})] = \mathbf{P}^{(n-1)}$ is transitive. Thus X is isomorphic to the cone over x; since x is a smooth point $X = \mathbf{P}^n$.

Lemma 2.3. *Let $G = SL(n - 1)$ acting with a dense open orbit on a n-fold X. Then there are no fixed points.*

Proof. If $n = 3$ this is the Lemma 1.2.2 in [20]. Therefore we assume that $n \geq 4$ and that, by contradiction, x is a fixed point. Then we have an induced linear action of G on $T_{X,x}$, i.e. an n-dimensional representation of G. These are of three types, namely if $A \in SL(n - 1)$

$$A \longrightarrow (A, 1), \text{ or } ({}^t A^{-1}, 1), \text{ or } I;$$

in particular there are no n-dimensional orbits on $T_{X,x}$ in any of these there cases.

On the other hand we can apply the Luna's étale slice theorem (see [16]); this says that there exists a G-stable affine subvariety V containing x and an étale G-equivariant morphism $V \longrightarrow T_{X,x}$. This is a contradiction since, by assumption, X has a n-dimensional orbit. $\qquad \square$

Actually the following more general result holds; it was proved for $n = 3$ in [22], here we adapt this proof (or the one of 1.2.2 in [20]) to the general case.

Lemma 2.4. *Let G be any reductive group acting with a dense open orbit on a projective variety Z and assume that x is a fixed point. Then m_x/m_x^2 does not have nonzero invariants.*

Proof. Assume by contradiction that there exists a non-zero invariants $f \in m_x/m_x^2$. Let $U = \mathrm{Spec}(A)$ be a G-invariant affine neighborhood of x. Let $\bar{f}$ be a lifting of f, i.e. $\bar{f} \in \Gamma(U, \mathcal{O}_U)$ is such that $\tau(\bar{f}) = f$ where $\tau : \Gamma(U, \mathcal{O}_U) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_x/m_x$. Let V be a finite dimensional G-invariant vector subspace of A con-

taining f; this exists by Borel [7] (it can be defined as the vector subspace of A generated by $\{g \circ \bar{f} | g \in G\}$ which is of finite dimension). Since $\bar{f}(x) = 0$ we have that $\tau(V) \subset m_x/m_x^2$. The image $\tau(V)$ contains a non zero G-invariant hence V contains a G-invariant. Since V and m_x/m_x^2 are finite dimensional, and G is linearly reductive, the image $\tau(V)$ is a direct summand of V; hence V, in particular A, contains a non zero G-invariant F. Since G has an open orbit the invariant F should be constant. Since its value on x is zero, it is constantly zero which is a contradiction. $\square$

3. A starting point.

Our main goal will be a classification of smooth connected projective varieties with a non trivial action of a simple group G which has the number r_G "big enough" with respect to the dimension of X. The following easy result seems to be a good starting point.

Proposition 3.1. *Let G be a connected simple Lie group acting on a connected projective variety X of dimension n. If the action is not trivial, then $n \geq r_G$; if moreover $r_G = n$, then X is homogeneous. In particular if $G = SL(m)$ or $\mathrm{Sp}(m)$ acts on a connected projective varieties X of dimension $n < m - 1$ then this action is trivial; if $n = m - 1$ then $X = \mathbf{P}^{(m-1)}$ and the action is the standard one apart for the case $G = \mathrm{Sp}(4)$ where we have both $\mathbf{P}^3$ and $\mathbf{Q}^3$ as homogeneous variety of dimension 3. If $G = \mathrm{Spin}(m)$ with $m \geq 7$ acts on a connected projective varieties X of dimension $n < m - 2$ then this action is trivial; if $n = m - 2$ then $X = \mathbf{Q}^{(m-2)}$ and the action is the standard one.*

Proof.   If X contains a non-trivial closed G-orbit, then $n \geq r_G$ with equality if and only if X is homogeneous. Thus we may assume that all closed orbits in X are fixed points; moreover there is at least one fixed point (see for instance [7], 1.8), call it $x \in X$. If X is a smooth variety, then by the first part of the Lemma 2.2 G acts trivially on X. In general if X is singular, replacing X by its normalization, we may assume that X is normal and we can prove that G acts trivially on X by induction on n as it follows. If $n = 1$ the only simple group acting non trivially on a projective curve is $SL(2)$ acting transitively on projective line. If $n > 1$, let $x \in X$ be a fixed point and let U be an open affine G-stable neighborhood of x in X. By the induction hypothesis, G acts trivially on the complement X' of U; because U is affine, each irreducible component of X' has codimension one in X. By normality of X, we can choose $x' \in X'$ which is a smooth point of X and of X'. Then G acts trivially on the tangent space $T_{X',x'}$, a subspace of codimension one in $T_{X,x'}$. Because G is simple, it acts trivially on $T_{X,x'}$; by the same argument used in the proof of 2.2, it acts trivially on X. $\square$

Remark 3.1.1. The special case $n = 3$ of the proposition gives the main theorem of [21].

4. Minimal Model Program on manifolds with a G-action.

In this section we use the notation and the approach of the previous one, passing to the next step; namely we assume that $r_G = n - 1$. We will prove the following theorem, the first part of which was proved in [19] with different methods.

Theorem 4.1. *Let X be a smooth projective manifold of dimension n and G a simple, simply connected and connected Lie group acting non trivially on X.*

If $G = SL(n)$ then X is isomorphic to one of the following varieties; the action of G is unique for each case and it is described in the course of the proof (see also [19]):

1) *the complex projective space $\mathbf{P}^n$,*
2) *$\mathbf{P}^{(n-1)} \times R$, where R is a smooth projective curve,*
3) *The projective bundles $\mathbf{P}(\mathcal{O}_{\mathbf{P}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{P}^{(n-1)}})$ with $m > 0$,*
4) *if $n = 2$ we have an extra action on $\mathbf{P}^1 \times \mathbf{P}^1$ and on $\mathbf{P}^2$,*
5) *if $n = 3$ we have moreover the projective bundle $\mathbf{P}(T_{\mathbf{P}^2})$,*
6) *if $n = 4$ we have moreover the smooth 4-dimensional quadric which is isomorphic to $Gr(2, 4)$, the Grassmannian of 2-planes in $\mathbf{C}^4$.*

If $G = \mathrm{Sp}(n)$ then X is isomorphic to one of the following varieties and the action of G is unique for each case.

1) *the complex projective space $\mathbf{P}^n$,*
2) *$\mathbf{P}^{(n-1)} \times R$, where R is a smooth projective curve,*
3) *The projective bundle $\mathbf{P}(\mathcal{O}_{\mathbf{P}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{P}^{(n-1)}})$ with $m > 0$,*
4) *if $n = 4$ we have moreover $\mathbf{Q}^4$, the homogeneous variety which is the quotient of $\mathrm{Sp}(4)$ by the Borel subgroup (which has two structure of a $\mathbf{P}^1$-bundle over $\mathbf{P}^3$ and over $\mathbf{Q}^3$), $\mathbf{Q}^3 \times R$, where R is a smooth projective curve and the projective bundles $\mathbf{P}(\mathcal{O}_{\mathbf{Q}^3}(m) \oplus \mathcal{O}_{\mathbf{Q}^3})$ with $m > 0$.*

If $G = \mathrm{Spin}(n + 1)$ with $n \geq 6$ then X is isomorphic to one of the following varieties and the action of G is unique for each case.

1) *the complex projective space $\mathbf{P}^n$,*
2) *the complex projective quadric $\mathbf{Q}^n \subset \mathbf{P}^{(n+1)}$,*
3) *$\mathbf{Q}^{(n-1)} \times R$, where R is a smooth projective curve,*
4) *The projective bundle $\mathbf{P}(\mathcal{O}_{\mathbf{Q}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{Q}^{(n-1)}})$ with $m > 0$.*

Proof. The proof of the theorem will be reached in a number of steps which are similar for all the three groups.

Lemma 4.2. *Let X and Y two manifolds on which a simple group G acts in the hypothesis of the theorem (i.e. $= r_G + 1 = \dim X = \dim Y$). Assume that X and Y have*

each a dense open orbit which are G isomorphic, then $X \simeq Y$ unless $G = SL(n)$ or $\mathrm{Sp}(n)$ *and* $Y = \mathbf{P}^n$, $X = \mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O})$.

Proof (See also the last part of the proof of 2.2). Since both X and Y are completion of the same open dense orbit there is a birational map $f : Y \: - -- > X$ induced by identifying the orbit. If $Y = \mathbf{P}^n$ let us consider the blow-up of the fixed point $\sigma : Y' \longrightarrow Y$ and take instead of f the composition $g = f \circ \sigma$. This map is defined in codimension 1, since both X and Y has minimal closed orbits of codimension 1 and no fixed point (see 2.2), thus it is an isomorphism. $\square$

Let us now run the Minimal Model Program to classify X; in the following $\rho(X)$ will denote the Picard number of X.

1-st Step. Assume that $\rho(X) \geq 2$ and let $\varphi : X \longrightarrow Z$ be the contraction of an extremal ray (which exists by Lemma 1.5).

a) If φ is birational then, by the G-equivariant property of φ and our assumption on r, it must be divisorial and the divisor has to be contracted to a point. Moreover the exceptional divisor E is isomorphic to $\mathbf{P}^{(n-1)}$, respectively to $\mathbf{Q}^{(n-1)}$; here the two cases depends on whether $G = SL(n)$, $\mathrm{Sp}(n = 2s)$ or if $G = \mathrm{Spin}(n+1)$, $n \geq 6$, unless $G = \mathrm{Sp}(4) \simeq \mathrm{Spin}(5)$ in which both are possible. Since it is an extremal contraction the conormal bundle of the exceptional locus is $N^* = \mathcal{O}(k)$ with $1 \leq k \leq n - 1$, respectively $1 \leq k \leq n - 2$.

We can thus apply the cone's Proposition 2.1 (to $z \in Z$); this gives that X is a completion of the open variety $V(E, N^*) = \mathrm{Spec}(\bigoplus_h \mathcal{O}(hk))$. Note that the open orbit is isomorphic to G/K where K is the kernel of the character map $\rho : P \longrightarrow \mathbf{C}^*$ associated to the homogeneous line bundle $\mathcal{O}(k)$, P is the parabolic subgroup associated to $\mathbf{P}^{n-1}$, resp. $\mathbf{Q}^{n-1}$.

One possible completion is $X_k = \mathbf{P}(N^* \oplus \mathcal{O})$ which has an open orbit isomorphic to G/K and two closed orbit isomorphic to $\mathbf{P}^{(n-1)}$, respectively $\mathbf{Q}^{(n-1)}$. But, by the above Lemma 4.2, this is actually the only one except if $k = 1$ and $G = SL(m)$ or $\mathrm{Sp}(m)$, where X_1 can be actually blow-down to $\mathbf{P}^n$. In this case there are thus two possible completions (actually $\rho(\mathbf{P}^n) = 1$ and therefore $\mathbf{P}^n$ will appear in the proper place in the second step).

b) Let φ be of fiber type and consider the induced action of G on Z. By our assumption either this action is trivial or $Z = \mathbf{P}^{(n-1)}$ if $G = SL(n)$ or $\mathrm{Sp}(n = 2s)$, respectively $\mathbf{Q}^{(n-1)}$ if $G = \mathrm{Spin}(n+1)$.

In the first case, since any fiber of φ is an orbit, we must have that $\dim Z = 1$ and $X = \mathbf{P}^{(n-1)} \times Z$, respectively $X = \mathbf{Q}^{(n-1)} \times Z$, with the G-action factorizing to the product of the standard homogeneous one on $\mathbf{P}^{(n-1)}$, respectively on $\mathbf{Q}^{(n-1)}$, and the trivial one on Z, except possibly for $n = 2$. This follows for instance by the more general Theorem 1.2.1 in [18]; for the reader's convenience we outline his proof in this case. Namely take a point $p_0 \in X$ and let H be the isotropy group of G at p_0. Let

$Z_1 = \{p \in X : G_p = H\}$, where G_p is the isotropy group of G at p. Then one can define a regular map $\tau : G/H \times Z_1 \longrightarrow X$ by $\tau(gH, p) = g \cdot p$. It is straightforward to see that this map is well defined, injective and G-equivariant. Moreover, by the Zariski's main Theorem, it is an algebraic G-equivariant isomorphism. This gives our claim after noticing that $G/H \simeq \mathbf{P}^{(n-1)}$, respectively $\mathbf{Q}^{(n-1)}$, and that $Z_1 = X/G = Z$.

If $n = 2$ and $G = SL(2)$ then we have another case which comes from the diagonal action of $SL(2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. It is straightforward to prove that there are no other actions of $SL(2)$ on the smooth two dimensional quadric.

In the second one φ is an equivariant $\mathbf{P}^1$-bundle over $\mathbf{P}^{(n-1)}$, respectively $\mathbf{Q}^{(n-1)}$: in fact the action on Z is homogeneous and thus the fibers are all equidimensional and there are no reducible or double fibers. Thus $X = \mathbf{P}(E)$ with E a rank 2 vector bundle on Z; E is homogeneous since the action is φ equivariant. Therefore either $E = \mathcal{O}(s) \oplus \mathcal{O}$ with $s \geq 0$, after normalizing if necessary, or $n = 3$, $G = SL(3)$ and $E = T_Z$, or $n = 4$, $G = Sp(4)$ and E is the nullcorrelation bundle on $\mathbf{P}^3$ or the spinor bundle on $\mathbf{Q}^3$.

If $E = \mathcal{O}(s) \oplus \mathcal{O}$ we have a decomposition of X into three orbits. Two isomorphic to $\mathbf{P}^{(n-1)}$, respectively $\mathbf{Q}^{(n-1)}$ (the section at infinity and the zero section) and an open dense orbit isomorphic to G/S where S is the kernel of the character map $\rho : P \longrightarrow \mathbf{C}^*$ associated to the homogeneous line bundle $\mathcal{O}(s)$, P being the parabolic subgroup associated to $\mathbf{P}^{n-1}$, resp. $\mathbf{Q}^{n-1}$. The fact that this is the unique action on X can be proved as above with the exception $n = 2$ and $s = 0$ (note that the section at infinity can be contracted so we can apply the cone's proposition).

If $n = 3$ and $E = T_Z$ it is well known that $X = \mathbf{P}(T_{\mathbf{P}^2})$ is the homogeneous variety G/B where B is a Borel subgroup of $SL(3)$ which corresponds in taking all the Dynkin diagram A_3 (or equivalently the kernel of the two dimensional representation of H associated to the tangent bundle); it is the unique closed orbit of the adjoint representation of $SL(3)$.

If $n = 4$ and E is either the nullcorrelation bundle on $\mathbf{P}^3$ or the spinor bundle on $\mathbf{Q}^3$ then $X = \mathbf{P}(E) = Sp(4)/B$ where B is a Borel subgroup.

2-nd Step. Assume finally that $\rho(X) = 1$, i.e., since it has an extremal ray, X is a Fano manifold.

If X is homogeneous then we can just look at the list of parabolic subgroups of codimension n corresponding to one node of the Dynkin diagram.

If $G = SL(n)$ we have only one possibility for $n = 4$, namely $X = SL(4)/Q$ where Q is the parabolic subgroup corresponding to the second node of the Dynkin diagram A_4. It is the unique orbit of the irreducible representation of $SL(4)$ into $\Lambda^2 \mathbf{C}^4$ and it is isomorphic to the Grassmanian of planes in $\mathbf{C}^4$, i.e. the smooth 4-dimensional quadric.

If $G = Sp(n)$ or $Spin(n+1)$ with $n \geq 6$ there is no homogeneous manifold of dimension n with $\rho(X) = 1$.

If X is not homogeneous and has no fixed points then it must have a closed orbit H which will be isomorphic to $\mathbf{P}^{(n-1)}$, respectively $\mathbf{Q}^{(n-1)}$. Let L be a positive gen-

erator of Pic(X); then $H = mL$. Since H is effective $m > 0$; then it is well known that a smooth projective variety with an ample section isomorphic to $\mathbf{P}^{(n-1)}$, respectively $\mathbf{Q}^{(n-1)}$, has to be isomorphic to $\mathbf{P}^n$ (if $n = 2$ we can have also $\mathbf{P}^1 \times \mathbf{P}^1$, this has however $\rho(X) = 2$ and thus it was considered above), respectively to $\mathbf{P}^n$ or to $\mathbf{Q}^n$.

So if $G = SL(n)$ or Sp(n), the last with $n \neq 4$, then X has to be $\mathbf{P}^n$ and it contains the closed orbit $H \simeq \mathbf{P}^{(n-1)}$ as a linear subspace except for $n = 2$ in which case the orbit can be a conic $\simeq \mathbf{P}^1$. If the orbit is linear then X contains as an open Zariski subset the total space of the normal bundle. Thus the action on this open subset is fixed (by the action on the orbit) and as discussed above it is unique (see the Lemma 4.2).

If $n = 2$ we have another non trivial action: namely the induced action on $\mathbf{P}^2 = \mathbf{P}(\mathbf{C}^3)$ by the 3-dimensional irreducible representation $\alpha_3 : SL(2, \mathbf{C}) \longrightarrow GL(3, \mathbf{C})$. It is straightforward to prove that there are no other actions of $SL(2)$ on $\mathbf{P}^2$.

If $G = $ Sp(4) then we have the above case when $H \simeq \mathbf{P}^3$ but we can have also $H \simeq \mathbf{Q}^3$. Then X can be either $\mathbf{P}^4$ or $\mathbf{Q}^4$; the action is described in the following if we think of G as Spin(5).

If $G = $ Spin($n + 1$) then we have an action on $X = \mathbf{Q}^n$ given by the embedding Spin($n+1$) $\longrightarrow$ Spin($n+2$) and one can prove that this is the only possible action; there is a closed orbit, isomorphic to the $(n - 1)$-dimensional quadric and an open orbit. If $X = \mathbf{P}^n$ the action is coming from the canonical action of G on $\mathbf{C}^{(n+1)}$ and X has two orbits: a closed one, isomorphic to the $(n - 1)$-dimensional quadric and an open one isomorphic to $X_2 = $ Spin($n + 1$)$/S(O(1) \times O(n))$.

5. Fourfolds which are quasi-homogeneous under the action of $SL(3)$.

The next step will be the case $r_G = n - 2$, so for instance $G = SL(n - 1)$.

If $n = 3$, $G = SL(2)$ and X quasi-homogeneous this was studied in a series of papers starting with the one of Mukai-Umemura (see [20] and [22]).

If $n = 4$ and $G = SL(3)$ Nakano proved the following theorem; his proof started by computing the closed subgroup of codimension 4 in $SL(3)$.

Theorem 5.1 ([23]). *Let X be a smooth 4-fold on which $G = SL(3)$ acts with an open orbit. Then X is isomorphic to one of the following*:

1) $X_{(p,q)} = \mathbf{P}(L_{p,q} \oplus \mathcal{O})$ where $L_{p,q}$ is a line bundle on $Z = SL(3)/B = \mathbf{P}(T_{\mathbf{P}^2})$ (*described in the point* d)).

2) $Y_{(a)} = \mathbf{P}(T_{\mathbf{P}^2}(a)) \oplus \mathcal{O})$

3) $X = \mathbf{P}(S^2 T_{\mathbf{P}^2})$

4) $X = \mathbf{P}^2 \times \mathbf{P}^2$ and $Bl_\Delta(\mathbf{P}^2 \times \mathbf{P}^2)$

5) $X = \mathbf{Q}^4 \subset \mathbf{P}^5$.

We will try now to reprove this result by applying the MMP; so from now on we assume that X is a smooth 4-fold, quasi-homogeneous with respect to a $G = SL(3)$-

action, and we will run the MMP on X.

Let first $\rho(X) \geq 2$ and let $\varphi : X \longrightarrow Z$ be the contraction of an extremal ray; if φ is birational let also E be its exceptional locus. As in the previous section, by the G-equivariant property of φ and the fact that $r_{SL(3)} = 2$, we can, a priori, have only the following cases.

a) φ is birational, $\dim E = 3$ and $\varphi(E) = z$ is one point; in this case E is a 3-dimensional del Pezzo variety with an $SL(3)$-action induced by the one of X. In particular E cannot have fixed point.

First note that E has to be smooth: in fact its singular locus is $SL(3)$-invariant and thus it has to be isomorphic to $\mathbf{P}^2$. The normal bundle of this $\mathbf{P}^2$ in X has to be homogeneous. But this cannot occur because there is a description of the possible non normal del Pezzo exceptional divisor by Fujita and the normal bundle of the singular locus (which is $\mathbf{P}^2$) in X is not homogeneous (see [8]).

Thus, being smooth, E has to be in the classification of the previous section: that is E can be either $\mathbf{P}^3$, either $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2})$ with conormal bundle $\xi \otimes H$ where ξ is the tautological bundle and H is the pull back of $\mathcal{O}(1)$ from $\mathbf{P}^2$, or $\mathbf{P}(T_{\mathbf{P}^2})$ with the conormal bundle $\mathcal{O}(1,1)$, the tensor of the two line bundles obtained by pulling back $\mathcal{O}(1)$ from the two projections into $\mathbf{P}^2$.

The case $E = \mathbf{P}^3$ cannot occur because it has a fixed point. In the second case we notice that the section at infinity of $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2})$ is an orbit $\simeq \mathbf{P}^2$ with conormal bundle $N^* = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Then we can G-equivariantly blow-up this orbit and contract the exceptional divisor into a compact (non projective) manifold which will then contains a 1-dimensional orbit, namely the image of the exceptional divisor isomorphic to $\mathbf{P}^1$; this is a contradiction since $SL(3)$ has no 1-dimensional homogeneous variety (see also the next point c) concerning small contractions).

The case $E = \mathbf{P}(T_{\mathbf{P}^2})$ can actually occur. We apply the cone's proposition, thus $X = \mathbf{P}(\mathcal{O}(1,1) \oplus \mathcal{O})$ and φ is the contraction of the zero section to a point. But this contraction is not elementary and it factors through a smooth blow down with center $\mathbf{P}^2$ (and then through a flop of this $\mathbf{P}^2$ to a point).

b) φ is birational, $\dim E = 3$ and $\dim(\varphi(E)) > 0$; thus $\dim(E) = 1$ or 2.

If $\dim(E) = 1$ then, by the usual arguments, $f(E)$ is a curve of fixed points and all fibers F are isomorphich to $\mathbf{P}^2$. Moreove one can prove that the normal bundle of F is either $\mathcal{O}(-1) \oplus \mathcal{O}$ or $\mathcal{O}(-2) \oplus \mathcal{O}$ (for more details on contractions of this type see the section 4 in [5]). In the first case all points in $f(E)$ are smooth points of Z (f is a smooth blow-up along $f(E)$), and this is a contradiction with 2.3. In the second case one can see that for every $z \in f(E) \subset Z$ we have $m_z/m_z^2 = H^0(\mathbf{P}^2, \mathcal{O}(-2) \oplus \mathcal{O})$ (see for instance 5.5 in [4]) and this is a contradiction to 2.4.

In the other case, by the $SL(3)$-equivariance of φ, we have that $\varphi(E) = \mathbf{P}^2$ and all non trivial fibers are one dimensional. We can thus apply a result of T. Ando ([2], see also [3]) which says that in this hypothesis the extremal contraction φ is an equivariant smooth blow-up of an orbit $\simeq \mathbf{P}^2$ in a smooth manifold Z.

c) φ is a small contractions, i.e. codim(E) ≥ 2. Thus E has to be of dimension 2 and isomorphic to $\mathbf{P}^2$ and with conormal bundle N^* homogeneous. It is immediate then to check that $N^* = \mathcal{O}(1) \oplus \mathcal{O}(1)$ (since $\det N^* = 2$ and N^* has to be ample); this follows also by a general theorem of Kawamata which describes all small contractions on a smooth 4-fold (see [11]). We blow-up the orbit $\mathbf{P}^2$ and we obtain a smooth variety with a G-action; but since N^* is ample we can blow-down the exceptional divisor in the other direction, i.e. consider the map supported by $-\hat{E} - \tau L$ where L is a φ-ample divisor and τ is a rational number such that $-\hat{E} - \tau L$ is nef but not ample (thus we can flip the contraction). We thus obtain a (smooth) projective variety with a G-action and a orbit of dimension one, the image of the exceptional divisor $\simeq \mathbf{P}^1$, a contradiction.

The above three steps prove the following

Proposition 5.2. *Let X be a smooth projective* 4-*fold which has an action of* $SL(3)$ *with an open orbit. If $\varphi : X \longrightarrow Z$ is a birational elementary Fano Mori contraction then Z is smooth and φ is the blow-up of an orbit isomorphic to $\mathbf{P}^2$ in Z.*

REMARK 5.3. The above proposition implies that we can run the Minimal Model Program *within the category of smooth varieties*. This is true also for the case of quasi-homogeneous 3-folds under the action of $SL(2)$ (see [20]) and we conjecture it should be true for quasi-homogeneous n-folds under the action of $SL(n-1)$.

Therefore we consider now the cases in which φ is of fiber type.

d) φ is a conic bundle.

There can be some isolated two dimensional fibers: then they have to be orbits isomorphic to $\mathbf{P}^2$ and with homogeneous normal bundle. By the results in [4] (in particular 5.9.6) there is only one possibility for the conormal bundle, namely $N^* = T_{\mathbf{P}^2}(-1)$. Moreover in this case Z is smooth thus we use the classification in the previous section which gives that $Z = \mathbf{P}^3$ (since the images of the isolated exceptional fibers are fixed points in Z). This will eventually give the case $X = \mathbf{P}(T_{\mathbf{P}^2}(-1) \oplus \mathcal{O})$, for instance using the results in [6], which is $Y_{(1)}$ in the Theorem 5.1.

With the above exception, we have thus that all fibers of the conic bundle φ are one dimensional; then this implies that Z is smooth, again by the results in [2], and we can use the classification in the previous section. Z cannot be $\mathbf{P}^3$ since otherwise we will have a one dimensional orbit (the fiber over the fixed point). Thus $Z = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(m) \oplus \mathcal{O})$ or $\mathbf{P}(T_{\mathbf{P}^2})$; the first cannot happen since in this case we will not have a dense orbit while in the second case $X_{(p,q)} = \mathbf{P}(L_{p,q} \oplus \mathcal{O})$ where $L_{p,q}$ is the line bundle which corresponds to the character defined on B, the Borel subgroup of

$SL(3)$, by

$$\begin{bmatrix} a & * & * \\ 0 & e & * \\ 0 & 0 & i \end{bmatrix} \longrightarrow a^p e^q \,;$$

this is the case 1) in 5.1.

e) φ is a Fano fibration over $\mathbf{P}^2$; thus it is actually an equivariant $\mathbf{P}^2$-bundle, i.e. $X = \mathbf{P}(\mathcal{E})$ with $\mathcal{E}$ an homogeneous bundle of rank 3 on $\mathbf{P}^2$. The homogeneous bundles $\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}$ don't give a quasi-homogeneous variety, i.e. there is no open orbit, except if $a = b = 0$ in which case we have the diagonal action on $\mathbf{P}^2 \times \mathbf{P}^2$ which has an open orbit. Therefore X is one of the manifolds $Y_a := \mathbf{P}(T_{\mathbf{P}^2}(a)) \oplus \mathcal{O})$ or $\mathbf{P}(S^2 T_{\mathbf{P}^2})$.

f) Since Z cannot be a curve the only remaining case is when $\dim Z = 0$, i.e. X is a Fano 4-fold with $\mathrm{Pic}(X) = \mathbf{Z}$. Note that there are no homogeneous such manifolds. From the Theorem 5.1 of Nakano it happens that $X = \mathbf{Q}^4 \subset \mathbf{P}^5$. We hope to find a direct proof of this last fact and in general we believe that the following holds.

Conjecture. *Let X be a smooth Fano manifold of dimension n which is quasi-homogeneous under a regular action of the group $SL(n-1)$; assume also that $\mathrm{Pic}(X) = \mathbf{Z}$. Then $n = 3$ and X is one of the examples found in* [20] *and* [22] *or $n = 4$ and X is the smooth quadric in $\mathbf{P}^5$.*

Thus a 4-fold X which is quasi-homogeneous with respect to $SL(3)$-action has to be one of the manifolds coming up in d), e), f) or the blow-up of one of them along a closed orbit isomorphic to $\mathbf{P}^2$ 5.2. So we also have $Bl_\Delta(\mathbf{P}^2 \times \mathbf{P}^2)$; note that the quadric in f) has two closed orbits isomorphic to $\mathbf{P}^2$ (and an open one); blowing up one of them we obtain a manifold in the class 2) of 5.1, then blowing up the other we obtain a manifold in the class 1) of 5.1.

Added in proof. The conjecture stated at the end of section 5 as well as the one in 5.2.1 have been recently proved. They follow from a more general result obtained by J. A. Wisniewski and the author in the preprint: "On quasihomogeneous manifolds-via Brion-Luna-Vust theorem".

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