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A construction of surface bundles over surfaces with non-zero signature

（符号数が0でない曲面上の曲面束の一構成法）

遠藤 久顕

A CONSTRUCTION OF SURFACE BUNDLES OVER SURFACES WITH NON-ZERO SIGNATURE

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1. Introduction

Let Σ_g (respectively Σ_h) be a closed oriented surface of genus g (respectively h), where g (respectively h) is a non-negative integer. Let $Diff_+\Sigma_h$ be the group of all orientation-preserving diffeomorphisms of Σ_h with C^∞-topology. A Σ_h-*bundle* over Σ_g (also called a *surface bundle* over a surface) is a fiber bundle $\xi = (E, \Sigma_g, p, \Sigma_h, Diff_+\Sigma_h)$ over Σ_g with total space E, fiber Σ_h, projection $p : E \longrightarrow \Sigma_g$ and structure group $Diff_+\Sigma_h$. Our main concern is the signature $\tau(E)$ of the total space E of ξ.

It is easily seen that if ξ is a trivial bundle then $\tau(E) = \tau(\Sigma_g)\tau(\Sigma_h) = 0$. Chern-Hirzebruch-Serre [5] proved that if the fundamental group $\pi(\Sigma_g)$ of Σ_g acts trivially on the cohomology ring $H^*(\Sigma_h; \mathbb{R})$ of Σ_h then $\tau(E) = 0$.

Kodaira [12] and Atiyah [1] gave examples of surface bundles over surfaces with non-zero signature. For each pair (m, t) of integers $m, t \in \mathbb{Z}\,(m \geq 2, t \geq 3)$, Kodaira constructed a surface bundle $\xi = \xi(m, t)$ with

$$\begin{aligned} g &= m^{2t}(t-1) + 1, \\ h &= mt, \\ \tau(E) &= \frac{4}{3} m^{2t-1}(t-1)(m^2-1). \end{aligned}$$

By setting $m = 2$ and $t = 3$, we obtain a surface bundle $\xi = \xi(2,3)$ with $g = 129$, $h = 6$ and $\tau(E) = 256$. The total space E of the bundle $\xi = \xi(m, t)$ is an m-fold branched covering of $\Sigma_g \times \Sigma_t$ and its signature $\tau(E)$ can be calculated by using G-signature theorem(see [9] and [11]).

Meyer [16][17] gave a signature formula for surface bundles over surfaces in terms of the *signature cocycle* τ_h, which is a 2-cocycle of the Siegel modular group $Sp(2h, \mathbb{Z})$ of degree h. Using the signature cocycle and Birman-Hilden's relations [3] of mapping class groups of surfaces, he showed that if $h = 1, 2$ or $g = 1$ then $\tau(E) = 0$. But he also showed that for every $h \geq 3$ and every $n \in \mathbb{Z}$ there exist an integer $g \geq 0$ and a Σ_h-bundle ξ over Σ_g such that $\tau(E) = 4n$.

We consider the following problem:

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Problem 1.1. *For each $h \geq 3$ and each $n \in \mathbb{Z}$, let $g(h,n)$ be the minimum value of the genus g such that there exists a Σ_h-bundle ξ over Σ_g with $\tau(E) = 4n$. Determine the value $g(h,n)$.*

In this paper, we estimate the value $g(h,n)$ by using Wajnryb's presentation [19] of the mapping class group $\mathcal{M}_h$ of Σ_h.

Our main result is:

Theorem 1.2. *For each $h \geq 3$ and each $n \in \mathbb{Z}(n \neq 0)$, the following inequality holds:*

$$\frac{|n|}{h-1} + 1 \leq g(h,n) \leq 111|n|.$$

We construct a Σ_h-bundle ξ over Σ_g with $g = 111$, $h = 3$ and $\tau(E) = -4$ to prove Theorem 1.2. The genus of the base space of this bundle and that of a fiber of it are smaller than those of any example constructed by Kodaira [12] and Atiyah [1].

In Section 2, we review Meyer's work [16][17] on signature of surface bundles over surfaces. And in Section 3, we calculate the values of Meyer's signature cocycle for the relators of Wajnryb's presentation [19] of the mapping class group $\mathcal{M}_h$ and characterize the 2-cycles of $\mathcal{M}_h$ as words in the generators of the presentation of $\mathcal{M}_h$. We prove our main theorem in Section 4 by using this characterization and a simple technique of the commutator collection process [7].

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2. Meyer's signature formula

In this section we review Meyer's signature cocycle and Meyer's signature formula [16][17] for surface bundles over surfaces.

For a pair (α, β) of symplectic matricies $\alpha, \beta \in Sp(2h, \mathbb{Z})$, the vector space $V_{\alpha,\beta}$ is defined by:

$$V_{\alpha,\beta} := \{(x,y) \in \mathbb{R}^{2h} \times \mathbb{R}^{2h} \mid (\alpha^{-1} - I)x + (\beta - I)y = 0\},$$

where I is the identity matrix. Consider the (possibly degenerate) symmetric bilinear form

$$<\ ,\ >_{\alpha,\beta} : V_{\alpha,\beta} \times V_{\alpha,\beta} \longrightarrow \mathbb{R}$$

on $V_{\alpha,\beta}$ defined by:

$$<(x_1,y_1),(x_2,y_2)>_{\alpha,\beta} := <x_1 + y_1, (I-\beta)y_2>,$$

$$(x_i, y_i) \in V_{\alpha,\beta} \quad (i = 1,2),$$

where $<\ ,\ >$ is the standard symplectic form on $\mathbb{R}^{2h}$ given by:

$$<x,y> = {}^t x J y \quad (x, y \in \mathbb{R}^{2h}),$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in M_{2h}(\mathbb{R}).$$

Meyer's signature cocycle [16][17]

$$\tau_h : Sp(2h,\mathbb{Z}) \times Sp(2h,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is defined by:

$$\tau_h(\alpha,\beta) := sign(V_{\alpha,\beta}, < \ , \ >_{\alpha,\beta})$$
$$(\alpha,\beta \in Sp(2h,\mathbb{Z})).$$

From the Novikov additivity, τ_h is a 2-cocycle of $Sp(2h,\mathbb{Z})$ and represents a cohomology class $[\tau_h] \in H^2(Sp(2h,\mathbb{Z}),\mathbb{Z})$.

Let $\mathcal{M}_h$ be the mapping class group of a surface Σ_h of genus h, namely it is the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_h. By choosing a symplectic basis on $H^1(\Sigma_h;\mathbb{Z}) \cong \mathbb{Z}^{\oplus 2h}$, the natural action of $\mathcal{M}_h$ on $H^1(\Sigma_h;\mathbb{Z})$ induces a representation $\sigma : \mathcal{M}_h \longrightarrow Sp(2h,\mathbb{Z})$.

Next, we define a homomorphism $k : H_2(\mathcal{M}_h;\mathbb{Z}) \longrightarrow \mathbb{Z}$ by using τ_h and σ. It is known that the group $\mathcal{M}_h$ is finitely presentable, so there exists an exact sequence:

$$1 \longrightarrow R \longrightarrow F \stackrel{\pi}{\longrightarrow} \mathcal{M}_h \longrightarrow 1,$$

where F is a free group of finite rank generated by a free basis $E = \{e_\lambda\}_{\lambda\in\Lambda}$. By well known Hopf's theorem (cf. [4]) the following isomorphism holds:

$$H_2(\mathcal{M}_h;\mathbb{Z}) \cong R \cap [F,F]/[R,F].$$

The map $c : F \longrightarrow \mathbb{Z}$ is defined by:

$$c(x) := \sum_{j=1}^{m} \tau_h(\sigma(\pi(\widetilde{x}_{j-1})), \sigma(\pi(x_j)))$$

$$(x = \prod_{j=1}^{m} x_i,\ x_i \in E \cup E^{-1},\ \widetilde{x}_j = \prod_{i=1}^{j} x_i).$$

It can be checked that the restriction $c\,|_R : R \longrightarrow \mathbb{Z}$ is actually a homomorphism and that $c([R,F]) = 0$. Hence $c\,|_R$ naturally induces a homomorphism $k : H_2(\mathcal{M}_h;\mathbb{Z}) \cong R \cap [F,F]/[R,F] \longrightarrow \mathbb{Z}$.

Now, we describe Meyer's signature formula for surface bundles over surfaces.

Let $\xi = (E, \Sigma_g, p, \Sigma_h, Diff_+\Sigma_h)$ be a Σ_h-bundle over Σ_g and $f : \Sigma_g \longrightarrow BDiff_+\Sigma_h$ its classifying map. The map f induces a homomorphism χ between fundamental groups:

$$\chi := f_\sharp : \pi_1(\Sigma_g) \longrightarrow \pi_1(BDiff_+\Sigma_h) \cong \pi_0(Diff_+\Sigma_h) \cong \mathcal{M}_h,$$

which is called the *holonomy homomorphism* of ξ (cf. [18]). By a theorem of Earle and Eells [6], which states that the connected component $Diff_0\Sigma_h$ of the identity of $Diff_+\Sigma_h$ is contractible if $h \geq 2$, the isomorphism class of ξ is completely determined by its holonomy homomorphism χ when $h \geq 2$ (see [16][17] and [18]). From now on, we suppose that $h \geq 2$ and $g \geq 1$.

The fundamental group $\pi_1(\Sigma_g)$ of Σ_g is finitely presented, so we have an exact sequence:

$$1 \longrightarrow \widetilde{R} \longrightarrow \widetilde{F} \stackrel{\widetilde{\pi}}{\longrightarrow} \pi_1(\Sigma_g) \longrightarrow 1,$$

where

$$\pi_1(\Sigma_g) = < a_1, \cdots, a_g, b_1, \cdots, b_g | \prod_{i=1}^{g} [a_i, b_i] (= \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}) = 1 >,$$

$$\widetilde{F} = < \widetilde{a}_1, \cdots, \widetilde{a}_g, \widetilde{b}_1, \cdots, \widetilde{b}_g >,$$

$$\widetilde{\pi} : \widetilde{a}_i \longmapsto a_i, \widetilde{b}_i \longmapsto b_i$$

and $\widetilde{R}$ is the normal closure of $\widetilde{r} := \prod_{i=1}^{g} [\widetilde{a}_i, \widetilde{b}_i] (= \prod_{i=1}^{g} \widetilde{a}_i \widetilde{b}_i \widetilde{a}_i^{-1} \widetilde{b}_i^{-1})$ in $\widetilde{F}$. Hopf's theorem allows us to identify $H_2(\pi_1(\Sigma_g); \mathbb{Z})$ with $\widetilde{R} \cap [\widetilde{F}, \widetilde{F}] / [\widetilde{R}, \widetilde{F}]$. For the holonomy homomorphism χ, we can choose a homomorphism $\widetilde{\chi} : \widetilde{F} \longrightarrow F$ so that $\pi \circ \widetilde{\chi} = \chi \circ \widetilde{\pi}$. Then the induced homomorphism $\chi_* : H_2(\pi_1(\Sigma_g); \mathbb{Z}) \longrightarrow H_2(\mathcal{M}_h; \mathbb{Z})$ is defined by:

$$\chi_*(\widetilde{x}[\widetilde{R}, \widetilde{F}]) := \widetilde{\chi}(\widetilde{x})[R, F] \quad (\widetilde{x} \in \widetilde{R} \cap [\widetilde{F}, \widetilde{F}])$$

and is not depend on a choice of $\widetilde{\chi}$.

Meyer proved the following theorem by using the Leray-Serre spectral sequence for ξ and the cohomology group $H^1(\Sigma_g; H_1(\Sigma_h; \mathbb{R}))$ of Σ_g with local coefficients.

Theorem 2.1 (Meyer [16][17]). *Let $\xi = (E, \Sigma_g, p, \Sigma_h, Diff_+\Sigma_h)$ be a Σ_h-bundle over Σ_g $(h \geq 2, g \geq 1)$ and $\chi : \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_h$ its holonomy homomorphism. Then the following equality holds:*

$$\tau(E) = -k(\chi_*(\widetilde{r}[\widetilde{R}, \widetilde{F}])) (= -c(\widetilde{\chi}(\widetilde{r}))).$$

3. Explicit description of 2-cycles of $\mathcal{M}_h$

In this section, we calculate values of the map $c : F \longrightarrow \mathbb{Z}$ for the relators of the finite presentation of $\mathcal{M}_h$ due to Wajnryb and give an explicit description of the homomorphism k defined in the preceding section in order to characterize the elements of $R \cap [F, F]$ as words of F.

Let $\mathcal{M}_h$ be the mapping class group of a surface Σ_h of genus h. A finite presentation of $\mathcal{M}_2$ was obtained by Birman-Hilden [3] and that of $\mathcal{M}_h (h \geq 3)$ by Hatcher-Thurston [8].

Wajnryb [19] simplified their presentation of $\mathcal{M}_h (h \geq 2)$ as follows. (We denote the commutator $xyx^{-1}y^{-1}$ of $x, y \in F$ by $[x, y]$.)

The generators, which are called the Lickorish-Humphries generators, of the presentation are:

$$y_1, y_2, u_1, u_2, \cdots, u_h, z_1, z_2, \cdots, z_{h-1}$$

and the relators of it are:

$$
\begin{aligned}
A^1 &:= [y_1, y_2],\\
A^2_{i,j} &:= [y_i, u_j] \quad (i = 1, 2, 1 \le j \le h, i \ne j),\\
A^3_{i,j} &:= [y_i, z_j] \quad (i = 1, 2, 1 \le j \le h-1),\\
A^4_{i,j} &:= [u_i, u_j] \quad (1 \le i < j \le h),\\
A^5_{i,j} &:= [u_i, z_j] \quad (1 \le i \le h, 1 \le j \le h-1, j \ne i, i+1),\\
A^6_{i,j} &:= [z_i, z_j] \quad (1 \le i < j \le h-1),\\
B^1_i &:= y_i u_i y_i u_i^{-1} y_i^{-1} u_i^{-1} \quad (i = 1, 2),\\
B^2_i &:= u_i z_i u_i z_i^{-1} u_i^{-1} z_i^{-1} \quad (1 \le i \le h-1),\\
B^3_i &:= z_i u_{i+1} z_i u_{i+1}^{-1} z_i^{-1} u_{i+1}^{-1} \quad (1 \le i \le h-1),\\
C^1 &:= (y_1 u_1 z_1)^{-4} y_2 (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2)^{-1} y_2 (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2),\\
D^1 &:= y_1 z_1 z_2 t_1 t_2 (y_2 t_2 y_2 t_2^{-1} t_1 t_2 y_2)^{-1} (w u_1 z_1 u_2 z_2 u_3)^{-1} v w u_1 z_1 u_2 z_2 u_3,\\
E^1 &:= [d, u_h z_{h-1} u_{h-1} \cdots z_1 u_1 y_1^2 u_1 z_1 \cdots u_{h-1} z_{h-1} u_h],
\end{aligned}
$$

where

$$
\begin{aligned}
t_1 &:= u_1 y_1 z_1 u_1,\\
t_i &:= u_i z_{i-1} z_i u_i \quad (2 \le i \le h-1),\\
v &:= y_1 u_1 z_1 u_2 y_2 (y_1 u_1 z_1 u_2)^{-1},\\
w &:= z_2 u_3 t_2 y_2 (z_2 u_3 t_2)^{-1},\\
v_1 &:= (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2)^{-1} y_2 (u_2 z_1 u_1 y_1^2 u_1 z_1 u_2),\\
v_i &:= t_{i-1} t_i v_{i-1} (t_{i-1} t_i)^{-1} \quad (2 \le i \le h-1),\\
w_1 &:= u_1 z_1 u_2 v_1 (u_1 z_1 u_2)^{-1},\\
w_i &:= u_i z_i u_{i+1} v_i (u_i z_i u_{i+1})^{-1} \quad (2 \le i \le h-1),\\
d &:= (w_1 w_2 \cdots w_{h-1})^{-1} y_1 w_1 w_2 \cdots w_{h-1}.
\end{aligned}
$$

Elements y_i, u_i, z_i can be interpreted as Dehn twists with respect to curves Y_i, U_i, Z_i in Fig.1 of [3] (see also [13] and [10]). For $h = 2$ we can omit the relator D^1.

By choosing a symplectic basis of $H^1(\Sigma_h; \mathbb{Z})$ as in [17], we fix an explicit representation $\sigma : \mathcal{M}_h \longrightarrow Sp(2h, \mathbb{Z})$ by:

$$
\begin{aligned}
\sigma : y_i &\longmapsto \begin{pmatrix} I & 0 \\ -E_{ii} & I \end{pmatrix} \quad (i = 1, 2),\\
\sigma : u_i &\longmapsto \begin{pmatrix} I & E_{ii} \\ 0 & I \end{pmatrix} \quad (1 \le i \le h),\\
\sigma : z_i &\longmapsto \begin{pmatrix} I & 0 \\ -E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} & I \end{pmatrix} \quad (1 \le i \le h-1),
\end{aligned}
$$

where $E_{ij} \in M_h(\mathbb{Z})$ is the (i,j)-matrix unit.

We also fix an exact sequence:

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} \mathcal{M}_h \longrightarrow 1,$$

where

$$F := < y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1} >$$

and R is the normal closure of the set of all relators $A^l_{i,j}, B^l_i, C^1, D^1, E^1$ in F. Let $c : F \longrightarrow \mathbb{Z}$ be the map defined as in Section 2 by using explicit homomorphisms σ and π fixed above.

Now we calculate values of the map $c : F \longrightarrow \mathbb{Z}$ for relators $A^l_{i,j}, B^l_i, C^1, D^1, E^1$ of the presentation and describe the homomorphism $c|_R : R \longrightarrow \mathbb{Z}$.

To compute values of c, Meyer showed the following lemma:

Lemma 3.1 (Meyer [16][17]). *The map $c : F \longrightarrow \mathbb{Z}$ satisfies the following properties:*

(1) $c(xy) = c(x) + c(y) + \tau_h(\sigma(\pi(x)), \sigma(\pi(y))) \quad (x, y \in F)$;
(2) $c(x^{-1}) = -c(x) \quad (x \in F)$;
(3) $c(xyx^{-1}) = c(y) \quad (x, y \in F)$;
(4) $c(xzyz^{-1}) = c(x) + c(y)$ *if* $\pi(xzyz^{-1}) = 1 \in \mathcal{M}_h \quad (x, y, z \in F)$.

Values of c for relators are computed by using Lemma 3.1.

Lemma 3.2. *The values of c for the relators of Wajnryb's presentation of $\mathcal{M}_h (h \geq 3)$ are calculated as follows:*

(1) $c(A^l_{i,j}) = 0$ *(for every l, i, j)*;
(2) $c(B^l_i) = 0$ *(for every l, i)*;
(3) $c(C^1) = -6$;
(4) $c(D^1) = 1$;
(5) $c(E^1) = 0$.

Proof. We denote $\tau_h(\sigma(\pi(x)), \sigma(\pi(y)))$ by $\widetilde{\tau}_h(x, y)$ for $x, y \in F$. By virtue of Lemma 3.1, it follows immediately that $c(A^l_{i,j}) = c(B^l_i) = c(E^1) = 0$. For example,

$$\begin{aligned} c(B^1_1) &= c(y_1 \cdot u_1 \cdot y_1 u_1^{-1} y_1^{-1} \cdot u_1^{-1}) \\ &= c(y_1) + c(y_1 u_1^{-1} y_1^{-1}) \\ &= c(y_1) + c(u_1^{-1}) = c(y_1) - c(u_1) \\ &= 0. \end{aligned}$$

Using Lemma 3.1 and calculating signature of symmetric bilinear forms concretely, we obtain values $c(C^1)$ and $c(D^1)$.

$$
\begin{aligned}
c(C^1) &= c((y_1u_1z_1)^{-4}y_2(u_2z_1u_1y_2^2u_1z_1u_2)^{-1}y_2(u_2z_1u_1y_2^2u_1z_1u_2)) \\
&= c((y_1u_1z_1)^{-4}y_2) \qquad (c(y_2)=0) \\
&= c((y_1u_1z_1)^{-4}) + \widetilde{\tau}_h((y_1u_1z_1)^{-4}, y_2) \qquad (c(y_2)=0) \\
&= 2c((y_1u_1z_1)^{-2}) + \widetilde{\tau}_h((y_1u_1z_1)^{-2}, (y_1u_1z_1)^{-2}) + \widetilde{\tau}_h((y_1u_1z_1)^{-4}, y_2) \\
&= 4(\widetilde{\tau}_h(1, z_1^{-1}) + \widetilde{\tau}_h(z_1^{-1}, u_1^{-1}) + \widetilde{\tau}_h(z_1^{-1}u_1^{-1}, y_1^{-1})) \\
&\quad + 2\widetilde{\tau}_h((y_1u_1z_1)^{-1}, (y_1u_1z_1)^{-1}) \\
&\quad + \widetilde{\tau}_h((y_1u_1z_1)^{-2}, (y_1u_1z_1)^{-2}) + \widetilde{\tau}_h((y_1u_1z_1)^{-4}, y_2) \\
&= 4(0+0+0) + 2\cdot(-3) + (-1) + 1 \\
&= -6.
\end{aligned}
$$

$$
\begin{aligned}
c(D^1) &= c(y_1z_1z_2t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}(wu_1z_1u_2z_2u_3)^{-1}vwu_1z_1u_2z_2u_3) \\
&= c(y_1z_1z_2t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}) \\
&\qquad (c(v) = c(y_1u_1z_1u_2y_2(y_1u_1z_1u_2)^{-1}) = c(y_1) = 0) \\
&= c(y_1z_1z_2) + c(t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}) \\
&\quad + \widetilde{\tau}_h(y_1z_1z_2, t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}) \\
&= \widetilde{\tau}_h(y_1, z_1) + \widetilde{\tau}_h(y_1z_1, z_2) + c(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}) + c(t_2y_2^{-1}t_2^{-1}y_2^{-1}) \\
&\quad + \widetilde{\tau}_h(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}, t_2y_2^{-1}t_2^{-1}y_2^{-1}) + \widetilde{\tau}_h(y_1z_1z_2, t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}) \\
&= \widetilde{\tau}_h(y_1, z_1) + \widetilde{\tau}_h(y_1z_1, z_2) + \widetilde{\tau}_h(t_2y_2^{-1}t_2^{-1}, y_2^{-1}) \\
&\quad + \widetilde{\tau}_h(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}, t_2y_2^{-1}t_2^{-1}y_2^{-1}) + \widetilde{\tau}_h(y_1z_1z_2, t_1t_2(y_2t_2y_2t_2^{-1}t_1t_2y_2)^{-1}) \\
&\qquad (c(t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}) = c(y_2^{-1}) = -c(y_2) = 0, \\
&\qquad\quad c(t_2y_2^{-1}t_2^{-1}y_2^{-1}) = c(t_2y_2^{-1}t_2^{-1}) + c(y_2^{-1}) + \widetilde{\tau}_h(t_2y_2^{-1}t_2^{-1}, y_2^{-1}) \\
&\qquad\qquad = \widetilde{\tau}_h(t_2y_2^{-1}t_2^{-1}, y_2^{-1})) \\
&= 0+0+0+0+1 \\
&= 1. \quad \square
\end{aligned}
$$

REMARK 3.3. All values of Meyer's signature cocycle τ_h calculated in Lemma 3.2 are independent of the genus $h(\geq 3)$ because all generators which appear in C^1 and D^1 are $y_1, y_2, u_1, u_2, u_3, z_1$ and z_2. We can easily check by using a computer that the values are correct in the case $h = 3$. (We used *Mathematica*).

DEFINITION 3.4. Let F_n be a free group of rank n. *Algebraic m copies* of an element $x \in F_n$ are m_+ copies of x and m_- copies of x^{-1}, where $m_+, m_- \geq 0$ and $m_+ - m_- = m$. The integer m is called the *algebraic number* of these algebraic copies.

For each generator $e = y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}$, the homomorphism $e^* : F \longrightarrow \mathbb{Z}$ is defined by:

$$e^*(x) := \begin{cases} +1 & (x = e), \\ 0 & (x : \text{other generators}). \end{cases}$$

An element $x \in F$ belongs to $[F, F]$ if and only if $e^*(x) = 0$ for every generator e. Combining this with Lemma 3.2, we characterize the elements of $R \cap [F, F]$ as words in y_i, u_i, z_i and calculate the value of c for each element $x \in R \cap [F, F]$.

Proposition 3.5. *Suppose that $h \geq 3$. For an element $x \in F$, the following two conditions are equivalent:*

(1) *$x \in R \cap [F, F]$ and $c(x) = 4n \, (n \in \mathbb{Z})$;*

(2) *x is equal to a product of conjugates of algebraic copies of relators and the algebraic number $m(R^1)$ of algebraic copies of a relator R^1 included in x is determined as follows:*

R^1	$A^l_{i,j}$	B^1_1	B^1_2	B^2_1	B^2_2	$B^2_i (i \geq 3)$
$m(R^1)$	$\forall$	$-6n$	$18n$	$-2n$	$10n$	0

B^3_1	$B^3_i (i \geq 2)$	C^1	D^1	E^1
$-8n$	0	n	$10n$	$\forall$,

where $\forall$ stands for arbitrary number of algebraic copies of R^1.

Proof. (1)$\Longrightarrow$(2) : Since R is the normal closure of the set $\{A^l_{i,j}, B^l_i, C^1, D^1, E^1\}$ of all relators, any $x \in R$ is a product of conjugates of algebraic copies of relators. For $x \in R \cap [F, F]$, let $a^l_{i,j}$ (respectively b^l_i, c^1, d^1, e^1) be the algebraic number of algebraic copies of $A^l_{i,j}$ (respectively B^l_i, C^1, D^1, E^1) included in x. These numbers must satisfy the following system of equations because x belongs to $[F, F]$.

$$\sum_{i=1}^{2} b^1_i e^*(B^1_i) + \sum_{i=1}^{h-1} b^2_i e^*(B^2_i) + \sum_{i=1}^{h-1} b^3_i e^*(B^3_i) + c^1 e^*(C^1) + d^1 e^*(D^1) = 0$$

$$(e = y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}).$$

($e^*(A^l_{i,j}) = e^*(E^1) = 0$ for every generator e because $A^l_{i,j}$ and E^1 belong to $[F, F]$. Values of e^* and c for other relators are exhibited in Table 3.6 below.) Solving this, we get

$$b^1_1 = -6n,\ b^1_2 = 18n,\ b^2_1 = -2n,\ b^2_2 = 10n,\ b^2_i = 0\,(3 \leq i \leq h-1),$$
$$b^3_1 = -8n,\ b^3_i = 0\,(2 \leq i \leq h-1),\ c^1 = n,\ d^1 = 10n,$$

where n is an integer, while $a^l_{i,j}$ and e^1 are arbitrary integers.

(2)$\Longrightarrow$(1) : Such an element x belongs to $R \cap [F, F]$ because $e^*(x) = 0$ for every generator e. The value $c(x)$ can be calculated by using Lemma 3.2:

$$\begin{aligned} c(x) &= n\, c(C^1) + 10n\, c(D^1) \\ &= -6n + 10n \\ &= 4n. \end{aligned}$$

This completes the proof of Proposition 3.5. $\square$

TABLE 3.6.

	y_1^*	y_2^*	u_1^*	u_2^*	$\cdots$	u_{h-2}^*	u_{h-1}^*	u_h^*	z_1^*	z_2^*	$\cdots$	z_{h-2}^*	z_{h-1}^*	c
B_1^1	1		-1											0
B_2^1		1		-1										0
B_1^2			1						-1					0
B_2^2				1						-1				0
$\vdots$					$\ddots$						$\ddots$			$\vdots$
B_{h-2}^2						1						-1		0
B_{h-1}^2							1						-1	0
B_1^3				-1					1					0
B_2^3					$\ddots$					1				0
$\vdots$						$\ddots$					$\ddots$			$\vdots$
B_{h-2}^3							-1					1		0
B_{h-1}^3								-1					1	0
C^1	-4	2	-4	0	$\cdots$	0	0	0	-4	0	$\cdots$	0	0	-6
D^1	1	-2	0	0	$\cdots$	0	0	0	1	1	$\cdots$	0	0	1

(The blanks in the table above mean that the corresponding value is equal to zero.)

REMARK 3.7. Proposion 3.5 implies that the 'signature' $c(x)$ of a '2-cycle' $x \in R \cap [F, F]$ of $\mathcal{M}_h$ is concentrated on relators $B_1^1, B_2^1, B_1^2, B_2^2, B_1^3, C^1, D^1$ of Wajnryb's presentation and the algebraic number $m(R^1)$ of a relator R^1 is independent of the genus $h(\geq 3)$.

4. A construction of holonomy homomorphisms

We now construct the holonomy homomorphism $\chi : \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_h$ of a surface bundle ξ over a surface Σ_g with non-zero signature. We use a simple technique of the commutator collection process (see [7][15]) to construct χ.

DEFINITION 4.1. Let F_n be the free group on the n free generators $e_1, \cdots, e_n$ and let a, b, u, v and w be words in $e_1, \cdots, e_n$. Two words u and v are called *freely equal* (denoted $u \approx v$) if they determine the same element of F_n.

The α-*skip* is the following sequence of free equalities:

$$\begin{aligned} uava^{-1}w &\approx u(ava^{-1}v^{-1})vw \\ &= u[a, v]vw \end{aligned}$$

and the β-*skip* is the following sequence of free equalities:

$$\begin{aligned} uavba^{-1}b^{-1}w &\approx u(avba^{-1}b^{-1}v^{-1})vw \\ &= u[a, vb]vw, \end{aligned}$$

where $[a, b] := aba^{-1}b^{-1}$. (We used the commutator relation $ba \approx [b, a]ab$.)

We apply α- and β-skips to elements of the free group F on the generators $y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}$ defined in the preceding section and prove the following lemma.

Lemma 4.2. *Suppose that $h \geq 3$. There exists a word W in $y_1, y_2, u_1, \cdots, u_h, z_1, \cdots, z_{h-1}$ with the following properties:*

(1) *W is a product of 111 commutators;*
(2) *W belongs to $R \cap [F, F]$ as an element of F;*
(3) *$c(W) = 4$.*

Proof. We set

$$\begin{aligned}
\widetilde{W_1} &:= (B_1^2)^{-1}(B_1^1)^{-3}B_2^1B_2^2D^1, \\
\widetilde{W_2} &:= B_2^1(B_1^3)^{-1}B_2^1B_2^2D^1, \\
\widetilde{W} &:= C^1\widetilde{W}_8^2\widetilde{W}_2^8.
\end{aligned}$$

Since the word $\widetilde{W}$ satisfies the condition (2) of Proposition 3.5 in case $n = 1$, $\widetilde{W}$ has the properties (2) and (3) above. We decompose $\widetilde{W}$ to a product W of 111 commutators by using α- and β-skips repeatedly.

We rewrite some of Wajnryb's relators as follows:

$$\begin{aligned}
B_1^1 &= y_1R_1u_1^{-1} \quad (R_1 = [u_1, y_1]), \\
B_2^1 &= y_2R_2u_2^{-1} \quad (R_2 = [u_2, y_2]), \\
B_1^2 &= u_1R_3z_1^{-1} \quad (R_3 = [z_1, u_1]), \\
B_2^2 &= u_2R_4z_2^{-1} \quad (R_4 = [z_2, u_2]), \\
B_1^3 &= z_1R_5u_2^{-1} \quad (R_5 = [u_2, z_1]), \\
C^1 &= (y_1u_1z_1)^{-4}y_2^2R_6 \quad (R_6 = [y_2^{-1}, (u_2z_1u_1y_1^2u_1z_1u_2)^{-1}]), \\
D^1 &= y_1z_1z_2t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (R_7 = [y_2^{-1}, y_1u_1z_1u_2], \quad R_8 = [v^{-1}, (wu_1z_1u_2z_2u_3)^{-1}]),
\end{aligned}$$

where $R_1, \cdots R_8$ are commutators.

$\widetilde{W_i}(i = 1, 2)$ is transformed into another word $W_i(i = 1, 2)$ by using α- and β-skips in the following way:

$$\begin{aligned}
\widetilde{W_1} &= (B_1^2)^{-1}(B_1^1)^{-3}B_2^1B_2^2D^1 \\
&\approx z_1R_3^{-1}R_1^{-1}y_1^{-1}(u_1R_1^{-1}y_1^{-1})^2y_2R_2R_4z_2^{-1}y_1z_1z_2t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\underset{(\beta)}{\approx} z_1R_3^{-1}R_1^{-1}y_1^{-1}(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}y_1z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (S_1 := [y_2, R_2R_4z_2^{-1}y_1z_1z_2t_1t_2]) \\
&\underset{(\alpha)}{\approx} z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (S_2 := [y_1^{-1}, (u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}]) \\
&=: W_1;
\end{aligned}$$

$$\begin{aligned}
\widetilde{W}_2 &= B_2^1(B_1^3)^{-1}B_2^1B_2^2D^1 \\
&\approx y_2R_2R_5^{-1}z_1^{-1}y_2R_2R_4z_2^{-1}y_1z_1z_2t_1t_2y_2^{-1}t_2^{-1}t_1^{-1}t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\underset{(\beta)}{\approx} y_2R_2R_5^{-1}z_1^{-1}S_3R_2R_4z_2^{-1}y_1z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8 \\
&\qquad (S_3 := [y_2, R_2R_4z_2^{-1}y_1z_1z_2t_1t_2]) \\
&\underset{(\beta)}{\approx} S_4R_2R_5^{-1}z_1^{-1}S_3R_2R_4z_2^{-1}y_1z_1z_2R_7R_8 \\
&\qquad (S_4 := [y_2, R_2R_5^{-1}z_1^{-1}S_3R_2R_4z_2^{-1}y_1z_2t_2]) \\
&\underset{(\alpha)}{\approx} S_4R_2R_5^{-1}S_5S_3R_2R_4z_2^{-1}y_1z_2R_7R_8 \\
&\qquad (S_5 := [z_1^{-1}, S_3R_2R_4z_2^{-1}y_1]) \\
&=: W_2.
\end{aligned}$$

The word W_1 obtained above naturally includes 10 commutators and the word W_2 9 ones. Hence the word $C^1W_1^2W_2^8$ naturally includes 93 commutators.

Furthermore we perform 6 α-skips and 4 β-skips to $C^1W_1^2$ and get a word $\widehat{W}$ in the following way:

$$\begin{aligned}
C^1W_1^2 &= (y_1u_1z_1)^{-4}y_2y_2R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2 \\
&\quad \cdot S_1R_2R_4z_2^{-1}z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8W_1 \\
&\underset{(\beta)}{\approx} (y_1u_1z_1)^{-3}z_1^{-1}u_1^{-1}y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2 \\
&\quad \cdot S_1R_2R_4z_2^{-1}z_1z_2R_7R_8W_1 \\
&\qquad (S_6 := [y_2, R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2t_2]) \\
&\underset{(\beta)}{\approx} (y_1u_1z_1)^{-3}S_7u_1^{-1}y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2 \\
&\quad \cdot S_1R_2R_4R_7R_8W_1 \\
&\qquad (S_7 := [z_1^{-1}, u_1^{-1}y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}]) \\
&\underset{(\alpha)}{\approx} (y_1u_1z_1)^{-2}z_1^{-1}u_1^{-1}y_1^{-1}S_7S_8y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1} \\
&\quad \cdot u_1R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8W_1 \\
&\qquad (S_8 := [u_1^{-1}, y_1^{-1}y_2S_6R_6z_1R_3^{-1}R_1^{-1}S_2]) \\
&\underset{(\alpha)}{\approx} (y_1u_1z_1)^{-2}S_9u_1^{-1}y_1^{-1}S_7S_8y_1^{-1}y_2S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1} \\
&\quad \cdot u_1R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8W_1 \\
&\qquad (S_9 := [z_1^{-1}, u_1^{-1}y_1^{-1}S_7S_8y_1^{-1}y_2S_6R_6])
\end{aligned}$$

$$\underset{(\alpha)}{\approx} (y_1u_1z_1)^{-2}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}y_2S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8$$
$$\cdot z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2t_2y_2^{-1}t_2^{-1}R_7R_8$$
$$(S_{10} := [u_1^{-1}, y_1^{-1}S_7S_8y_1^{-1}y_2S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}])$$
$$\underset{(\beta)}{\approx} (z_1^{-1}u_1^{-1}y_1^{-1})^2S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2R_7R_8$$
$$(S_{11} := [y_2, S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8$$
$$\cdot z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}z_1z_2t_2])$$
$$\underset{(\beta)}{\approx} z_1^{-1}u_1^{-1}y_1^{-1}S_{12}u_1^{-1}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4R_7R_8$$
$$(S_{12} := [z_1^{-1}, u_1^{-1}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2(u_1R_1^{-1}y_1^{-1})^2S_1R_2R_4z_2^{-1}])$$
$$\underset{(\alpha)}{\approx} z_1^{-1}u_1^{-1}y_1^{-1}S_{12}S_{13}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}u_1R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8$$
$$(S_{13} := [u_1^{-1}, y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2])$$
$$\underset{(\alpha)}{\approx} z_1^{-1}S_{14}y_1^{-1}S_{12}S_{13}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8$$
$$(S_{14} := [u_1^{-1}, y_1^{-1}S_{12}S_{13}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8z_1R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}])$$
$$\underset{(\alpha)}{\approx} S_{15}S_{14}y_1^{-1}S_{12}S_{13}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8$$
$$(S_{15} := [z_1^{-1}, S_{14}y_1^{-1}S_{12}S_{13}y_1^{-1}S_9S_{10}y_1^{-1}S_7S_8y_1^{-1}S_{11}S_6R_6R_3^{-1}R_1^{-1}S_2R_1^{-1}y_1^{-1}$$
$$\cdot R_1^{-1}y_1^{-1}S_1R_2R_4R_7R_8])$$
$$=: \widehat{W}$$

The word $\widehat{W}$ is a product of 31 commutators and 8 copies of y_1^{-1}. The word W_2^8 is a product of 72 commutators and 8 copies of $z_1^{-1}y_1z_1$.

We perform 8 β-skips to the word $\widehat{W}W_2^8$ repeatedly by setting $a = y_1^{-1}$ and $b = z_1^{-1}$ in Definition 4.1. Then we obtain a word W which is a product of 111(=31+72+8) commutators and is freely equal to $\widetilde{W}$. This completes the proof of Lemma 4.2. □

By virtue of Lemma 4.2, we can show the following theorem.

Theorem 4.3. *There exists a Σ_h-bundle $\xi = (E, \Sigma_g, p, \Sigma_h, Diff_+\Sigma_h)$ over Σ_g with $g = 111, h = 3$ and $\tau(E) = -4$.*

Proof. Set $g = 111$ and $h = 3$. We choose a word W which satisfies conditions (1)-(3) of Lemma 4.2 and write

$$W = \prod_{i=1}^{g}[\alpha_i, \beta_i] \quad (\alpha_i, \beta_i \in F(i = 1, \cdots, g)).$$

Let $\widetilde{\chi} : \widetilde{F} \longrightarrow F$ the homomorphism defined by:

$$\widetilde{\chi}(\widetilde{a}_i) = \alpha_i, \quad \widetilde{\chi}(\widetilde{b}_i) = \beta_i \quad (i = 1, \cdots, g),$$

where $\widetilde{F} =< \widetilde{a}_1, \cdots, \widetilde{a}_g, \widetilde{b}_1, \cdots, \widetilde{b}_g >$. Since $\widetilde{\chi}(\widetilde{r}) = W \in R \cap [F, F]$, $\widetilde{\chi}$ induces the homomorphism $\chi : \pi_1(\Sigma_g) \longrightarrow \mathcal{M}_h$ (i.e., $\pi \circ \widetilde{\chi} = \chi \circ \widetilde{\pi}$) as in Section 2. For the Σ_h-bundle ξ over Σ_g which has χ as its holonomy homomorphism, we calculate the signature of its total space E:

$$\begin{aligned}\tau(E) &= -c(\widetilde{\chi}(\widetilde{r})) \\ &= -c(W) \\ &= -4.\end{aligned}$$

We have thus proved the theorem. □

Finally, we prove our main theorem (Theorem 1.2) by using Lemma 4.2 and results of Lück [14] concerning about L^2-Betti numbers of groups.

Proof of Theorem 1.2. Let W be the word constructed in the proof of Lemma 4.2. For every $h \geq 3$ and each $n \in \mathbb{Z}(n \neq 0)$, we can construct a Σ_h-bundle $\xi = \hat{\xi}(h, n)$ with $g = 111|n|$ and $\tau(E) = 4n$ by using the word W^{-n} as in the proof of Theorem 4.3 (see Remark 3.7). Therefore we have

$$g(h, n) \leq 111|n|.$$

On the other hand, for every Σ_h-bundle ξ over Σ_g with $g \geq 1, h \geq 3$ and $\tau(E) = 4n$, the associated exact sequence:

$$1 \longrightarrow \pi_1(\Sigma_h) \longrightarrow \pi_1(E) \xrightarrow{p_\sharp} \pi_1(\Sigma_g) \longrightarrow 1$$

of fundamental groups satisfies the assumption of [14,Theorem 4.1]. Then the first L^2-Betti number $b_1(\pi_1(E))$ of $\pi_1(E)$ is equal to zero and the Winkelnkemper-type inequality $\chi(E) \geq |\tau(E)|$ holds from [14,Theorem 5.1]. By substituting

$$\chi(E) = \chi(\Sigma_h)\chi(\Sigma_g) = 4(h-1)(g-1), \quad \tau(E) = 4n$$

for the inequality, we obtain

$$g(h, n) \geq \frac{|n|}{h-1} + 1$$

and this completes the proof of our theorem. □

REMARK 4.4. The Σ_h-bundle $\xi = \hat{\xi}(h,n)$ over Σ_g constructed in the first half of the proof of Theorem 1.2 has $g = 111|n|$, $\tau(E) = 4n$, $b_1(E) = 2(111|n| + h - 3)$, $b_2(E) = 2(222|n|h - 5)$ and $\chi(E) = 4(111|n| - 1)(h-1)$, where $h(\geq 3)$ and $n \in \mathbb{Z}(n \neq 0)$. If the total space E admits a complex structure, E is an algebraic surface of general type and satisfies the Noether condition, the Noether inequality and the Bogomolov-Miyaoka-Yau inequality (cf. [2]). But E cannot be a geometric 4-manifold in the sense of Thurston [20], in particular, a compact Kähler surface covered by the unit ball in $\mathbb{C}^2$.

Let $\Gamma(h,n)$ be the fundamental group of the total space of $\xi = \hat{\xi}(h,n)(h \geq 3, n \geq 1)$ constructed in the first half of the proof of Theorem 1.2. Computing an invariant defined by Johnson [11], we obtain the following result.

Corollary 4.5. *The family $\{\Gamma(h,n)\}_{h\geq 3, n\geq 1}$ contains infinitely many commensurability classes of discrete groups. In particular, $\{\Gamma(h,n)\}_{n\geq 1}$ is a family of infinitely many non-commensurable discrete groups for each $h(\geq 3)$.*

Proof. The commensurability invariant $\gamma(\Gamma)$ [11] for $\Gamma = \Gamma(h,n)$ is

$$\gamma(\Gamma(h,n)) = \frac{n}{(111n-1)(h-1)} \quad (h \geq 3, n \geq 1),$$

which runs over infinitely many rational numbers. □

REMARK 4.6. Although the author attempted to show that the value $g(h,n)$ does not depend on the genus $h(\geq 3)$ of fiber Σ_h for each $n \in \mathbb{Z}(n \neq 0)$, it was not achieved because of some serious transformation problems on words in free generators.

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