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A REMARK ON JAMES NUMBERS OF STIEFEL MANIFOLDS

Dedicated to Professor Nobuo Shimada on his 60th birthday

HIDEAKI ŌSHIMA

(Received July 13, 1983)

1. Introduction

The purpose of this note is to supply a few relations between the unstable and stable James numbers of Stiefel manifolds.

Let \( F \) be the field \( H \) of the quaternions or the field \( C \) of the complex numbers, and \( d \) the dimension of \( F \) over the field of the real numbers. Let \( G(F^n) \) be the symplectic group \( Sp(n) \) or the unitary group \( U(n) \) according as \( F \) is \( H \) or \( C \). The stunted quasi-projective space \( Q_{n,k} = Q_n/Q_{n-k} \) is a subspace of the Stiefel manifold \( O_{n,k} = G(F^n)/G(F^{n-k}) \) (see e.g. [8]). There exist the quotient maps \( q_r : Q_{n,k} \rightarrow Q_{n,k-r} \) and \( p_r : O_{n,k} \rightarrow O_{n,k-r} \). Let \( i' : Q_{n,k} \rightarrow O_{n,k} \) be the inclusion map. Then \( i'^* q_r = p_r i'' \) and \( i'^* O_{n,k} = O_{n,k} \) is the identity map of the \((dn-1)\)-dimensional sphere \( S^{dn-1} \).

Applying the homotopy functor \( \pi_{dn-1}(\ ) \) and the stable homotopy functor \( \pi_{dn-1}(\ ) \) to \( q_{k-1} \) and \( p_{k-1} \), we define the unstable James numbers (see [7]) \( Q\{n,k\} = Q_F\{n,k\}, O\{n,k\} = O_F\{n,k\} \) and the stable James numbers \( Q^*\{n,k\} = Q_F^*\{n,k\}, O^*\{n,k\} = O_F^*\{n,k\} \) by the following equations:

\[
q_{k-1}\pi_{dn-1}(Q_{n,k}) = Q\{n,k\} \pi_{dn-1}(S^{dn-1}),
\]
\[
p_{k-1}\pi_{dn-1}(O_{n,k}) = O\{n,k\} \pi_{dn-1}(S^{dn-1}),
\]
\[
q_{k-1}\pi_{dn-1}(Q_{n,k}) = Q^*\{n,k\} \pi_{dn-1}(S^{dn-1}),
\]
\[
p_{k-1}\pi_{dn-1}(O_{n,k}) = O^*\{n,k\} \pi_{dn-1}(S^{dn-1});
\]

whenever \( 1 \leq k \leq n \). As easily seen (see e.g. [12]), we have

\[
Q^*\{n,k\} | Q\{n,k\}, O^*\{n,k\} | O\{n,k\}, O\{n,k\} | Q\{n,k\}, O\{n,k\} | Q\{n,k\}, O\{n,k\} | O\{n,k\}, O\{n,k\} | O\{n, k+1\} \text{ and } O\{n,k\} | O\{n,k+1\};
\]

where \( a | b \) means that \( b \) is a multiple of \( a \). In [12] we proved

\[
Q^*\{n,k\} = O^*\{n,k\}.
\]

The stable James number \( O^*\{n,k\} \) has been investigated by various au-
thors, but the unstable ones $Q\{n, k\}$, $O\{n, k\}$ have been done not so much (see e.g. [7], [13], [15], [18]). By [2], [3], [4] we have

\[
\begin{align*}
O^H\{n, n\} &= \begin{cases} 
2 \cdot (2n-1)! & \text{if } n \text{ is even} \\
(2n-1)! & \text{if } n \text{ is odd}; 
\end{cases} \\
O^c\{n, n\} &= O^c\{n, n-1\} = (n-1)!.
\end{align*}
\]

Our first result is an easy consequence of the results of Mukai [10], [11].

**Theorem 1.** (i) $O^H\{n, n\} = a \cdot Q^H\{n, n\}$, where $a=1$ if $n$ is even, $a=1$ or $1/2$ if $n$ is odd.

(ii) $O^c\{n, n\} = O^c\{n, n-1\} = O^c\{n, n-1\} = Q^c\{n, n-1\}$.

Let $E^\ast : \pi_r( ) \to \pi_s( )$ be the stabilization homomorphism. Since $Q^s_{n,k}$ and $O^s_{n,k}$ are $(d(n-k+1)-2)$-connected (see e.g. [8]), it follows from Freudenthal suspension theorem that $E^\ast : \pi_{d-1}(Q_{n,k}) \to \pi_{d-1}(O_{n,k})$ and $E^\ast : \pi_{d-1}(O_{n,k}) \to \pi_{d-1}(O_{n,k})$ are surjective whenever $n \geq 2k-1$. Thus $Q\{n, k\} = Q^\ast\{n, k\}$ and $O\{n, k\} = O^\ast\{n, k\}$ if $n \geq 2k-1$. As seen in [13], if $n<2k-1$, then $O\{n, k\} \neq O^\ast\{n, k\}$ in general.

We consider the case $n=2k-2$. Since $(O_{2k-2,k}, Q_{2k-2,k})$ is $(2dk-d-3)$-connected (see e.g. [8]) and $d(2k-2)-1 \leq 2dk-d-3$, it follows that $i^\ast : \pi_{d(2k-2)-1}(O_{2k-2,k}) \to \pi_{d(2k-2)-1}(O_{2k-2,k})$ is surjective, so that

\[
\begin{align*}
O\{2k-2, k\} &= Q\{2k-2, k\}.
\end{align*}
\]

Our second result is

**Theorem 2.** (iii) If $F=H$ or $F=C$ and $k$ is odd, then $O\{2k-2, k\} = O^\ast\{2k-2, k\}$.

(iv) If $F=H$ and $k$ is even, then $O\{2k-2, k\}/O^\ast\{2k-2, k\} = 1$ or 2.

**Remark 1.** In [13] we proved (iv) by a different method from the one in this note, and showed that $O^c\{2k-2, k\}/O^c\{2k-2, k\}$ is 1 if $k=2, 6$ and it is 2 if $k=4, 8$.

**Remark 2.** In [13] we did not determine $O_H\{8, 5\}$. Now (iii) says that $O_H\{8, 5\} = O^\ast\{8, 5\}$ which was calculated in [12].

**Remark 3.** I know of no case where $O^H\{n, k\} \neq O^H\{n, k\}$.

2. **Proof of Theorem 1**

The assertions are trivial when $n=1$. So we assume that $n \geq 2$.

Let $E$ denote both the reduced suspension functor in the category of pointed spaces and the suspension homomorphism in homotopy groups. For a continuous map $f : S^r \to X$, we denote the order of $f$ in $\pi_r(X)$ and $\pi^r(X)$ by $\# f$
and \( \#E^{-f} \), respectively.

As well known (see e.g. [8]), we have a \( CW \)-decomposition

\[
Q_n = Q_{n,n} = e^0 \cup e^{d-1} \cup e^{2d-1} \cup \cdots \cup e^{dn-1}
\]

such that \( Q_m \) is a subcomplex of \( Q_n \) provided \( m < n \), so

\[
(2.1) \quad Q_{n,k} = e^0 \cup e^{(n-k+1)-1} \cup \cdots \cup e^{dn-1}.
\]

Let \( T'_{n-1}: (B^{dn-1}, S^{dn-2}) \to (Q_n, Q_{n-1}) \) be a characteristic map of the top cell, and let \( T_{n-1}: S^{dn-2} \to Q_{n-1} \) be the restriction of \( T'_{n-1} \) to \( S^{dn-2} \), the boundary of the disk \( B^{dn-1} \). Let also \( T'_{n-1,k-1}=\xi_{n-k} \circ T'_{n-1}: (B^{dn-1}, S^{dn-2}) \to (Q_{n,k}, Q_{n-1,k-1}) \) and \( T_{n-1,k-1}=q_{n-k} \circ T_{n-1}: S^{dn-2} \to Q_{n-1,k-1} \).

Applying \( \pi_* \) to the cofibre sequence

\[
S^{dn-2} \xrightarrow{T_{n-1,k-1}} Q_{n-1,k-1} \xrightarrow{Q_{n,k}} S^{dn-1},
\]

we obtain the exact sequence

\[
\pi_{dn-1}(Q_{n,k}) \xrightarrow{q_{k-1}^*} \pi_{dn-1}(S^{dn-1}) \xrightarrow{(ET_{n-1,k-1})_k^*} \pi_{dn-1}(E\Omega_{n-1,k-1}).
\]

It follows from the cell structure of \( Q_{n-1,k-1} \) that \( \pi_{dn-1}(E\Omega_{n-1,k-1}) \) is finite, so \( \#E^{-f}T_{n-1,k-1} \) is finite. Hence the exactness implies that

\[
(2.2) \quad Q'\{n, k\} = \#E^{-f}T_{n-1,k-1}.
\]

Next we see the unstable case. Consider the homotopy exact sequence of the pair \((Q_{n,k}, Q_{n-1,k-1})\):

\[
\pi_{dn-1}(Q_{n,k}) \xrightarrow{\partial} \pi_{dn-1}(Q_{n,k}) \xrightarrow{\partial} \pi_{dn-1}(Q_{n,1}) \xrightarrow{\partial} \pi_{dn-1}(Q_{n,1}).
\]

By definition \( \partial(T'_{n-1,k-1})=T_{n-1,k-1}. \) Let \( q'(Q_{n,k}, Q_{n-1,k-1}) \to (S^{dn-1}, *) \) be the collapsing map. Then \( q'^*(T'_{n-1,k-1}) \) generates \( \pi_{dn-1}(S^{dn-1}) \). If \( n > k \) or \( F=H \), then, by Blakers–Massey [1], \( q'^*: \pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1}) \to \pi_{dn-1}(S^{dn-1}) \) is an isomorphism, so \( T'_{n-1,k-1} \) generates \( \pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1}) \). Since \( q^* \circ j^* = q_{k-1} \), it follows that the order of \( T_{n-1,k-1} \) is equal to the order of the cokernel of \( q_{k-1}^* \):

\[
\pi_{dn-1}(Q_{n,k}) \to \pi_{dn-1}(S^{dn-1}) \text{ provided } n > k \text{ or } F=H.
\]

Hence the following lemma implies that

\[
(2.3) \quad Q\{n, k\} = \#T_{n-1,k-1} \text{ if } n > k \text{ or } F=H.
\]

**Lemma (2.4).** The order of \( T_{n-1,k-1} \) is finite if \( n > k \) or \( F=H \).

Since \( T_{n-1,k-1}=q_{n-k} \circ T_{n-1} \) and since \( T_{n-1,k-1}=q_{n-k-1} \circ T_{n-1,n-2} \) if \( n > k \), it is sufficient for proving (2.4) to show that \( \#T_{n-1} \) is finite if \( F=H \), and \( \#T_{n-1,n-2} \) is finite if \( F=C \).
The rest of this section is devoted to the proofs of (2.4) and Theorem 1.

We consider the case \( F = H \). In [11] Mukai proved that \( \#T_{n-1} = \#E^rT_{n-1} = 2 \cdot (2n-1)! \) if \( n \) is even; \( \#E^rT_{n-1} = (2n-1)! \) and \( \#T_{n-1}/(2n-1)! = 1 \) or 2 if \( n \) is odd. Hence we obtain (2.4) and (1) follows from (1.1), (1.2), (1.3), (2.2), (2.3).

We see the case \( F = C \). Let \( P_n \) be the \((n-1)\)-dimensional complex projective space, and let \( P^*_n \) be the union of \( P_n \) and a base point. We then have \( Q_n = E(P^*_n) \) and \( Q_{n,n-1} = EP_n \) (see e.g. [8]). Note that there is a homotopy equivalence \( E(P^*_n) \cong EP_n \vee S^1 \) which makes the following triangle commutative up to homotopy:

\[
\begin{array}{ccc}
Q_n = E(P^*_n) & \cong & EP_n \vee S^1 \\
\downarrow q \downarrow \ \\
Q_{n,n-1} = EP_n \\
\end{array}
\]

where \( p \) is the projection. Hence \( q \) has a left homotopy inverse, so

\[
(2.5) \quad q_! : \pi_{2n-1}(Q_n) \rightarrow \pi_{2n-1}(Q_{n,n-1}) \text{ is surjective.}
\]

Let \( SU(n) \) be the special unitary group and let \( h : O_{n,n-1} = U(n)/(U(1) \times 1_{n-1}) \rightarrow SU(n) \) be the homeomorphism defined by

\[
h(A \mod U(1) \times 1_{n-1}) = A \left( \begin{array}{ccc} 1/|A| & & \\
1 & & \\
& \ddots & \\
& & 1 \end{array} \right).
\]

Note that \( h \circ i^*_\ast : EP_n = Q_{n,n-1} \rightarrow SU(n) \) is the inclusion map defined in [20]. Hence in the following commutative diagram \( h \circ i^*_\ast \) is surjective by Proposition 4.2 of [16].

\[
\begin{array}{ccc}
\pi_{2n-1}(Q_n) & \xrightarrow{i^*_\ast} & \pi_{2n-1}(U(n)) \\
\downarrow q_! \downarrow & \cong & \downarrow p_! \downarrow \\
\pi_{2n-1}(Q_{n,n-1}) & \xrightarrow{i^*_\ast} & \pi_{2n-1}(O_{n,n-1}) \rightarrow \pi_{2n-1}(SU(n)) \\
\end{array}
\]

It follows that the lower \( i^*_\ast \) is surjective and so is the upper \( i^*_\ast \) from (2.5). Thus we have

\[
(2.6) \quad Q_c \{n, n\} = O_c \{n, n\}.
\]

On the other hand we can take \( T_{n-1,n-2} = E\gamma_{n-1} \) where \( \gamma_{n-1} : S^{2n-3} \rightarrow P_{n-1} \) is the canonical \( S^1 \)-fibration. It is well known (see e.g. [10]) that \( \#E\gamma_{n-1} = \#E^{r}\gamma_{n-1} = (n-1)! \). Thus we have (2.4) and

\[
(2.7) \quad Q_c \{n, n-1\} = Q_c \{n, n-1\} = (n-1)!
\]
by (2.2), (2.3). Therefore (ii) follows from (1.1), (1.2), (1.3), (2.6), (2.7). This completes the proofs of (2.4) and Theorem 1.

3. EHP-sequence

Let \( X, Y \) be \( r \)-connected \( CW \)-complexes which have exactly one vertex \(*\), and let \( f: X \to Y \) be a continuous map with \( f(*)=\ast \). We then have a diagram consisting of the exact \( E{\Pi}_j \)-sequences for \( i \leq 3r+1 \) (see e.g. [9], [19]):

\[
\begin{array}{cccc}
\pi_i(X) & \to & \pi_{i+1}(EX) & \to & \pi_{i+1}(E(X \wedge X)) & \to & \pi_{i-1}(X) & \to & \cdots \\
\downarrow f_* & & \downarrow (Ef)_* & & \downarrow (Ef\wedge f)_* & & \downarrow f_* & & \downarrow \\
\pi_i(Y) & \to & \pi_{i+1}(EY) & \to & \pi_{i+1}(E(Y \wedge Y)) & \to & \pi_{i-1}(Y) & \to & \cdots
\end{array}
\]

In the next section we shall use \( \text{Lemma (3.1).} \) The above diagram commutes.

By using Theorem 5.3 of [6] and following faithfully the construction of the \( E{\Pi}_j \)-sequence, we can prove (3.1). We omit the details.

4. Proof of Theorem 2

For an abelian group \( A \), \( A/\text{Tor} \) denotes the quotient group of \( A \) by its torsion subgroup, and \( \pi: A \to A/\text{Tor} \) denotes the quotient homomorphism. Let \( Z \) be the infinite cyclic group.

By (2.1) we have

\[
\pi_{2n-1}(Q_{n,k})/\text{Tor} \cong Z.
\]

It follows that \( Q^r\{n, k\} \neq 0 \) from (2.2) and that \( Q\{n, k\} \neq 0 \) from (2.3), (2.6), (1.3). Thus we have

\[ \text{Lemma (4.2).} \quad \pi \circ E^\circ \neq 0: \pi_{2n-1}(Q_{n,k}) \to \pi_{2n-1}(Q_{n,k})/\text{Tor}. \]

From now on we denote \( Q_{2k-2,k} \) by \( Q \). By (1.1), (1.2), (1.4), (4.1) and (4.2), Theorem 2 is equivalent to

\[ \text{Proposition (4.3).} \quad \text{Let } n=2k-2. \text{ Then the image of } \pi \circ E^\circ: \pi_{2n-1}(Q) \to \pi_{2n-1}(Q)/\text{Tor} \text{ is } a \cdot \pi_{2n-1}(Q)/\text{Tor}, \text{ where } a=1 \text{ if } F=H \text{ or } k \text{ is odd, } a=1 \text{ or } 2 \text{ if } F=C \text{ and } k \text{ is even.} \]

Proof. We consider the case \( F=C \) only, because we can prove the assertion for the case \( F=H \) by a similar but slightly easier method to the following one.

If \( k=2 \), then the assertion is trivial by Theorem 1. So we assume that \( k \geq 3 \). By (2.1) we have

\[
Q = e^0 \cup e^{2k-3} \cup e^{2k-1} \cup \cdots \cup e^{4k-5},
\]
and so

\[(4.4) \quad Q \wedge Q = \varepsilon^8 \cup e^{4k-6} \cup e^{4k-4} \cup \cdots \cup e^{4k-10}.\]

Let \(i: S^{2k-3} \rightarrow e^6 \cup e^{2k-3} = Q_{2k-1,1} \rightarrow Q\) be the inclusion. Since \(Q\) is \((2k-4)\)-connected, it follows that

\[(4.5) \quad E^\infty: \pi_{4k-3}(\mathbb{E}^3Q) \rightarrow \pi_{4k-5}(Q) \text{ is an isomorphism, and}\]

\[(4.6) \quad E: \pi_{4k-3}(\mathbb{E}^3Q) \rightarrow \pi_{4k-5}(\mathbb{E}^3Q) \text{ is surjective.}\]

By (3.1) we have the commutative diagram:

\[
\begin{array}{ccc}
\pi_{4k-1}(\mathbb{E}^3S^{2k-3}) & \xrightarrow{H} & \pi_{4k-1}(\mathbb{E}^3(S^{2k-3} \wedge S^{2k-3})) \\
\downarrow{(E^i)_*} & & \downarrow{(E^3(i \wedge i))_*} \\
\pi_{4k-1}(\mathbb{E}^3Q) & \xrightarrow{H} & \pi_{4k-1}(\mathbb{E}^3(Q \wedge Q)) \\
\end{array}
\]

\[(4.7) \quad E: \pi_{4k-3}(\mathbb{E}^3Q)/\text{Tor} \simeq \pi_{4k-5}(\mathbb{E}^3Q)/\text{Tor}.\]

Consider the EHP-sequence:

\[
\pi_{4k-3}(\mathbb{E}^3Q \wedge Q) \rightarrow \pi_{4k-4}(\mathbb{E}^3Q) \rightarrow \pi_{4k-3}(\mathbb{E}^3Q) \rightarrow \pi_{4k-3}(\mathbb{E}^3Q \wedge Q).
\]

By (4.4) and Blakers-Massey [1], \(\pi_{4k}(\mathbb{E}^3(Q \wedge Q), S^{4k-1}) \simeq \pi_{4k-1}(\mathbb{E}^3(Q \wedge Q), S^{4k-1}) \simeq 0\), so the above \((\mathbb{E}^3(i \wedge i))_*\) is an isomorphism. As well known (see e.g. Proposition 2.7 of [17]), the upper \(H\) is not zero, hence so is the lower \(H\). Thus the image of \(P\) is finite, so that, by (4.6), \(E\) induces an isomorphism

\[(4.8) \quad \pi_{4k-3}(\mathbb{E}^3(Q \wedge Q)) \text{ is finite, and}\]

\[(4.9) \quad \pi_{4k-3}(\mathbb{E}^3(Q \wedge Q)) \simeq Z.\]

The kernel of \(E\) is finite by (4.8). The cokernel of \(E\) is torsion free by (4.9), while it is finite by (4.1), (4.2), (4.5), (4.7), hence it is zero, so \(E\) is surjective. Thus \(E\) induces an isomorphism

\[(4.10) \quad E: \pi_{4k-3}(\mathbb{E}^3Q)/\text{Tor} \simeq \pi_{4k-5}(\mathbb{E}^3Q)/\text{Tor}.\]

By (3.1) we have the following commutative diagram:

\[
\begin{array}{cccc}
\pi_{4k-3}(S^{2k-3}) & \xrightarrow{E} & \pi_{4k-4}(S^{2k-3}) & \xrightarrow{H} & \pi_{4k-6}(S^{2k-3}) \\
\downarrow{i_*} & & \downarrow{(E^i)_*} & & \downarrow{(E(i \wedge i))_*} \\
\pi_{4k-3}(Q) & \xrightarrow{E} & \pi_{4k-4}(\mathbb{E}^3Q) & \xrightarrow{H} & \pi_{4k-6}(\mathbb{E}^3Q) \\
\end{array}
\]
Here $\eta_2: S^3 \to S^2$ is the Hopf map and $\eta_m = E^{m-2} \eta_2: S^{m+1} \to S^m$ for $m \geq 2$. By (4.4) and Blakers–Massey [1], $\pi_{4k-4}(E(Q \land Q), S^{4k-5}) \simeq 0$. Thus $(E(i \wedge i))_*$ is surjective, so $\pi_{4k-4}(E(Q \land Q)) \cong \mathbb{Z}_2$ or 0. Hence the cokernel of the lower $E$ is $\mathbb{Z}_2$ or 0. Since $\pi_{4k-4}(EQ)/Tor \cong \mathbb{Z}$ by (4.1), (4.5), (4.7), (4.10), it follows that the image of the homomorphism $\text{mod} \ E: \pi_{4k-5}(Q)/Tor \to \pi_{4k-4}(EQ)/Tor$ induced by $E$ is $a \pi_{4k-4}(EQ)/Tor$, where $a = 1$ or 2. Thus the assertion of (4.3) for $k$ even follows from (4.5), (4.7) and (4.10). We can prove (4.3) for $k$ odd by showing that $\pi_{4k-4}(E(Q \land Q))$ is $\mathbb{Z}_2$ if $k$ is even and 0 if $k$ is odd. But we will take a different method which can be applied to the case $F = H$.

As well known (see e.g. [19]), $P(\eta_{4k-3}) = [l_{2k-3}, \eta_{2k-3}]$, the Whitehead product, where $l_{2k-3}$ is the identity map of $S^{2k-3}$. It follows from [5] that $[l_{2k-3}, \eta_{2k-3}] = 0$ if and only if $k$ is odd. We show that $\tilde{E}: \pi_{4k-5}(Q)/Tor \to \pi_{4k-4}(EQ)/Tor$ is surjective if $k$ is odd. Then the assertion of (4.3) for $k$ odd follows from (4.5), (4.7) and (4.10).

Let $k$ be odd. Then there is an element $x$ in $\pi_{4k-4}(S^{2k-2})$ such that $H(x) = \eta_{4k-3}$ by exactness. Hence $H((Ei)_*(x)) = E(i \wedge i)_*(H(x)) = E(i \wedge i)_*(\eta_{4k-3})$ which generates $\pi_{4k-4}(E(Q \land Q))$. Choose $y$ in $\pi_{4k-4}(EQ)$ such that $\pi(y)$ generates the infinite cyclic group $\pi_{4k-4}(EQ)/Tor$. If $H(y) = 0$, then there exists $y'$ in $\pi_{4k-5}(Q)$ such that $E(y') = y$, so $\tilde{E}(\pi(y')) = \pi(y)$ and $\tilde{E}$ is surjective. If $H(y) \neq 0$, then $\pi_{4k-4}(E(Q \land Q))/Tor \cong \mathbb{Z}_2$ which is generated by $H(y)$. Hence $H(y) = H((Ei)_*(x))$ and there exists $y''$ in $\pi_{4k-5}(Q)$ such that $E(y'') = y - (Ei)_*(x)$. Since $\pi_{4k-4}(S^{2k-2})$ is finite as seen in [14], it follows that $(Ei)_*(x)$ has a finite order and $\tilde{E}(\pi(y'')) = \pi(y - (Ei)_*(x)) = \pi(y)$, so that $\tilde{E}$ is surjective. This completes the proofs of (4.3) and hence of Theorem 2.

References


