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A REMARK ON JAMES NUMBERS OF STIEFEL MANIFOLDS

Dedicated to Professor Nobuo Shimada on his 60th birthday

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1. Introduction

The purpose of this note is to supply a few relations between the unstable and stable James numbers of Stiefel manifolds.

Let F be the field H of the quaternions or the field C of the complex numbers, and d the dimension of F over the field of the real numbers. Let $G(F^n)$ be the symplectic group $Sp(n)$ or the unitary group $U(n)$ according as F is H or C. The stunted quasi-projective space $Q_{n,k}=Q_n/Q_{n-k}$ is a subspace of the Stiefel manifold $O_{n,k}=G(F^n)/G(F^{n-k})$ (see e.g. [8]). There exist the quotient maps $q_r\colon Q_{n,k}\to Q_{n,k-r}$ and $p_r\colon O_{n,k}\to O_{n,k-r}$. Let $i'\colon Q_{n,k}\to O_{n,k}$ be the inclusion map. Then $i'\circ q_r=p_r\circ i'$ and $i'\colon Q_{n,1}\to O_{n,1}$ is the identity map of the $(dn-1)$-dimensional sphere S^{dn-1}.

Applying the homotopy functor $\pi_{dn-1}(\ )$ and the stable homotopy functor $\pi^s_{dn-1}(\ )$ to q_{k-1} and p_{k-1}, we define the unstable James numbers (see [7]) $Q\{n, k\}=Q_F\{n, k\}$, $O\{n, k\}=O_F\{n, k\}$ and the stable James numbers $Q^s\{n, k\}=Q^s_F\{n, k\}$, $O^s\{n, k\}=O^s_F\{n, k\}$ by the following equations:

$$q_{k-1*}\pi_{dn-1}(Q_{n,k}) = Q\{n, k\}\pi_{dn-1}(S^{dn-1}),$$
$$p_{k-1*}\pi_{dn-1}(O_{n,k}) = O\{n, k\}\pi_{dn-1}(S^{dn-1}),$$
$$q_{k-1*}\pi^s_{dn-1}(Q_{n,k}) = Q^s\{n, k\}\pi^s_{dn-1}(S^{dn-1}),$$
$$p_{k-1*}\pi^s_{dn-1}(O_{n,k}) = O^s\{n, k\}\pi^s_{dn-1}(S^{dn-1});$$

whenever $1\leqq k\leqq n$. As easily seen (see e.g. [12]), we have

(1.1) $\quad Q^s\{n, k\}\,|\,Q\{n, k\}$, $O^s\{n, k\}\,|\,O\{n, k\}$, $O\{n, k\}\,|\,Q\{n, k\}$, $Q^s\{n, k\}\,|\,Q^s\{n, k+1\}$, $Q\{n, k\}\,|\,Q\{n, k+1\}$ *and* $O\{n, k\}\,|\,O\{n, k+1\}$;

where $a\,|\,b$ means that b is a multiple of a. In [12] we proved

(1.2) $$Q^s\{n, k\} = O^s\{n, k\}.$$

The stable James number $O^s\{n, k\}$ has been investigated by various authors, but the unstable ones $Q\{n, k\}$, $O\{n, k\}$ have been done not so much (see e.g. [7], [13], [15], [18]). By [2], [3], [4] we have

$$(1.3) \qquad O_H\{n, n\} = \begin{cases} 2\cdot(2n-1)! & \text{if } n \text{ is even} \\ (2n-1)! & \text{if } n \text{ is odd;} \end{cases}$$
$$O_C\{n, n\} = O_C\{n, n-1\} = (n-1)!\,.$$

Our first result is an easy consequence of the results of Mukai [10], [11].

Theorem 1. (i) *$O_H\{n, n\}=O^s_H\{n, n\}=a\cdot Q_H\{n, n\}$, where $a=1$ if n is even, $a=1$ or $1/2$ if n is odd.*

(ii) $O_C\{n, n\}=O^s_C\{n, n\}=Q_C\{n, n\}=O_C\{n, n-1\}=O^s_C\{n, n-1\}=Q_C\{n, n-1\}$.

Let E^∞: $\pi_r(\ )\rightarrow\pi^s_r(\ )$ be the stabilization homomorphism. Since $Q_{n,k}$ and $O_{n,k}$ are $(d(n-k+1)-2)$-connected (see e.g. [8]), it follows from Freudenthal suspension theorem that E^∞: $\pi_{dn-1}(Q_{n,k})\rightarrow\pi^s_{dn-1}(Q_{n,k})$ and E^∞: $\pi_{dn-1}(O_{n,k})\rightarrow\pi^s_{dn-1}(O_{n,k})$ are surjective whenever $n\geqq 2k-1$. Thus $Q\{n, k\}=Q^s\{n, k\}$ and $O\{n, k\}=O^s\{n, k\}$ if $n\geqq 2k-1$. As seen in [13], if $n<2k-1$, then $O\{n, k\}\neq O^s\{n, k\}$ in general.

We consider the case $n=2k-2$. Since $(O_{2k-2,k}, Q_{2k-2,k})$ is $(2dk-d-3)$-connected (see e.g. [8]) and $d(2k-2)-1\leqq 2dk-d-3$, it follows that i'_*: $\pi_{d(2k-2)-1}(Q_{2k-2,k})\rightarrow\pi_{d(2k-2)-1}(O_{2k-2,k})$ is surjective, so that

$$(1.4) \qquad O\{2k-2, k\} = Q\{2k-2, k\}\,.$$

Our second result is

Theorem 2. (iii) *If $F=H$ or $F=C$ and k is odd, then $O\{2k-2, k\}=O^s\{2k-2, k\}$.*

(iv) *If $F=C$ and k is even, then $O\{2k-2, k\}/O^s\{2k-2, k\}=1$ or 2.*

REMARK 1. In [13] we proved (iv) by a different method from the one in this note, and showed that $O_C\{2k-2, k\}/O^s_C\{2k-2, k\}$ is 1 if $k=2, 6$ and it is 2 if $k=4, 8$.

REMARK 2. In [13] we did not determine $O_H\{8, 5\}$. Now (iii) says that $O_H\{8, 5\}=O^s_H\{8, 5\}$ which was calculated in [12].

REMARK 3. I know of no case where $O_H\{n, k\}\neq O^s_H\{n, k\}$.

2. Proof of Theorem 1

The assertions are trivial when $n=1$. So we assume that $n\geqq 2$.

Let E denote both the reduced suspension functor in the category of pointed spaces and the suspension homomorphism in homotopy groups. For a continuous map f: $S^r\rightarrow X$, we denote the order of f in $\pi_r(X)$ and $\pi^s_r(X)$ by $\sharp f$

and $\# E^\infty f$, respectively.

As well known (see e.g. [8]), we have a CW-decomposition

$$Q_n = Q_{n,n} = e^0 \cup e^{d-1} \cup e^{2d-1} \cup \cdots \cup e^{dn-1}$$

such that Q_m is a subcomplex of Q_n provided $m<n$, so

$$Q_{n,k} = e^0 \cup e^{d(n-k+1)-1} \cup \cdots \cup e^{dn-1}. \tag{2.1}$$

Let $T'_{n-1}: (B^{dn-1}, S^{dn-2}) \to (Q_n, Q_{n-1})$ be a characteristic map of the top cell, and let $T_{n-1}: S^{dn-2} \to Q_{n-1}$ be the restriction of T'_{n-1} to S^{dn-2}, the boundary of the disk B^{dn-1}. Let also $T'_{n-1,k-1} = q_{n-k} \circ T'_{n-1}: (B^{dn-1}, S^{dn-2}) \to (Q_{n,k}, Q_{n-1,k-1})$ and $T_{n-1,k-1} = q_{n-k} \circ T_{n-1}: S^{dn-2} \to Q_{n-1,k-1}$.

Applying $\pi^s_*(\ \ )$ to the cofibre sequence

$$S^{dn-2} \xrightarrow{T_{n-1,k-1}} Q_{n-1,k-1} \to Q_{n,k} \xrightarrow{q_{k-1}} S^{dn-1},$$

we obtain the exact sequence

$$\pi^s_{dn-1}(Q_{n,k}) \xrightarrow{q_{k-1*}} \pi^s_{dn-1}(S^{dn-1}) \xrightarrow{(ET_{n-1,k-1})_*} \pi^s_{dn-1}(EQ_{n-1,k-1}).$$

It follows from the cell structure of $Q_{n-1,k-1}$ that $\pi^s_{dn-1}(EQ_{n-1,k-1})$ is finite, so $\# E^\infty T_{n-1,k-1}$ is finite. Hence the exactness implies that

$$Q^s\{n, k\} = \# E^\infty T_{n-1,k-1}. \tag{2.2}$$

Next we see the unstable case. Consider the homotopy exact sequence of the pair $(Q_{n,k}, Q_{n-1,k-1})$:

$$\pi_{dn-1}(Q_{n,k}) \xrightarrow{j_*} \pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1}) \xrightarrow{\partial} \pi_{dn-2}(Q_{n-1,k-1}).$$

By definition $\partial(T'_{n-1,k-1}) = T_{n-1,k-1}$. Let $q': (Q_{n,k}, Q_{n-1,k-1}) \to (S^{dn-1}, *)$ be the collapsing map. Then $q'_*(T'_{n-1,k-1})$ generates $\pi_{dn-1}(S^{dn-1})$. If $n>k$ or $F=H$, then, by Blakers–Massey [1], $q'_*: \pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1}) \to \pi_{dn-1}(S^{dn-1})$ is an isomorphism, so $T'_{n-1,k-1}$ generates $\pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1})$. Since $q'_* \circ j_* = q_{k-1*}$, it follows that the order of $T_{n-1,k-1}$ is equal to the order of the cokernel of q_{k-1*}: $\pi_{dn-1}(Q_{n,k}) \to \pi_{dn-1}(S^{dn-1})$ provided $n>k$ or $F=H$. Hence the following lemma implies that

$$Q\{n, k\} = \# T_{n-1,k-1} \quad \textit{if } n>k \textit{ or } F=H. \tag{2.3}$$

Lemma (2.4). *The order of $T_{n-1,k-1}$ is finite if $n>k$ or $F=H$.*

Since $T_{n-1,k-1} = q_{n-k} \circ T_{n-1}$ and since $T_{n-1,k-1} = q_{n-k-1} \circ T_{n-1,n-2}$ if $n>k$, it is sufficient for proving (2.4) to show that $\# T_{n-1}$ is finite if $F=H$, and $\# T_{n-1,n-2}$ is finite if $F=C$.

The rest of this section is devoted to the proofs of (2.4) and Theorem 1.

We consider the case $F=H$. In [11] Mukai proved that $\sharp T_{n-1}=\sharp E^{\infty}T_{n-1}=2\cdot(2n-1)!$ if n is even; $\sharp E^{\infty}T_{n-1}=(2n-1)!$ and $\sharp T_{n-1}/(2n-1)!=1$ or 2 if n is odd. Hence we obtain (2.4) and (i) follows from (1.1), (1.2), (1.3), (2.2), (2.3).

We see the case $F=C$. Let P_n be the $(n-1)$-dimensional complex projective space, and let P_n^+ be the union of P_n and a base point. We then have $Q_n=E(P_n^+)$ and $Q_{n,n-1}=EP_n$ (see e.g. [8]). Note that there is a homotopy equivalence $E(P_n^+)\simeq EP_n\vee S^1$ which makes the following triangle commutative up to homotopy:

$$\begin{array}{ccc} Q_n = E(P_n^+) & \simeq & EP_n\vee S^1 \\ q_1\searrow & & \swarrow p \\ & Q_{n,n-1} = EP_n & \end{array}$$

where p is the projection. Hence q_1 has a left homotopy inverse, so

(2.5) $\quad q_{1*}\colon \pi_{2n-1}(Q_n)\to\pi_{2n-1}(Q_{n,n-1})$ *is surjective.*

Let $SU(n)$ be the special unitary group and let $h\colon O_{n,n-1}=U(n)/(U(1)\times 1_{n-1})\to SU(n)$ be the homeomorphism defined by

$$h(A \bmod U(1)\times 1_{n-1}) = A\begin{pmatrix} 1/|A| & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{pmatrix}.$$

Note that $h\circ i'\colon EP_n=Q_{n,n-1}\to SU(n)$ is the inclusion map defined in [20]. Hence in the following commutative diagram $h_*\circ i'_*$ is surjective by Proposition 4.2 of [16].

$$\begin{array}{ccccc} \pi_{2n-1}(Q_n) & \xrightarrow{i'_*} & \pi_{2n-1}(U(n)) & & \\ q_{1*}\downarrow & & \cong\downarrow p_{1*} & & \\ \pi_{2n-1}(Q_{n,n-1}) & \xrightarrow{i'_*} & \pi_{2n-1}(O_{n,n-1}) & \underset{\cong}{\xrightarrow{h_*}} & \pi_{2n-1}(SU(n)). \end{array}$$

It follows that the lower i'_* is surjective and so is the upper i'_* from (2.5). Thus we have

(2.6) $\quad Q_C\{n, n\} = O_C\{n, n\}$.

On the other hand we can take $T_{n-1,n-2}=E\gamma_{n-1}$ where $\gamma_{n-1}\colon S^{2n-3}\to P_{n-1}$ is the canonical S^1-fibration. It is well known (see e.g. [10]) that $\sharp E\gamma_{n-1}=\sharp E^{\infty}\gamma_{n-1}=(n-1)!$. Thus we have (2.4) and

(2.7) $\quad Q_C\{n, n-1\} = Q_C^s\{n, n-1\} = (n-1)!$

by (2.2), (2.3). Therefore (ii) follows from (1.1), (1.2), (1.3), (2.6), (2.7). This completes the proofs of (2.4) and Theorem 1.

3. EHP-sequence

Let X, Y be r-connected CW-complexes which have exactly one vertex $*$, and let f: $X\to Y$ be a continuous map with $f(*)=*$. We then have a diagram consisting of the exact EHP-sequences for $i\leqq 3r+1$ (see e.g. [9], [19]):

$$\begin{array}{ccccccccc} \pi_i(X) & \xrightarrow{E} & \pi_{i+1}(EX) & \xrightarrow{H} & \pi_{i+1}(E(X\wedge X)) & \xrightarrow{P} & \pi_{i-1}(X) & \xrightarrow{E} & \cdots \\ \downarrow f_* & & \downarrow (Ef)_* & & \downarrow (E(f\wedge f))_* & & \downarrow f_* & & \\ \pi_i(Y) & \xrightarrow{E} & \pi_{i+1}(EY) & \xrightarrow{H} & \pi_{i+1}(E(Y\wedge Y)) & \xrightarrow{P} & \pi_{i-1}(Y) & \xrightarrow{E} & \cdots \end{array}$$

In the next section we shall use

Lemma (3.1). *The above diagram commutes.*

By using Theorem 5.3 of [6] and following faithfully the construction of the EHP-sequence, we can prove (3.1). We omit the details.

4. Proof of Theorem 2

For an abelian group A, A/Tor denotes the quotient group of A by its torsion subgroup, and π: $A\to A/Tor$ denotes the quotient homomorphism. Let Z be the infinite cyclic group.

By (2.1) we have

$$\pi^s_{dn-1}(Q_{n,k})/Tor\cong Z\,. \tag{4.1}$$

It follows that $Q^s\{n, k\}\neq 0$ from (2.2) and that $Q\{n, k\}\neq 0$ from (2.3), (2.6), (1.3). Thus we have

Lemma (4.2). $\pi\circ E^\infty\neq 0$: $\pi_{dn-1}(Q_{n,k})\to\pi^s_{dn-1}(Q_{n,k})/Tor$.

From now on we denote $Q_{2k-2,k}$ by Q. By (1.1), (1.2), (1.4), (4.1) and (4.2), Theorem 2 is equivalent to

Proposition (4.3). *Let $n=2k-2$. Then the image of $\pi\circ E^\infty$: $\pi_{dn-1}(Q)\to\pi^s_{dn-1}(Q)/Tor$ is $a\cdot\pi^s_{dn-1}(Q)/Tor$, where $a=1$ if $F=H$ or k is odd, $a=1$ or 2 if $F=C$ and k is even.*

Proof. We consider the case $F=C$ only, because we can prove the assertion for the case $F=H$ by a similar but slightly easier method to the following one.

If $k=2$, then the assertion is trivial by Theorem 1. So we assume that $k\geqq 3$. By (2.1) we have

$$Q=e^0\cup e^{2k-3}\cup e^{2k-1}\cup\cdots\cup e^{4k-5}\,,$$

and so

(4.4) $$Q\wedge Q=e^0\cup e^{4k-6}\cup e^{4k-4}\cup e^{4k-4}\cup\cdots\cup e^{8k-10}.$$

Let i: $S^{2k-3}=e^0\cup e^{2k-3}=Q_{k-1,1}\to Q$ be the inclusion. Since Q is $(2k-4)$-connected, it follows that

(4.5) $E^\infty\colon \pi_{4k-2}(E^3Q)\to\pi^s_{4k-5}(Q)$ *is an isomorphism, and*

(4.6) $E\colon \pi_{4k-3}(E^2Q)\to\pi_{4k-2}(E^3Q)$ *is surjective.*

By (3.1) we have the commutative diagram:

$$\begin{array}{ccccccc}\pi_{4k-1}(E^3S^{2k-3}) & \xrightarrow{H} & \pi_{4k-1}(E^5(S^{2k-3}\wedge S^{2k-3}))\cong\pi_{4k-1}(S^{4k-1})\cong Z & & & & \\ \downarrow(E^3i)_* & & \downarrow(E^5(i\wedge i))_* & & & & \\ \pi_{4k-1}(E^3Q) & \xrightarrow[H]{} & \pi_{4k-1}(E^5(Q\wedge Q)) & \xrightarrow[P]{} & \pi_{4k-3}(E^2Q) & \xrightarrow[E]{} & \pi_{4k-2}(E^3Q).\end{array}$$

By (4.4) and Blakers–Massey [1], $\pi_{4k}(E^5(Q\wedge Q),S^{4k-1})\cong\pi_{4k-1}(E^5(Q\wedge Q),S^{4k-1})\cong0$, so the above $(E^5(i\wedge i))_*$ is an isomorphism. As well known (see e.g. Proposition 2.7 of [17]), the upper H is not zero, hence so is the lower H. Thus the image of P is finite, so that, by (4.6), E induces an isomorphism

(4.7) $$\tilde{E}\colon \pi_{4k-3}(E^2Q)/Tor\cong\pi_{4k-2}(E^3Q)/Tor.$$

Consider the EHP-sequence:

$$\pi_{4k-2}(E^3(Q\wedge Q))\xrightarrow{P}\pi_{4k-4}(EQ)\xrightarrow{E}\pi_{4k-3}(E^2Q)\xrightarrow{H}\pi_{4k-3}(E^3(Q\wedge Q)).$$

By (4.4) and Blakers–Massey [1], $\pi_{4k-3}(E^3(Q\wedge Q),S^{4k-3})\cong\pi_{4k-2}(E^3(Q\wedge Q),S^{4k-3})\cong0$, so $E^3(i\wedge i)$ induces a surjection $(Z_2\cong)\ \pi_{4k-2}(S^{4k-3})\to\to\pi_{4k-2}(E^3(Q\wedge Q))$ and an isomorphism $(Z\cong)\ \pi_{4k-3}(S^{4k-3})\cong\pi_{4k-3}(E^3(Q\wedge Q))$. Thus it follows that

(4.8) $\pi_{4k-2}(E^3(Q\wedge Q))$ *is finite, and*

(4.9) $$\pi_{4k-3}(E^3(Q\wedge Q))\cong Z.$$

The kernel of E is finite by (4.8). The cokernel of E is torsion free by (4.9), while it is finite by (4.1), (4.2), (4.5), (4.7), hence it is zero, so E is surjective. Thus E induces an isomorphism

(4.10) $$\tilde{E}\colon \pi_{4k-4}(EQ)/Tor\cong\pi_{4k-3}(E^2Q)/Tor.$$

By (3.1) we have the following commutative diagram:

$$\begin{array}{ccccccc} & & & & Z_2\{\eta_{4k-5}\} & & \\ & & & & \| & & \\ \pi_{4k-5}(S^{2k-3}) & \xrightarrow{E} & \pi_{4k-4}(S^{2k-2}) & \xrightarrow{H} & \pi_{4k-4}(S^{4k-5}) & \xrightarrow{P} & \pi_{4k-6}(S^{2k-3}) \\ \downarrow i_* & & \downarrow(Ei)_* & & \downarrow(E(i\wedge i))_* & & \downarrow i_* \\ \pi_{4k-5}(Q) & \xrightarrow{E} & \pi_{4k-4}(EQ) & \xrightarrow{H} & \pi_{4k-4}(E(Q\wedge Q)) & \xrightarrow{P} & \pi_{4k-6}(Q).\end{array}$$

Here η_2: $S^3 \rightarrow S^2$ is the Hopf map and $\eta_m = E^{m-2}\eta_2$: $S^{m+1} \rightarrow S^m$ for $m \geqq 2$. By (4.4) and Blakers–Massey [1], $\pi_{4k-4}(E(Q \wedge Q), S^{4k-5}) \cong 0$. Thus $(E(i \wedge i))_*$ is surjective, so $\pi_{4k-4}(E(Q \wedge Q)) \cong Z_2$ or 0. Hence the cokernel of the lower E is Z_2 or 0. Since $\pi_{4k-4}(EQ)/Tor \cong Z$ by (4.1), (4.5), (4.7), (4.10), it follows that the image of the homomorphism $\tilde{E}$: $\pi_{4k-5}(Q)/Tor \rightarrow \pi_{4k-4}(EQ)/Tor$ induced by E is $a \cdot \pi_{4k-4}(EQ)/Tor$, where $a=1$ or 2. Thus the assertion of (4.3) for k even follows from (4.5), (4.7) and (4.10). We can prove (4.3) for k odd by showing that $\pi_{4k-4}(E(Q \wedge Q))$ is Z_2 if k is even and 0 if k is odd. But we will take a different method which can be applied to the case $F=H$.

As well known (see e.g. [19]), $P(\eta_{4k-5}) = [l_{2k-3}, \eta_{2k-3}]$, the Whitehead product, where l_{2k-3} is the identity map of S^{2k-3}. It follows from [5] that $[l_{2k-3}, \eta_{2k-3}] = 0$ if and only if k is odd. We show that $\tilde{E}$: $\pi_{4k-5}(Q)/Tor \rightarrow \pi_{4k-4}(EQ)/Tor$ is surjective if k is odd. Then the assertion of (4.3) for k odd follows from (4.5), (4.7) and (4.10).

Let k be odd. Then there is an element x in $\pi_{4k-4}(S^{2k-2})$ such that $H(x) = \eta_{4k-5}$ by exactness. Hence $H((Ei)_*(x)) = E(i \wedge i)_*(H(x)) = E(i \wedge i)_*(\eta_{4k-5})$ which generates $\pi_{4k-4}(E(Q \wedge Q))$. Choose y in $\pi_{4k-4}(EQ)$ such that $\pi(y)$ generates the infinite cyclic group $\pi_{4k-4}(EQ)/Tor$. If $H(y)=0$, then there exists y' in $\pi_{4k-5}(Q)$ such that $E(y')=y$, so $\tilde{E}(\pi(y')) = \pi(y)$ and $\tilde{E}$ is surjective. If $H(y) \neq 0$, then $\pi_{4k-4}(E(Q \wedge Q)) \cong Z_2$ which is generated by $H(y)$. Hence $H(y) = H((Ei)_*(x))$ and there exists y'' in $\pi_{4k-5}(Q)$ such that $E(y'') = y - (Ei)_*(x)$. Since $\pi_{4k-4}(S^{2k-2})$ is finite as seen in [14], it follows that $(Ei)_*(x)$ has a finite order and $\tilde{E}(\pi(y'')) = \pi(y - (Ei)_*(x)) = \pi(y)$, so that $\tilde{E}$ is surjective. This completes the proofs of (4.3) and hence of Theorem 2.

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