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EQUIVARIANT CRITICAL POINT THEORY AND IDEAL-VALUED COHOMOLOGICAL INDEX

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Introduction

We develop an equivariant critical point theory for differentiable G-functions on a Banach G-manifold with the aid of ideal-valued cohomological index theory, where G is a compact Lie group. We obtain a lower bound for the number of critical orbits with values in a given interval $(a,b]=\{t\in\boldsymbol{R}\,|\,a<t\leq b\}$ and for the number of critical values in $(a,b]$. We also obtain cohomological information about the topology of the critical set K of a G-function, which says a lot more about K than that obtained by using the Lusternik-Schnirelmann category.

The Lusternik-Schnirelmann category is a numerical homotopical invariant which gives a lower bound for the number of critical points (see for example [16], [17]), and this category is successfully extended to the equivariant setting [2], [3], [5], [6], [7], [15]. Ideal-valued cohomological index theory also gives important information about the existence of critical points [8], [9], [10]. The index theory in these papers is a priori in the equivariant setting and contains the nonequivariant (absolute) setting as trivial case.

In their paper [6] M. Clapp and D. Puppe developed an equivariant critical point theory using an equivariant Lusternik-Schnirelmann category. In the present paper we will develop one using an ideal-valued cohomological index theory which contains the nonequivariant setting as nontrivial case. We will obtain a type of results corresponding to their Theorem 1.1 of [6] and further results which are derived only from our theory.

Throughout this paper G always denotes a compact Lie group, and spaces considered are all paracompact Hausdorff. Let M be a Banach G-manifold of class at least C^1, i.e., M is a C^1 Banach manifold and G acts differentiably by diffeomorphisms. Let f: $M\rightarrow\boldsymbol{R}$ be a C^1 G-function, i.e., f is of class C^1 and satisfies $f(gx)=f(x)$ for all $x\in M$ and $g\in G$. Let $K=\{x\in M\,|\,df_x=0\}$ the critical set of f, $M_c=f^{-1}(-\infty,c]$ and $K_c=K\cap f^{-1}(c)$ for any $c\in\boldsymbol{R}$.

If $x\in M$ is a critical point of f, then every point of $Gx=\{gx\,|\,g\in G\}$

is also a critical point, and Gx is called a *critical orbit* of f. Note that Gx is diffeomorphic to the homogeneous space G/G_x where G_x is the isotropy subgroup at x.

Consider the following *deformation conditions* (D_0)-(D_2) for f: $M \to \boldsymbol{R}$ at $c \in \boldsymbol{R}$:

(D_0) *There is an* $\varepsilon > 0$ *such that* $M_{c+\varepsilon}$ *is G-deformable to* M_c, *i.e., there is a G-homotopy* φ_t: $M_{c+\varepsilon} \to M_{c+\varepsilon}$ $(0 \leq t \leq 1)$ *such that* $\varphi_0 = \mathrm{id}$ *and* $\varphi_1(M_{c+\varepsilon}) \subseteq M_c$.

(D_1) K_c *is compact.*

(D_2) *For every* $\delta > 0$ *and every G-invariant neighborhood* U *of* K_c *there is an* ε *with* $0 < \varepsilon < \delta$ *such that* $M_{c+\varepsilon} - U$ *is G-deformable to* $M_{c-\varepsilon}$ *relative to* $M_{c-\delta}$.

A C^1 Banach G-manifold M admits a G-invariant Finsler structure $\| \ \|$: $TM \to \boldsymbol{R}$ (see Palais [16], Krawcewicz-Marzantowicz [14]). The *Palais-Smale condition* (or (PS) *condition* for abbreviation) for f is:

(PS) *Any sequence* $\{x_n\}$ *in* M *such that* $\{f(x_n)\}$ *is bounded and* $\{\|df_{x_n}\|\}$ *converges to* 0 *contains a convergent subsequence.*

As is well-known, (D_1) and (D_2) at any $c \in \boldsymbol{R}$ is a consequence of (PS) under suitable assumptions on differentiability and completeness. See for the proof Palais [16; Theorem 5.11], [17; Theorem 4.6] for the nonequivariant case, and Clapp-Puppe [6; Appendix A], Krawcewicz-Marzantowicz [14; Lemma 1.9] for the equivariant case. If c is a regular value of f, (D_0) is also a consequence of (PS) (see [6; Appendix A]). Even if c is not a regular value we can see that (D_0) follows from (PS) under the assumption that c is an isolated critical value.

By a *G-pair* (X,A) we mean a G-space X together with a G-invariant subspace A. A G-map f: $(X,A) \to (Y,B)$ means a G-map f: $X \to Y$, i.e., $f(gx) = gf(x)$ *for* $g \in G$ and $x \in X$, such that $f(A) \subseteq B$. Let $\mathscr{P}$ be the category of such G-pairs and G-maps. Let h^* be a generalized G-cohomology theory on $\mathscr{P}$, i.e., h^* is a contravariant functor into graded moudles and h^* is equipped with long exact sequences, excision and homotopy property. In this paper, moreover we require h^* to be continuous and multiplicative with unit. See section 1 for the definition of the terms.

For $(X,A) \in \mathscr{P}$ the *ideal-valued index* of A in X, denoted $\mathrm{ind}(A,X)$, is defined to be the kernel of the homomorphism i^*: $h^*(X) \to h^*(A)$ where i: $A \to X$ is the inclusion and $h^*(X) = h^*(X,\emptyset)$. Then $\mathrm{ind}(A,X)$ is an ideal of $h^*(X)$.

We can now state our first theorem, which corresponds to Theorem 2.3 in section 2.

Theorem 0.0. *Let M be a C^1 Banach G-manifold with $h^*(M)$ Noetherian, and f: $M \to \boldsymbol{R}$ a C^1-function. For given $-\infty < a < b \leq \infty$, assume that f satisfies* (D_0) *at a and* (D_1), (D_2) *at every $c \in (a,b]$ $(c \neq \infty)$. If $b = \infty$, assume in addition that $f(K)$ is bounded above. Then there are a finite number of critical values $c_1, \cdots, c_k \in (a,b]$ of f such that*

$$\mathrm{ind}(M_a,M) \cdot \mathrm{ind}(K_{c_1},M) \cdot \cdots \cdot \mathrm{ind}(K_{c_k},M) \subseteq \mathrm{ind}(M_b,M),$$

where $\cdot$ represents the products of ideals [1].

A ring R is said to be *nilpotent* if $R^n = 0$ for some integer $n > 0$. The least such integer n is called the *index of nilpotency* and written $\mathrm{nil}(R)$. If no such integer n exists we put $\mathrm{nil}(R) = \infty$.

REMARK. See Marzantowicz [15] for the relation between the index of nilpotency of $\tilde{h}^*(X)$ of a G-space X, the cup-length of $\tilde{h}^*(M)$ and the G-category of X.

If $-\infty < a < b \leq \infty$, we see $\mathrm{ind}(M_b,M) \subseteq \mathrm{ind}(M_a, M)$ *in* $h^*(M)$ since $M_a \subseteq M_b$. Define for any integer $s \geq 0$,

$$s\text{-}\mathrm{nil}(M_a,M_b) := \mathrm{nil}(\mathrm{ind}^{\geq s}(M_a,M)/\mathrm{ind}^{\geq s}(M_b,M)),$$

where

$$\mathrm{ind}^{\geq s}(A,M) = \mathrm{ind}(A,M) \cap h^{\geq s}(M), \ h^{\geq s}(M) = \bigoplus_{n \geq s} h^n(M).$$

Note that if $s \leq t$ then $t\text{-}\mathrm{nil}(M_a,M_b) \leq s\text{-}\mathrm{nil}(M_a,M_b)$, and if $b = \infty$ then $s\text{-}\mathrm{nil}(M_a,M_b) = \mathrm{nil}(\mathrm{ind}^{\geq s}(M_a,M))$ since $M_b = M$ and $\mathrm{ind}(M_b,M) = 0$.

Using a suitable G-cohomology theory h^*, we will derive the following theorem from Theorem 0.0, which summarizes Theorems 3.4, 3.5, 3.6 and 3.9 in section 3.

Theorem 0.1. *Let f: $M \to \boldsymbol{R}$ be as in Theorem* 0.0 *except that $f(K)$ is bounded if $b = \infty$.*

(1) *f has at least* $1\text{-}\mathrm{nil}(M_a,M_b) - 1$ *critical orbits in $M_{(a,b]} = f^{-1}(a,b]$.*

(2) *If $h^{\geq s}(M) \subseteq \mathrm{ind}(K_c,M)$ for all critical values $c \in (a,b]$, then f has at least* $s\text{-}\mathrm{nil}(M_a,M_b) - 1$ *critical values in $(a,b]$.*

(3) *If* $s\text{-}\mathrm{nil}(M_a,M_b) - 1$ *is greater than the number of critical values of f in $(a,b]$, then there is a critical value $c \in (a,b]$ of f such that $h^{\geq s}(K_c) \neq 0$.*

(4) *If* 1-nil$(M_a,M_\infty)=\infty$ *for some* $a\in\boldsymbol{R}$, *then there is an unbounded sequence of critical values of* f.

If in the above theorem f is bounded below and $a<\inf f(M)$, then we will obtain a bit better results (see Theorem 3.7).

We will also obtain the following theorem more precisely than in Theorem 0.1 (3).

Theorem 0.2. *Assume that* f *has* k *critical values* $c_1,\cdots,c_k$ *in* $(a,b]$, *and that there are* $x_0\in \operatorname{ind}(M_a,M)$ *and* $x_1,\cdots,x_k\in h^*(M)$ *such that* $x_0x_1\cdots x_k\notin \operatorname{ind}(M_b,M)$. *If each of* $x_1,\cdots,x_k$ *is homogeneous, then*

$$h^{d_1}(K_{c_1})\oplus\cdots\oplus h^{d_k}(K_{c_k})\neq 0,$$

where $d_i=\deg x_i$.

This theorem corresponds to Theorem 3.11, and the following corollary corresponds to Corollary 3.13 in section 3.

Corollary 0.3. *Assume that* f *is bounded* (*above and below*) *and has* k *critical values. Then* $h^{ml}(K)\neq 0$ *for any integers* $m,l\geq 0$ *with* $kl\leq \operatorname{cup}_m(h^*(M))$.

Here $\operatorname{cup}_m(h^*(M))$ is the cup_m-*length* of $h^*(M)$ defined to be the largest integer t such that $(h_m(M))^t\neq 0$ in $h^*(M)$. Corollary 0.3 roughly says that the smaller the number of critical values is, the higher the dimension of the nonzero cohomology of K is.

1. Ideal-valued cohomological index

Let h^* be a generalized G-cohomology theory on $\mathscr{P}$. h^* is said to be *multiplicative* if it has products

$$h^p(X,A)\times h^q(X,B)\to h^{p+q}(X,A\cup B)$$

for any (X,A), $(X,B)\in\mathscr{P}$ with $\{A,B\}$ excisive and any $p,q\in\boldsymbol{Z}$, which is natural, bilinear, associative, commutative (up to the sign $(-1)^{pq}$). h^* is said to be *continuous* if for any $(X,A)\in\mathscr{P}$ with A closed,

$$h^*(A)\cong\varinjlim h^*(U)$$

where the direct limit is taken over all G-invariant neighborhoods U of A in X, and the isomorphism is induced by the inclusions.

EXAMPLE 1.1. Let H^* be the Alexander-Spanier cohomology theory with coefficients in a field $\boldsymbol{F}$. The following (1) and (2) are both generalized cohomology theories on $\mathscr{P}$ which are continuous and multiplicative with unit in $h^0(X)$.

(1) The Borel G-cohomology based on H^*,

$$h^*(X,A):=H^*(EG\times_G X,\ EG\times_G A;\boldsymbol{F}),$$

where EG is a universal G-space.

(2)

$$h^*(X,A):=H^*(X/G,A/G;\boldsymbol{F}).$$

REMARK 1.2. The equivariant stable cohomotopy theory and the equivariant K-theory are also examples of a generalized G-cohomology theory. The former is employed in Bartsch-Clapp-Puppe [4].

In what follows we assume h^* is a generalized G-cohomology theory on $\mathscr{P}$ which is continuous and multiplicative with unit. For $(X,A)\in\mathscr{P}$ the ideal-valued index $\mathrm{ind}(A,X)$ is defined as in the Introduction. We summarize its properties in the following.

Proposition 1.3. *Let* (X,A), (X,A_1), $(X,A_2)\in\mathscr{P}$.

(1) *Monotonicity: If there is a G-map* $\varphi:A_1\to A_2$ *such that* $i_2\varphi$ *is G-homotopic to* i_1 *where* $i_1: A_1\to X$ *and* $i_2: A_2\to X$ *are the inclusions, then*

$$\mathrm{ind}(A_2,X)\subseteq\mathrm{ind}(A_1,X).$$

(2) *Subadditivity: If* $\{A_1,A_2\}$ *is an excisive pair, then*

$$\mathrm{ind}(A_1,X)\cdot\mathrm{ind}(A_2,X)\subseteq\mathrm{ind}(A_1\cup A_2,X).$$

(3) *Continuity: If A is closed in X and* $\mathrm{ind}(A,X)$ *is a finitely generated ideal of* $h^*(X)$, *then there is a G-invariant neighborhood U of A in X such that*

$$\mathrm{ind}(A,X)=\mathrm{ind}(U,X).$$

Proof. (1) Easy by the definition of the index.

(2) It suffices to show that if $x_n\in\mathrm{ind}(A_n,X), n=1,2$, *then* $x_1x_2\in\mathrm{ind}(A_1\cup A_2,X)$. *Consider the following commutative diagram.*

$$
\begin{tikzcd}
 & h^*(A_1) & & h^*(A_1\cup A_2,A_2) \arrow[dl, "j_2^*"'] \\
h^*(X) \arrow[ur, "i_1^*"] \arrow[rr, "i_3^*"] \arrow[dr, "i_2^*"'] & & h^*(A_1\cup A_2) \arrow[ul, "k_1^*"'] \arrow[dl, "k_2^*"] & h^*(A_1\cup A_2,A_1\cup A_2)=0 \arrow[l, "j_3^*"'] \\
 & h^*(A_2) & & h^*(A_1\cup A_2,A_1) \arrow[ul, "j_1^*"]
\end{tikzcd}
$$

where the homomorphisms are all induced from the inclusions. Note that the two sequences $\{j_1^*,k_1^*\}$ *and* $\{j_2^*,k_2^*\}$ are both exact. By the commutativity of the diagram we see $k_n^* i_3^* x_n=0$ *in* $h^*(A_n)$ *for* $n=1,2$, and by the exactness we see that for $n=1,2$ there are $y_n\in h^*(A_1\cup A_2,A_n)$ such that $j_n^* y_n=i_3^* x_n$. Hence

$$i_3^*(x_1x_2)=i_3^*x_1\cdot i_3^*x_2=j_1^*y_1\cdot j_2^*y_2=j_3^*(y_1y_2)=0.$$

This implies $x_1x_2\in \mathrm{ind}(A_1\cup A_2,X)$.

(3) *Let* $x_1,\cdots,x_k$ be generators of $\mathrm{ind}(A,X)$. Since $x_n|A=i^*x_n=0$ in $h^*(A)(n=1,2,\cdots,k)$, by the continuity there is a G-invariant neighborhood U_n of A in X such that $x_n|U=0$ in $h^*(U_n)$. Then $U=U_1\cap\cdots\cap U_n$ is also a G-invariant neighborhood of A, and $x_n|U=0$, i.e., $x_n\in\mathrm{ind}(U,X)$. Hence $\mathrm{ind}(A,X)\subseteq\mathrm{ind}(U,X)$. On the other hand we see $\mathrm{ind}(A,X)\supseteq\mathrm{ind}(U,X)$ by the monotonicity of index. $\square$

Remark 1.4. In (3) of the above proposition $\mathrm{ind}(A,X)$ is finitely generated if $h^*(X)$ is Noetherian. One can find in Fadell [8; §3] some sufficient conditions for $h^*(X)$ to be Noetherian.

2. Indices of critical sets

Lemma 2.1. *Let M be a C^1 Banach G-manifold and $f:M\to \boldsymbol{R}$ a C^1 G-function. For given $-\infty<a<b\le\infty$, assume that f satisfies (D_0) at a and (D_2) at every $c\in(a,b](c\ne\infty)$. If f has no critical value in $(a,b]$, then*

$$\mathrm{ind}(M_a,M)=\mathrm{ind}(M_b,M).$$

Proof. By the conditions $(D_0),(D_2)$ we can see that M_b is

G-deformable to M_a. By the monotonicity of index we see $\operatorname{ind}(M_a,M)\subseteq \operatorname{ind}(M_b,M)$. Conversely, by the monotonicity again we see $\operatorname{ind}(M_a,M)\supseteq \operatorname{ind}(M_b,M)$ since $M_a\subseteq M_b$. Thus the lemma is proved. □

Lemma 2.2. *Let M be a C^1 Banach G-manifold with $h^*(M)$ Noetherian. If a C^1 G-function $f:M\to \boldsymbol{R}$ satisfies (D_1) and (D_2) at c, then there is an $\varepsilon>0$ such that*

$$\operatorname{ind}(M_{c-\varepsilon},M)\cdot\operatorname{ind}(K_c,M)\subseteq\operatorname{ind}(M_{c+\varepsilon},M).$$

In particular, if $M_{c-\varepsilon}=\emptyset$ then

$$\operatorname{ind}(K_c,M)=\operatorname{ind}(M_{c+\varepsilon},M),$$

and if $K_c=\emptyset$ then

$$\operatorname{ind}(M_{c-\varepsilon},M)=\operatorname{ind}(M_{c+\varepsilon},M).$$

Proof. By the assumptions, K_c is compact and $h^*(M)$ is Noetherian. So by the continuity of index there is a G-invariant neighborhood U of K_c such that $\operatorname{ind}(K_c,M)=\operatorname{ind}(U,M)$. There is also a G-invariant neighborhood V of K_c such that $K_c\subseteq V\subseteq \bar{V}\subseteq \mathring{U}$. By the monotonicity we see $\operatorname{ind}(K_c,M)=\operatorname{ind}(V,M)$. *Take an* $\varepsilon>0$ satisfying (D_2) for this V. Then we have

$$\begin{aligned}\operatorname{ind}(M_{c+\varepsilon},M)&=\operatorname{ind}((M_{c+\varepsilon},-V)\cup U,M)\\&\supseteq\operatorname{ind}(M_{c+\varepsilon}-V,M)\cdot\operatorname{ind}(U,M)\ \text{by subadditivity}\\&=\operatorname{ind}(M_{c+\varepsilon}-V,M)\cdot\operatorname{ind}(K_c,M)\\&\supseteq\operatorname{ind}(M_{c-\varepsilon},M)\cdot\operatorname{ind}(K_c,M)\ \text{by } (D_2) \text{ and monotonicity.}\end{aligned}$$

Thus the first half of the lemma is proved. If $A=\emptyset$ then $\operatorname{ind}(A,M)=h^*(M)$. This fact and the monotonicity implies the second half. □

We will obtain the following theorem:

Theorem 2.3. *Let M be a C^1 Banach G-manifold with $h^*(M)$ Noetherian. For given $-\infty<a<b\leq\infty$, assume that C^1 G-function $f:M\to\boldsymbol{R}$ satisfies (D_0) at a and (D_1),(D_2) at every $c\in(a,b](c\neq\infty)$. If $b=\infty$, assume in addition that $f(K)$ is bounded above. Then there are a finite number of critical values $c_1,\cdots,c_k\in(a,b]$ of f such that*

$$\operatorname{ind}(M_a,M)\cdot\operatorname{ind}(K_{c_1},M)\cdot\cdots\cdot\operatorname{ind}(K_{c_k},M)\subseteq\operatorname{ind}(M_b,M).$$

Proof. First assume $b<\infty$. Let $\varepsilon(a)$ be such an $\varepsilon>0$ as in (D_0) at a. For any $c\in(a,b]$ let $\varepsilon(c)$ be such an $\varepsilon>0$ as in Lemma 2.2, i.e.,

$$\mathrm{ind}(M_{c-\varepsilon(c)},M)\cdot\mathrm{ind}(K_c,M)\subseteq\mathrm{ind}(M_{c+\varepsilon(c)},M).$$

Let V_c denote the open interval $(c-\varepsilon(c),c+\varepsilon(c))$ for any $c\in[a,b]$. Then $\{V_c|c\in[a,b]\}$ is an open covering of $[a,b]$. Since $[a,b]$ is compact, there are a finite number of $d_1,\cdots,d_m\in[a,b]$ such that

$$[a,b]\subseteq V_{d_1}\cup\cdots\cup V_{d_m}.$$

By the monotonicity and Lemma 2.2 we have

$$\begin{aligned}\mathrm{ind}(M_b,M)&\supseteq\mathrm{ind}(M_{b+\varepsilon(b)},M)\\&\supseteq\mathrm{ind}(K_b,M)\cdot\mathrm{ind}(M_{b-\varepsilon(b)},M).\end{aligned}$$

$b-\varepsilon(b)$ is contained in V_d for some $d\in\{d_1,\cdots,d_m\}$. Since $b-\varepsilon(b)<d+\varepsilon(d)$ we have

$$\begin{aligned}\mathrm{ind}(M_{b-\varepsilon(b)},M)&\supseteq\mathrm{ind}(M_{d+\varepsilon(d)},M)\\&\supseteq\mathrm{ind}(K_d,M)\cdot\mathrm{ind}(M_{d-\varepsilon(d)},M)\text{ by Lemma 2.2.}\end{aligned}$$

By the above we have

$$\mathrm{ind}(M_b,M)\supseteq\mathrm{ind}(K_b,M)\cdot\mathrm{ind}(K_d,M)\cdot\mathrm{ind}(M_{d-\varepsilon(d)},M)$$

Repeating this we have

$$\mathrm{ind}(M_b,M)\supseteq\mathrm{ind}(K_{c_1},M)\cdot\cdots\cdot\mathrm{ind}(K_{c_k},M)\cdot\mathrm{ind}(M_a,M)\tag{2.4}$$

for some $c_1,\cdots,c_k\in(a,b]$. If c is not a critical value then $K_c=\emptyset$ and $\mathrm{ind}(K_c,M)=h^*(M)\ni 1$. So we may ssume that $c_1,\cdots,c_k$ in (2.4) are all critical values. Thus the theorem is proved for the case $b<\infty$.

Now assume $b=\infty$. Take an $r>0$ such that $\sup f(K)<r<\infty$. By the above we see that there are a finite number of critical values $c_1,\cdots,c_k\in(a,r]$ *such that*

$$\mathrm{ind}(M_a,M)\cdot\mathrm{ind}(K_{c_1},M)\cdot\cdots\cdot\mathrm{ind}(K_{c_k},M)\subseteq\mathrm{ind}(M_r,M).$$

Since there is no critical value in $[r,\infty)$ we can see by (D_2) *that* $M_b=M$ is G-deformable to M_r. Thus $\mathrm{ind}(M_r,M)=\mathrm{ind}(M_b,M)$ $(=0)$. Thus the theorem is also proved for the case $b=\infty$. □

If f is bounded below and $a<\inf f(M)$, then $M_a=\emptyset$ and $\mathrm{ind}(M_a,M)=h^*(M)\ni 1$. Thus we obtain the following corollary from Theorem 2.3.

Corollary 2.4. *If f is bounded below and $a<\inf f(M)$ in Theorem 2.3, then there are a finite number of critical values $c_1,\cdots,c_k\leq b$ of f such that*

$$\mathrm{ind}(K_{c_1},M)\cdot\cdots\cdot\mathrm{ind}(K_{c_k},M)\subseteq\mathrm{ind}(M_b,M).$$

In particular, if $b=\infty$ then

$$\mathrm{ind}(K_{c_1},M)\cdot\cdots\cdot\mathrm{ind}(K_{c_k},M)=0.$$

3. The number of critical orbits and values

In this section we will derive some results from Theorem 2.3. Before doing that we need a lemma.

Lemma 3.1. *Let $\mathfrak{A}\supseteq\mathfrak{B}$ be two ideals of a ring R. If $\mathfrak{A}\cdot R^k\subseteq\mathfrak{B}$ for some $k\geq 0$, then* $\mathrm{nil}(\mathfrak{A}/\mathfrak{B})\leq k+1$.

Proof. Assume to the contrary that $k+1<\mathrm{nil}(\mathfrak{A}/\mathfrak{B})$. Then there were $k+1$ elements $x_0,x_1,\cdots,x_k\in\mathfrak{A}$ such that $[x_0]\cdot[x_1]\cdots[x_k]\neq 0$ in $\mathfrak{A}/\mathfrak{B}$, i.e., $x_0x_1\cdots x_k\notin\mathfrak{B}$. This contradicts the assumption $\mathfrak{A}\cdot R^k\subseteq\mathfrak{B}$. $\square$

For a function f: $M\to\boldsymbol{R}$ and a subset $S\subseteq\boldsymbol{R}$ define $M_s:=f^{-1}(S)$ and $K_s:=K\cap M_s$. In the theorems below we will assume (3.2) and (3.3).

ASSUMPTION 3.2. *A generalized G-cohomology theory h^* is continuous and multiplicative with unit and satisfies $h^{\geq 1}(G/H)=0$ for all closed subgroups H of G.*

The G-cohomology theory of Example 1.1 (2) satisfies Assumption 3.2. Note that if K is a disjoint union of a finite number of orbits $G/H_1,\cdots,G/H_m$ in M then

$$\mathrm{ind}(K,M)=\bigcap_{i=1}^{m}\mathrm{ind}(G/H_i,M)\supset h^{\geq 1}(M)$$

under Assumption 3.2.

ASSUMPTION 3.3. *M is a C^1 Banach G-manifold with $h^*(M)$ Noetherian. For given $-\infty<a<b\leq\infty$, a C^1 G-function f: $M\to\boldsymbol{R}$ satisfies (D_0) at a and (D_1), (D_2) at every $c\in(a,b]$ $(c\neq\infty)$.*

Theorem 3.4. *f has at least* $1\text{-nil}(M_a,M_b)-1$ *critical orbits in $M_{(a,b]}$. In particular, if* $1\text{-nil}(M_a,,M_b)=\infty$ *then f has infinitely many critical* *orbits in* $M_{(a,b]}$.

Proof. It suffices to consider only the case where the number of critical values in $(a,b]$ is finite. Let $c_1,\cdots,c_k\in(a,b]$ be such critical values. It also suffices to consider the case where K_{c_i} is a finite union of orbits for all $1\leq i\leq k$. In this case we see $h^{\geq 1}(M)\subseteq \mathrm{ind}(K_{c_i},M)$. Thus by Theorem 2.3 we have

$$\mathrm{ind}(M_a,M)\cdot(h^{\geq 1}(M))^k\subseteq \mathrm{ind}(M_b,M).$$

By Lemma 3.1 we see $1\text{-nil}(M_a,M_b)\leq k+1$. This implies that the number of critical orbits in $\mathrm{M}_{(a,b]}$ is at least $1\text{-nil}(M_a,M_b)-1$.

A similar proof to above also shows the following.

Theorem 3.5. *If* $h^{\geq s}(M)\subseteq \mathrm{ind}(K_c,M)$ *for all critical values* $c\in(a,b]$ *and for some integer* $s\geq 0$, *then* f *has at least* $s\text{-nil}(M_a,M_b)-1$ *critical values in* $(a,b]$.

The contrapsotion of this theorem is:

Theorem 3.6. *If* $s\text{-nil}(M_a,M_b)\text{-}1$ *is greater than the number of critical values of* f *in* $(a,b]$, *then there is a critical value* $c\in(a,b]$ *of* f *such that*

$$h^{\geq s}(M)\nsubseteq \mathrm{ind}(K_c,M)$$

and hence $h^{\geq s}(K_c)\neq 0$.

If f is bounded below and $a<\inf f(M)$, then we may use Corollary 2.4 instead of Theorem 2.3 in the proofs of Theorems 3.4, 3.5, 3.6, and obtain

Theorem 3.7. *Assume that* f *is bounded below and* $a<\inf f(M)$. Then

(1) f *has at least* $1\text{-nil}(\emptyset,M_b)$ *critical orbits in* M_b,

(2) *if* $h^{\geq s}(M)\subseteq \mathrm{ind}(K_c,M)$ *for all critical values* $c\leq b$ *of* f, *then* f *has at least* $s\text{-nil}(\emptyset,M_b)$ *critical values in* $(-\infty,b]$,

(3) *if* $s\text{-nil}(\emptyset,M_b)$ *is greater than the number of critical values of* f *in* $(-\infty,b]$, *then there is a critical value* $c\leq b$ *of* f *such that* $h^{\geq s}(K_c)\neq 0$.

Note that $s\text{-nil}(\emptyset,M_b)=\mathrm{nil}(h^{\geq s}(M)/\mathrm{ind}^{\geq s}(M_b,M))$.

Lemma 3.8. *If* A *is a* G*-invariant compact subspace of a* G*-space* X *with* $h^*(X)$ *Noetherian, then*

$$(h^{\geq 1}(X))^k \subseteq \mathrm{ind}(A,X)$$

for some integer $k>0$.

Proof. Since A is compact, there are a finite number of orbits in A, say G/H_i $(1 \leq i \leq k)$, and G-invariant open neighborhoods U_i of G/H_i such that A is covered by $U_i (1 \leq i \leq k)$ and $\mathrm{ind}(G/H_i,X)=\mathrm{ind}(U_i,X)$. This fact shows

$$\mathrm{ind}(G/H_1,X)\cdots\cdots\mathrm{ind}(G/H_k,X) \subseteq \mathrm{ind}(A,X)$$

by the monotonicity and subadditivity of index. Then Assumption 3.2 implies the lemma.□

Theorem 3.9. *If* $1\text{-nil}(M_a,M_b)=\infty$ *and* $b=\infty$, *then* $f(K)$ *is not bounded, i.e., there is an unbounded sequence of critical values of* f.

Proof. If $f(K)$ were bounded, then by Theorem 2.3 there were a finite number of critical values $c_1,\cdots,c_k>a$ such that

$$\mathrm{ind}(M_a,M)\cdot\mathrm{ind}(K_{c_1},M)\cdots\cdots\mathrm{ind}(K_{c_k},M)=0. \tag{3.10}$$

Since $\mathrm{nil}(\mathrm{ind}^{\geq 1}(M_a,M))=1\text{-nil}(M_a,M)=\infty$, for every $n>0$ there are $x_1,\cdots,x_n \in \mathrm{ind}^{\geq 1}(M_a,M)$ with $x_1\cdots x \neq 0$. Since $K_{c_i}(1 \leq i \leq k)$ is compact, Lemma 3.8 shows that for a sufficiently large n there is an $m<n$ such that

$$x_1\cdots x_m \in \mathrm{ind}(K_{c_1},M)\cdots\cdots\mathrm{ind}(K_{c_k},M).$$

Then (3.10) implies $x_1\cdots x_m\cdots x_n=0$. This is a conradiction. So $f(K)$ is not bounded.□

Theorem 3.11. *Assume that* f *has* k *critical values* $c_1,\cdots,c_k$ in $(a,b]$, and *that there are* $x_0 \in \mathrm{ind}(M_a,M)$ *and* $x_1,\cdots,x_k \in h^*(M)$ *such that* $x_0x_1\cdots x_k \notin \mathrm{ind}(M_b,M)$. *If each of* $x_1,\cdots,x_k$ *is homogeneous, then*

$$h^{d_1}(K_{c_1})\oplus\cdots\oplus h^{d_k}(K_{c_k})\neq 0, \tag{3.12}$$

where $d_i=\deg\, x_i$.

Proof. If the left hand side of (3.12) were zero, then $x_i \in \mathrm{ind}(K_{c_i},M)$ for all $1 \leq i \leq k$. This implies

$$x_0x_1\cdots x_k \in \mathrm{ind}(M_a,M)\cdot\mathrm{ind}(K_{c_1},M)\cdots\cdots\mathrm{ind}(K_{c_k},M),$$

and by Theorem 2.3 we see $x_0x_1\cdots x_k \in \mathrm{ind}(M_b,M)$. This conradicts the assumption of the theorem. $\square$

Corollary 3.13. *Assume that f is bounded (above and below) and has k critical values. Then $h^{ml}(K)\neq 0$ for any integers $m,l\geq 0$ with $kl\leq \mathrm{cup}_m(h^*(M))$.*

Proof. If $\mathrm{cup}_m(h^*(M))<k$, then the corollary is trivial since $l=0$ can only be taken. So assume $k\leq \mathrm{cup}_m(h^*(M))=t$. Then there are $y_i\in h^m(M)$ for $i=1,\cdots,t$ such that $y_1\cdots y_t\neq 0$. If we take a and b such that $-\infty<a<\inf f(M)\leq \sup f(M)<b<\infty$, then $\mathrm{ind}(M_a,M)=h^*(M)$ and $\mathrm{ind}(M_b,M)=0$. Thus we can take $x_0,x_1,\cdots,x_k$ in Theorem 3.11 so as

$$x_0=1,\ x_i=y_{(i-1)l+1}\cdot y_{(i-1)l+2}\cdots y_{il}\ (1\leq i\leq k).$$

Since $\deg x_i=ml$ for all i with $1\leq i\leq k$, Theorem 3.11 shows $h^{ml}(K)\neq 0$. $\square$

Finally, we give an application of Corollary 3.13. Let $\boldsymbol{K}$ be the reals $\boldsymbol{R}$, the complexes $\boldsymbol{C}$, or the quaternions $\boldsymbol{H}$, and according to that G be the group $\boldsymbol{Z}_2$, S^1 or S^3 of $g\in\boldsymbol{K}$ with $|g|=1$. Then G acts on $\boldsymbol{K}^n$ by coordinate-wise multiplication, and the unit sphere $S(\boldsymbol{K}^n)$ of $\boldsymbol{K}^n$ is a G-invariant submanifold with the orbit space $S(\boldsymbol{K}^n)/G=\boldsymbol{K}P^{n-1}$, the projective space. Let $h^*(X)=H^*(X/G;\boldsymbol{F})$ where H^* is the Alexander-Spanier cohomology and $\boldsymbol{F}=\boldsymbol{Z}_2$, $\boldsymbol{Q}$ or $\boldsymbol{Q}$ according to $\boldsymbol{K}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. Then

$$h^*(S(\boldsymbol{K}^n))\cong \boldsymbol{F}[x]/(x^n),\ d=\deg x=1,2 \text{ or } 4,$$

and we see $\mathrm{cup}_d(h^*(S(k)))=n-1$. Thus Corollary 3.13 shows that if a C^1 *G-function* $f\colon S(\boldsymbol{K}^n)\to\boldsymbol{R}$ has k critical values, then $h^{dl}(K)\neq 0$ for any integer l with $0\leq kl\leq n-1$. This says a lot more about the cohomology of K than in Clapp-Puppe [5; §2].

For many spaces other than $S(\boldsymbol{K}^n)$ we already know the cup_1-length or a lower bound of that. See for example Fadell-Husseini[10; Theorem 3.16], Hiller [11], Jaworowski [12; §5] and Komiya [13; Remark 5.10]. So we can apply Corollary 3.13 to functions on such spaces.

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