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EQUIVARIANT CRITICAL POINT THEORY AND IDEAL-VALUED COHOMOLOGICAL INDEX

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Introduction

We develop an equivariant critical point theory for differentiable $G$-functions on a Banach $G$-manifold with the aid of ideal-valued cohomological index theory, where $G$ is a compact Lie group. We obtain a lower bound for the number of critical orbits with values in a given interval $(a,b] = \{ t \in \mathbb{R} | a < t \leq b \}$ and for the number of critical values in $(a,b]$. We also obtain cohomological information about the topology of the critical set $K$ of a $G$-function, which says a lot more about $K$ than that obtained by using the Lusternik-Schnirelmann category.

The Lusternik-Schnirelmann category is a numerical homotopical invariant which gives a lower bound for the number of critical points (see for example [16], [17]), and this category is successfully extended to the equivariant setting [2], [3], [5], [6], [7], [15]. Ideal-valued cohomological index theory also gives important information about the existence of critical points [8], [9], [10]. The index theory in these papers is a priori in the equivariant setting and contains the nonequivariant (absolute) setting as trivial case.

In their paper [6] M. Clapp and D. Puppe developed an equivariant critical point theory using an equivariant Lusternik-Schnirelmann category. In the present paper we will develop one using an ideal-valued cohomological index theory which contains the nonequivariant setting as nontrivial case. We will obtain a type of results corresponding to their Theorem 1.1 of [6] and further results which are derived only from our theory.

Throughout this paper $G$ always denotes a compact Lie group, and spaces considered are all paracompact Hausdorff. Let $M$ be a Banach $G$-manifold of class at least $C^1$, i.e., $M$ is a $C^1$ Banach manifold and $G$ acts differentiably by diffeomorphisms. Let $f: M \to \mathbb{R}$ be a $C^1$ $G$-function, i.e., $f$ is of class $C^1$ and satisfies $f(gx) = f(x)$ for all $x \in M$ and $g \in G$. Let $K = \{ x \in M | df_x = 0 \}$ the critical set of $f$, $M_c = f^{-1}(\langle -\infty, c \rangle)$ and $K_c = K \cap f^{-1}(c)$ for any $c \in \mathbb{R}$.

If $x \in M$ is a critical point of $f$, then every point of $Gx = \{ gx | g \in G \}$
is also a critical point, and \( G_x \) is called a \textit{critical orbit} of \( f \). Note that \( G_x \) is diffeomorphic to the homogeneous space \( G/G_x \) where \( G_x \) is the isotropy subgroup at \( x \).

Consider the following \textit{deformation conditions} \((D_0)-(D_2)\) for \( f: M \to \mathbb{R} \) at \( c \in \mathbb{R} \):

\begin{align*}
(D_0) & \text{ There is an } \varepsilon > 0 \text{ such that } M_{c+\varepsilon} \text{ is } G \text{-deformable to } M_c, \text{ i.e., there is a } G \text{-homotopy } \varphi: M_{c+\varepsilon} \to M_{c+\varepsilon} \text{ (} 0 \leq t \leq 1 \text{) such that } \varphi_0 = \text{id} \text{ and } \varphi_1(M_{c+\varepsilon}) \subseteq M_c. \\
(D_1) & \text{ } K_c \text{ is compact.} \\
(D_2) & \text{ For every } \delta > 0 \text{ and every } G \text{-invariant neighborhood } U \text{ of } K_c \text{ there is an } \varepsilon \text{ with } 0 < \varepsilon < \delta \text{ such that } M_{c+\varepsilon} - U \text{ is } G \text{-deformable to } M_{c-\varepsilon} \text{ relative to } M_{c-\delta}. 
\end{align*}

A \( C^1 \) Banach \( G \)-manifold \( M \) admits a \( G \)-invariant Finsler structure \( || \cdot ||: TM \to \mathbb{R} \) (see Palais [16], Krawcewicz-Marzantowicz [14]). The \textit{Palais-Smale condition} (or \( (PS) \) \textit{condition} for abbreviation) for \( f \) is:

\[ (PS) \text{ Any sequence } \{x_n\} \text{ in } M \text{ such that } \{f(x_n)\} \text{ is bounded and } \{||df_{x_n}||\} \text{ converges to } 0 \text{ contains a convergent subsequence.} \]

As is well-known, \( (D_1) \) and \( (D_2) \) at any \( c \in \mathbb{R} \) is a consequence of \( (PS) \) under suitable assumptions on differentiability and completeness. See for the proof Palais [16; Theorem 5.11], [17; Theorem 4.6] for the nonequivariant case, and Clapp-Puppe [6; Appendix A], Krawcewicz-Marzantowicz [14; Lemma 1.9] for the equivariant case. If \( c \) is a regular value of \( f \), \( (D_0) \) is also a consequence of \( (PS) \) (see [6; Appendix A]). Even if \( c \) is not a regular value we can see that \( (D_0) \) follows from \( (PS) \) under the assumption that \( c \) is an isolated critical value.

By a \( G \)-pair \((X,A)\) we mean a \( G \)-space \( X \) together with a \( G \)-invariant subspace \( A \). A \( G \)-map \( f: (X,A) \to (Y,B) \) means a \( G \)-map \( f: X \to Y \), i.e., \( f(gx) = gf(x) \) for \( g \in G \) and \( x \in X \), such that \( f(A) \subseteq B \). Let \( \mathcal{P} \) be the category of such \( G \)-pairs and \( G \)-maps. Let \( h^* \) be a generalized \( G \)-cohomology theory on \( \mathcal{P} \), i.e., \( h^* \) is a contravariant functor into graded modules and \( h^* \) is equipped with long exact sequences, excision and homotopy property. In this paper, moreover we require \( h^* \) to be continuous and multiplicative with unit. See section 1 for the definition of the terms.

For \((X,A) \in \mathcal{P}\) the \textit{ideal-valued index} of \( A \) in \( X \), denoted \( \text{ind}(A,X) \), is defined to be the kernel of the homomorphism \( i^*: h^*(X) \to h^*(A) \) where \( i: A \to X \) is the inclusion and \( h^*(X) = h^*(X,\emptyset) \). Then \( \text{ind}(A,X) \) is an ideal of \( h^*(X) \).
We can now state our first theorem, which corresponds to Theorem 2.3 in section 2.

**Theorem 0.0.** Let $M$ be a $C^1$ Banach $G$-manifold with $h^*(M)$ Noetherian, and $f: M \to \mathbb{R}$ a $C^1$-function. For given $-\infty < a < b \leq \infty$, assume that $f$ satisfies $(D_0)$ at $a$ and $(D_1)$, $(D_2)$ at every $c \in (a,b)$ ($c \neq \infty$). If $b = \infty$, assume in addition that $f(K)$ is bounded above. Then there are a finite number of critical values $c_1, \ldots, c_k \in (a,b]$ of $f$ such that

$$\text{ind}(M_a, M) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \leq \text{ind}(M_b, M),$$

where $\cdot$ represents the products of ideals [1].

A ring $R$ is said to be nilpotent if $R^n = 0$ for some integer $n > 0$. The least such integer $n$ is called the index of nilpotency and written $\text{nil}(R)$. If no such integer $n$ exists we put $\text{nil}(R) = \infty$.

**Remark.** See Marzantowicz [15] for the relation between the index of nilpotency of $h^*(X)$ of a $G$-space $X$, the cup-length of $\Lambda^*(M)$ and the $G$-category of $X$.

If $-\infty < a < b \leq \infty$, we see $\text{ind}(M_a, M) \leq \text{ind}(M_b, M)$ in $h^*(M)$ since $M_a \subseteq M_b$. Define for any integer $s \geq 0$,

$$\text{s-nil}(M_a, M_b) := \text{nil}(\text{ind}^{\geq s}(M_a, M)/\text{ind}^{\geq s}(M_b, M)),$$

where

$$\text{ind}^{\geq s}(A, M) = \text{ind}(A, M) \cap h^{\geq s}(M), \quad h^{\geq s}(M) = \bigoplus h^n(M).$$

Note that if $s \leq t$ then $t\text{-nil}(M_a, M_b) \leq s\text{-nil}(M_a, M_b)$, and if $b = \infty$ then $s\text{-nil}(M_a, M_b) = \text{nil}(\text{ind}^{\geq s}(M_a, M))$ since $M_b = M$ and $\text{ind}(M_b, M) = 0$.

Using a suitable $G$-cohomology theory $h^*$, we will derive the following theorem from Theorem 0.0, which summarizes Theorems 3.4, 3.5, 3.6 and 3.9 in section 3.

**Theorem 0.1.** Let $f: M \to \mathbb{R}$ be as in Theorem 0.0 except that $f(K)$ is bounded if $b = \infty$.

1. $f$ has at least $1\text{-nil}(M_a, M_b) - 1$ critical orbits in $M_{[a,b]} = f^{-1}(a,b]$.
2. If $h^{\geq s}(M) \subseteq \text{ind}(K_c, M)$ for all critical values $c \in (a,b]$, then $f$ has at least $s\text{-nil}(M_a, M_b) - 1$ critical values in $(a,b]$.
3. If $s\text{-nil}(M_a, M_b) - 1$ is greater than the number of critical values of $f$ in $(a,b]$, then there is a critical value $c \in (a,b]$ of $f$ such that $h^{\geq s}(K_c) \neq 0$. 

(4) If $1$-nil$(M_a,M_a)=\infty$ for some $a \in \mathbb{R}$, then there is an unbounded sequence of critical values of $f$.

If in the above theorem $f$ is bounded below and $a<\inf f(M)$, then we will obtain a bit better results (see Theorem 3.7).

We will also obtain the following theorem more precisely than in Theorem 0.1 (3).

**Theorem 0.2.** Assume that $f$ has $k$ critical values $c_1, \ldots, c_k$ in $(a,b]$, and that there are $x_0 \in \text{ind}(M_a,M)$ and $x_1, \ldots, x_k \in h^*(M)$ such that $x_0 x_1 \cdots x_k \notin \text{ind}(M_b,M)$. If each of $x_1, \ldots, x_k$ is homogeneous, then

$$h^d(K_{c_1}) \oplus \cdots \oplus h^d(K_{c_k}) \neq 0,$$

where $d_i = \deg x_i$.

This theorem corresponds to Theorem 3.11, and the following corollary corresponds to Corollary 3.13 in section 3.

**Corollary 0.3.** Assume that $f$ is bounded (above and below) and has $k$ critical values. Then $h^m(K) \neq 0$ for any integers $m, l \geq 0$ with $kl \leq \text{cup}_m (h^*(M))$.

Here $\text{cup}_m(h^*(M))$ is the $\text{cup}_m$-length of $h^*(M)$ defined to be the largest integer $t$ such that $(h_m(M))^t \neq 0$ in $h^*(M)$. Corollary 0.3 roughly says that the smaller the number of critical values is, the higher the dimension of the nonzero cohomology of $K$ is.

1. Ideal-valued cohomological index

Let $h^*$ be a generalized $G$-cohomology theory on $\mathcal{P}$. $h^*$ is said to be *multiplicative* if it has products

$$h^p(X,A) \times h^q(X,B) \to h^{p+q}(X,A \cup B)$$

for any $(X,A), (X,B) \in \mathcal{P}$ with $\{A,B\}$ excisive and any $p, q \in \mathbb{Z}$, which is natural, bilinear, associative, commutative (up to the sign $(-1)^{pq}$). $h^*$ is said to be *continuous* if for any $(X,A) \in \mathcal{P}$ with $A$ closed,

$$h^*(A) \cong \lim_{U} h^*(U)$$

where the direct limit is taken over all $G$-invariant neighborhoods $U$ of $A$ in $X$, and the isomorphism is induced by the inclusions.
Example 1.1. Let $H^*$ be the Alexander-Spanier cohomology theory with coefficients in a field $F$. The following (1) and (2) are both generalized cohomology theories on $\mathcal{P}$ which are continuous and multiplicative with unit in $h^0(X)$.

(1) The Borel $G$-cohomology based on $H^*$,

$$h^*(X,A):=H^*(EG \times_G X, EG \times_G A;F),$$

where $EG$ is a universal $G$-space.

(2) $h^*(X,A):=H^*(X/G, A/G;F)$.

Remark 1.2. The equivariant stable cohomotopy theory and the equivariant $\mathbb{K}$-theory are also examples of a generalized $G$-cohomology theory. The former is employed in Bartsch-Clapp-Puppe [4].

In what follows we assume $h^*$ is a generalized $G$-cohomology theory on $\mathcal{P}$ which is continuous and multiplicative with unit. For $(X,A) \in \mathcal{P}$ the ideal-valued index $\text{ind}(A,X)$ is defined as in the Introduction. We summarize its properties in the following.

Proposition 1.3. Let $(X,A)$, $(X,A_1)$, $(X,A_2) \in \mathcal{P}$.

(1) Monotonicity: If there is a $G$-map $\phi:A_1 \to A_2$ such that $i_2 \phi$ is $G$-homotopic to $i_1$ where $i_1: A_1 \to X$ and $i_2: A_2 \to X$ are the inclusions, then

$$\text{ind}(A_2,X) \leq \text{ind}(A_1,X).$$

(2) Subadditivity: If $\{A_1, A_2\}$ is an excisive pair, then

$$\text{ind}(A_1,X) \cdot \text{ind}(A_2,X) \leq \text{ind}(A_1 \cup A_2,X).$$

(3) Continuity: If $A$ is closed in $X$ and $\text{ind}(A,X)$ is a finitely generated ideal of $h^*(X)$, then there is a $G$-invariant neighborhood $U$ of $A$ in $X$ such that

$$\text{ind}(A,X) = \text{ind}(U,X).$$

Proof. (1) Easy by the definition of the index.

(2) It suffices to show that if $x_n \in \text{ind}(A_n,X), n=1,2$, then $x_1 x_2 \in \text{ind}(A_1 \cup A_2,X)$. Consider the following commutative diagram.
where the homomorphisms are all induced from the inclusions. Note
that the two sequences \( \{ i^*_1, k^*_1 \} \) and \( \{ j^*_2, k^*_2 \} \) are both exact. By the
commutativity of the diagram we see \( h^*(X) = 0 \) in \( h^*(A_n) \) for \( n = 1, 2 \), and
by the exactness we see that for \( n = 1, 2 \) there are \( y_n \in h^*(A_1 \cup A_2, A_n) \) such
that \( j^*_n y_n = i^*_n x_n \). Hence
\[
i^*_n(x_1 x_2) = i^*_n x_1 \cdot i^*_n x_2 = j^*_n y_1 \cdot j^*_n y_2 = j^*_n(y_1 y_2) = 0.
\]
This implies \( x_1 x_2 \in \text{ind}(A_1 \cup A_2, X) \).
(3) Let \( x_1, \cdots, x_k \) be generators of \( \text{ind}(A, X) \). Since \( x_n | A = i^* x_n = 0 \) in
\( h^*(A)(n = 1, 2, \cdots, k) \), by the continuity there is a \( G \)-invariant neighborhood
\( U_n \) of \( A \) in \( X \) such that \( x_n | U = 0 \) in \( h^*(U_n) \). Then \( U = U_1 \cap \cdots \cap U_n \) is
also a \( G \)-invariant neighborhood of \( A \), and \( x_n | U = 0 \), i.e., \( x_n \in \text{ind}(U, X) \).
Hence \( \text{ind}(A, X) \subseteq \text{ind}(U, X) \). On the other hand we see \( \text{ind}(A, X) \supseteq
\text{ind}(U, X) \) by the monotonicity of index.

**Remark 1.4.** In (3) of the above proposition \( \text{ind}(A, X) \) is finitely
generated if \( h^*(X) \) is Noetherian. One can find in Fadell [8; §3] some
sufficient conditions for \( h^*(X) \) to be Noetherian.

2. Indices of critical sets

**Lemma 2.1.** Let \( M \) be a \( C^1 \) Banach \( G \)-manifold and \( f : M \to R \) a \( C^1 \)
\( G \)-function. For given \(-\infty < a < b \leq \infty \), assume that \( f \) satisfies \((D_0)\) at \( a \) and
\((D_2)\) at every \( c \in (a, b) \) \( c \neq \infty \). If \( f \) has no critical value in \( (a, b) \), then
\[
\text{ind}(M_a, M) = \text{ind}(M_b, M).
\]
Proof. By the conditions \((D_0),(D_2)\) we can see that \( M_b \) is
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G-deformable to $M_a$. By the monotonicity of index we see $\text{ind}(M_a, M) \subseteq \text{ind}(M_b, M)$. Conversely, by the monotonicity again we see $\text{ind}(M_a, M) \supseteq \text{ind}(M_b, M)$ since $M_a \subseteq M_b$. Thus the lemma is proved.

Lemma 2.2. Let $M$ be a $C^1$ Banach $G$-manifold with $h^*(M)$ Noetherian. If a $C^1$ $G$-function $f: M \to \mathbb{R}$ satisfies $(D_1)$ and $(D_2)$ at $c$, then there is an $\varepsilon > 0$ such that

$$\text{ind}(M_{c-\varepsilon}, M) \cdot \text{ind}(K_c, M) \subseteq \text{ind}(M_{c+\varepsilon}, M).$$

In particular, if $M_{c-\varepsilon} = \emptyset$ then

$$\text{ind}(K_c, M) = \text{ind}(M_{c+\varepsilon}, M),$$

and if $K_c = \emptyset$ then

$$\text{ind}(M_{c-\varepsilon}, M) = \text{ind}(M_{c+\varepsilon}, M).$$

Proof. By the assumptions, $K_c$ is compact and $h^*(M)$ is Noetherian. So by the continuity of index there is a $G$-invariant neighborhood $U$ of $K_c$ such that $\text{ind}(K_c, M) = \text{ind}(U, M)$. There is also a $G$-invariant neighborhood $V$ of $K_c$ such that $K_c \subseteq V \subseteq \bar{V} \subseteq \bar{U}$. By the monotonicity we see $\text{ind}(K_c, M) = \text{ind}(V, M)$. Take an $\varepsilon > 0$ satisfying $(D_2)$ for this $V$. Then we have

$$\text{ind}(M_{c+\varepsilon}, M) = \text{ind}((M_{c+\varepsilon} - V) \cup U, M)$$

$$\supseteq \text{ind}(M_{c+\varepsilon} - V, M) \cdot \text{ind}(U, M) \quad \text{by subadditivity}$$

$$= \text{ind}(M_{c+\varepsilon} - V, M) \cdot \text{ind}(K_c, M)$$

$$\supseteq \text{ind}(M_{c-\varepsilon}, M) \cdot \text{ind}(K_c, M) \quad \text{by} \quad (D_2) \quad \text{and monotonicity}.$$ 

Thus the first half of the lemma is proved. If $A = \emptyset$ then $\text{ind}(A, M) = h^*(M)$. This fact and the monotonicity implies the second half.

We will obtain the following theorem:

Theorem 2.3. Let $M$ be a $C^1$ Banach $G$-manifold with $h^*(M)$ Noetherian. For given $-\infty < a < b \leq \infty$, assume that $C^1$ $G$-function $f: M \to \mathbb{R}$ satisfies $(D_0)$ at $a$ and $(D_1), (D_2)$ at every $c \in (a, b)(c \neq \infty)$. If $b = \infty$, assume in addition that $f(K)$ is bounded above. Then there are a finite number of critical values $c_1, \cdots, c_k \in (a, b]$ of $f$ such that

$$\text{ind}(M_a, M) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \subseteq \text{ind}(M_b, M).$$
Proof. First assume $b<\infty$. Let $\varepsilon(a)$ be such an $\varepsilon>0$ as in $(D_0)$ at $a$. For any $c\in(a,b]$ let $\varepsilon(c)$ be such an $\varepsilon>0$ as in Lemma 2.2, i.e.,

$$\text{ind}(M_{c-\varepsilon(c)},M) \cdot \text{ind}(K_{c,M}) \subseteq \text{ind}(M_{c+\varepsilon(c)},M).$$

Let $V_c$ denote the open interval $(c-\varepsilon(c),c+\varepsilon(c))$ for any $c\in[a,b]$. Then \( \{V_c|c\in[a,b]\}\) is an open covering of $[a,b]$. Since $[a,b]$ is compact, there are a finite number of $d_1,\ldots,d_m\in[a,b]$ such that

$$[a,b] \subseteq V_{d_1} \cup \cdots \cup V_{d_m}.$$  

By the monotonicity and Lemma 2.2 we have

$$\text{ind}(M_b,M) \geq \text{ind}(M_{b+\varepsilon(b)},M) \geq \text{ind}(K_{b,M}) \cdot \text{ind}(M_{b-\varepsilon(b)},M).$$

$b-\varepsilon(b)$ is contained in $V_d$ for some $d\in\{d_1,\ldots,d_m\}$. Since $b-\varepsilon(b)<d+\varepsilon(d)$ we have

$$\text{ind}(M_{b-\varepsilon(b)},M) \geq \text{ind}(M_{d+\varepsilon(d)},M) \geq \text{ind}(K_{d,M}) \cdot \text{ind}(M_{d-\varepsilon(d)},M) \text{ by Lemma 2.2.}$$

By the above we have

$$\text{ind}(M_b,M) \geq \text{ind}(K_b,M) \cdot \text{ind}(K_d,M) \cdot \text{ind}(M_{d-\varepsilon(d)},M)$$

Repeating this we have

$$(2.4) \quad \text{ind}(M_b,M) \geq \text{ind}(K_{c_1,M}) \cdots \text{ind}(K_{c_k,M}) \cdot \text{ind}(M_{a,M})$$

for some $c_1,\ldots,c_k\in(a,b]$. If $c$ is not a critical value then $K_c=\emptyset$ and $\text{ind}(K_{c,M})=h^*(M)\geq 1$. So we may assume that $c_1,\ldots,c_k$ in (2.4) are all critical values. Thus the theorem is proved for the case $b<\infty$.

Now assume $b=\infty$. Take an $r>0$ such that $\sup f(K)<r<\infty$. By the above we see that there are a finite number of critical values $c_1,\ldots,c_k\in(a,r]$ such that

$$\text{ind}(M_{a,M}) \cdot \text{ind}(K_{c_1,M}) \cdots \text{ind}(K_{c_k,M}) \subseteq \text{ind}(M_r,M).$$

Since there is no critical value in $[r,\infty)$ we can see by $(D_2)$ that $M_b=M$ is $G$-deformable to $M_r$. Thus $\text{ind}(M_r,M)=\text{ind}(M_b,M) (=0)$. Thus the theorem is also proved for the case $b=\infty$.  

If $f$ is bounded below and $a<\inf f(M)$, then $M_a=\emptyset$ and $\text{ind}(M_a,M)=h^*(M)\geq 1$. Thus we obtain the following corollary from Theorem 2.3.
Corollary 2.4. If $f$ is bounded below and $a < \inf f(M)$ in Theorem 2.3, then there are a finite number of critical values $c_1, \cdots, c_k \leq b$ of $f$ such that

$$\text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) \subseteq \text{ind}(M_b, M).$$

In particular, if $b = \infty$ then

$$\text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) = 0.$$

3. The number of critical orbits and values

In this section we will derive some results from Theorem 2.3. Before doing that we need a lemma.

Lemma 3.1. Let $U \supseteq B$ be two ideals of a ring $R$. If $U \cdot R^k \subseteq B$ for some $k \geq 0$, then $\text{nil}(U/B) \leq k + 1$.

Proof. Assume to the contrary that $k + 1 < \text{nil}(U/B)$. Then there were $k + 1$ elements $x_0, x_1, \cdots, x_k \in U$ such that $[x_0] \cdot [x_1] \cdots [x_k] \neq 0$ in $U/B$, i.e., $x_0 x_1 \cdots x_k \notin B$. This contradicts the assumption $U \cdot R^k \subseteq B$. \qed

For a function $f: M \to R$ and a subset $S \subseteq R$ define $M_S := f^{-1}(S)$ and $K_S := \overline{K \cap M_S}$. In the theorems below we will assume (3.2) and (3.3).

Assumption 3.2. A generalized $G$-cohomology theory $h^*$ is continuous and multiplicative with unit and satisfies $h^{-1}(G/H) = 0$ for all closed subgroups $H$ of $G$.

The $G$-cohomology theory of Example 1.1 (2) satisfies Assumption 3.2. Note that if $K$ is a disjoint union of a finite number of orbits $G/H_1, \cdots, G/H_m$ in $M$ then

$$\text{ind}(K, M) = \bigcap_{i=1}^m \text{ind}(G/H_i, M) \supseteq h^{\geq 1}(M)$$

under Assumption 3.2.

Assumption 3.3. $M$ is a $C^1$ Banach $G$-manifold with $h^*(M)$ Noetherian. For given $-\infty < a < b \leq \infty$, a $C^1$ $G$-function $f: M \to R$ satisfies ($D_0$) at $a$ and ($D_1$), ($D_2$) at every $c \in (a, b]$ ($c \neq \infty$).

Theorem 3.4. $f$ has at least $1 - \text{nil}(M_a, M_b) - 1$ critical orbits in $M_{(a, b]}$. In particular, if $1 - \text{nil}(M_a, M_b) = \infty$ then $f$ has infinitely many critical
orbits in $M_{(a,b)}$.

Proof. It suffices to consider only the case where the number of critical values in $(a,b)$ is finite. Let $c_1, \cdots, c_k \in (a,b]$ be such critical values. It also suffices to consider the case where $K_{c_i}$ is a finite union of orbits for all $1 \leq i \leq k$. In this case we see $h^{\geq 1}(M) \subseteq \text{ind}(K_{c_i}, M)$. Thus by Theorem 2.3 we have

$$\text{ind}(M_a, M) \cdot (h^{\geq 1}(M))^k \equiv \text{ind}(M_b, M).$$

By Lemma 3.1 we see $1\text{-nil}(M_a, M_b) \leq k + 1$. This implies that the number of critical orbits in $M_{(a,b]}$ is at least $1\text{-nil}(M_a, M_b) - 1$.

A similar proof to above also shows the following.

**Theorem 3.5.** If $h^{\geq s}(M) \equiv \text{ind}(K_{c_i}, M)$ for all critical values $c \in (a,b]$ and for some integer $s \geq 0$, then $f$ has at least $s\text{-nil}(M_a, M_b) - 1$ critical values in $(a,b]$.

The contrapositive of this theorem is:

**Theorem 3.6.** If $s\text{-nil}(M_a, M_b) - 1$ is greater than the number of critical values of $f$ in $(a,b]$, then there is a critical value $c \in (a,b]$ of $f$ such that

$$h^{\geq s}(M) \not\equiv \text{ind}(K_c, M)$$

and hence $h^{\geq s}(K_c) \neq 0$.

If $f$ is bounded below and $a < \inf f(M)$, then we may use Corollary 2.4 instead of Theorem 2.3 in the proofs of Theorems 3.4, 3.5, 3.6, and obtain

**Theorem 3.7.** Assume that $f$ is bounded below and $a < \inf f(M)$. Then

1. $f$ has at least $1\text{-nil}(0, M_b)$ critical orbits in $M_b$,
2. if $h^{\geq s}(M) \equiv \text{ind}(K_c, M)$ for all critical values $c \leq b$ of $f$, then $f$ has at least $s\text{-nil}(0, M_b)$ critical values in $(-\infty, b]$,
3. if $s\text{-nil}(0, M_b)$ is greater than the number of critical values of $f$ in $(-\infty, b]$, then there is a critical value $c \leq b$ of $f$ such that $h^{\geq s}(K_c) \neq 0$.

Note that $s\text{-nil}(0, M_b) = \text{nil}(h^{\geq s}(M)/\text{ind}^{\geq s}(M_b, M))$.

**Lemma 3.8.** If $A$ is a $G$-invariant compact subspace of a $G$-space $X$ with $h^*(X)$ Noetherian, then
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\[(h^{\geq 1}(X))^k \leq \text{ind}(A, X)\]

for some integer \(k > 0\).

Proof. Since \(A\) is compact, there are a finite number of orbits in \(A\), say \(G/H_i (1 \leq i \leq k)\), and \(G\)-invariant open neighborhoods \(U_i\) of \(G/H_i\) such that \(A\) is covered by \(U_i (1 \leq i \leq k)\) and \(\text{ind}(G/H_i, X) = \text{ind}(U_i, X)\). This fact shows

\[\text{ind}(G/H_1, X) \cdots \text{ind}(G/H_k, X) \leq \text{ind}(A, X)\]

by the monotonicity and subadditivity of index. Then Assumption 3.2 implies the lemma. \(\square\)

**Theorem 3.9.** If \(1\)-nil\((M_a, M_b) = \infty\) and \(b = \infty\), then \(f(K)\) is not bounded, i.e., there is an unbounded sequence of critical values of \(f\).

Proof. If \(f(K)\) were bounded, then by Theorem 2.3 there were a finite number of critical values \(c_1, \ldots, c_k > a\) such that

\[(3.10) \quad \text{ind}(M_a, M) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M) = 0.\]

Since \(\text{nil}(\text{ind}^{\geq 1}(M_a, M)) = 1\)-nil\((M_a, M) = \infty\), for every \(n > 0\) there are \(x_1, \ldots, x_n \in \text{ind}^{\geq 1}(M_a, M)\) with \(x_1 \cdots x \neq 0\). Since \(K_{c_i} (1 \leq i \leq k)\) is compact, Lemma 3.8 shows that for a sufficiently large \(n\) there is an \(m < n\) such that

\[x_1 \cdots x_m \in \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M).\]

Then (3.10) implies \(x_1 \cdots x_m \cdots x_n = 0\). This is a contradiction. So \(f(K)\) is not bounded. \(\square\)

**Theorem 3.11.** Assume that \(f\) has \(k\) critical values \(c_1, \ldots, c_k\) in \((a, b]\), and that there are \(x_0 \in \text{ind}(M_a, M)\) and \(x_1, \ldots, x_k \in h^*(M)\) such that \(x_0 x_1 \cdots x_k \notin \text{ind}(M_b, M)\). If each of \(x_1, \ldots, x_k\) is homogeneous, then

\[(3.12) \quad h^{d_1}(K_{c_1}) \oplus \cdots \oplus h^{d_k}(K_{c_k}) \neq 0,\]

where \(d_i = \text{deg } x_i\).

Proof. If the left hand side of (3.12) were zero, then \(x_i \in \text{ind}(K_{c_i}, M)\) for all \(1 \leq i \leq k\). This implies

\[x_0 x_1 \cdots x_k \in \text{ind}(M_a, M) \cdot \text{ind}(K_{c_1}, M) \cdots \text{ind}(K_{c_k}, M),\]
and by Theorem 2.3 we see $x_0 x_1 \cdots x_k \in \text{ind}(M_b, M)$. This contradicts the assumption of the theorem.

**Corollary 3.13.** Assume that $f$ is bounded (above and below) and has $k$ critical values. Then $h^{ml}(K) \neq 0$ for any integers $m, l \geq 0$ with $kl \leq \cup_m (h^*(M))$.

Proof. If $\cup_m (h^*(M)) < k$, then the corollary is trivial since $l = 0$ can only be taken. So assume $k \leq \cup_m (h^*(M)) = t$. Then there are $y_1, \ldots, y_t \in h^m(M)$ for $i = 1, \ldots, t$ such that $y_1, \ldots, y_t \neq 0$. If we take $a$ and $b$ such that $-\infty < a < \inf f(M) \leq \sup f(M) < b < \infty$, then $\text{ind}(M_a, M) = h^*(M)$ and $\text{ind}(M_b, M) = 0$. Thus we can take $x_0, x_1, \ldots, x_k$ in Theorem 3.11 so as

$$x_0 = 1, x_i = y_{(i-1)l+1} \cdots y_{(i-1)l+2} \cdots y_{il} \quad (1 \leq i \leq k).$$

Since $\deg x_i = ml$ for all $i$ with $1 \leq i \leq k$, Theorem 3.11 shows $h^{ml}(K) \neq 0$. □

Finally, we give an application of Corollary 3.13. Let $K$ be the reals $R$, the complexes $C$, or the quaternions $H$, and according to that $G$ be the group $Z_2$, $S^1$ or $S^3$ of $g \in K$ with $|g| = 1$. Then $G$ acts on $K^n$ by coordinate-wise multiplication, and the unit sphere $S(K^n)$ of $K^n$ is a $G$-invariant submanifold with the orbit space $S(K^n)/G = K P^{n-1}$, the projective space. Let $h^*(X) = H^*(X/G; F)$ where $H^*$ is the Alexander-Spanier cohomology and $F = Z_2$, $Q$ or $Q$ according to $K = R$, $C$ or $H$. Then

$$h^*(S(K^n)) \cong F[x]/(x^n), \quad d = \deg x = 1, 2 \text{ or } 4,$$

and we see $\cup_d (h^*(S(k))) = n - 1$. Thus Corollary 3.13 shows that if a $C^1 G$-function $f: S(K^n) \to R$ has $k$ critical values, then $h^{dl}(K) \neq 0$ for any integer $l$ with $0 \leq kl \leq n - 1$. This says a lot more about the cohomology of $K$ than in Clapp-Puppe [5; §2].

For many spaces other than $S(K^n)$ we already know the $\cup_1$-length or a lower bound of that. See for example Fadell-Husseini[10; Theorem 3.16], Hiller [11], Jaworowski [12; §5] and Komiya [13; Remark 5.10]. So we can apply Corollary 3.13 to functions on such spaces.

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**References**


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