Title	The probability of two integers to be co-prime, revisited : on the behavior of CLT-scaling limit
Author(s)	Sugita, Hiroshi; Takanobu, Satoshi
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THE PROBABILITY OF TWO INTEGERS
TO BE CO-PRIME, REVISITED
— ON THE BEHAVIOR OF CLT-SCALING LIMIT

Hiroshi SUGITA* and Satoshi TAKANOBU†

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1. Introduction

Let $\gcd(x, y)$ denote the greatest common divisor of integers x and y. Define functions X and S_N on $\mathbb{Z}^2$ by

$$
(1) \qquad X(x, y) := \begin{cases} 1, & \text{if } \gcd(x, y) = 1, \\ 0, & \text{if } \gcd(x, y) > 1, \end{cases}
$$

$$
(2) \qquad S_N(x, y) := \frac{1}{N^2} \sum_{m,m'=1}^{N} X(x + m, y + m'), \quad N \in \mathbb{N}.
$$

The following number-theoretic limit theorem is due to Dirichlet [4] (cf. [6, Theorem 332]):

$$
(3) \qquad \lim_{N \to \infty} S_N(x, y) = \frac{6}{\pi^2}, \quad (x, y) \in \mathbb{Z}^2.
$$

Regarding (3) as a law of large numbers (LLN for short), it is natural to ask if a central limit theorem (CLT for short) holds for X. That is, for sufficiently large N, is the scaled function

$$
(4) \qquad Y_N(x, y) := N \left(S_N(x, y) - \frac{6}{\pi^2} \right)
$$

approximately normally "distributed"? Here we consider "distribution" of Y_N, at the suggestion of [2], [5] and [9], as follows: If the limit

$$
(5) \qquad \lim_{M \to \infty} \frac{1}{M^2} \sum_{m,m'=1}^{M} \exp \left(\sqrt{-1}\, t\, Y_N(m, m') \right), \quad t \in \mathbb{R},
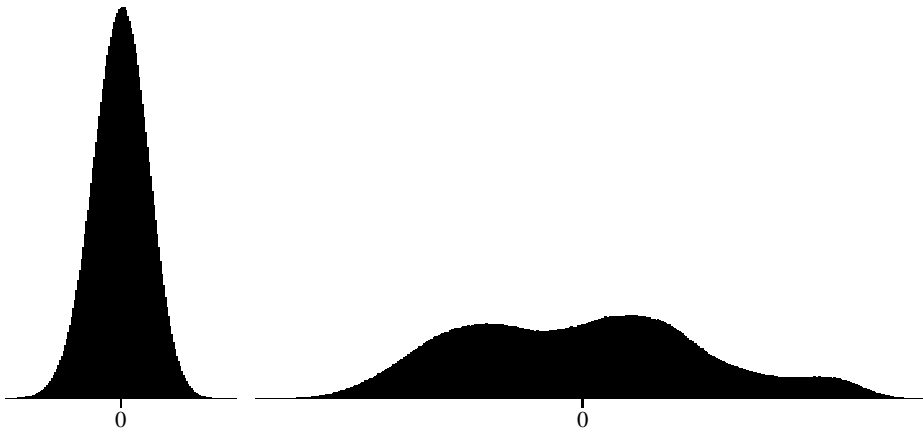$$

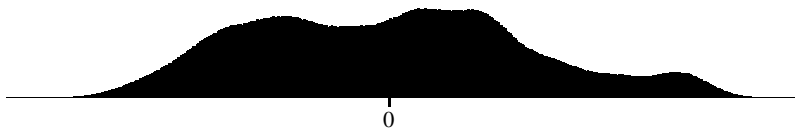Fig. 1.   The "distributions" of Y_{210} (left) and Y_{211} (right).

Fig. 2.   The "distribution" of Y_{2311}.

exists and it coincides with the characteristic function of some probability distribution on $\mathbb{R}$, then we call it the "distribution" of Y_N.

In order to sketch the "distribution" of Y_N, we made some numerical experiments to compute the relative frequency distribution of $Y_N(x, y)$ by picking random 10^7 samples of (x, y) from a big square $\{0, 1, \ldots, M = 2^{31} - 1\}^2 \subset \mathbb{Z}^2$. (In this numerical experiment, we used the pseudo-random number generator proposed by [20].)

For each N, a "distribution" surely appears, however it sharply depends on N. For example, the left picture of Fig. 1 shows the "distribution" of Y_{210}, which looks like a Gaussian distribution (the variance is approximately 6.26×10^{-3}), but if we increase N by 1, that is, $N = 211$, then the "distribution" becomes as illustrated by the right picture of Fig. 1, which is far from Gaussian (the variance is approximately 1.23×10^{-1}). Thus, the "distribution" of Y_N does not converge by simply letting $N \to \infty$. On the other hand, we can find very close "distributions". For example, as we see in Fig. 2, the "distribution" of Y_{2311} (the variance is approximately 1.21×10^{-1}) is very close to that of Y_{211}.

Then, our aim of this paper is to give a complete description of this mysterious behavior of Y_N when $N \to \infty$.

To this end, we discovered that the formulation by means of *the ring of finite in-*

tegral adeles $\widehat{\mathbb{Z}}$ (see Definition 1 below), which is a well-known compactification of $\mathbb{Z}$ in number theory, is indispensable.

Since $\widehat{\mathbb{Z}}$ is a compact group with respect to addition, there exists a unique normalized Haar measure λ on $\widehat{\mathbb{Z}}$. We first extend X, S_N and Y_N to random variables on the probability space $(\widehat{\mathbb{Z}}^2, \lambda^2)$ (Definitions 2 and 3). Of course, the distributions of those extended random variables coincide with the "distributions" of the original functions on $\mathbb{Z}^2$ in the sense of (5) (Theorem 1), respectively. This extension makes indirect probabilistic discussions, such as (3) and (5), into real and straightforward probabilistic ones, and hence it enables us to use all tools provided by probability theory.

In Theorem 4, we formulate Dirichlet's theorem (3) in this framework as a rigorous strong LLN. By this result, it becomes clear that (3) is just a cross-section of the LLN intersected with $\mathbb{Z}^2$ (cf. [12]).

Next, we study the limit behavior of Y_N, our main target. To mention our main result, Theorem 6, we must introduce a quotient ring $\widehat{\mathbb{Z}}/\sim$ of $\widehat{\mathbb{Z}}$. $\widehat{\mathbb{Z}}/\sim$ can be said as the ring obtained by completing $\mathbb{Z}$ by a metric

$$(6) \qquad \widetilde{d}(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbf{1}_{\{x \not\equiv y \,(\mathrm{mod}\ p_i)\}}, \quad x, y \in \mathbb{Z},$$

where $\{p_i\}_{i=1}^{\infty}$ is the sequence of all prime numbers in the increasing order. (For the precise definition of $\widehat{\mathbb{Z}}/\sim$ as a quotient ring, see Definition 5.) We then describe completely the set of all limit points of $\{Y_N\}_{N=1}^{\infty}$ in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$ by parametrizing them continuously in terms of elements of $\widehat{\mathbb{Z}}/\sim$ (Theorem 6). Thus, the notion of adeles is essentially needed in this theorem.

According to Theorem 6, the phenomena seen in Fig. 1 and Fig. 2 are explained in the following way: Since 210 and 211 are far away from each other in the metric of $\widehat{\mathbb{Z}}/\sim$, the corresponding distributions are quite different, on the other hand, since 211 and 2311 are close to each other in the metric of $\widehat{\mathbb{Z}}/\sim$, the corresponding distributions look very similar. Indeed, we have

$$210 = 2 \cdot 3 \cdot 5 \cdot 7 = p_1 p_2 p_3 p_4,$$
$$211 = 2 \cdot 3 \cdot 5 \cdot 7 + 1 = p_1 p_2 p_3 p_4 + 1,$$
$$2311 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = p_1 p_2 p_3 p_4 p_5 + 1,$$

so that $\widetilde{d}(210, 211) = 1$, which is equal to the diameter of $\widehat{\mathbb{Z}}/\sim$, while $\widetilde{d}(211, 2311) = \sum_{i=1}^{\infty} \mathbf{1}_{\{211 \not\equiv 2311 \,(\mathrm{mod}\ p_i)\}} 2^{-i} = \sum_{i=5}^{\infty} 2^{-i} = 2^{-4}$.

Furthermore, according to Theorem 6 and Theorem 7 (i), if $\mathbb{N} \ni N_k \neq 0$ and $N_k \to 0$ in $\widehat{\mathbb{Z}}/\sim$, then Y_{N_k} converges to 0 in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$. This explains that Y_{210} has a very small variance, since 210 is close to 0 in $\widehat{\mathbb{Z}}/\sim$ ($\widetilde{d}(210, 0) = 2^{-4}$).

REMARK 1.   We here give a couple of philological notes: The function $X(x, y)$ defined by (1) is one of the instances from the class of two-variable multiplicative func-

tions introduced by [3], in which Dirichlet's theorem (3) was presented as one of the consequences of a general mean value theorem for the functions of that class. The method of compactification of $\mathbb{Z}$ for the investigation of the mean value or distribution problems of arithmetic functions was initiated by [15], and it has been studied by several papers and books, such as [7, 10, 11, 12, 14, 18].

We would like to thank the referee for letting us know these references.

2. Basic notions and summary of theorems

Let us introduce our basic framework and the theorems obtained in this paper. Proofs will be given later.

For a prime p, the p-adic metric d_p is defined by

$$d_p(x, y) := \inf \left\{ p^{-k} ; p^k \mid (x - y) \right\}, \quad x, y \in \mathbb{Z}.$$

The completion of $\mathbb{Z}$ by d_p is denoted by $\mathbb{Z}_p$. By extending the algebraic operations '+' and '×' in $\mathbb{Z}$ continuously to those in $\mathbb{Z}_p$, the compact metric space $(\mathbb{Z}_p, d_p)$ becomes a ring, called the ring of p-adic integers. In particular, $(\mathbb{Z}_p, d_p)$ is a compact abelian group with respect to '+'. According to the general theory of compact groups (for instance, [17, Theorem 5.14]), there is a unique normalized Haar measure λ_p with respect to '+' on the measurable space $(\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p))$, where $\mathcal{B}(\mathbb{Z}_p)$ denotes the Borel field of $\mathbb{Z}_p$.

DEFINITION 1. (i) Let $\{p_i\}_{i=1}^{\infty}$, $2 = p_1 < p_2 < \cdots$, be the sequence of all primes.
(ii) Put

$$\widehat{\mathbb{Z}} := \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}, \qquad \lambda := \prod_{i=1}^{\infty} \lambda_{p_i}.$$

For $x = (x_i)$, $y = (y_i) \in \widehat{\mathbb{Z}}$, we define

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} d_{p_i}(x_i, y_i), \qquad x + y := (x_i + y_i), \qquad xy := (x_i y_i).$$

By these definitions, $\widehat{\mathbb{Z}}$ becomes a ring, called *the ring of finite integral adeles*. $(\widehat{\mathbb{Z}}, d)$ is again a compact metric space, and both '+' and '×' are continuous. In particular, $(\widehat{\mathbb{Z}}, d)$ is a compact abelian group with respect to '+' and its normalized Haar measure is nothing but λ.

In the modern number theory, *adeles* are treated in much more abstract way than they are presented here. For details, see [8, Chapter III]. For $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$, [21, Chapter 9] is a good reference.

REMARK 2. Throughout the present paper, p is always a prime. For the sake of simple notations, we often write $\sum_p f(p)$ or $\prod_p f(p)$ instead of $\sum_{i=1}^{\infty} f(p_i)$ or $\prod_{i=1}^{\infty} f(p_i)$, respectively. $\sum_{p \le p_L} f(p)$ means $\sum_{i=1}^{L} f(p_i)$, etc.

DEFINITION 2. (i) We identify $\mathbb{Z}$ with the diagonal set $\{(n, n, \ldots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots\} \subset \widehat{\mathbb{Z}}$.

(ii) For $\mathbb{N} \ni m \ge 2$ and $k \in \{0, 1, \ldots, m-1\}$, we define $m\widehat{\mathbb{Z}} + k := \{mx + k \,;\, x \in \widehat{\mathbb{Z}}\}$. Then we have $\widehat{\mathbb{Z}} = \bigcup_{k=0}^{m-1}(m\widehat{\mathbb{Z}} + k)$, which is a disjoint union (Lemma 3 (iii) in Section 3). So, for $x \in \widehat{\mathbb{Z}}$ and $\mathbb{N} \ni m \ge 2$, there exists a unique $k \in \{0, 1, \ldots, m-1\}$ such that $x - k \in m\widehat{\mathbb{Z}}$. This k is denoted by $x \bmod m$. For $m = 1$, we always set $x \bmod m := 0$. Obviously, if $x \in \mathbb{Z}$, this definition coincides with the usual modulo operation.

(iii) For $x, y \in \widehat{\mathbb{Z}}$, we define

$$(7) \qquad \gcd(x, y) := \sup\{m \in \mathbb{N} \,;\, (x \bmod m) = (y \bmod m) = 0\}.$$

Obviously, for $x, y \in \mathbb{Z}$, this definition coincides with the usual gcd.

REMARK 3. It is easy to see that $\lambda(\mathbb{Z}) = 0$.

DEFINITION 3. We define random variables X, S_N and Y_N on $\widehat{\mathbb{Z}}^2$ by

$$X(x, y) := \begin{cases} 1 & \text{if } \gcd(x, y) = 1, \\ 0 & \text{if } \gcd(x, y) > 1, \end{cases}$$

$$S_N(x, y) := \frac{1}{N^2} \sum_{m,n=1}^{N} X(x + m, y + n), \quad N \in \mathbb{N},$$

$$Y_N(x, y) := N\left(S_N(x, y) - \frac{6}{\pi^2}\right), \quad N \in \mathbb{N}.$$

Obviously, for $x, y \in \mathbb{Z}$, these definitions coincide with the original functions (1), (2) and (4).

As is naturally expected, the distributions of X, S_N and Y_N coincide with the "distributions" of the original functions on $\mathbb{Z}^2$ in the sense of (5), respectively. Namely, we have:

Theorem 1 (Section 6). *Let $U = X$, $= S_N$ or $= Y_N$. Then for each $t \in \mathbb{R}$, it holds that*

$$(8) \qquad \lim_{M \to \infty} \frac{1}{M^2} \sum_{m,m'=1}^{M} \exp\left(\sqrt{-1}\, t\, U(m, m')\right) = \mathbf{E}^{\lambda^2}\left[\exp\left(\sqrt{-1}\, t\, U(x, y)\right)\right].$$

Here (*and hereafter*) $\mathbf{E}^{\lambda^2}$ *stands for the expectation with respect to* λ^2.

In other words, although $\lambda^2(\mathbb{Z}^2) = 0$ (Remark 3), the distributions of X, S_N and Y_N, which are realized on $(\widehat{\mathbb{Z}}^2, \lambda^2)$, can be seen by observing their behaviors on a large square $\{0, 1, \ldots, M - 1\}^2 \subset \mathbb{Z}^2$.

Theorem 2 (Section 3). *It holds that* $\mathbf{E}^{\lambda^2}[X] = 6/\pi^2$.

Let $\{\Sigma^{(m,n)} ; (m, n) \in \mathbb{Z}^2\}$ be the shift transformations on $(\widehat{\mathbb{Z}}^2, \mathcal{B}(\widehat{\mathbb{Z}}^2), \lambda^2)$ defined by

$$\Sigma^{(m,n)} : \widehat{\mathbb{Z}}^2 \ni (x, y) \mapsto (x + m, y + n) \in \widehat{\mathbb{Z}}^2.$$

Then $\Sigma^{(m,n)}$ is $\mathcal{B}(\widehat{\mathbb{Z}}^2)$-measurable, $\Sigma^{(0,0)} = $ identity, $\Sigma^{(m,n)} \circ \Sigma^{(m',n')} = \Sigma^{(m+m',n+n')}$, and it preserves λ^2, i.e.,

$$\lambda^2 \left(\Sigma^{(m,n)^{-1}}(\Gamma) \right) = \lambda^2(\Gamma), \quad \forall (m, n) \in \mathbb{Z}^2, \ \forall \Gamma \in \mathcal{B}(\widehat{\mathbb{Z}}^2).$$

Theorem 3 (Section 4). $\{\Sigma^{(m,n)} ; (m, n) \in \mathbb{Z}^2\}$ *is ergodic*.

Theorem 2 and Theorem 3 imply an LLN for $X(x, y)$, Theorem 4 below, which is a natural probability-theoretic extension of Dirichlet's theorem (3).

Theorem 4.   *For* λ^2-*a.e.* $(x, y) \in \widehat{\mathbb{Z}}^2$, *we have*

$$(9) \qquad\qquad \lim_{N \to \infty} S_N(x, y) = \mathbf{E}^{\lambda^2}[X] = \frac{6}{\pi^2}.$$

In contrast to Dirichlet's limit theorem (3), there are exceptional points $(x, y) \in \widehat{\mathbb{Z}}^2$ for which (9) does not hold (Remark 7 in Section 4).

DEFINITION 4 (Frequently used arithmetic functions).   (i)   Let $\mu \colon \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function, i.e.,

$$\mu(n) := \begin{cases} 1, & (n = 1), \\ 0, & (\exists m \geq 2, \ m^2 \mid n), \\ (-1)^k, & (n \text{ is the product of } k \text{ distinct primes}). \end{cases}$$

(ii) Let $\phi \colon \mathbb{N} \to \mathbb{N}$ be the Euler function, i.e., $\phi(n)$ denotes the number of positive integers not exceeding n and relatively prime to n. In other words, $\phi(n) = \sum_{j=1}^{n} X(n, j)$.
(iii) Define a function $\rho_k \colon \widehat{\mathbb{Z}} \to \{0, 1\}$ by

$$\rho_k(x) := \mathbf{1}_{k\widehat{\mathbb{Z}}}(x) = \begin{cases} 1, & (x \bmod k = 0), \\ 0, & (x \bmod k > 0), \end{cases} \qquad k \in \mathbb{N}, \ x \in \widehat{\mathbb{Z}}.$$

Next, we consider the CLT-scaling limit of $X(x, y)$, that is, the limit behavior of $Y_N(x, y)$. We have the following explicit expression.

Theorem 5 (Section 5). *Let $N \in \mathbb{N}$. As an equality in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$, or as an equality for each $(x, y) \in \mathbb{Z}^2$ with $\max(x, y) \geq 0$, the following holds*:

$$
\begin{aligned}
Y_N(x, y) = &-\sum_{u=1}^{\infty} \frac{\mu(u)}{u} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&- \sum_{u=1}^{\infty} \frac{\mu(u)}{u} \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) \\
&+ \frac{1}{N} \sum_{u=1}^{\infty} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&\qquad\qquad \times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right).
\end{aligned}
$$
(10)

For $(x, y) \in \mathbb{Z}^2$ with $\max(x, y) < 0$, (10) holds if we replace its last infinite sum by

$$
\lim_{\substack{\mathbb{M} \nearrow \mathbb{N} \\ \text{with } \sum_{u \in \mathbb{M}} \mu(u)=0, \\ \sum_{u \in \mathbb{M}} \mu(u)/u \to 0}} \frac{1}{N} \sum_{u \in \mathbb{M}} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right).
$$

REMARK 4. In the numerical experiments in Section 1, we used the right-hand side of (10) to evaluate $Y_N(x, y)$ to save the computation time. Of course, we cannot compute the infinite sums of (10), instead, we approximated them by $\sum_{u=1}^{3900}$.

Let us write the right-hand side of (10) as $-T(x; N) - T(y; N) + R(x, y; N)$, i.e.,

$$
-T(x; N) := -\sum_{u=1}^{\infty} \frac{\mu(u)}{u} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right),
$$
(11)

$$
\begin{aligned}
R(x, y; N) := &\frac{1}{N} \sum_{u=1}^{\infty} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&\times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right).
\end{aligned}
$$
(12)

Then as $N \to \infty$, we have $R(x, y; N) \to 0$ in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$, but $-T(\cdot; N)$ does not converge, and hence Y_N does not, either.

DEFINITION 5. (i) For $z, z' \in \widehat{\mathbb{Z}}$, we define an equivalence relation

$$
z \sim z' \iff \forall p: \text{prime}, \ (z - z') \bmod p = 0.
$$

For each $z \in \widehat{\mathbb{Z}}$, we let $[z]$ denote the equivalence class to which z belongs. We endow $\widehat{\mathbb{Z}}/\sim$ with the quotient topology, by which $\widehat{\mathbb{Z}}/\sim$ becomes again a compact metrizable ring. One of the metrics that are consistent with the topology is:

$$(13) \qquad \widetilde{d}([x], [y]) := \sum_{i=1}^{\infty} 2^{-i}(1 - \rho_{p_i}(x - y)), \qquad x, y \in \widehat{\mathbb{Z}}.$$

(ii) For each $z \in \widehat{\mathbb{Z}}$, we define $-T(x; z)$ by replacing N in (11) by z. If $z \sim z'$, we see $-T(x; z) = -T(x; z')$ (Lemma 11 (i) in Section 7), and hence we write it as $-T(x; [z])$.

Now, the following theorem completely describes the limit behavior of Y_N.

Theorem 6 (Section 7.1). *The set of all limit points of* $\{Y_N(x, y)\}_{N=1}^{\infty}$ *in* $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$ *is*

$$(14) \qquad \{-T(x; [z]) - T(y; [z]) \, ; \, [z] \in \widehat{\mathbb{Z}}/\sim\}.$$

Moreover, it holds that for each $[z] \in \widehat{\mathbb{Z}}/\sim$,

$$\lim_{\substack{[N] \to [z] \ in \ \widehat{\mathbb{Z}}/\sim \\ with \ [N] \neq [z]}} Y_N(x, y) = -T(x; [z]) - T(y; [z]), \quad in \ L^2(\widehat{\mathbb{Z}}^2, \lambda^2).$$

Since $\mathbb{N}$ is dense in $\widehat{\mathbb{Z}}/\sim$ (Lemma 1 in Section 3), we can let Y_N converge to any limit point of (14). About the random variable $-T(\,\cdot\,; z)$, the following facts are known.

Theorem 7 (Section 7.2). (i) $-T(x; [z]) = 0$, λ-*a.e.* x, *if and only if* $z \sim 0$.
(ii) *The mapping* $\widehat{\mathbb{Z}}/\sim \ni [z] \mapsto T(\,\cdot\,; [z]) \in L^2(\widehat{\mathbb{Z}}, \lambda)$ *is continuous and injective.*
(iii) *If* $N \in \mathbb{N}$, *it holds that for* λ-*a.e.* x *or* $x \in \{0, 1, 2, \ldots\}$,

$$-T(x; N) = -\sum_{m=0}^{N-1} T(x + m; 1) = \sum_{m=1}^{N} \prod_{p} \left(1 - \frac{\rho_p(x + m)}{p}\right) - N\frac{6}{\pi^2}.$$

In particular, if $x \in \mathbb{N}$, *we have*

$$-T(x; 1) = \frac{\phi(x + 1)}{x + 1} - \frac{6}{\pi^2}.$$

(iv) *The distribution of* $-T(x; 1)$ *is continuous, but it is singular with respect to the Lebesgue measure.*
(v) *For* $N \in \mathbb{N}$, *the distribution of* $-T(x; N)$ *is supported by a compact set.*

REMARK 5. As a by-product, we can refine the following limit theorem:

$$(15) \qquad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\phi(n+m)}{n+m} = \frac{6}{\pi^2}, \qquad m \in \mathbb{N}.$$

In [9, Chapter 4 section 2], we can find (15) for $m = 0$, from which (15) easily follows for any $m \in \mathbb{N}$. According to Theorem 7 (iii), the adelic counterpart of this limit theorem is the following LLN:

$$-\frac{1}{M} \sum_{m=0}^{M-1} T(x+m;1) = \frac{1}{M} \sum_{m=1}^{M} \prod_p \left(1 - \frac{\rho_p(x+m)}{p}\right) - \frac{6}{\pi^2} \to 0$$

$$\text{as } M \to \infty, \quad \lambda\text{-a.e. } x.$$

This LLN follows from the ergodicity of the shift $\widehat{\mathbb{Z}} \ni x \mapsto (x+1) \in \widehat{\mathbb{Z}}$, which is shown similarly as Theorem 3, and from the fact $\mathbf{E}^\lambda[-T(\cdot;1)] = 0$. So, the usual CLT-scaling gives $\{-M^{-1/2} \sum_{m=0}^{M-1} T(x+m;1)\}_{M \in \mathbb{N}}$, but its limit is degenerate. The proper scaling is "doing nothing", that is,

$$(16) \qquad \left\{-\sum_{m=0}^{M-1} T(x+m;1)\right\}_{M \in \mathbb{N}},$$

which has now uncountably many limit distributions. Indeed, Theorem 7 (iii) shows that the sequence (16) is nothing but $\{-T(x;M); M \in \mathbb{N}\}$, hence the set of all limit points of (16) in $L^2(\widehat{\mathbb{Z}}, \lambda)$ is $\{-T(\cdot;[z]); [z] \in \widehat{\mathbb{Z}}/\sim\}$ by Lemma 11 in Section 7.

3. Preliminary lemmas

In this section, we prove some fundamental properties on $\widehat{\mathbb{Z}}$ and λ.

Lemma 1. $\mathbb{N}' := \{(n, n, \ldots) \in \widehat{\mathbb{Z}}; n \in \mathbb{N}\}$ *is dense in* $\widehat{\mathbb{Z}}$.

Proof. The Chinese remainder theorem ([6, Theorem 121]) implies that for any $k \in \mathbb{N}$ and any $n_1, \ldots, n_k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $n = n_i \mod p_i^m$, $i = 1, \ldots, k$. This means that $\mathbb{N}'$ is dense in $\mathbb{Z} \times \mathbb{Z} \times \cdots$ with respect to the metric d. $\square$

As we identify $\mathbb{Z}$ with $\mathbb{Z}' := \{(n, n, \ldots) \in \widehat{\mathbb{Z}}; n \in \mathbb{Z}\}$ (Definition 2 (i)) by a bijection $\mathbb{Z} \ni n \mapsto (n, n, \ldots) \in \mathbb{Z}'$, Lemma 1 implies that $\mathbb{Z}$ is a dense subring of $\widehat{\mathbb{Z}}$. Thus $\widehat{\mathbb{Z}}$ is a compactification of $\mathbb{Z}$.

Lemma 2. (i) *Let p be a prime and $j \in \mathbb{N}$. Then $p^j \mathbb{Z}_p$ is closed and open.* (ii) *Let p, q be distinct primes and $j \in \mathbb{N}$. Then we have $p^j \mathbb{Z}_q = \mathbb{Z}_q$.*

Proof. (i) It is easy to see that $p^j\mathbb{Z}_p = \{x \in \mathbb{Z}_p \,;\, d_p(x,0) \le p^{-j}\}$, and hence it is closed. Since $d_p(x,0) \in \{p^{-a} \,;\, a = 0,1,\dots,\infty\}$ for any $x \in \mathbb{Z}_p$, we may write $p^j\mathbb{Z}_p = \{x \in \mathbb{Z}_p \,;\, d_p(x,0) < p^{-j+1}\}$, which implies it is open.

(ii) $p^j\mathbb{Z}_q \subset \mathbb{Z}_q$ is obvious, so let us prove $p^j\mathbb{Z}_q \supset \mathbb{Z}_q$. To this end, it is enough to show that there is an $x \in \mathbb{Z}_q$ such that $p^j x = 1$. For any $m \in \mathbb{N}$, there exists an $x_m \in \mathbb{N}$ such that $x_m p^j = 1 \bmod q^m$. Then for any $n > m$, we have $(x_n - x_m)p^j = 0 \bmod q^m$. Since $\gcd(p^j, q^m) = 1$, we see $x_n - x_m = 0 \bmod q^m$, which means that $\{x_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Z}_q$. Then putting $x := \lim_{m \to \infty} x_m$, we have $p^j x = 1$ in $\mathbb{Z}_q$. $\qquad\square$

Lemma 3. *Let $m \in \mathbb{N}$ and $k \in \{0,1,\dots,m-1\}$.*
(i) *The set $(m\widehat{\mathbb{Z}}+k)$ is closed and open.*
(ii) *$\rho_m : \widehat{\mathbb{Z}} \to \{0,1\}$ is continuous.*
(iii) *$\widehat{\mathbb{Z}} = \bigcup_{k=0}^{m-1}(m\widehat{\mathbb{Z}}+k)$, which is a disjoint union.*

Proof. (i) Let $m = \prod_p p^{\alpha_p(m)}$ be the factorization of m into primes, where $\alpha_p(m) = 0$ except for finitely many primes p. Then, Lemma 2 implies that

$$m\widehat{\mathbb{Z}} = \prod_p m\mathbb{Z}_p = \prod_p p^{\alpha_p(m)}\mathbb{Z}_p,$$

and that each $p^{\alpha_p(m)}\mathbb{Z}_p$ is closed and open. Therefore, $m\widehat{\mathbb{Z}}$ is also closed and open in $\widehat{\mathbb{Z}}$. Finally, since the shift $\widehat{\mathbb{Z}} \ni x \mapsto (x+k) \in \widehat{\mathbb{Z}}$ is a homeomorphism, $(m\widehat{\mathbb{Z}}+k)$ is also closed and open.

(ii) Since (i) implies that $\rho_m^{-1}(\{1\}) = m\widehat{\mathbb{Z}}$ is closed and open, the statement is obvious.

(iii) From the denseness of $\mathbb{Z}$ in $\widehat{\mathbb{Z}}$, and from the continuity and closedness of the mapping $x \mapsto mx+k$, it follows that $\overline{m\mathbb{Z}+k} = m\widehat{\mathbb{Z}}+k$. Since $\mathbb{Z} = \bigcup_{k=0}^{m-1}(m\mathbb{Z}+k)$, this implies

$$\widehat{\mathbb{Z}} = \bigcup_{k=0}^{m-1}(m\widehat{\mathbb{Z}}+k).$$

Next we check the disjointness of this union. Let $m \ge 2$ and $k, k' \in \{0,\dots,m-1\}$ be distinct integers. By (i), $A := (m\widehat{\mathbb{Z}}+k) \cap (m\widehat{\mathbb{Z}}+k')$ is open. If $A \ne \emptyset$, then $\mathbb{Z} \cap A \ne \emptyset$, because $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$. But then, taking an $l \in \mathbb{Z} \cap A$, we see from the observation of (i) that

$$d_p(l-k,0) \le p^{-\alpha_p(m)}, \quad d_p(l-k',0) \le p^{-\alpha_p(m)}, \quad \forall p: \text{prime}.$$

This implies that $p^{\alpha_p(m)} \mid k - k'$ for each prime p, that is, $m \mid k - k'$, which is impossible. Thus A should be empty. $\qquad\square$

Corollary 1. *For any $l \in \mathbb{Z}$, the mapping*

$$\widehat{\mathbb{Z}} \ni x \mapsto \frac{(l+x) \bmod m}{m} \in [0, 1)$$

is continuous.

Lemma 4. *For any $k \in \mathbb{Z} \setminus \{0\}$ and any $A \in \mathcal{B}(\widehat{\mathbb{Z}})$, we have $kA \in \mathcal{B}(\widehat{\mathbb{Z}})$ and*

$$(17) \qquad \lambda(kA) = \frac{1}{|k|}\lambda(A).$$

Proof.   Since $\widehat{\mathbb{Z}}$ is complete separable metric space and the map $\widehat{\mathbb{Z}} \ni x \mapsto kx \in \widehat{\mathbb{Z}}$ is one-to-one and measurable, we have $kA \in \mathcal{B}(\widehat{\mathbb{Z}})$ (cf. [16, Chapter I Theorem 3.9]). Let ν be a Borel probability measure on $\widehat{\mathbb{Z}}$ defined by

$$\nu(A) = \frac{\lambda(kA)}{\lambda(|k|\widehat{\mathbb{Z}})}, \quad A \in \mathcal{B}(\widehat{\mathbb{Z}}).$$

Then ν is clearly shift invariant, and hence $\nu = \lambda$, so that $\lambda(kA) = \lambda(|k|\widehat{\mathbb{Z}})\lambda(A)$. By Lemma 3 and the shift invariance of λ, we see

$$1 = \lambda(\widehat{\mathbb{Z}}) = \sum_{i=0}^{|k|-1} \lambda(|k|\widehat{\mathbb{Z}} + i) = |k|\lambda(|k|\widehat{\mathbb{Z}}),$$

from which, (17) immediately follows.   □

Proof of Theorem 2.   Since $\gcd(x, y) = 1$ if and only if $x \bmod p \neq 0$ or $y \bmod p \neq 0$ for any prime p, we see

$$(18) \qquad X(x, y) = \prod_p \left(1 - \rho_p(x)\rho_p(y)\right).$$

Therefore, noting that $\mathbf{E}^\lambda[\rho_p] = \lambda(p\widehat{\mathbb{Z}}) = 1/p$ by Lemma 4, we have

$$\mathbf{E}^{\lambda^2}[X] = \prod_p \mathbf{E}^{\lambda^2}\left[1 - \rho_p(x)\rho_p(y)\right]$$

$$= \prod_p \left(1 - \mathbf{E}^\lambda[\rho_p]^2\right)$$

$$= \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}. \qquad \square$$

Since $\widehat{\mathbb{Z}}$ is compact and $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$ (Lemma 1), each continuous function defined on $\widehat{\mathbb{Z}}$ is uniquely determined by the values $f(n, n, \ldots) =: f(n)$ on $\mathbb{Z}$.

Lemma 5. (i) *Let* $f \colon \widehat{\mathbb{Z}} \to \mathbb{C}$ *be a continuous function. Then* $\{f(n)\}_{n \in \mathbb{Z}}$ *is an almost periodic sequence, i.e.,*

$$(19) \quad \forall \varepsilon > 0, \ \exists l_0, \ m_0 \in \mathbb{N} \ such \ that \ \left| f(n) - f\left(n \bmod \prod_{i=1}^{l_0} p_i^{m_0}\right) \right| < \varepsilon, \quad \forall n \in \mathbb{Z}.$$

(ii) *Conversely, if* $\{f(n)\}_{n \in \mathbb{Z}}$ *is an almost periodic sequence, i.e., it satisfies (19), then there exists a unique continuous function* $\widetilde{f} \colon \widehat{\mathbb{Z}} \to \mathbb{C}$ *such that* $\widetilde{f}(n) = f(n)$ *for each* $n \in \mathbb{Z}$.

Proof. (i) Obvious by the definition of the metric of $\widehat{\mathbb{Z}}$.
(ii) If f is a periodic sequence with period $m \in \mathbb{N}$, it is of the form $f(n) = \sum_{i=1}^{m} f(i) \rho_m(n - i)$, $n \in \mathbb{Z}$. Then $\widetilde{f}(x) := \sum_{i=1}^{m} f(i) \rho_m(x - i)$, $x \in \widehat{\mathbb{Z}}$, is the continuous function with the property $\widetilde{f}|_{\mathbb{Z}} = f$. Note that a general f satisfying (19) is a uniformly convergent limit of a sequence of periodic sequences, and hence it has again a continuous extension $\widetilde{f}$. Since $\mathbb{Z}$ is densely embedded in $\widehat{\mathbb{Z}}$, the uniqueness of $\widetilde{f}$ is obvious. □

For a periodic sequence $\{g(n)\}_{n \in \mathbb{Z}}$ with period m and its unique continuous extension $\widetilde{g}(x)$, it is easy to see that

$$(20) \qquad \int_{\widehat{\mathbb{Z}}} \widetilde{g}(x) \lambda(dx) = \sum_{n=1}^{m} g(n) \int_{\widehat{\mathbb{Z}}} \rho_m(x - n) \lambda(dx)$$

$$= \frac{1}{m} \sum_{n=1}^{m} g(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(n).$$

In general, we have the following lemma.

Lemma 6. *If* $f \colon \widehat{\mathbb{Z}} \to \mathbb{C}$ *is continuous, then*

$$(21) \qquad \int_{\widehat{\mathbb{Z}}} f(x) \lambda(dx) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} f(n), \quad \forall n_0 \in \mathbb{Z}.$$

The convergence is uniform in $n_0 \in \mathbb{Z}$.

Proof. Let $f \colon \widehat{\mathbb{Z}} \to \mathbb{C}$ be continuous and set $f_{n_0}(x) = f(n_0 + x)$, $n_0 \in \mathbb{Z}$. By the uniform continuity of f, a family $\{f_{n_0}; n_0 \in \mathbb{Z}\}$ satisfy (19). For simplicity set

$m := \prod_{i=1}^{l_0} p_i^{m_0}$. Then we see for any $n_0 \in \mathbb{Z}$,

$$\left| \int_{\widehat{\mathbb{Z}}} f_{n_0}(x) \lambda(dx) - \int_{\widehat{\mathbb{Z}}} f_{n_0}(x \bmod m) \lambda(dx) \right| \leq \varepsilon,$$

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n) - \frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n \bmod m) \right| < \varepsilon, \quad \forall N \in \mathbb{N}.$$

By (20),

$$\int_{\widehat{\mathbb{Z}}} f_{n_0}(x \bmod m) \lambda(dx) = \frac{1}{m} \sum_{r=0}^{m-1} f_{n_0}(r).$$

Also, by a simple calculation

$$\frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n \bmod m) = \frac{1}{N} \left(\left\lfloor \frac{N-1}{m} \right\rfloor \sum_{r=0}^{m-1} f_{n_0}(r) + \sum_{r=0}^{(N-1) \bmod m} f_{n_0}(r) \right)$$

$$= \frac{1}{m} \sum_{r=0}^{m-1} f_{n_0}(r) - \frac{1}{N} \left(\frac{1}{m} + \frac{(N-1) \bmod m}{m} \right) \sum_{r=0}^{m-1} f_{n_0}(r)$$

$$+ \frac{1}{N} \sum_{r=0}^{(N-1) \bmod m} f_{n_0}(r).$$

In the above and in what follows, the symbol $\lfloor t \rfloor$ stands for the largest integer not exceeding $t \in \mathbb{R}$.

From these, it follows that

$$\left| \int_{\widehat{\mathbb{Z}}} f_{n_0}(x \bmod m) \lambda(dx) - \frac{1}{N} \sum_{n=0}^{N-1} f_{n_0}(n \bmod m) \right| \leq \frac{2}{N} m \|f\|_\infty.$$

Therefore, choosing an $N_0 \in \mathbb{N}$ so large that $(2/N_0)m\|f\|_\infty < \varepsilon$, we have that for any $N \geq N_0$ and any $n_0 \in \mathbb{N}$,

$$\left| \int_{\widehat{\mathbb{Z}}} f(x) \lambda(dx) - \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} f(n) \right| < 3\varepsilon. \qquad \square$$

REMARK 6. As a matter of fact, (21) is a consequence of the following general theorem: *Let G be a compact group, and let $x \in G$. Then, if the sequence $\{x^n\}_{n=1}^\infty$ is dense in G, it is uniformly distributed, that is, $N^{-1} \sum_{n=1}^N \delta_{x^n}$ converges weakly to the normalized Haar measure of G as $N \to \infty$.* For details, see [13, Chapter 4 Theorem 4.2].

Corollary 2. *If $f: \widehat{\mathbb{Z}}^2 \to \mathbb{C}$ is continuous, then*

$$\int_{\widehat{\mathbb{Z}}^2} f(x, y)\lambda^2\,(dxdy) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{m_1 \leq m \leq m_1+N-1, \\ n_1 \leq n \leq n_1+N-1}} f(m, n), \quad \forall (m_1, n_1) \in \mathbb{Z}^2.$$

The convergence is uniform in $(m_1, n_1) \in \mathbb{Z}^2$.

4. Ergodicity and LLN

In this section, we prove Theorem 3, the ergodicity of the shift $\{\Sigma^{(m,n)}; (m, n) \in \mathbb{Z}^2\}$. Here, we show the following: *If $\Gamma \in \mathcal{B}(\widehat{\mathbb{Z}}^2)$ is Σ-invariant, i.e.,*

$$\lambda^2\left(\Gamma \triangle \Sigma^{(m,n)^{-1}}(\Gamma)\right) = 0, \quad \forall (m, n) \in \mathbb{Z}^2,$$

where $\triangle$ stands for the symmetric difference, then $\lambda^2(\Gamma) = 0$ or 1.

STEP 1.   For any $F \in L^1(\widehat{\mathbb{Z}}^2, \lambda^2)$, we have

(22) $$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(\cdot + m, * + n) = \mathbf{E}^{\lambda^2}[F] \quad \text{in } L^1(\widehat{\mathbb{Z}}^2, \lambda^2).$$

To show this, for any $\varepsilon > 0$, take a continuous function F_ε so that $\mathbf{E}^{\lambda^2}[|F - F_\varepsilon|] < \varepsilon$. By Corollary 2,

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F_\varepsilon(x + m, y + n) = \mathbf{E}^{\lambda^2}[F_\varepsilon(x + \cdot, y + *)]$$

$$= \mathbf{E}^{\lambda^2}[F_\varepsilon], \quad \forall (x, y) \in \widehat{\mathbb{Z}}^2.$$

By virtue of the bounded convergence theorem, this convergence takes place also in $L^1(\widehat{\mathbb{Z}}^2, \lambda^2)$. By the property of F_ε,

$$\mathbf{E}^{\lambda^2}\left[\left|\frac{1}{N^2} \sum_{m,n=1}^{N} F(\cdot + m, * + n) - \frac{1}{N^2} \sum_{m,n=1}^{N} F_\varepsilon(\cdot + m, * + n)\right|\right]$$

$$\leq \frac{1}{N^2} \sum_{m,n=1}^{N} \mathbf{E}^{\lambda^2}\left[|F(\cdot + m, * + n) - F_\varepsilon(\cdot + m, * + n)|\right] < \varepsilon.$$

Consequently,

$$\mathbf{E}^{\lambda^2}\left[\left|\frac{1}{N^2} \sum_{m,n=1}^{N} F(\cdot + m, * + n) - \mathbf{E}^{\lambda^2}[F]\right|\right]$$

$$\leq 2\varepsilon + \mathbf{E}^{\lambda^2}\left[\left|\frac{1}{N^2}\sum_{m,n=1}^{N}F_\varepsilon(\cdot + m, * + n) - \mathbf{E}^{\lambda^2}[F_\varepsilon]\right|\right] \underset{\substack{\text{first } N \to \infty \\ \text{second } \varepsilon \to 0}}{\longrightarrow} 0,$$

which implies (22).

STEP 2. By Step 1, for any $\Gamma \in \mathcal{B}(\widehat{\mathbb{Z}}^2)$, we see

$$\lim_{N \to \infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\mathbf{1}_\Gamma(\cdot + m, * + n) = \lambda^2(\Gamma) \quad \text{in } L^1(\widehat{\mathbb{Z}}^2, \lambda^2).$$

If Γ is Σ-invariant, then

$$\mathbf{1}_\Gamma(\cdot + m, * + n) = \mathbf{1}_\Gamma \circ \Sigma^{(m,n)}(\cdot, *)$$
$$= \mathbf{1}_{\Sigma^{(m,n)-1}(\Gamma)}(\cdot, *) = \mathbf{1}_\Gamma(\cdot, *), \quad \lambda^2\text{-a.s.},$$

and hence,

$$\frac{1}{N^2}\sum_{m,n=1}^{N}\mathbf{1}_\Gamma(\cdot + m, * + n) = \mathbf{1}_\Gamma(\cdot, *) \quad \lambda^2\text{-a.s.}$$

Therefore we have $\mathbf{1}_\Gamma(\cdot, *) = \lambda^2(\Gamma)$, λ^2-a.s., which implies that $\lambda^2(\Gamma) \in \{0, 1\}$. $\quad\square$

By the individual ergodic theorem (cf. [19, Theorem 6.1.8]), Theorem 2 and Theorem 3 imply an LLN for $X(x, y)$, that is, Theorem 4.

REMARK 7. There exist many $(x, y) \in \widehat{\mathbb{Z}}^2$ for which (9) does not hold. The following is one of such examples: Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be an injective mapping. For each $N \in \mathbb{N}$, we consider a system of equations

$$\begin{array}{ll} (x + m) \bmod p_{f(m,n)} = 0, & \\ (y + n) \bmod p_{f(m,n)} = 0, & m, n = 1, 2, \ldots, N, \end{array}$$

with unknown variable $(x, y) \in \mathbb{Z}^2$. By the Chinese remainder theorem, the solution (x, y), say (x_N, y_N), uniquely exists in the set $\{1, 2, \ldots, \prod_{m,n=1}^{N} p_{f(m,n)}\}^2$. Let $[x_N]$ and $[y_N]$ be the equivalence classes in $\widehat{\mathbb{Z}}/\sim$ to which x_N and y_N belong, respectively (see Definition 5). It is easy to see by the definition of the metric (13) that the sequence $\{([x_N], [y_N])\}_{N=1}^{\infty}$ is convergent in $(\widehat{\mathbb{Z}}/\sim)^2$. Let $(x_\infty, y_\infty) \in \widehat{\mathbb{Z}}^2$ be one of the representatives of the equivalence class to which $\{([x_N], [y_N])\}_{N=1}^{\infty}$ converges. Then it holds that

$$\begin{array}{ll} (x_\infty + m) \bmod p_{f(m,n)} = 0, & \\ (y_\infty + n) \bmod p_{f(m,n)} = 0, & m, n \in \mathbb{N}. \end{array}$$

Clearly, we have $X(x_\infty + m, y_\infty + n) = 0$, $m, n \in \mathbb{N}$, and so, $S_N(x_\infty, y_\infty) = 0$, $N \in \mathbb{N}$. Thus we have $\lim_{N \to \infty} S_N(x_\infty, y_\infty) = 0 \neq 6/\pi^2$.

5.   Explicit formula for S_N

In this section, we prove Theorem 5.
Let

$$(23) \qquad X_L(x, y) := \prod_{p \le p_L} \left(1 - \rho_p(x)\rho_p(y)\right),$$

$$(24) \qquad S_{N,L}(x, y) := \frac{1}{N^2} \sum_{m,n=1}^{N} X_L(x + m, y + n),$$

and let

$$(25) \qquad \mathbb{M}_L := \left\{ u = p_1^{\alpha_1} \cdots p_L^{\alpha_L} \in \mathbb{N} \,;\, 0 \le \alpha_1, \dots, \alpha_L \le L \right\}.$$

Lemma 7.

$$(26) \qquad S_N(x, y) = \lim_{L \to \infty} S_{N,L}(x, y) \quad \textit{(pointwise convergence)},$$

$$
\begin{aligned}
S_{N,L}(x, y) = {} & \sum_{u \in \mathbb{M}_L} \frac{\mu(u)}{u^2} - \frac{1}{N} \sum_{u \in \mathbb{M}_L} \frac{\mu(u)}{u} \left(\frac{(N + x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
& - \frac{1}{N} \sum_{u \in \mathbb{M}_L} \frac{\mu(u)}{u} \left(\frac{(N + y) \bmod u}{u} - \frac{y \bmod u}{u} \right) \\
& + \frac{1}{N^2} \sum_{u \in \mathbb{M}_L} \mu(u) \left(\frac{(N + x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
& \qquad\qquad \times \left(\frac{(N + y) \bmod u}{u} - \frac{y \bmod u}{u} \right).
\end{aligned}
\tag{27}
$$

Proof.   (26) is obvious. Next,

$$
\begin{aligned}
S_{N,L}(x, y) & \\
= {} & \frac{1}{N^2} \sum_{m,n=1}^{N} \prod_{p \le p_L} \left(1 - \rho_p(x + m)\rho_p(y + n)\right) \\
= {} & \frac{1}{N^2} \sum_{m,n=1}^{N} \left(1 + \sum_{r=1}^{L} \sum_{1 \le i_1 < \cdots < i_r \le L} (-1)^r \rho_{p_{i_1}}(x + m) \cdots \rho_{p_{i_r}}(x + m) \right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \left. \rho_{p_{i_1}}(y + n) \cdots \rho_{p_{i_r}}(y + n) \right) \\
= {} & \frac{1}{N^2} \sum_{m,n=1}^{N} \left(1 + \sum_{r=1}^{L} \sum_{1 \le i_1 < \cdots < i_r \le L} (-1)^r \rho_{p_{i_1} \cdots p_{i_r}}(x + m) \rho_{p_{i_1} \cdots p_{i_r}}(y + n) \right)
\end{aligned}
$$

$$= \frac{1}{N^2} \sum_{m,n=1}^{N} \sum_{u \mid p_1 \cdots p_L} \mu(u) \rho_u(x+m) \rho_u(y+n)$$

$$= \sum_{u \mid p_1 \cdots p_L} \mu(u) \left(\frac{1}{N} \sum_{m=1}^{N} \rho_u(x+m) \right) \left(\frac{1}{N} \sum_{n=1}^{N} \rho_u(y+n) \right).$$

Here we have

$$(28) \qquad \frac{1}{N} \sum_{m=1}^{N} \rho_u(x+m) = \frac{1}{N} \left\lfloor \frac{N + x \bmod u}{u} \right\rfloor$$

$$= \frac{1}{N} \left(\frac{N + x \bmod u}{u} - \frac{(N+x) \bmod u}{u} \right)$$

$$= \frac{1}{u} - \frac{1}{N} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right),$$

so that

$$S_{N,L}(x,y) = \sum_{u \mid p_1 \cdots p_L} \mu(u) \left(\frac{1}{u} - \frac{1}{N} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \right)$$

$$\times \left(\frac{1}{u} - \frac{1}{N} \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) \right)$$

$$= \sum_{u \mid p_1 \cdots p_L} \frac{\mu(u)}{u^2} - \frac{1}{N} \sum_{u \mid p_1 \cdots p_L} \frac{\mu(u)}{u} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right)$$

$$- \frac{1}{N} \sum_{u \mid p_1 \cdots p_L} \frac{\mu(u)}{u} \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right)$$

$$+ \frac{1}{N^2} \sum_{u \mid p_1 \cdots p_L} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right)$$

$$\times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right)$$

$$= \text{the right-hand side of (27)}. \qquad \qquad \square$$

5.1. Convergence for $(x,y) \in \mathbb{Z}^2$. We here prove that $N(S_{N,L}(x,y) - 6/\pi^2)$ converges to the right-hand side of (10) as $L \to \infty$.

If $u > |x|$,

$$x \bmod u = \begin{cases} x & \text{if } x \geq 0, \\ x+u & \text{if } x < 0, \end{cases}$$

so that if $u > N + |x|$, we have

$$\frac{(N + x) \bmod u}{u} - \frac{x \bmod u}{u} = \frac{N}{u} - \mathbf{1}_{N \geq x^- > 0},$$

where $x^- := (-x) \vee 0$. Hence, (27) is rewritten as

$$
\begin{aligned}
S_{N,L}(x, y) = &\sum_{u \in \mathbb{M}_L} \frac{\mu(u)}{u^2} \\
&- \frac{1}{N} \Bigg[\sum_{u \in \mathbb{M}_L \cap [1, N+|x|]} \frac{\mu(u)}{u} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&\quad + N \sum_{u \in \mathbb{M}_L \cap (N+|x|, \infty)} \frac{\mu(u)}{u^2} - \left(\sum_{u \in \mathbb{M}_L \cap (N+|x|, \infty)} \frac{\mu(u)}{u} \right) \mathbf{1}_{N \geq x^- > 0} \Bigg] \\
&- \frac{1}{N} \Bigg[\sum_{u \in \mathbb{M}_L \cap [1, N+|y|]} \frac{\mu(u)}{u} \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) \\
&\quad + N \sum_{u \in \mathbb{M}_L \cap (N+|y|, \infty)} \frac{\mu(u)}{u^2} - \left(\sum_{u \in \mathbb{M}_L \cap (N+|y|, \infty)} \frac{\mu(u)}{u} \right) \mathbf{1}_{N \geq y^- > 0} \Bigg] \\
&+ \frac{1}{N^2} \Bigg[\sum_{u \in \mathbb{M}_L \cap [1, N+|x| \vee |y|]} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&\qquad\qquad\qquad \times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) \\
&\quad + N^2 \sum_{u \in \mathbb{M}_L \cap (N+|x| \vee |y|, \infty)} \frac{\mu(u)}{u^2} \\
&\quad - N \left(\sum_{u \in \mathbb{M}_L \cap (N+|x| \vee |y|, \infty)} \frac{\mu(u)}{u} \right) (\mathbf{1}_{N \geq x^- > 0} + \mathbf{1}_{N \geq y^- > 0}) \\
&\quad + \left(\sum_{u \in \mathbb{M}_L \cap (N+|x| \vee |y|, \infty)} \mu(u) \right) \mathbf{1}_{N \geq x^- > 0} \mathbf{1}_{N \geq y^- > 0} \Bigg] \\
=: &\sum_{u \in \mathbb{M}_L} \frac{\mu(u)}{u^2} - \frac{1}{N} T_{N,L}(x) - \frac{1}{N} T_{N,L}(y) + \frac{1}{N^2} R_{N,L}(x, y).
\end{aligned}
$$

Here we note that

$$\sum_{u=1}^{\infty} \frac{\mu(u)}{u^2} = \frac{6}{\pi^2} \quad \text{(absolute convergence)},$$

$$\sum_{u=1}^{\infty} \frac{\mu(u)}{u} = 0 \quad \text{(conditional convergence)}$$

and

$$\sum_{u \in \mathbb{M}_L} \frac{\mu(u)}{u} \to 0 \quad \text{as } L \to \infty,$$

$$\sum_{u \in \mathbb{M}_L} \mu(u) = 0, \quad \forall L.$$

If, generally, $\mathbb{M} \nearrow \mathbb{N}$ so that $\sum_{u \in \mathbb{M}} \mu(u) = 0$ and $\sum_{u \in \mathbb{M}} \mu(u)/u \to 0$, then, for $K \in \mathbb{N}$

$$\sum_{u \in \mathbb{M} \cap (K, \infty)} \frac{\mu(u)}{u} = \sum_{u \in \mathbb{M}} \frac{\mu(u)}{u} - \sum_{u \in \mathbb{M} \cap [1, K]} \frac{\mu(u)}{u} \to - \sum_{u \leq K} \frac{\mu(u)}{u} = \sum_{u > K} \frac{\mu(u)}{u},$$

$$\sum_{u \in \mathbb{M} \cap (K, \infty)} \mu(u) = \sum_{u \in \mathbb{M}} \mu(u) - \sum_{u \in \mathbb{M} \cap [1, K]} \mu(u) \to - \sum_{u \leq K} \mu(u).$$

Therefore

$$\lim_{L \to \infty} T_{N,L}(x)$$

$$= \sum_{u \leq N + |x|} \frac{\mu(u)}{u} \left(\frac{(N + x) \bmod u}{u} - \frac{x \bmod u}{u} \right)$$

$$+ N \sum_{u > N + |x|} \frac{\mu(u)}{u^2} - \left(\sum_{u > N + |x|} \frac{\mu(u)}{u} \right) \mathbf{1}_{N \geq x^- > 0}$$

$$= \lim_{K \to \infty} \sum_{u=1}^{K} \frac{\mu(u)}{u} \left(\frac{(N + x) \bmod u}{u} - \frac{x \bmod u}{u} \right),$$

$$\lim_{L \to \infty} R_{N,L}(x, y)$$

$$= \sum_{u \leq N + |x| \vee |y|} \mu(u) \left(\frac{(N + x) \bmod u}{u} - \frac{x \bmod u}{u} \right)$$

$$\times \left(\frac{(N + y) \bmod u}{u} - \frac{y \bmod u}{u} \right)$$

$$+ N^2 \sum_{u > N + |x| \vee |y|} \frac{\mu(u)}{u^2} - N \left(\sum_{u > N + |x| \vee |y|} \frac{\mu(u)}{u} \right) \left(\mathbf{1}_{N \geq x^- > 0} + \mathbf{1}_{N \geq y^- > 0} \right)$$

$$- \left(\sum_{u \leq N + |x| \vee |y|} \mu(u) \right) \mathbf{1}_{N \geq x^- > 0} \mathbf{1}_{N \geq y^- > 0}$$

$$
= \begin{cases}
\displaystyle \lim_{K \to \infty} \sum_{u=1}^{K} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
\qquad\qquad \times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) & \text{if } \max(x, y) \geq 0, \\[2em]
\displaystyle \lim_{\substack{\mathbb{M} \nearrow \mathbb{N} \\ \text{with } \sum_{u \in \mathbb{M}} \mu(u)=0, \\ \sum_{u \in \mathbb{M}} \mu(u)/u \to 0}} \sum_{u \in \mathbb{M}} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
\qquad\qquad \times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) & \text{if } \max(x, y) < 0,
\end{cases}
$$

so that, we have that

$$
\begin{aligned}
S_N(x, y) = &\frac{6}{\pi^2} - \frac{1}{N} \sum_{u=1}^{\infty} \frac{\mu(u)}{u} \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&- \frac{1}{N} \sum_{u=1}^{\infty} \frac{\mu(u)}{u} \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right) \\
&+ \frac{1}{N^2} \sum_{u=1}^{\infty} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \\
&\qquad \times \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u} \right),
\end{aligned}
$$

where, when $\max(x, y) < 0$, the last series of the right-hand side should be understood as above. □

REMARK 8. The convergence $\sum_{u=1}^{\infty} \mu(u)/u = 0$, which we used in the above proof, is equivalent to the prime number theorem. See [1, Chapter II section 7].

5.2. Convergence in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$. We prove that $S_{N,L}$ converges to S_N in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$ as $L \to \infty$, by showing several lemmas below.

Lemma 8. *For $u, v \in \mathbb{N}$ and $z \in \widehat{\mathbb{Z}}$, we have*

$$
\begin{aligned}
&\mathbf{E}^\lambda \left[\left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \left(\frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v} \right) \right] \\
&= \frac{(u, v)}{\{u, v\}} \frac{z \bmod (u, v)}{(u, v)} \left(1 - \frac{z \bmod (u, v)}{(u, v)} \right),
\end{aligned}
$$

where the expectation $\mathbf{E}^\lambda$ works on x, and

$$
(u, v) = \gcd(u, v),
$$
$$
\{u, v\} = \text{lcm}(u, v) = \text{the least common multiple of } u \text{ and } v.
$$

Proof. We divide the proof into four steps.

STEP 1. For a, b, $c \in \mathbb{N}$ with $(b, c) = 1$ and for $x \in \widehat{\mathbb{Z}}$, it holds that

$$(29) \qquad \frac{1}{b} \sum_{s=0}^{b-1} \frac{(x + sac) \bmod ab}{ab} = \frac{x \bmod a}{ab} + \frac{b-1}{2b}.$$

Since $(b, c) = 1$, by a similar argument of [6, Theorem 56], we have

$$\{(x + sac) \bmod ab \, ; \, s = 0, 1, \ldots, b - 1\}$$
$$= \{(x + sa) \bmod ab \, ; \, s = 0, 1, \ldots, b - 1\}.$$

Thus, it is enough to prove (29) only for $c = 1$. Moreover, we have

$$\{(x + sa) \bmod ab \, ; \, s = 0, \ldots, b - 1\}$$
$$= \{(x + a + sa) \bmod ab \, ; \, s = 0, \ldots, b - 1\},$$

so that we have only to prove (29) for $x = 0, 1, \ldots, a - 1$. But then, for $s = 0, 1, \ldots, b - 1$, we have $(x + sa) \bmod ab = x + sa$, consequently,

$$\frac{1}{b} \sum_{s=0}^{b-1} \frac{(x + sa) \bmod ab}{ab} = \frac{1}{b} \sum_{s=0}^{b-1} \frac{x + sa}{ab} = \frac{x}{ab} + \frac{b-1}{2b}.$$

Thus (29) is valid.

STEP 2. By Step 1, it is easy to see that for a, b, $c \in \mathbb{N}$ with $(b, c) = 1$ and x, $z \in \widehat{\mathbb{Z}}$,

$$\frac{1}{b} \left(\sum_{s=0}^{b-1} \frac{(z + x + sac) \bmod ab}{ab} - \frac{(x + sac) \bmod ab}{ab} \right)$$
$$= \frac{1}{b} \left(\frac{(z + x) \bmod a}{a} - \frac{x \bmod a}{a} \right).$$

Therefore, for any periodic function $f \colon \widehat{\mathbb{Z}} \to \mathbb{R}$ with period ac, we have

$$\mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod ab}{ab} - \frac{x \bmod ab}{ab} \right) f(x) \right]$$
$$= \frac{1}{b} \sum_{s=0}^{b-1} \mathbf{E}^\lambda \left[\left(\frac{(z + x + sac) \bmod ab}{ab} - \frac{(x + sac) \bmod ab}{ab} \right) f(x + sac) \right]$$
$$= \mathbf{E}^\lambda \left[\frac{1}{b} \sum_{s=0}^{b-1} \left(\frac{(z + x + sac) \bmod ab}{ab} - \frac{(x + sac) \bmod ab}{ab} \right) f(x) \right]$$
$$= \frac{1}{b} \mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod a}{a} - \frac{x \bmod a}{a} \right) f(x) \right].$$

STEP 3.   Set $a := (u, v)$, $b := u/a$, $c := v/a$ and f to be

$$f(x) := \frac{(z + x) \bmod v}{v} - \frac{x \bmod v}{v}.$$

Then Step 2 implies that

$$\mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod u}{u} - \frac{x \bmod u}{u} \right) \left(\frac{(z + x) \bmod v}{v} - \frac{x \bmod v}{v} \right) \right]$$

$$= \mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod ab}{ab} - \frac{x \bmod ab}{ab} \right) \left(\frac{(z + x) \bmod ac}{ac} - \frac{x \bmod ac}{ac} \right) \right]$$

$$= \frac{1}{b} \mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod a}{a} - \frac{x \bmod a}{a} \right) \left(\frac{(z + x) \bmod ac}{ac} - \frac{x \bmod ac}{ac} \right) \right].$$

By letting b and c in Step 2 be c and 1, respectively, we see that the last line above is equal to

$$(30) \qquad \frac{1}{bc} \mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod a}{a} - \frac{x \bmod a}{a} \right) \left(\frac{(z + x) \bmod a}{a} - \frac{x \bmod a}{a} \right) \right]$$

$$= \frac{1}{bc} \mathbf{E}^\lambda \left[\left(\frac{(z + x) \bmod a}{a} - \frac{x \bmod a}{a} \right)^2 \right].$$

STEP 4.   By Corollary 1, the integrand of the right-hand side of (30) is continuous, and it is periodic with period a. Therefore (20) implies that

$$(30) = \frac{1}{a} \sum_{s=0}^{a-1} \frac{1}{bc} \left(\frac{(z + s) \bmod a}{a} - \frac{s}{a} \right)^2$$

$$= \frac{1}{bc} \frac{1}{a} \sum_{s=0}^{a-1} \left(\frac{z \bmod a}{a} \mathbf{1}_{\frac{z \bmod a}{a} + \frac{s}{a} < 1} + \left(\frac{z \bmod a}{a} - 1 \right) \mathbf{1}_{\frac{z \bmod a}{a} + \frac{s}{a} \geq 1} \right)^2$$

$$= \frac{1}{bc} \frac{1}{a} \sum_{s=0}^{a-1} \left(\left(\frac{z \bmod a}{a} \right)^2 \mathbf{1}_{s < a - (z \bmod a)} + \left(1 - \frac{z \bmod a}{a} \right)^2 \mathbf{1}_{s \geq a - (z \bmod a)} \right)$$

$$= \frac{1}{bc} \frac{1}{a} \left(\left(\frac{z \bmod a}{a} \right)^2 (a - (z \bmod a)) + \left(1 - \frac{z \bmod a}{a} \right)^2 (z \bmod a) \right)$$

$$= \frac{1}{bc} \frac{z \bmod a}{a} \left(1 - \frac{z \bmod a}{a} \right)$$

$$= \frac{(u, v)}{\{u, v\}} \frac{z \bmod (u, v)}{(u, v)} \left(1 - \frac{z \bmod (u, v)}{(u, v)} \right). \qquad \Box$$

Lemma 9. *For any bounded function* $H: \mathbb{N} \to \mathbb{R}$, *it holds that*

$$\sum_{u,v=1}^{\infty} \frac{|\mu(u)\mu(v)|}{\{u,v\}^2} |H((u,v))| = \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \prod_{p \nmid n} \left(1 + \frac{2}{p^2} \right) < \infty,$$

$$\sum_{u,v=1}^{\infty} \frac{\mu(u)\mu(v)}{\{u,v\}^2} H((u,v)) = \sum_{n=1}^{\infty} \frac{|\mu(n)|H(n)}{n^2} \prod_{p \nmid n} \left(1 - \frac{2}{p^2} \right).$$

Proof. The first identity is seen in the following way:

$$\sum_{u,v=1}^{\infty} \frac{|\mu(u)\mu(v)|}{\{u,v\}^2} |H((u,v))|$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{u',v' \in \mathbb{N}; \\ (u',v')=1}} \frac{|\mu(nu')\mu(nv')|}{n^2(u'v')^2} |H(n)|$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{u',v' \in \mathbb{N}; \\ (u',v')=1, \\ (n,u')=1, \\ (n,v')=1}} \frac{|\mu(n)^2\mu(u')\mu(v')|}{n^2(u'v')^2} |H(n)|$$

$$\left[\begin{array}{l} \text{because } (n,u') > 1 \text{ implies } \mu(nu') = 0, \text{ and } (n,u') = 1 \text{ implies} \\ \mu(nu') = \mu(n)\mu(u'), \end{array} \right]$$

(31) $$= \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \sum_{\substack{u',v' \in \mathbb{N}; \\ (u',v')=1,(n,u'v')=1}} \frac{|\mu(u'v')|}{(u'v')^2}$$

$$\left[\text{because } (n,u') = (n,v') = 1 \underset{\text{iff}}{\Longleftrightarrow} (n,u'v') = 1, \right]$$

$$= \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \sum_{u',v' \in \mathbb{N}} \frac{|\mu(u'v')|}{(u'v')^2} \mathbf{1}_{(n,u'v')=1}$$

$$\left[\text{because } (u',v') > 1 \text{ implies } \mu(u'v') = 0, \right]$$

$$= \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \sum_{m=1}^{\infty} \frac{|\mu(m)|}{m^2} \mathbf{1}_{(n,m)=1} \sum_{\substack{u',v' \in \mathbb{N}; \\ u'v'=m}} 1$$

$$= \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \sum_{m=1}^{\infty} \frac{|\mu(m)|}{m^2} \mathbf{1}_{(n,m)=1} d(m),$$

where

$$d(m) := \sum_{\substack{u',v' \in \mathbb{N}; \\ u'v'=m}} 1 = \text{the number of divisors of } m.$$

Since the arithmetic function $m \mapsto |\mu(m)|\mathbf{1}_{(n,m)=1}d(m)$ is multiplicative with value 1 at $m = 1$, and since

$$\sum_{m=1}^{\infty} \frac{|\mu(m)|}{m^2} d(m) < \infty,$$

we can apply [6, Theorem 286] to get

$$\text{the last line of (31)} = \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \prod_p \left(1 + \frac{2\mathbf{1}_{(n,p)=1}}{p^2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^2} \prod_{p \nmid n} \left(1 + \frac{2}{p^2}\right).$$

For $\sum_{u,v=1}^{\infty} (\mu(u)\mu(v)/\{u,v\}^2)H((u,v))$, a similar argument works.   □

Lemma 10. (i) *For each* $z \in \widehat{\mathbb{Z}}$,

$$\sum_{u=1}^{\infty} \frac{\mu(u)}{u} \left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u}\right) =: T(x; z)$$

is convergent in $L^2(\widehat{\mathbb{Z}}, \lambda)$, *and the mapping* $\widehat{\mathbb{Z}} \ni z \mapsto T(\,\cdot\,; z) \in L^2(\widehat{\mathbb{Z}}, \lambda)$ *is continuous.*
(ii) *For each* $N \in \mathbb{N}$,

$$\sum_{u=1}^{\infty} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u}\right) \left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u}\right)$$

$$=: N R(x, y; N)$$

is convergent in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$, *and* $\lim_{N \to \infty} \mathbf{E}^{\lambda^2}[R(x, y; N)^2] = 0$.

Proof. (i) Fix any $z \in \widehat{\mathbb{Z}}$. For finite sets $\mathbb{L}$ and $\mathbb{M}$ such that $\mathbb{L} \subset \mathbb{M} \subset \mathbb{N}$, Lemma 8 and Lemma 9 imply that

$$\mathbf{E}^{\lambda}\left[\left(\sum_{u \in \mathbb{M}} \frac{\mu(u)}{u} \left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right.\right.$$

$$\left.\left. - \sum_{u \in \mathbb{L}} \frac{\mu(u)}{u} \left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right)^2\right]$$

$$= \mathbf{E}^{\lambda}\left[\left(\sum_{u \in \mathbb{M}\backslash \mathbb{L}} \frac{\mu(u)}{u}\left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right)^2\right]$$

$$= \sum_{u,v \in \mathbb{M}\backslash \mathbb{L}} \frac{\mu(u)\mu(v)}{uv}\mathbf{E}^{\lambda}\left[\left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right.$$

$$\left. \times \left(\frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v}\right)\right]$$

$$= \sum_{u,v \in \mathbb{M}\backslash \mathbb{L}} \frac{\mu(u)\mu(v)}{uv}\frac{(u,v)}{\{u,v\}}\frac{z \bmod (u,v)}{(u,v)}\left(1 - \frac{z \bmod (u,v)}{(u,v)}\right)$$

$$\leq \frac{1}{4}\sum_{u,v \in \mathbb{M}\backslash \mathbb{L}} \frac{|\mu(u)\mu(v)|}{\{u,v\}^2}$$

$$\leq \frac{1}{4}\sum_{u,v \in \mathbb{N}\backslash \mathbb{L}} \frac{|\mu(u)\mu(v)|}{\{u,v\}^2} \to 0 \quad \text{as } \mathbb{L} \nearrow \mathbb{N}.$$

From this, the convergence of

$$\sum_{u=1}^{\infty} \frac{\mu(u)}{u}\left(\frac{(z+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)$$

in $L^2(\widehat{\mathbb{Z}}, \lambda)$ follows.

Next, we show the continuity of $\widehat{\mathbb{Z}} \ni z \mapsto T(\cdot\,; z) \in L^2(\widehat{\mathbb{Z}}, \lambda)$. For any $w \in \widehat{\mathbb{Z}}$, we have $w \bmod u + x \bmod u \equiv (x+w) \bmod u \pmod{u}$, and hence

$$\mathbf{E}^{\lambda}[(T(x; z') - T(x; z))^2]$$

$$= \lim_{U \to \infty} \mathbf{E}^{\lambda}\left[\left(\sum_{u \leq U} \frac{\mu(u)}{u}\left(\frac{(z'+x) \bmod u}{u} - \frac{(z+x) \bmod u}{u}\right)\right)^2\right]$$

$$= \lim_{U \to \infty} \mathbf{E}^{\lambda}\left[\left(\sum_{u \leq U} \frac{\mu(u)}{u}\left(\frac{(x+z') \bmod u}{u} - \frac{(x+z) \bmod u}{u}\right)\right)^2\right]$$

(32)
$$= \lim_{U \to \infty} \mathbf{E}^{\lambda}\left[\left(\sum_{u \leq U} \frac{\mu(u)}{u}\left(\frac{(x+z'-z) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right)^2\right]$$

[because λ is shift-invariant,]

$$= \mathbf{E}^{\lambda}\left[\left(\sum_{u=1}^{\infty} \frac{\mu(u)}{u}\left(\frac{(x+z'-z) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right)^2\right]$$

$$= \sum_{u,v \in \mathbb{N}} \frac{\mu(u)\mu(v)}{\{u,v\}^2}\frac{(z'-z) \bmod (u,v)}{(u,v)}\left(1 - \frac{(z'-z) \bmod (u,v)}{(u,v)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^2} \frac{(z'-z) \bmod n}{n} \left(1 - \frac{(z'-z) \bmod n}{n}\right) \prod_{p \nmid n} \left(1 - \frac{2}{p^2}\right)$$

$$\left[\text{by Lemma 9}\right].$$

Now let $\{z^{(k)}\}_{k=1}^{\infty} \subset \widehat{\mathbb{Z}}$ be such a sequence that $d(z^{(k)}, z) \to 0$ as $k \to \infty$. Then for any $n \in \mathbb{N}$, we see $\lim_{k \to \infty}((z^{(k)} - z) \bmod n)/n = 0$. Therefore the bounded convergence theorem implies that

$$\lim_{k \to \infty} \mathbf{E}^{\lambda}\left[(T(x; z^{(k)}) - T(x; z))^2\right] = 0,$$

which shows the desired continuity.

(ii) Fix any $N \in \mathbb{N}$. For any finite set $\mathbb{L}$ and $\mathbb{M}$ such that $\mathbb{L} \subset \mathbb{M} \subset \mathbb{N}$, by using Lemma 8 and Lemma 9, we have

$$\mathbf{E}^{\lambda^2}\left[\left(\left(\sum_{u \in \mathbb{M}} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u}\right)\right.\right.\right.$$

$$\left.\left.\left. - \sum_{u \in \mathbb{L}} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u}\right)\right)^2\right)\right]$$

$$= \sum_{u,v \in \mathbb{M} \setminus \mathbb{L}} \mu(u)\mu(v)\mathbf{E}^{\lambda}\left[\left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\right.$$

$$\left. \times \left(\frac{(N+x) \bmod v}{v} - \frac{x \bmod v}{v}\right)\right]^2$$

$$= \sum_{u,v \in \mathbb{M} \setminus \mathbb{L}} \mu(u)\mu(v)\left(\frac{(u,v)}{\{u,v\}} \frac{N \bmod (u,v)}{(u,v)}\left(1 - \frac{N \bmod (u,v)}{(u,v)}\right)\right)^2$$

$$= N^2 \sum_{u,v \in \mathbb{M} \setminus \mathbb{L}} \frac{\mu(u)\mu(v)}{\{u,v\}^2}\left(\frac{(u,v)}{N} \frac{N \bmod (u,v)}{(u,v)}\left(1 - \frac{N \bmod (u,v)}{(u,v)}\right)\right)^2$$

$$\leq N^2 \sum_{u,v \in \mathbb{M} \setminus \mathbb{L}} \frac{|\mu(u)\mu(v)|}{\{u,v\}^2}$$

$$\leq N^2 \sum_{u,v \in \mathbb{N} \setminus \mathbb{L}} \frac{|\mu(u)\mu(v)|}{\{u,v\}^2} \to 0 \quad \text{as } \mathbb{L} \nearrow \mathbb{N}.$$

This shows the convergence of

$$\sum_{u=1}^{\infty} \mu(u) \left(\frac{(N+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\left(\frac{(N+y) \bmod u}{u} - \frac{y \bmod u}{u}\right)$$

in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$. Moreover, the above computation shows that

$$\mathbf{E}^{\lambda^2}[R(x, y; N)^2]$$

$$= \sum_{u,v \in \mathbb{N}} \frac{\mu(u)\mu(v)}{\{u,v\}^2} \left(\frac{(u,v)}{N} \frac{N \bmod (u,v)}{(u,v)} \left(1 - \frac{N \bmod (u,v)}{(u,v)} \right) \right)^2$$

$$= \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^2} \left(\frac{n}{N} \frac{N \bmod n}{n} \left(1 - \frac{N \bmod n}{n} \right) \right)^2 \prod_{p \nmid n} \left(1 - \frac{2}{p^2} \right)$$

$$\left[\text{by Lemma 9,} \right]$$

$$\to 0 \quad \text{as } N \to \infty. \qquad \square$$

6. Coincidence of "distribution" and distribution

In this section, we prove Theorem 1. But we deal with the case $U = S_N$ only, because the other cases easily follow from this case.

Let P_M be a probability measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, $\mathcal{P}(\mathbb{N})$ being the set of all subsets of $\mathbb{N}$, defined by

$$P_M(A) = \frac{1}{M} \# A \cap \{1, \dots, M\}, \quad A \in \mathcal{P}(\mathbb{N}).$$

We let $\mathbf{E}_M$ denote the expectation with respect to P_M. Finally, let $P_M^2 := P_M \otimes P_M$ and $\mathbf{E}_M^2 := \mathbf{E}^{P_M^2}$.

Since we have

$$|X(x, y) - X_L(x, y)| = X_L(x, y) \left(1 - \prod_{p > p_L} \left(1 - \rho_p(x)\rho_p(y) \right) \right)$$

$$\leq \sum_{p > p_L} \rho_p(x)\rho_p(y),$$

we see that

$$|S_N(x, y) - S_{N,L}(x, y)| \leq \frac{1}{N^2} \sum_{m,n=1}^{N} |X(x+m, y+n) - X_L(x+m, y+n)|$$

$$\leq \sum_{p > p_L} \left(\frac{1}{N} \sum_{m=1}^{N} \rho_p(x+m) \right) \left(\frac{1}{N} \sum_{n=1}^{N} \rho_p(y+n) \right).$$

Hence

$$\left| \mathbf{E}_M^2 \left[e^{\sqrt{-1}\, t S_N(x,y)} \right] - \mathbf{E}_M^2 \left[e^{\sqrt{-1}\, t S_{N,L}(x,y)} \right] \right| \leq \mathbf{E}_M^2 \left[\left| e^{\sqrt{-1}\, t S_N(x,y)} - e^{\sqrt{-1}\, t S_{N,L}(x,y)} \right| \right]$$

$$\leq |t| \mathbf{E}_M^2 \left[|S_N(x, y) - S_{N,L}(x, y)| \right]$$

$$\leq |t| \sum_{p > p_L} \left(\mathbf{E}_M \left[\frac{1}{N} \sum_{m=1}^{N} \rho_p(x+m) \right] \right)^2$$

$$= |t| \sum_{p > p_L} \left(\frac{1}{N} \sum_{m=1}^{N} \mathbf{E}_M \left[\rho_p(x+m) \right] \right)^2$$

$$\le |t| \sum_{p > p_L} \frac{1}{p^2} \left(1 + \frac{N+1}{2M} \right)^2,$$

where the last line comes from $\mathbf{E}_M[\rho_p(x+m)] \le (1/p)(1+m/M)$ (cf. (28)). On the other hand, since $S_{N,L}$ is continuous on $\widehat{\mathbb{Z}}^2$, Lemma 6 implies that

$$\lim_{M \to \infty} \mathbf{E}_M^2 \left[e^{\sqrt{-1}\, t S_{N,L}(x,y)} \right] = \mathbf{E}^{\lambda^2} \left[e^{\sqrt{-1}\, t S_{N,L}(x,y)} \right].$$

Since $S_{N,L}$ converges to S_N in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$ as $L \to \infty$ (Section 5.2), it is clear that

$$\mathbf{E}^{\lambda^2} \left[e^{\sqrt{-1}\, t S_{N,L}(x,y)} \right] \to \mathbf{E}^{\lambda^2} \left[e^{\sqrt{-1}\, t S_N(x,y)} \right] \quad \text{as } L \to \infty.$$

Therefore, collecting all the above, we see

$$\limsup_{M \to \infty} \left| \mathbf{E}_M^2 \left[e^{\sqrt{-1}\, t S_N(x,y)} \right] - \mathbf{E}^{\lambda^2} \left[e^{\sqrt{-1}\, t S_N(x,y)} \right] \right|$$

$$\le |t| \sum_{p > p_L} \frac{1}{p^2} + \left| \mathbf{E}^{\lambda^2} \left[e^{\sqrt{-1}\, t S_{N,L}(x,y)} \right] - \mathbf{E}^{\lambda^2} \left[e^{\sqrt{-1}\, t S_N(x,y)} \right] \right| \underset{L \to \infty}{\to} 0,$$

which completes the proof. $\square$

7.   Limit distributions of CLT scaling

In Lemma 10 (i), we showed the continuity of $\widehat{\mathbb{Z}} \ni z \mapsto T(\cdot\,; z) \in L^2(\widehat{\mathbb{Z}}, \lambda)$. About $T(\cdot\,; z)$, we further assert the following lemma.

Lemma 11.   (i) *For z, $z' \in \widehat{\mathbb{Z}}$,*

$$T(\cdot\,; z) = T(\cdot\,; z') \text{ in } L^2(\widehat{\mathbb{Z}}, \lambda) \underset{\text{iff}}{\Longleftrightarrow} z \sim z'.$$

(ii) *For $\{z^{(k)}\}_{k=1}^{\infty} \subset \widehat{\mathbb{Z}}$ and $z \in \widehat{\mathbb{Z}}$, the following three statements are equivalent:*
   (a) $\lim_{k \to \infty} \|T(\cdot\,; z^{(k)}) - T(\cdot\,; z)\|_{L^2(\widehat{\mathbb{Z}}, \lambda)} = 0$.
   (b) *For any prime p, there exists $k_p \in \mathbb{N}$ such that $(z^{(k)} - z) \bmod p = 0$ for all $k \ge k_p$.*
   (c) $\lim_{k \to \infty} [z^{(k)}] = [z]$ *in $\widehat{\mathbb{Z}}/\sim$.*

Proof.   (i) is a special case of the equivalence between (a) and (b) for $z^{(k)} = z' \in \widehat{\mathbb{Z}}$ ($\forall k$) and $z \in \widehat{\mathbb{Z}}$. Therefore we prove only (ii).

First, by (32) and noting that

$$0 < \prod_p \left(1 - \frac{2}{p^2}\right) \le \prod_{p \nmid n} \left(1 - \frac{2}{p^2}\right) \le 1, \quad \forall n \in \mathbb{N},$$

we have

(33)
$$\prod_p \left(1 - \frac{2}{p^2}\right) \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^2} \frac{(z - z') \bmod n}{n} \left(1 - \frac{(z - z') \bmod n}{n}\right)$$

$$\le \|T(\,\cdot\,; z) - T(\,\cdot\,; z')\|^2_{L^2(\widehat{\mathbb{Z}}, \lambda)}$$

$$\le \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^2} \frac{(z - z') \bmod n}{n} \left(1 - \frac{(z - z') \bmod n}{n}\right).$$

Proof of (a) $\Rightarrow$ (b). (33) implies for any prime p that

$$\|T(\,\cdot\,; z^{(k)}) - T(\,\cdot\,; z)\|^2_{L^2(\widehat{\mathbb{Z}}, \lambda)}$$

$$\ge \frac{1}{p^2} \frac{(z^{(k)} - z) \bmod p}{p} \left(1 - \frac{(z^{(k)} - z) \bmod p}{p}\right) \prod_p \left(1 - \frac{2}{p^2}\right)$$

$$\ge \frac{1}{p^4} ((z^{(k)} - z) \bmod p) \prod_p \left(1 - \frac{2}{p^2}\right).$$

Therefore (a) implies that $\lim_{k \to \infty} (z^{(k)} - z) \bmod p = 0$ for any prime p. From this, (b) easily follows.

Proof of (b) $\Rightarrow$ (a). Conversely, assume (b). Let $n := q_1 \cdots q_L$, where $q_1 < \cdots < q_L$ are primes. Then

$$\exists k_0 \in \mathbb{N} \quad \text{such that} \quad (z^{(k)} - z) \bmod q_i = 0, \quad \forall k \ge k_0, \ 1 \le \forall i \le L.$$

Since $q_1, \ldots, q_L$ are clearly co-prime to each other, we have

$$(z^{(k)} - z) \bmod n = 0, \quad \forall k \ge k_0.$$

Therefore, for any $n \in \mathbb{N}$ with $|\mu(n)| = 1$, we have

$$\lim_{k \to \infty} (z^{(k)} - z) \bmod n = 0.$$

By the bounded convergence theorem, we see

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^2} \frac{(z^{(k)} - z) \bmod n}{n} \left(1 - \frac{(z^{(k)} - z) \bmod n}{n}\right) = 0,$$

which shows (a) with the help of (33).

Proof of (b) $\Rightarrow$ (c). For any $\varepsilon > 0$, take $L \in \mathbb{N}$ so that $2^{-L} < \varepsilon$. Let $\nu_L :=$ $\max_{p \le p_L} k_p$. Then if $k \ge \nu_L$, $(z^{(k)} - z) \bmod p = 0$ for $p \le p_L$. Thus if $k \ge \nu_L$,

$$\widetilde{d}([z^{(k)}], [z]) \le \sum_{i>L} 2^{-i} = 2^{-L} < \varepsilon,$$

where $\widetilde{d}$ is the metric of $\widehat{\mathbb{Z}}/\sim$ defined by (13). This means that (b) implies (c).

Proof of (c) $\Rightarrow$ (b). (c) means that for each prime p, $\lim_{k\to\infty} \rho_p(z^{(k)} - z) = 1$. From this (b) follows. $\square$

7.1. Proof of Theorem 6. Let $z \in \widehat{\mathbb{Z}}$ and a sequence $\{N_k\}_{k=1}^{\infty} \subset \mathbb{N}$ be such that $[N_k] \ne [z]$ and $[N_k] \to [z]$ in $\widehat{\mathbb{Z}}/\sim$ as $k \to \infty$. Then by Lemma 11 (ii), we have $\lim_{k\to\infty} T(x; N_k) = T(x; [z])$ in $L^2(\widehat{\mathbb{Z}}, \lambda)$. And by Lemma 10 (ii), we have $\lim_{k\to\infty} R(x, y; N_k) = 0$ in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$. Therefore Theorem 5 implies that

$$\lim_{k\to\infty} Y_{N_k}(x, y) = -T(x; [z]) - T(y; [z]) \quad \text{in } L^2(\widehat{\mathbb{Z}}^2, \lambda^2).$$

Now, from Lemma 1 it follows that for any $z \in \widehat{\mathbb{Z}}$, there exists a sequence $\{N_k\}_{k=1}^{\infty}$ of distinct positive integers such that $N_k \to z$ in $\widehat{\mathbb{Z}}$ as $k \to \infty$. Then we have $[N_k] \to [z] \in \widehat{\mathbb{Z}}/\sim$ as $k \to \infty$. Conversely, by compactness of $\widehat{\mathbb{Z}}/\sim$, for any sequence $\{N_k\}_{k=1}^{\infty}$ of distinct positive integers, there exists a subsequence $\{N_{k_i}\}_{i=1}^{\infty} \subset \{N_k\}_{k=1}^{\infty}$ such that $[N_{k_i}] \to [z]$ for some $[z] \in \widehat{\mathbb{Z}}/\sim$. From these facts, Theorem 6 follows. $\square$

7.2. Proof of Theorem 7. Theorem 7 (i) and (ii) are clear from Lemma 11. Since we can show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{\sqrt{-1}\,tT(n;1)} = \mathbf{E}^{\lambda}\left[e^{\sqrt{-1}\,tT(\cdot\,;1)}\right], \quad t \in \mathbb{R},$$

in a similar way as Theorem 1, Theorem 7 (iv) is reduced to the results of [5] and [9]. Theorem 7 (v) is clear from Theorem 7 (iii). So, we have only to show Theorem 7 (iii).

We know by Theorem 5,

$$Y_N(x, y) = -T(x; N) - T(y; N) + R(x, y; N)$$

as an identity in $L^2(\widehat{\mathbb{Z}}^2, \lambda^2)$. Integrating both sides by $\lambda(dy)$, we have

$$-T(x; N) = \int_{\widehat{\mathbb{Z}}} Y_N(x, y)\lambda(dy)$$

$$\left[\text{because} \quad \int_{\widehat{\mathbb{Z}}} T(y; N)\lambda(dy) = \int_{\widehat{\mathbb{Z}}} R(x, y; N)\lambda(dy) = 0\right]$$

$$= \int_{\widehat{\mathbb{Z}}} N\left(\frac{1}{N^2}\sum_{m,n=1}^{N}\prod_p \left(1 - \rho_p(x+m)\rho_p(y+n)\right) - \frac{6}{\pi^2}\right)\lambda(dy)$$

$$\left[\text{by (18)}\right]$$

$$= N\left(\frac{1}{N^2}\sum_{m,n=1}^{N}\prod_p \left(1 - \rho_p(x+m)\int_{\widehat{\mathbb{Z}}}\rho_p(y+n)\lambda(dy)\right) - \frac{6}{\pi^2}\right)$$

$$= N\left(\frac{1}{N^2}\sum_{m,n=1}^{N}\prod_p \left(1 - \frac{\rho_p(x+m)}{p}\right) - \frac{6}{\pi^2}\right)$$

$$= \sum_{m=1}^{N}\prod_p \left(1 - \frac{\rho_p(x+m)}{p}\right) - N\frac{6}{\pi^2}.$$

In particular, setting $N = 1$, we have

$$-T(x; 1) = \prod_p \left(1 - \frac{\rho_p(x+1)}{p}\right) - \frac{6}{\pi^2}.$$

Consequently, we have $-T(x; N) = -\sum_{m=0}^{N-1} T(x+m; 1)$. $\square$

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H. Sugita
Faculty of Mathematics
Kyushu University
Fukuoka 812-8581, Japan

Current address:
Department of Mathematics
Graduate School of Science
Osaka University
Osaka 560-0043, Japan
e-mail: sugita@math.sci.osaka-u.ac.jp

S. Takanobu
Department of Mathematics
Faculty of Science
Kanazawa University
Kanazawa 920-1192, Japan
e-mail: takanob@kenroku.kanazawa-u.ac.jp