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GENERALIZED KÄHLER METRICS AND PROPER MODIFICATIONS

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1. Introduction

Although proper modifications modify the geometry of the space only along a rare analytic set it is enough to "disturb" important analytic and geometric properties.

For instance, Moishezon [8] proved by an example that for a surjective, proper modification $p\colon X \to Y$ between compact complex spaces such that Y has a Kähler metric it does not follow necessarily that X is also Kähler.

Among the compact complex manifolds the Kähler manifolds enjoy a number of remarkable properties. Kähler spaces were first introduced by Grauert [6] and their study was continued by Moishezon [8]. (It is known that the definition of Moishezon of a Kähler metric coincides with that of Grauert at least for normal spaces.)

The example of Moishezon gives naturally rise to the question which special properties proper modifications of compact Kähler spaces nevertheless still might have, in particular, how far away is X from being Kähler?

In the manifold case there are several results in this direction. For example, Alessandrini and Bassanelli introduced the notion of a balanced metric. Every Kähler metric is balanced and they proved that balanced metrics are invariant under proper modifications.

In this paper we study this problem in the singular case. We introduce in Definition 2.4 the notion of a generalized Kähler metric and prove that this notion is invariant under proper modifications (Theorem 2.5).

Our notion of a generalized Kähler metric differs only a little bit from the definition of Moishezon: we admit $-\infty$ as value for the system of defining functions.

Using the Stein factorization Theorem we prove that Theorem 2.5 admits a generalization to the more general context of Theorem 3.1.

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2. Setup and main result

Throughout this paper all complex spaces are assumed to have countable topology, unless it is otherwise stated.

DEFINITION 2.1. A holomorphic map $p\colon X \to Y$ is called a proper modification if it is proper and there exists a rare analytic set A in Y such that $p^{-1}(A)$ is rare in X and such that $p|_{X\setminus p^{-1}(A)}\colon X\setminus p^{-1}(A) \to Y\setminus A$ is biholomorphic.

DEFINITION 2.2. A reduced complex space Y is called Kähler (in the sense of Moishezon) if there exists a covering $(V_i)_{i\in I}$ of Y with open sets such that for each index i there exists a strongly plurisubharmonic function $\lambda_i\colon V_i \to \mathbb{R}$ which is regular of class $\mathcal{C}^\infty$ and such that on each nonempty intersection $V_i \cap V_j$ we have the pluriharmonic compatibility condition: $\lambda_i - \lambda_j = \operatorname{Re} g_{ij}$, locally on $V_i \cap V_j$ for some holomorphic function g_{ij}.

Two such collections $(V_i, \lambda_i)_{i\in I}$ and $(W_j, \psi_j)_{j\in J}$ define the same Kähler metric on Y if each $\lambda_i - \psi_j$ is pluriharmonic (i.e. is locally the real part of a holomorphic function) on $V_i \cap W_j \neq \emptyset$.

REMARK 2.3. If Y is a complex manifold such a collection $(V_i, \lambda_i)_{i\in I}$ defines indeed a metric on Y, by endowing Y with the $(1,1)$-form given locally (on each open set V_i) by $\partial\bar{\partial}\lambda_i$.

We want to generalize the above concept of Kähler metrics.

DEFINITION 2.4. We say that the reduced complex space X has a generalized Kähler metric if there exists a covering of X with open sets $(U_i)_i$ such that on each set U_i there exists a function $\varphi_i\colon U_i \to [-\infty, \infty)$, $\varphi_i \not\equiv -\infty$ on each irreducible component of U_i, which is strongly plurisubharmonic, regular of class $\mathcal{C}^\infty$ outside the set $\{\varphi_i = -\infty\}$ and such that on each nonempty intersection $U_i \cap U_j$ we have (locally) the pluriharmonic compatibility condition $\varphi_i = \varphi_j + \operatorname{Re} f_{ij}$ for some holomorphic function f_{ij}.

The main result of this paper is the following:

Theorem 2.5. *Let X and Y be two reduced, compact, complex spaces (with singularities) and $p\colon X \to Y$ a surjective, holomorphic map, which is a proper modification. Suppose that Y is Kähler. Then X has a generalized Kähler metric.*

PROOF. Consider the covering of Y given by Definition 2.2 and the covering of X given by $U_i := p^{-1}(V_i)$, $i \in I$ and on each U_i the function $\tilde{\varphi}_i = \lambda_i \circ p$. Then it follows at once that $\tilde{\varphi}_i \in \mathcal{C}^\infty(U_i)$ and that $\tilde{\varphi}_i$ is plurisubharmonic on U_i but not necessarily strongly plurisubharmonic. The idea in what follows is to modify in a first step $\tilde{\varphi}_i$ such that they become strongly plurisubharmonic. But then we destroy the "pluriharmonic compatibility condition" $\tilde{\varphi}_i = \tilde{\varphi}_j + \mathrm{Re}(g_{ij} \circ p)$ locally on $U_i \cap U_j$. In a second step we also get this condition back.

First step of the proof. To modify $\tilde{\varphi}_i$ such that they become strongly plurisubharmonic we use a technique from an article of Colţoiu-Mihalache [3]. We look at the following commutative diagram given by Chow's lemma (see for instance [7] and [9] or [5, p.171]):

(1)
$$\begin{array}{ccc} Y^* & \xrightarrow{F} & X \\ & \overset{\pi}{\searrow} & \downarrow p \\ & & Y \end{array}$$

More precisely, given the proper modification p and so implicitly the rare analytic set A, the lemma of Chow ensures the existence of a coherent ideal $\mathcal{J}$ on Y, with $\mathrm{supp}(\mathcal{O}_Y/\mathcal{J}) = A$ such that, denoting by $\pi\colon Y^* \to Y$ the blowing-up of Y with center $(A, (\mathcal{O}_Y/\mathcal{J})|_A)$, it follows the existence of a holomorphic, proper and surjective map F making the above diagram commutative. The ideal $\mathcal{J}$ is called the ideal of Hironaka.

Without loss of generality we can suppose that the open sets of the covering of Y given by the definition of the Kähler metric are all Stein open sets. Fix now for the moment an arbitrary Stein open set V_j of the finite covering $(V_i)_{i\in I}$ of Y.

There exist sections $f_{1,j}, \ldots, f_{s,j} \in \mathcal{J}(V_j)$ generating each fiber of $\mathcal{J}$ such that

$$A \cap V_j = \{x \in V_j \mid f_{1,j}(x) = \cdots = f_{s,j}(x) = 0\}.$$

It then follows for the map

$$f_j := (f_{1,j}, \ldots, f_{s,j})\colon V_j \longrightarrow \mathbb{C}^s$$

that we have:

$$f_j^{-1}(0) = (A \cap V_j, (\mathcal{O}_Y/\mathcal{J})|_{A\cap V_j}).$$

Now consider the function

$$\psi_j\colon V_j \longrightarrow [-\infty, \infty)$$

given by

$$\psi_j = \lambda_j + \log\left(\sum_{k=1}^{s} |f_{k,j}|^2\right).$$

It is clear that ψ_j is strongly plurisubharmonic on V_j, $\{\psi_j = -\infty\} = A \cap V_j$ and that $\psi_j|_{V_j \backslash A} \in \mathcal{C}^\infty(V_j \backslash A)$. Considering now the composed function $\psi_j \circ p$ we have that $\psi_j \circ p$ is plurisubharmonic on $U_j = p^{-1}(V_j)$, $\mathcal{C}^\infty$ on $U_j \backslash p^{-1}(A)$ and $\{\psi_j \circ p = -\infty\} = p^{-1}(A) \cap U_j$. We will see below that $\psi_j \circ p$ is even strongly plurisubharmonic.

We use the following lemma which is true for all reduced complex spaces (not necessarily compact). For a proof see [3], [2].

Lemma 2.6. *Let X and Y be reduced complex spaces and $p\colon X \to Y$ a proper, holomorphic, surjective map. Let $\Phi\colon Y \to [-\infty, \infty)$ be an upper semicontinuous function such that $\Phi \circ p$ is* (*strongly*) *plurisubharmonic on X. Then Φ is* (*strongly*) *plurisubharmonic on Y.*

Using diagram (1) we can conclude with help of this lemma that, in order to show that $\psi_j \circ p$ is strongly plurisubharmonic, it is enough to prove that $\psi_j \circ \pi = (\psi_j \circ p) \circ F$ is strongly plurisubharmonic on $\pi^{-1}(V_j)$.

For this we need the explicit description of the analytic blowing-up. Let $\mathfrak{m} \subset \mathcal{O}_{\mathbb{C}^s}$ denote the maximal ideal of the origin in $\mathbb{C}^s$. One has then an exact sequence (the syzygy-theorem) of the form:

$$\mathcal{O}_{\mathbb{C}^s}^{\binom{s}{2}} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^s}^{s} \xrightarrow{\beta} \mathfrak{m} \longrightarrow 0$$

where β is given by multiplication with the coordinates $(z_1, \dots, z_s)$ of $\mathbb{C}^s$ and α is given by the $s \times \binom{s}{2}$ matrix:

$$\begin{pmatrix}
z_2 & z_3 & z_4 & \cdots & z_s & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
-z_1 & 0 & 0 & \cdots & 0 & z_3 & z_4 & \cdots & z_s & \cdots & 0 & \cdots & 0 \\
0 & -z_1 & 0 & \cdots & 0 & -z_2 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & -z_1 & \cdots & 0 & 0 & -z_2 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & z_j & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & -z_i & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & z_s \\
0 & 0 & 0 & \cdots & -z_1 & 0 & 0 & \cdots & -z_2 & \cdots & 0 & \cdots & -z_{s-1}
\end{pmatrix}$$

Via the analytic inverse image this gives rise to an exact sequence on V_j (remark that here $f_j^* \mathfrak{m} = \mathcal{J}|_{V_j}$):

$$\mathcal{O}_{V_j}^{\binom{s}{2}} \xrightarrow{f_j^* \alpha} \mathcal{O}_{V_j}^{s} \xrightarrow{f_j^* \beta} \mathcal{J}|_{V_j} \longrightarrow 0.$$

Let $\mathbb{P}(\mathcal{J})$ denote the projective space over Y associated to the coherent ideal sheaf $\mathcal{J}$ (see for instance [5]).

By construction of the blowing-up we have the following commutative diagram

$$\begin{array}{ccccc} V_j^* & \overset{i}{\hookrightarrow} & \mathbb{P}(f_j^*\mathfrak{m}) & \hookrightarrow & V_j \times \mathbb{P}^{s-1}(\mathbb{C}) \\ & {\scriptstyle \pi|_{V_j^*}} \searrow & \downarrow {\scriptstyle \xi} & \swarrow {\scriptstyle pr_1} & \\ & & V_j & & \end{array}$$

Therefore it is enough to prove that

$$\psi_j \circ \xi \colon \mathbb{P}(f_j^*\mathfrak{m}) \longrightarrow [-\infty, \infty)$$

is strongly plurisubharmonic.

But in this form the advantage is that for the closed subspace

$$\mathbb{P}(f_j^*\mathfrak{m}) \hookrightarrow V_j \times \mathbb{P}^{s-1}(\mathbb{C})$$

we can give the defining equations explicitly. They are

$$f_{k,j}(y)z_m - f_{m,j}(y)z_k = 0, \quad \forall 1 \leq m < k \leq s$$

where $(z_1 : \cdots : z_s)$ are the homogeneous coordinates on $\mathbb{P}^{s-1}(\mathbb{C})$.

Let

$$V_j \times \tilde{V}_\nu := \{(y, z) \in V_j \times \mathbb{P}^{s-1}(\mathbb{C}) \mid z_\nu \neq 0\} \quad \text{for } \nu \in \{1, \ldots, s\}$$

and

$$\alpha_\nu \colon V_j \times \tilde{V}_\nu \longrightarrow V_j \times \mathbb{C}^{s-1}$$

be the biholomorphic map given by

$$\alpha_\nu(y, z) = \left(y, \frac{z_1}{z_\nu}, \ldots, \frac{z_{\nu-1}}{z_\nu}, \frac{z_{\nu+1}}{z_\nu}, \ldots, \frac{z_s}{z_\nu}\right),$$

and define

$$\tau_\nu^j \colon V_j \times \mathbb{C}^{s-1} \longrightarrow [-\infty, \infty)$$

given by

$$\tau_\nu^j(y, t_1, \ldots, t_{s-1}) = \lambda_j(y) + \log |f_{\nu,j}(y)|^2 + \log \left(1 + \sum_{k=1}^{s-1} |t_k|^2\right)$$

where $(t_1, \dots, t_{s-1})$ denote the coordinates on $\mathbb{C}^{s-1}$.

It is then clear that τ_ν^j is strongly plurisubharmonic on $V_j \times \mathbb{C}^{s-1}$. Because α_ν is biholomorphic it follows that $\tau_\nu^j \circ \alpha_\nu$ is strongly plurisubharmonic on $V_j \times \tilde{V}_\nu$. But on $(V_j \times \tilde{V}_\nu) \cap \xi^{-1}(V_j)$ we have that

$$\tau_\nu^j \circ \alpha_\nu = \psi_j \circ \xi$$

so that finally it follows that $\psi_j \circ \xi$ is strongly plurisubharmonic on $\xi^{-1}(V_j)$. So we also obtain that $\psi_j \circ \pi$ is strongly plurisubharmonic on V_j^*. As already seen above this implies that $\psi_j \circ p$ is strongly plurisubharmonic on U_j.

As a conclusion of the first step of the proof we obtain the following properties for $\psi_j \circ p$: it is strongly plurisubharmonic on U_j, regular of class $\mathcal{C}^\infty$ on $U_j \setminus p^{-1}(A)$ and $\{\psi_j \circ p = -\infty\} = p^{-1}(A) \cap U_j$. But we have destroyed the pluriharmonic-compatibility condition, because now

$$\psi_j \circ p = \lambda_j \circ p + \log\left(\sum_{k=1}^{s} |f_{k,j} \circ p|^2\right)$$

the last term being a “perturbation factor”.

Second step of the proof. In order to obtain on X a collection of strongly plurisubharmonic functions with the pluriharmonic compatibility condition we proceed as follows.

Let

$$a_j := |f_{1,j}|^2 + \cdots + |f_{s,j}|^2 \quad \text{on } V_j$$

and

$$a_k := |f_{1,k}|^2 + \cdots + |f_{l,k}|^2 \quad \text{on } V_k.$$

Consider now a relatively compact subcover of Y with open subsets $V_j' \subset V_j$, $\forall j \in I$. Then the quotient

$$\frac{a_j}{a_k} = \frac{|f_{1,j}|^2 + \cdots + |f_{s,j}|^2}{|f_{1,k}|^2 + \cdots + |f_{l,k}|^2}$$

remains bounded (upper and lower) on $(V_j' \cap V_k') \setminus A$. The problem is only in small neighbourhoods of A in $(V_j' \cap V_k') \setminus A$. But we know that on $V_j \cap V_k$ the sections in $\mathcal{J}(V_j \cap V_k)$ are generated by $(f_{1,j}, \dots, f_{s,j})|_{V_j \cap V_k}$ and also by $(f_{1,k}, \dots, f_{l,k})|_{V_j \cap V_k}$ because the respective germs generate $\mathcal{J}_y$ for all $y \in V_j \cap V_k$ and $V_j \cap V_k$ is Stein. So the boundedness is clear and therefore we also know that $\log a_j - \log a_k$ is bounded on $(V_j' \cap V_k') \setminus A$.

In what follows we apply a glueing technique of Demailly [4] for a collection of certain functions, which has the advantage that the glueing result is of class $\mathcal{C}^\infty$.

More precisely, we can suppose, without loss of generality, from the begining that the open sets V'_j are isomorphic with analytic sets in open balls $B(0, r_j) \subset \mathbb{C}^{N_j}$.

Let $\Phi_j \colon V'_j \to B(0, r_j)$ denote the chart. We can assume that $0 \in \Phi_j(V'_j)$. Consider for each j the function

$$\mathfrak{v}_j \colon V'_j \longrightarrow [-\infty, \infty)$$

given by

$$\mathfrak{v}_j(z) = \log a_j(z) - \frac{1}{r_j^2 - |\Phi_j(z)|^2} =: \log a_j(z) - \theta_j(z).$$

One sees at once that $\mathfrak{v}_j \in \mathcal{C}^\infty(V'_j \backslash A)$ and $\mathfrak{v}_j(z) \to -\infty$ for $z \to \partial V'_j$, $z \in V'_j$ (we also have that $\mathfrak{v}_j(z) = -\infty$ for $z \in A \cap V'_j$).

In order to get a $\mathcal{C}^\infty$-glueing of the functions $\mathfrak{v}_i$ on $Y \backslash A$, to overcome the fact that the function $\max(\mathfrak{v}_i)_i$ is only continuous, one proceeds as follows:

Let $\varrho \colon \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ with $\varrho \geq 0$, $\operatorname{supp} \varrho \subset [-1/2, 1/2]$ and with $\int_{\mathbb{R}} \varrho(u)\, du = 1$ and let m denote the function

$$m \colon \mathbb{R}^q \longrightarrow \mathbb{R}$$

given by

$$m(t_1, \ldots, t_q) = \int_{\mathbb{R}^q} \max\{t_1 + u_1, \ldots, t_q + u_q\} \prod_{1 \leq n \leq q} \varrho(u_n)\, du_n$$

(in our case q will be the number of open sets of the finite covering $(V'_j)_j$ of Y).

It is clear that m is increasing in each variable, that it is convex and of class $\mathcal{C}^\infty$ and that the following property holds:

$$m(t_1, \ldots, \hat{t}_j, \ldots, t_q) = m(t_1, \ldots, t_j, \ldots, t_q) \tag{2}$$

whenever

$$t_j < \max\{t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_q\} - 1$$

(where $\hat{\ }$ denotes, as usual, that the respective variable is missing).

Let now $\mathfrak{v}$ denote the function on Y given by

$$\mathfrak{v}(z) = m(\mathfrak{v}_1(z), \ldots, \mathfrak{v}_q(z)).$$

We then have that $\mathfrak{v} \in \mathcal{C}^\infty(Y \backslash A)$. However, written in this form we have to ignore the $\mathfrak{v}_i$'s for which $z \notin V'_i$. This can be done because of the following: the maximum is taken over the $\mathfrak{v}_i$'s with $z \in V_i'$, so for different positions of z we have a different number of functions over which we take the maximum. But we have that $\mathfrak{v}_i(z) \to -\infty$, for $z \to \partial V_i'$, $z \in V_i'$, i.e. the values of $\mathfrak{v}_i(z)$ with z near the boundary of V_i' doesn't play an effective role in the maximum. This fact together with (2) shows that $\mathfrak{v}$ is globally well defined.

At the same time it allows us to choose coverings $(V_j'')_j$ and $(W_j)_j$ of Y, $V_j'' \Subset W_j \Subset V_j'$ such that already each $\mathfrak{v}_j(z)$ for $z \in V_j' \setminus V_j''$ does not play an effective role in the maximum, in particular we have $m(\mathfrak{v}_{1|_{W_1}}(z), \ldots, \mathfrak{v}_{q|_{W_q}}(z)) = m(\mathfrak{v}_1(z), \ldots, \mathfrak{v}_q(z))$ for each $z \in Y$. We will need the covering $(W_j)_j$ in what follows.

Remark first that we have $\{z \in Y \mid \mathfrak{v}(z) = -\infty\} = A$.

The listed properties of the function m imply that $m(\eta_1, \ldots, \eta_q)$ is still plurisubharmonic if $\eta_1, \ldots, \eta_q$ are plurisubharmonic. Because of the special form of m it follows that it also preserves the strongly plurisubharmonicity.

Indeed, we have to check that for any strongly plurisubharmonic functions (such that the composition makes sense) $\eta_1, \ldots, \eta_q$ and for each $\theta \in \mathcal{C}_0^\infty$ there exists $\varepsilon_0 > 0$ such that $m(\eta) + \varepsilon\theta$ is plurisubharmonic for all $0 \le \varepsilon \le \varepsilon_0$.

But this follows at once from:

$$\begin{aligned} m(\eta) + \varepsilon\theta &= m(\eta) + \int_{\mathbb{R}^q} \varepsilon\theta \prod_{1 \le n \le q} \varrho(u_n)\, du_n \\ &= \int_{\mathbb{R}^q} \max(\eta_1 + u_1 + \varepsilon\theta, \ldots, \eta_q + u_q + \varepsilon\theta) \prod_{1 \le n \le q} \varrho(u_n)\, du_n \\ &= m(\eta + \varepsilon\theta). \end{aligned}$$

Now consider on V_j, for each index j, the function

$$M\lambda_j + \mathfrak{v}|_{V_j}.$$

We will show that, if M is a sufficiently big constant then

$$\varphi_j = (M\lambda_j + \mathfrak{v}) \circ p|_{p^{-1}(V_j)}$$

is strongly plurisubharmonic on $p^{-1}(V_j)$.

To do this consider first the function $M\lambda_j - \theta_i$ on $V_j \cap W_i$. Because θ_i and its derivatives are bounded on W_i and λ_j is strongly plurisubharmonic on V_j it follows that there exists a constant M such that $M\lambda_j - \theta_i$ is strongly plurisubharmonic on $V_j \cap W_i$.

Now look on $p^{-1}(V_j)$ at

$$\begin{aligned}\varphi_j &= (M\lambda_j \circ p + \mathfrak{v} \circ p)|_{p^{-1}(V_j)} \\ &= M\lambda_j \circ p + \int_{\mathbb{R}^q} \max(\mathfrak{v}_1 \circ p + u_1, \dots, \mathfrak{v}_q \circ p + u_q) \prod_{1\le n\le q} \varrho(u_n)\, du_n \\ &= \int_{\mathbb{R}^q} \max(M\lambda_j \circ p + \mathfrak{v}_1 \circ p + u_1, \dots, M\lambda_j \circ p + \mathfrak{v}_q \circ p + u_q) \prod_{1\le n\le q} \varrho(u_n)\, du_n\end{aligned}$$

(where $M\lambda_j \circ p + \mathfrak{v}_i \circ p = M\lambda_j \circ p + \log a_i \circ p - \theta_i \circ p$ is defined on $p^{-1}(V_j \cap W_i)$).

We have shown in the first step that

$$\psi_j \circ p = \lambda_j \circ p + \log\left(\sum_{k=1}^{s} |f_{k,j}|^2\right) \circ p = \lambda_j \circ p + \log a_j \circ p$$

is strongly plurisubharmonic on $U_j := p^{-1}(V_j)$. Concerning λ_j, in the proof of the first step it is only important that λ_j is strongly plurisubharmonic on V_j. So we can replace it by any other strongly plurisubharmonic function, for example by $M\lambda_j - \theta_i$ on $V_j \cap W_i$, to obtain by the same type of argumentation the analogue conclusion, namely that

$$M\lambda_j \circ p + \log a_i \circ p - \theta_i \circ p$$

is strongly plurisubharmonic on $p^{-1}(V_j \cap W_i)$, $\forall j$, $\forall i$.

So, it finally follows from the above listed properties of m that the function φ_j is strongly plurisubharmonic on $U_j := p^{-1}(V_j)$.

In conclusion, we obtained a covering

$$(U_j := p^{-1}(V_j))_{j\in I}$$

of X and on each open set U_j a strongly plurisubharmonic function

$$\varphi_j \colon U_j \longrightarrow [-\infty, \infty)$$

with the property that φ is regular of class $\mathcal{C}^\infty$ outside the rare set $\{x \in U_j \mid \varphi_j(x) = -\infty\} = U_j \cap p^{-1}(A)$.

This collection of functions also satisfies the desired pluriharmonic-compatibility condition, that is we have, on each non-empty intersection $U_i \cap U_j$, locally that

$$\begin{aligned}\varphi_i &= M\lambda_i \circ p + \mathfrak{v} \circ p|_{p^{-1}(V_i)\cap p^{-1}(V_j)} \\ &= M\lambda_j \circ p + M\,\mathrm{Re}(f_{ij} \circ p) + \mathfrak{v} \circ p|_{p^{-1}(V_i)\cap p^{-1}(V_j)} = \varphi_j + \mathrm{Re}\, g_{ij}\end{aligned}$$

with g_{ij} holomorphic. This completes the proof of our Theorem 2.5. □

REMARK 2.7. With almost the same proof it follows that Theorem 2.5 also holds if one only supposes that Y is generalized Kähler.

3. A generalization of Theorem 2.5

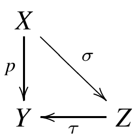
Now we can extend our result to the following more general context:

Theorem 3.1. *Let $p: X \to Y$ be a holomorphic and surjective map between two reduced, compact, complex spaces with singularities and with the property that p sends each irreducible component C_X of X (surjective) onto an irreducible component C_Y of Y of the same dimension, $\dim C_X = \dim C_Y$. If Y is Kähler, then X has a generalized Kähler metric.*

REMARK 3.2. 1. In the context of the above theorem it follows that $\dim X = \dim Y$.
2. The hypothesis of the above theorem concerning the irreducible components of X and Y is satisfied for example in the following special cases:
(a) X and Y are irreducible (and therefore pure dimensional) and $\dim X = \dim Y$.
(b) X and Y are pure dimensional with $\dim X = \dim Y$ and they have the same number of irreducible components.

The idea of the proof is to reduce this problem to the now known context of a proper modification between compact complex spaces, where the "base" space is Kähler. This is possible with help of the following "Stein factorization theorem" (see for instance [5, p. 70, Theorem 1.24]).

Theorem 3.3. *Let $p: X \to Y$ be a proper holomorphic map. Then there is a commutative diagram*



of complex spaces and holomorphic maps with the following properties:
1. *τ is finite.*
2. *σ is proper, surjective, has connected fibers and the canonical map $\sigma^0: \mathcal{O}_Z \to \sigma_* \mathcal{O}_X$ is an isomorphism.*

REMARK 3.4. In our context we also have the following supplementary properties:
1. Z is compact.
2. τ is surjective.
3. Z is reduced: Indeed, if there would exist an open set $V \subset Z$ such that $\mathcal{O}_Z(V)$

contains a nilpotent element, then because of $\mathcal{O}_Z(V) \simeq \mathcal{O}_X(\sigma^{-1}(V))$ it would follow that $\mathcal{O}_X(\sigma^{-1}(V))$ contains nilpotent elements, which is a contradiction to the fact that X is reduced.
4. τ being finite and surjective it also follows that $\dim Y = \dim Z$, so that $\dim X = \dim Z$.
5. Y being Kähler and τ being finite it follows that Z is also Kähler (see for instance [1] or [11]).

Proof of Theorem 3.1. In order to prove Theorem 3.1 our goal is to show that σ is a proper modification.

The subsets

$$\begin{aligned} &\operatorname{Sing}(Z) \hookrightarrow Z \\ &\operatorname{Sing}(X) \hookrightarrow X \\ &\sigma(\operatorname{Sing}(X)) \hookrightarrow Z \\ &\sigma^{-1}(\operatorname{Sing}(Z)) \hookrightarrow X \end{aligned}$$

and

$$\sigma^{-1}(\sigma(\operatorname{Sing}(X))) \hookrightarrow X$$

are all rare analytic sets.

Consider now $D = \operatorname{Sing} Z \cup \sigma(\operatorname{Sing} X)$, $D \hookrightarrow Z$ which is a rare analytic set in Z. For each irreducible component C_X of X we then have a surjective map between two connected manifolds

$$\sigma|_{C_X\setminus\sigma^{-1}(D)} \colon C_X\setminus\sigma^{-1}(D) \longrightarrow C_Z\setminus D$$

where C_Z is chosen such that $\sigma(C_X) = C_Z$ (in particular by our hypothesis we then have that $\dim C_X = \dim C_Z$). Applying Sard's Theorem it follows that there exists a regular point $a \in C_X \setminus \sigma^{-1}(D)$ where the linear tangent map of σ, $T_a(\sigma|_{C_X\setminus\sigma^{-1}(D)})$ is surjective. Because of the same finite dimension of the spaces it follows that the linear tangent map $T_a(\sigma|_{C_X\setminus\sigma^{-1}(D)})$ is in fact bijective. But this tells us that the set

$$\{x \in C_X\setminus\sigma^{-1}(D) \mid \det J_\sigma(x) = 0\}$$

is a rare analytic set in $C_X\setminus\sigma^{-1}(D)$, where $J_\sigma(x)$ denotes the Jacobian matrix. This being true for all irreducible components C_X of X it follows that

$$B := \{x \in X\setminus\sigma^{-1}(D) \mid \det J_\sigma(x) = 0\}$$

is a rare analytic set in $X\setminus\sigma^{-1}(D)$. Note that we do not know whether $\overline{B}$ is analytic in X (where $\overline{B}$ denotes the closure of B in X).

It is enough for us to find a rare analytic set C in X such that $\overline{B} \subset C$. To find C we make use of some known notions and results about the tangent space and the tangent map for complex spaces with singularities, namely that the set

$$C := \mathrm{Sing}^0(\sigma) := \{x \in X \mid \mathrm{corank}_x\, \sigma > 0\} = \{x \in X \mid \dim\ker T_x\sigma > 0\}$$

is analytic in X.

Moreover because $C \cap (X \setminus \sigma^{-1}(D)) = B$, this set is also rare.

Let $A := D \cup \sigma(C)$ which is rare in Z and consider the surjective map

$$\sigma|_{X\setminus\sigma^{-1}(A)} \colon X \setminus \sigma^{-1}(A) \longrightarrow Z \setminus A$$

We have for all $x \in X \setminus \sigma^{-1}(A)$ that $x \in \mathrm{Reg}(X)$ and $x \notin C$. Therefore $x \notin B$, so that $\det J_\sigma(x) \neq 0$ for each $x \in X \setminus \sigma^{-1}(A)$.

But this means that σ is locally biholomorphic on $X \setminus \sigma^{-1}(A)$. Because $\sigma|_{X\setminus\sigma^{-1}(A)}$ has connected fibers it therefore follows that $\sigma|_{X\setminus\sigma^{-1}(A)}$ is injective, so we finally deduce that the map

$$\sigma|_{X\setminus\sigma^{-1}(A)} \colon X \setminus \sigma^{-1}(A) \longrightarrow Z \setminus A$$

is biholomorphic. So $\sigma\colon X \to Z$ is indeed a proper modification.

As we mentioned above this is enough to conclude, as desired, that X has a generalized Kähler metric. The proof of Theorem 3.1 is now complete. □

REMARK 3.5. Of course the statement of Theorem 3.1 remains true when Y is only required to be generalized Kähler.

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