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COMPACTIFICATIONS OF MARTIN TYPE OF HARMONIC SPACES

Dedicated to Professor Yukio Kusunoki on his 60th birthday

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Introduction

In the classical theory of harmonic functions the Martin boundary or the Martin compactification, originated by R.S. Martin, is a very important and interesting material. After Martin some significant results such as resolutivity, minimal thinness and fine limit theorems were developped by many authors. Meanwhile, in the course of axiomatization of potential theory, it turned out that a relevant topological notion, ensuring the representation of *all* positive harmonic functions is the completion rather than the compactification. Recently, P.A. Loeb [10] has succeeded in constructing a compactification on which the Martin representation is possible if we restrict ourselves to *every bounded* harmonic function. Inspired by Loeb's paper, we want to identify some resolutive compactifications of a harmonic space on which we can develop analogous theory obtained in the classical case [12].

Let X be a $\mathcal{P}$-harmonic space in the sense of Constantinescu-Cornea [2] with countable base where 1 is superharmonic. Compactifications of Martin type are defined to be a quartet $(X^*, k(x,z), \Delta_1, \mu)$ consisting of the space, the kernel function, a part of the minimal boundary and a boundary measure, which are related to each other as described in the definition in § 1. It is worth to note that all harmonic measures λ_x are absolutely continuous with respect to μ and one may find the kernel function $k(x, z)$ as a density. Also, every quasi-bounded harmonic function $u(x)$ is the Dirichlet solution $H_f(x)$ and the solution is represented by $k(x,z)$ and $f\, d\mu$.

Martin compactifications in the classical case are of Martin type. As for Bauer spaces, Loeb's compactification, though we will discuss it in a slightly general context here, is of Martin type which differs essentially from Martin's. Another examples listed in § 2 are a sort of modifications (Example 4 and 5) and a simple one which illustrates the difference between Martin kernels and those of Martin type. (Example 6).

Like in the Martin space we can define the notion of minimal thinness in the compactification of Martin type. We considered fine filters in § 3 and obtained a theorem of fundamental importance which states that μ-almost all fine filters are convergent with respect to the topology of X^* (Theorem 3.3).

In § 4, following the idea of L. Naïm [12], the Dirichlet problem associated with fine filters is discussed to get the Fatou-Doob-Naïm theorem in § 5. In virtue of Fatou-Doob-Naïm theorem, we can establish an isomorphism between the space of harmonic functions on X and that of boundary functions. We then, as an application, restrict our consideration to a Hilbert space of harmonic functions and discussed a minimizing problem to reveal the extremal property of kernel functions (Corollary 6.7).

The last two sections are devoted to clear up the structure of Martin type compactifications. Unlike Martin spaces we can claim the representation by boundary measure only for quasi-bounded harmonic functions and not for *all* positive. As a result, we state, in Theorem 8.3, that in all compactifications of Martin type of a harmonic space the kernel functions are essentially same on harmonic boundaries.

1. Definition and simple properties

In the sequel, let X be a $\mathcal{P}$-harmonic space in the sense of Constantinescu-Cornea [2] with countable base, and suppose that 1 is superharmonic. A compactification X^* of X is defined to be of Martin type if

1) X^* is metrizable and resolutive, i.e,. for every $f \in \boldsymbol{C}(\Delta)$ (the set of all finite continuous functions on $\Delta = X^* \backslash X$) there exists the Dirichlet solution H_f by means of Perron-Brelot-Wiener's method [11], and

$$H_f(x) = \int f d\lambda_x ,$$

where λ_x is the harmonic measure of x;

2) there exists a finite continuous function $k(x, z)$ defined on $X \times \Delta$ such that $x \to k(x, z)$ is non-negative and harmonic on X for every $z \in \Delta$;

3) there exists a non-negative Borel measure μ on Δ and a boundary set $\Delta_1 \subset \{z \in \Delta;\, x \to k(x, z)$ is minimal harmonic$\}$ satisfying

i) $\mu(\Delta \backslash \Delta_1) = 0$,

ii) $\mu(T) = 0$ if T is negligible, i.e., $\lambda_x(T) = 0$ for every $x \in X$;

4) for every $u \in HB(X)$ (the set of all bounded harmonic functions on X) there exists a resolutive function f on Δ such that

$$u(x) = H_f(x) = \int k(x, z) f(z) d\mu(z) \qquad \forall x \in X .$$

Let τ be a reference measure, i.e., τ is a non-negative Borel measure on X and X is the smallest absorbent set containing the support of τ. We call μ the dilation of τ if $\mu = \int \lambda_x d\tau(x)$.

Before giving some examples of Martin type, we shall remark a few simple results which are derived immediately from the definition. We denote by $k^x(z)$ (resp. $k_z(x)$) the function $k(x, z)$ if we consider it as a function of z (resp. a function of x).

1. $d\lambda_x(z)=k(x, z)d\mu(z)$. For every $g\in \boldsymbol{C}(\Delta)$, by 4) of the definition, there is a resolutive function g' such that $H_g(x)=H_{g'}(x)=\int k^x(z)g'(z)d\mu(z)$ for every $x\in X$. This implies that $g=g'$ $d\lambda_x$-a.e. on Δ. We have thus, $H_g(x)=\int g(z)d\lambda_x(z)=\int k^x(z)g(z)d\mu(z)$. Similarly we know that every bounded μ-measurable function coincides with a resolutive function μ-a.e..

2. $\mu(T)=0$ if and only if T is negligible. This is an immediate consequence of the defintion 3), ii) and the above 1.

3. $L^1(d\mu)\subset\{f; f \textit{ is resolutive}\}$, and $\{H_f; f \textit{ is resolutive}, f\geq 0\}=MHB^+(X)$, where $MHB^+(X)$ denotes the class of all harmonic functions which are limits of increasing sequences of non-negative bounded harmonic functions.

4. $k_z(x)\not\equiv 0$ for μ-almost every $z\in\Delta$. For, letting $T=\bigcap_{x\in X}\{z\in\Delta; k(x, z)=0\}$, we have for every $x\in X$ $H_{\chi_T}(x)=\int \chi_T d\lambda_x=\int \chi_T k^x d\mu=\int_T k^x d\mu=0$, where χ_T is the characteristic function of T. This implies $\lambda_x(T)=0$ and thus $\mu(T)=0$.

REMARK. In a compactification of Martin type the function $k(x, z)$ is closely related to the measure μ, that is, $d\lambda_x(z)=k(x, z)d\mu(z)$. Thus if we introduce a strictly positive, finite continuous function $\rho(z)$ on Δ then the same compactification X^* equipped with $\rho(z)k(x, z)$ and $(1/\rho(z))\mu$ is also of Martin type. However, $(1/\rho(z))\mu$ is not a dilation in general even if μ is a dilation of a reference measure.

2. Examples

In this section we give some examples of compactifications of Martin type.

EXAMPLE 1. [12] Let X be a Green space in the sense of Brelot-Choquet [1]. Then, we can construct the Martin compactification X^*, which is metrizable and resolutive, and the Martin kernel $k(x, z)$. Every positive harmonic function u is represented by a positive Borel measure μ_u on Δ:

$$u(x)=\int k(x, z)d\mu_u(z),$$

with $\mu_u(\Delta\backslash\Delta_1)=0$, where $\Delta_1=\{z\in\Delta; x\to k(x, z) \textit{ is minimal harmonic}\}$. Then, denoting by μ_1 the representing measure of 1, $(X^*, k(x, z), \Delta_1, \mu_1)$ satisfies the requirements of our definition.

In an axiomatic framework, T. Kôri considered the Martin compactification of a Brelot harmonic space. More precisely:

EXAMPLE 2. Let X be a locally compact Hausdorff space with countable base, locally connected, connected, non-compact and satisfying Brelot's axioms. We suppose that there exists a positive potential on X, 1 is superharmonic and further that the axiom of proportionality is satisfied [4]. For every $y \in X$ we have a potential p_y with harmonic support at y and such that the mapping $p_y(x)$: $X \times X \to \bar{\boldsymbol{R}}$ is lower semi-continuous. Let S^+ be the cone of all non-negative superharmonic functions on X. By Hervé's topology [4], S^+ has a metrizable and compact base K_0. Denoting by P the set of extreme potentials in K_0 we know that X is homeomorphic to P. We may construct a positive continuous function $\alpha(y)$ such that $\alpha(y)p_y \in P$. We define $k(x, y) = \alpha(y)p_y(x)$. Consider $\mathcal{U}^*$, the coarsest uniformity making $y \to k(x, y)$ uniformly continuous, and let X^* be the completion of X by $\mathcal{U}^*$. X^* is homeomorphic to $\bar{P}$ (the closure of P in S^+) and this is a compactification of X which is resolutive. For every $z \in \Delta = X^* \backslash X$ we have $k_z(x)$ which is harmonic on X. $k(x, z) = k_z(x)$ is continuous on $X \times \Delta$. There exists a unique measure μ_1 on Δ with $\mu_1(\Delta \backslash \Delta_1) = 0$, where $\Delta_1 = \{z \in \Delta;\ k_z \textit{ is minimal}\}$ such that

$$H_1(x) = \int_{\Delta_1} k(x, z) d\mu_1(z) .$$

Then, $(X^*, k(x, z), \Delta_1, \mu_1)$ satisfies the conditions of our definition.

Recently, P.A. Loeb [10] gave an interesting compactification of a strict harmonic space satisfying Bauer's axioms. We can see that Loeb's compactification is of Martin type. Although in the next example we are going to discuss it in a slightly general situation, the essential part is due to Loeb.

EXAMPLE 3. Let X be a $\mathcal{P}$-harmonic space in the sense of Constantinescu-Cornea with countable base and we suppose that 1 is superharmonic. In the following we shall show that if the harmonic sheaf $\mathcal{H}$ on X has the property of nuclearity ([2], p. 276), X affords a compactification of Martin type.

Lemma 2.1. *There exists a finite reference measure τ on X satisfying*

$$\text{(2.1)} \qquad \left\{ \begin{array}{l} \textit{for every compact subset } K \textit{ of } X \textit{ there is } \alpha_K > 0 \\ \textit{such that } \sup_K |u| \leq \alpha_K \int |u| d\tau \textit{ for every } u \in H(X) . \end{array} \right\}$$

This is an immediate consequence of [2], Theorem 11.1.2.

In the following, we shall fix such a measure τ and assume that $\tau(X) = 1$. Let us consider the Wiener compactification X^W of X ([5]). Denoting by λ_x^W the harmonic measure of X^W at x and forming $\sigma^W = \int \lambda_x^W d\tau(x)$, we have $\lambda_x^W \ll \sigma^W$, i.e., λ_x^W is absolutely continuous with respect to σ^W for every $x \in X$. For, if a bounded function $\varphi \geq 0$ is σ^W-integrable then φ is $d\lambda_x^W$-integrable for τ-almost every x and $\int \varphi d\sigma^W = \int \lambda_x^W(\varphi) d\tau(x)$, and hence we have $\int \lambda_x^W(A) d\tau(x) = 0$ whenever $\sigma^W(A) = 0$. This implies that $\lambda_x^W(A) = 0$ for every $x \in X$ since $x \to \lambda_x^W(A)$ is non-negative harmonic and vanishes τ-almost everywhere. If we write $d\lambda_x^W / d\sigma^W = \kappa(x, \zeta)$, we can prove as in [10]

Lemma 2.2. $\kappa(x, \zeta) \in L^\infty(d\sigma^W)$.

In fact, suppose on the contrary that $\sigma^W(A_m) > 0$ for every integer m, where $A_m = \{\zeta \in \Delta^W = X^W \setminus X;\ \kappa(x, \zeta) \geq 2^m\}$. Letting $f_m = (2^m \sigma^W(A_m))^{-1} \chi_{A_m}$ we define $h_{f_m}(y) := \int f_m d\lambda_y^W$. Then, $h_{f_m}(x) = \int f_m d\lambda_x^W = \int f_m \kappa(x, \zeta) d\sigma^W \geq 1$. On the other hand, since $h_{f_m} \in H(X)$, by (2.1)

$$\sup_K h_{f_m} \leq \alpha_K \int h_{f_m} d\tau = \alpha_K \int \lambda_y^W(f_m) d\tau(y) = \alpha_K \int f_m d\sigma^W = 2^{-m} \alpha_K .$$

Hence, $h_n := \sum_{m=1}^n h_{f_m}$, $n = 1, 2, \cdots$ are locally uniformly bounded and $h := \lim_{n\to\infty} h_n \in H(X)$. However this contradicts $h(x) = \sum_{m=1}^\infty h_{f_m}(x) = +\infty$.

From $\kappa(x, \zeta)\kappa(y, \zeta) \in L^1(d\sigma^W)$ we have $\kappa(x, \zeta) \in L^1(d\lambda_y^W)$ for every $y \in X$. The function $y \to \int \kappa(x, \zeta) d\lambda_y^W$ is bounded and harmonic on X. We write this function $k_x(y)$. $k_x(y)$ is harmonic as a function of x. For $k_x(y) = \int \kappa(x, \zeta)\kappa(y, \zeta) d\sigma^W = \int \kappa(y, \zeta) d\lambda_x^W = k_y(x)$. Thus k_x can be extended continuously on X^W. If we denote by $\bar{k}_x$ the restriction of this extension on the harmonic boundary Γ^W ([5]), we have

$$k_x(y) = H_{\bar{k}_x}^W(y) = \int \bar{k}_x d\lambda_y^W = \int \kappa(x, \zeta) d\lambda_y^W,$$

where $H_{\bar{k}_x}^W$ is the Perron-Brelot-Wiener solution of the Dirichlet problem on X^W. Hence $\bar{k}_x(\zeta) = \kappa(x, \zeta)$ $d\lambda_y^W$-a.e. for every $y \in X$, and therefore each class $\kappa(x, \zeta)$ of $L^\infty(d\sigma^W)$ has a continuous representative, and in the following we assume that $\kappa(x, \zeta)$ is continuous on Δ^W.

Let $\{a_n\}$ be a sequence of points of X which is dense in X, and let $Q = \boldsymbol{C}_0(X) \cup \{k_{a_n};\ n \in \boldsymbol{N}\}$, where $\boldsymbol{C}_0(X)$ is the space of continuous functions on X with compact support and $\boldsymbol{N}$ denotes the set of all positive integers. Then the Q-compactification ([5]) $X^* = X^Q$ is metrizable and resolutive.

Lemma 2.3. *Every k_x has a continuous extension on X^*.*

Proof. Let $\{a_n'\}$ be a subsequence of $\{a_n\}$ tending to x, $z \in \Delta = X^* \setminus X$

and $\{y_m\}$ be a sequence of X tending to z. For every $y\in X$, $\sup_{x'\in K} k_y(x')\leq \alpha_K\int k_y(x)d\tau(x)=\alpha_K\int \lambda_x^W(k_y)d\tau(x)=\alpha_K\int k_y d\sigma^W=\alpha_K\int d\lambda_y^W=\alpha_K H_1^W(y)\leq\alpha_K$, i.e., the family of harmonic functions $\{k_y;\, y\in X\}$ is locally uniformly bounded and this implies that $\{k_y(x');\, y\in X\}$ is equi-continuous ([2], Theorem 11.1.1). Using $k_{x'}(y)=k_y(x')$, we deduce from this together with the inequality

$$|k_x(y_m)-k_x(y_{m'})|\leq|k_x(y_m)-k_{a'_n}(y_m)|+|k_{a'_n}(y_m)-k_{a'_n}(y_{m'})|+|k_{a'_n}(y_{m'})-k_x(y_{m'})|$$

that $\{k_x(y_m);\, m\in N\}$ is a Cauchy sequence.

We shall write this extended function k_x again.

From the proof of Lemma 2.3, we can easily see that the function $k(x, z)=k_x(z)$ defined on $X\times X^*$ is continuous on $X\times\Delta$ and $z\to k(x, z)$ is bounded and continuous for every $x\in X$, and $x\to k(x, z)$ is non-negative and harmonic for every $z\in\Delta$.

Now we define $\mathcal{H}^1=\{h\in H(X);\, h\geq 0, \int hd\tau\leq 1\}$, and $\mathcal{E}=\{h\in\mathcal{H}^1;$ *h is minimal harmonic,* $\int hd\tau=1.\}$ Then it is easily checked that

$$\mathcal{H}^1 \textit{ is a Choquet simplex; ext } \mathcal{H}^1=\mathcal{E}\cup\{0\}\,;\ \mathcal{E} \textit{ is a } G_\delta\textit{-set}.$$

We can prove also $k(x, y)\in\mathcal{H}^1$ for every $y\in X^*$. In fact, for $y\in X$,

$$\int k(x, y)d\tau(x)=\int\lambda_x^W(\kappa(y, \zeta))d\tau(x)=\int\kappa(y, \zeta)d\sigma^W(\zeta)=\int 1d\lambda_y^W=H_1^W(y)\leq 1;$$

for $z\in\Delta$, letting $\{y_n\}\subset X$, $y_n\to z$ we have $\int k(x, z)d\tau(x)\leq\limsup_{n\to\infty}\int k(x, y_n)d\tau(x)\leq 1$.

The mapping $\Omega\colon\Delta\to\mathcal{H}^1$ defined by $\Omega(z)=k(x, z)$ is a continuous injection ($\mathcal{H}^1$ is endowed with the compact convergence topology). This is a consequence of the facts that $k(x, z)$ is continuous on $X\times\Delta$ and that Q separates points of Δ. We define $\Delta_1=\{z\in\Delta;\, \Omega(z)\in\mathcal{E}\}$. Δ_1 is a G_δ-subset of Δ and Δ_1 is homeomorphic to $\Omega(\Delta)\cap\mathcal{E}$. Let λ_x be the harmonic measure of x with respect to X^*, μ be a dilation of τ, i.e., $\mu=\int\lambda_x d\tau(x)$ and π be the canonical mapping of X^W onto X^*, i.e., π is continuous and $\pi(x)=x$ for every $x\in X$.

Lemma 2.4. $\kappa(x, \zeta)=k(x, \pi(\zeta))$ *for every* $\zeta\in\Gamma^W$.

In fact, from the definition of $k(x, z)$, $H_{\kappa_x}^W(y)=k(x, y)=H_{k_x}(y)=H_{k_x\circ\pi}^W(y)$, therefore $\kappa_x=k_x\circ\pi$ $d\lambda_y^W$-a.e. for every $y\in X$. Since both functions are continuous on Γ^W and $\Gamma^W=\overline{\bigcup_{y\in X}\operatorname{supp}\lambda_y^W}$, we have $\kappa_x(\zeta)=k(x, \pi(\zeta))$ for every $\zeta\in\Gamma^W$.

Let $\mathcal{K}=\{u\in HB(X);\, u\geq 0, \int ud\tau=1\}$. For $u\in\mathcal{K}$ and $A\in\mathcal{B}(\Delta)$ (the set of all Borel sets on Δ) we define

$$\mu_u(A):=\int_{\pi^{-1}(A)}\bar{u}d\sigma^W, \text{ i.e., } \pi(\bar{u}d\sigma^W)=d\mu_u,$$

where $\bar{u}$ is the continuous extension of u on X^W. Quite in the same way as [10] Theorem 2, we can see that μ_u is the unique representating measure of u on $\mathscr{H}^1$, i.e., $\mu_u(\Delta\backslash\Delta_1)=0$ and $u(x)=\int k(x, z)d\mu_u(z)$. We have $\mu_u\ll\mu$, for if $A\in \mathscr{B}(\Delta)$ and $\mu(A)=0$ then $\sigma^W(\pi^{-1}(A))=\int\lambda_x^W(\pi^{-1}(A))d\tau(x)=\int\lambda_x(A)d\tau(x)=\mu(A)=0$, thus $\mu_u(A)=\int_{\pi^{-1}(A)}\bar{u}d\sigma^W=0$. Let $f=d\mu_u/d\mu$. f is obviously bounded. Since for every $g\in \boldsymbol{C}(\Delta)$, $\int gd\lambda_x=H_g(x)=H^W_{g\circ\pi}(x)=\int g\circ\pi d\lambda_x^W=\int(g\circ\pi)\kappa_x d\sigma^W=\int(g\circ\pi)(k^x\circ\pi)d\sigma^W=\int gk^x d(\pi\sigma^W)=\int gk^x d\mu$, we have $d\lambda_x=k^x d\mu$.

From the above result we conclude that for $u\in\mathscr{K}$

$$u(x)=H^W_{\bar{u}}(x)=\int\bar{u}d\lambda_x^W=\int\bar{u}(\zeta)\kappa(x, \zeta)d\sigma^W(\zeta)=\int(k^x\circ\pi)\bar{u}d\sigma^W=$$
$$=\int k^x\pi(\bar{u}d\sigma^W)=\int k^x d\mu_u=\int k^x f d\mu=\int fd\lambda_x=H_f(x),$$

since in a metrizable resolutive compactification X^*, f is resolutive if and only if f is $d\lambda_x$-integrable for every $x\in X$.

Finally we have $\alpha=\int H_1(x)d\tau(x)>0$, $(1/\alpha)H_1=H_{(1/\alpha)}\in\mathscr{K}$ and $\mu_{H_{(1/\alpha)}}=(1/\alpha)\mu$; hence $\mu(\Delta\backslash\Delta_1)=0$.

REMARK. If a harmonic space satisfies the Doob convergence axiom, then for every finite reference measure τ on X we can construct a compactification X^* of Martin type on which μ is a delation of τ.

EXAMPLE 4. Let $(X^*, k(x, z), \Delta_1, \mu)$ be a compactification of Martin type and μ be the dilation of a reference measure τ. We define $K(x, y)=\int k(x, z)k(y, z)d\mu(z)$. $K(x, y)=K(y, x)=H_{k^y}(x)=H_{k^x}(y)$. Let $\{a_n\}$ be a countable dense set of X and $Q=\{F\mid X; F\in \boldsymbol{C}(X^*)\}\cup\{K(x, a_n); n\in \boldsymbol{N}\}$, where $F\mid X$ denotes the restriction of F to X. We construct the Q-compactification $\hat{X}=X^Q$. $\hat{X}$ is metrizable and resolutive. In the same way as Example 3, we can show that the function $K(x, a)=K(a, x)$ of x extends continuously on $\hat{X}$. In the present example, however, the local uniform boundedness of $K(x, y)$ is proved from the continuity of $k(x, z)$ on $X\times\Delta$. In fact, $\sup[\sup\{K(x, y); x\in X\}; y\in L]=\sup[\sup\{H_{k^y}(x); x\in X\}; y\in L]\leq\sup\{\|k^y\|_\infty; y\in L\}<\infty$ for every compact subset L of X. Thus we have functions $K(x,y)$ defined on $X\times\hat{X}$ and $\hat{k}(x, \hat{z}):=K(x, z)$ defined on $X\times\hat{\Delta}$, where $\hat{\Delta}=\hat{X}\backslash X$. It is easily checked that

$\hat{k}(x, \hat{z})$ is continuous on $X\times\hat{\Delta}$. Let

$$\mathcal{A}=\{u\in MHB^{+}(X);\ \tau(u)\leq 1\}.$$

$\mathcal{A}$ is relatively compact in $H^{+}(x)$ with respect to the compact convergence topology. In fact, $\mathcal{A}$ is equi-continuous; for every $u\in\mathcal{A}$ is the Dirichlet solution H_f of a non-negative resolutive function f and $0\leq\int fd\mu=\int\lambda_x(f)d\tau(x)=\int H_f(x)d\tau(x)=\tau(u)\leq 1$. We have then

$$|u(x)-u(x')|=|\int(k^x-k^{x'})fd\mu|\leq\sup_{\Delta}|k^x-k^{x'}|,$$

and, by the continuity of $k(x, z)$, the last term is arbitrarily small if x' is near x. Furthermore $u(x)=\int k(x, z)f(z)d\mu(z)\leq\|k^x\|_\infty$ implies that $\{u(x);\ u\in\mathcal{A}\}$ is bounded for every $x\in X$. We note

$$\{\hat{k}(x, \hat{z});\ \hat{z}\in\hat{\Delta}\}\subset\overline{\mathcal{A}}\subset\{u\in H^{+}(X);\ \tau(u)\leq 1\},$$

since $K(x, y)\in\mathcal{A}$ for every $y\in X$ and $\hat{k}(x, \hat{z})$ is the local uniform limit of $K(x, y_m)$ whenever $y_m\to\hat{z}$. $\overline{\mathcal{A}}$ is compact, convex and metrizable, i.e., a Choquet simplex. Therefore, for every $u\in\overline{\mathcal{A}}$ there is a unique measure ν on $ext\ \overline{\mathcal{A}}$ such that $u=\int hd\nu(h)$. As in Example 3, we see that $ext\ \overline{\mathcal{A}}=\mathcal{E}\cup\{0\}$, where $\mathcal{E}=\{u\in H^{+}(X);\ minimal,\ \tau(u)=1\}$, $\mathcal{E}$ is a G_δ-set and the mapping $\hat{z}\to\hat{k}(x, \hat{z})$ of $\hat{\Delta}$ into $\overline{\mathcal{A}}$ is a continuous injection. We define

$$\hat{\Delta}_1=\{\hat{z}\in\hat{\Delta};\ \hat{k}(x, \hat{z})\in\mathcal{E}\},$$

that is, $\hat{k}(x, \hat{z})$ is minimal and $\int k(x, \hat{z})d\tau(x)=1$ if and only if $\hat{z}\in\hat{\Delta}_1$. We proceed as in the previous example. Let, for each $u\in HB^{+}(X)$ with $\tau(u)=1$, $\mathcal{M}_u=\{\nu;\ probability\ measure\ on\ \overline{\mathcal{A}},\ u=\int hd\nu(h)\}$. Denoting by $\hat{\mu}$ a dilation of τ on, $\hat{X}$ i.e., $\hat{\mu}=\int\hat{\lambda}_x d\tau(x)$, we can see that $\mu_u=(f\circ\pi)\hat{\mu}\in\mathcal{M}_u$, where $u=H_f$ and π is the canonical mapping of $\hat{X}$ onto X^*. This is an easy consequence of $\pi\hat{\mu}=\mu$, $H_{k^x}=H^{\hat{X}}_{\hat{k}^x}$, and $\hat{k}^x=k^x\circ\pi$ $\hat{\mu}$-a.e.. Moreover, we can prove, as in Example 3, that μ_u is minimal with respect to the Choquet order. Thus, $\mu_u(\overline{\mathcal{A}}\backslash ext\ \overline{\mathcal{A}})=0$ and, with the obvious identification, this is expressed as $\mu_u(\hat{\Delta}\backslash\hat{\Delta}_1)=0$. If we take, in particular, $u=h_0/\tau(h_0)$, where $h_0=H_1$ we have $[\tau(h_0)]^{-1}\hat{\mu}(\hat{\Delta}\backslash\hat{\Delta}_1)=0$. Hence $(\hat{X}, \hat{k}(x, \hat{z}), \hat{\Delta}_1\ \hat{\mu})$ is of Martin type.

Example 5. Let X^* be a compactification of Martin type and let Q be

$\{F|X; F\in \boldsymbol{C}(X^*)\}\subset Q\subset \{F|X; F\in \boldsymbol{C}(X^*)\}\cup\{H_f; f\in \boldsymbol{C}(\Delta)\}$. Then the Q-compactification X^Q is of Martin type, for the canonical mapping π of X^Q onto X^* yields the homeomorphism π_0 of harmonic boundaries Γ^Q and Γ ([8]). $\tilde{k}(x, \tilde{z}):=k(x, \pi_0(\tilde{z}))$, $\tilde{\Delta}_1:=\pi_0^{-1}(\Delta_1\cap\Gamma)$ and $\tilde{\mu}(\tilde{A}):=\mu(\pi_0(\tilde{A})\cap\Gamma^Q))$ for every $\tilde{A}\in \mathcal{B}(\Delta^Q)$ fulfill the conditions of the definition of Martin type. When $Q=\{F|X; F\in \boldsymbol{C}(X^*)\}\cup\{H_f; f\in \boldsymbol{C}(\Delta)\}$, it is known that all points of Γ^Q are regular with respect to the Dirichlet problem on X^Q ([7]). Also, for $Q=\{F|X; F\in \boldsymbol{C}(X^*)\}\cup\{H_{k^y}; y\in X\}$, $\tilde{\mu}$-almost all points are regular when μ is the dilation of a reference measure τ. (Cf. Example 4 and [10]). In this case, the compactification $\hat{X}$ coincides with X^Q, but kernel functions may different since the continuous extensions $\hat{k}(x, \hat{z})$ can separate the boundary points.

EXAMPLE 6. Let X be a punctured unit disc in the complex plane, e.g., $X=\{x\in \boldsymbol{C};\ |x|<1\}\setminus\{1/2\}$. The Martin boundary of X is just the topological boundary ∂X. Let

$$k(x, z)=\begin{cases}\mathrm{Re}\{(z+x)/(z-x)\} & \text{if } |z|=1\\ u_0 & \text{if } z=1/2\end{cases}$$

where u_0 is an arbitrary non-negative harmonic function, $\bar{X}=X\cup\partial X$, $\Delta_1=\{z\in \boldsymbol{C};\ |z|=1\}$ and μ be the Lebesgue measure on Δ_1. Then $(\bar{X}, k(x, z), \Delta_1, \mu)$ is of Martin type.

3. Fine filters

Let X^* be of Martin type. We use the convention $k_z(x)=k^x(z)=k(x, z)$. For $z\in\Delta_1$, a set $E\subset X$ is called *thin at* z if $R^E_{k_z}\not\equiv k_z$, where $R^E_{k_z}=\inf\{w;$ *non-negative hyperharmonic on* $X, w\geq k_z$ *on* $E\}$. The lower semicontinuous regularization of $R^E_{k_z}$ is denoted by $\hat{R}^E_{k_z}$, and $\hat{R}^E_{k_z}(x)=\int k_z(y)d\varepsilon^E_x(y)$ ([2], p. 160). It is trivially seen that if E is thin at z and $E_1\subset E$ then E_1 is thin at z. As in [3], we can prove

Proposition 3.1. *For every $z\in\Delta_1$ and for every $E\subset X$, $\hat{R}^E_{k_z}$ is either k_z or a potential.*

Proposition 3.2. *If E_i $(i=1,2)$ is thin at $z\in\Delta_1$ then $E_1\cup E_2$ is thin at z.*

For $z\in\Delta_1$ we define $\mathcal{G}_z=\{E\subset X;\ X\setminus E$ *is thin at* $z\}$. Since every compact subset of X is thin at every $z\in\Delta_1$, we see that $\mathcal{G}_z$ is a filter possessing no limit points in X. The next theorem is fundamental to our further consideration.

Theorem 3.3. *$\mathcal{G}_z$ converges to z for μ-almost all z, i.e., there exists $N\subset\Delta$ with $\mu(N)=0$ such that $U(z)\cap X\in\mathcal{G}_z$ for every $z\in\Delta_1\setminus N$ and for every neighborhood $U(z)$ of z in X^*.*

To prove the theorem we require some lemmas.

Lemma 3.4. *For $E\subset X$, $T(E)=\{z\in\Delta_1; E$ is not thin at $z\}$ is μ-measurable. Therefore $\{z\in\Delta_1; E$ is thin at $z\}$ is also μ-measurable.*

Proof. Let $\{a_k\}$ be a countable dense subset of X and $\{U_n\}$ be a base of X. We call (k, n) a pair if $a_k\in U_n$. For a pair (k, n) we set $T(k, n)=\{z\in\Delta; k_z(a_k)-H^{U_n}_{\hat{R}^E_{k_z}}(a_k)=0\}$. From $H^{U_n}_{\hat{R}^E_{k_z}}(a_k)=\int[\int k_z(x)d\varepsilon^E_y(x)]d\varepsilon^{CU_n}_{a_k}(y)$ and the continuity of k^z we see that $z\to H^{U_n}_{\hat{R}^E_{k_z}}(a_k)$ is lower semicontinuous, therefore $T(k, n)$ is a G_δ-set. Our lemma is derived from $T=[\cap T(k, n)]\cap\Delta_1$, where $\cap$ is taken over all pairs (k, n).

Now, for $A\subset\Delta$ and for a non-negative hyperharmonic function v on X we define

$$v_A(x) = \inf\{R_v^{G\cap X}(x);\ G\supset A,\ \textit{open in } X^*\}.$$

If v is a non-negative superharmonic function then $v_A\in H(X)$ and $(k_z)_A$ is either k_z or 0 for every $z\in\Delta_1$. We note that if A is closed and if $\{G_n\}$ is a decreasing sequence of open sets in X^* with $\bigcap_{n=1}^{\infty} G_n=A$, then $v_A(x)=\lim_{n\to\infty} R_v^{G_n\cap X}(x)$.

Lemma 3.5. *Let $u\in HB(X)$, $u(x)=\int_{\Delta_1} k_z(x)d\mu_u(z)$ and A be a closed subset of Δ. Then*

$$u_A(x) = \int_{\Delta_1}(k_z)_A(x)d\mu_u(z).$$

Proof. This is derived easily from the above remark and

$$\begin{aligned} R_u^{G\cap X}(x) &= \int[\int_{\Delta_1} k_z(y)d\mu_u(z)]d\varepsilon_x^{G\cap X}(y) \\ &= \int_{\Delta_1}[\int k_z(y)d\varepsilon_x^{G\cap X}(y)]d\mu_u(z) = \int_{\Delta_1} R_{k_z}^{G\cap X}(x)d\mu_u(z). \end{aligned}$$

In the following we write $h_0=\boldsymbol{H}_1$.

Lemma 3.6. *$(h_0)_A=\bar{H}_{\chi_A}$ for every $A\subset\Delta$, where χ_A denotes the characteristic function of A and $\bar{H}$ is the upper PBW solution.*

Proof. Let G be an open set in X^* with $A\subset G$ and let $1=h_0+p$ be the Riesz decomposition. From $R_1^{G\cap X}\leq R_{h_0}^{G\cap X}+p$ it follows that $\bar{H}_{\chi_A}\leq R_1^{G\cap X}\leq R_{h_0}^{G\cap X}+p$. Since G is arbitrary, $\bar{H}_{\chi_A}\leq(h_0)_A+p$. $\bar{H}_{\chi_A}$ being harmonic $\bar{H}_{\chi_A}\leq(h_0)_A$. Conversely, let v be a hyperharmonic function such that $\liminf v\geq\chi_A$ on Δ. For every $\varepsilon>0$ we set $G=\{\hat{v}>1-\varepsilon\}$, where $\hat{v}$ is the lower semicontinuous extension of v to X^*. Since G is open and $A\subset G$, $(h_0)_A\leq R_{h_0}^{G\cap X}\leq R_1^{G\cap X}\leq$

$v/(1-\varepsilon)$. v and ε being arbitrary, $(h_0)_A \leq \bar{H}_{\chi_A}$.

If $A \subset \Delta$ is μ-measurable, then χ_A is resolution (§1, 3). Thus

$$(h_0)_A(x) = H_{\chi_A}(x) = \int k_z(x)\chi_A(z)d\mu(z)\,.$$

Lemma 3.7. *If $A \subset \Delta$ is closed then $\mu(\{z \in A \cap \Delta_1;\ (k_z)_A = 0\}) = 0$, i.e., $(k_z)_A = k_z$ for μ-almost all z on A.*

Proof. By Lemma 3.5 and the above remark

$$(h_0)_A(x) = \int k_z(x)\chi_A(z)d\mu(z) = \int (k_z)_A(x)d\mu(z)\,.$$

Let $\{G_n\}$ be a decreasing sequence of open sets with $\bigcap_{n=1}^{\infty} G_n = A$. Then, by Lemma 3.4, $\{z \in \Delta_1;\ R_{k_z}^{G_n \cap X} = k_z\}$ is μ-measurable, and hence $B = \{z \in \Delta_1;\ (k_z)_A = k_z\} = \bigcap_{n=1}^{\infty} \{z \in \Delta_1;\ R_{k_z}^{G_n \cap X} = k_z\}$ is μ-measurable. Therefore

$$\int (k_z)_A(x)d\mu(z) = \int_B k_z(x)d\mu(z) = \int k_z(x)\chi_B(z)d\mu(z) = H_{\chi_B}(x)$$

and hence $H_{\chi_A} = H_{\chi_B}$, which means $\chi_A = \chi_B$ μ-almost everywhere. Hence $\mu(A \backslash B) = 0$.

Let $\{V_n\}$ be a countable base of open subsets of Δ. We call (n, m) a pair if $\bar{V}_n \subset V_m$. For every pair (n, m), we define

$$A_{n,m} = \{z \in V_n \cap \Delta_1;\ (k_z)_{\Delta \backslash V_m} = k_z\}\,.$$

Lemma 3.8. $\mu(A_{n,m}) = 0$.

Proof. By Lemma 3.5,

$$H_{\chi_{A_{n,m}}}(x) = \int_{A_{n,m}} k_z(x)d\mu(z) \leq \int (k_z)_{\Delta \backslash V_m}(x)d\mu(z) = (h_0)_{\Delta \backslash V_m}(x) = H_{\chi_{\Delta \backslash V_m}}(x)\,.$$

On the other hand,

$$H_{\chi_{A_{n,m}}}(x) = \int_{A_{n,m}} k_z(x)d\mu(z) \leq \int_{V_n} k_z(x)d\mu(z) = \int k_z(x)\chi_{V_n}(z)d\mu(z) = H_{\chi_{V_n}}(x)\,.$$

Hence

$$H_{\chi_{A_{n,m}}}(x) \leq (H_{\chi_{\Delta \backslash V_m}} \wedge H_{\chi_{V_n}})(x) = H_{\min(\chi_{\Delta \backslash V_m}, \chi_{V_n})}(x) = 0\,,$$

i.e., $A_{n,m}$ is negligible and therefore $\mu(A_{n,m}) = 0$ (§1, 2).

Now we shall prove the theorem. We set $N = \bigcup A_{n,m}$, where the union covers all pairs (n, m). Then, by Lemma 3.8, $\mu(N)=0$. For $z\in\Delta_1\backslash N$, let $U(z)$ be a neighborhood of z in X^*. Consider a pair (n, m) such that $z\in V_n\subset\bar{V}_n\subset V_m\subset\bar{V}_m\subset U(z)\cap\Delta$. $z\notin A_{n,m}$ implies that $(k_z)_{\Delta\backslash V_m}=0$, which means also that there is an open set G such that $\Delta\backslash V_m\subset G$ and $R_{k_z}^{G\cap X}$ is a potential. Since the set $K=X\backslash[G\cup U(z)]$ is a compact subset of X, $\hat{R}_{k_z}^K$ is also a potential. From $X\backslash U(z)\subset K\cup G$ we deduce that $\hat{R}_{k_z}^{X\backslash U(z)}\leq\hat{R}_{k_z}^{K\cup(G\cap X)}\leq\hat{R}_{k_z}^K+R_{k_z}^{G\cap X}$, and hence $X\backslash U(z)$ is thin at z, i.e., $U(z)\cap X\in\mathcal{G}_z$, q.e.d..

4. The Dirichlet problem associated with the fine filters

We define $\Delta_2=\{z\in\Delta_1;\ \mathcal{G}_z$ *converges to* $z\}$. By Theorem 3.3 we have $\mu(\Delta_1\backslash\Delta_2)=0$.

Following the idea of L. Naïm ([12]), we shall consider the Dirichlet problem associated with fine filters $\mathcal{G}_z$. The limits by the fine filters $\mathcal{G}_z$ will be denoted by lim sup$_{\mathcal{G}_z}$ etc.

Proposition 4.1 ([12] Théorème 22). *Let w be an upper bounded hypoharmonic function on X and let T be a subset of Δ with $\underline{H}_{\chi_T}=0$. If for every $z\in\Delta_2\backslash T$ there exists a set $E_z\subset X$ which is not thin at z and for which* $\limsup_{x\to z, x\in E_z} w(x)\leq 0$ *then $w\leq 0$.*

Proof. $w^+=\max(w, 0)$ is an upper bounded, non-negative and subharmonic on X. Let h be the least harmonic majorant of w^+. We have a representation of h:

$$h(x)=\int k(x, z)f(z)d\mu(z)\,,$$

where $f\in L^\infty(d\mu)$. For $\varepsilon>0$ we set $B_\varepsilon=\{x\in X;\ w^+\leq\varepsilon\}$. B_ε is not thin at $z\in\Delta_2\backslash T$. We have

$$\hat{R}_h^{B_\varepsilon}(x)=\int\hat{R}_{k_z}^{B_\varepsilon}(x)f(z)d\mu(z)=\int k_z(x)f(z)d\mu(z)=h(x)\,.$$

In fact, let $\Sigma=\{z\in\Delta_1;\ B_\varepsilon$ *is not thin at* $z\}$. By Lemma 3.4, Σ is μ-measurable, and $\Delta_2\backslash T\subset\Sigma$ thus $\Delta_2\backslash\Sigma\subset T$. Since $\Delta_2\backslash\Sigma$ is μ-measurable, $\chi_{\Delta_2\backslash\Sigma}$ is μ-integrable and is resolutive (§1, 3). $H_{\chi_{\Delta_2\backslash\Sigma}}\leq H_{\chi_T}$ implies $\mu(\Delta_2\backslash\Sigma)=0$. Therefore

$$\int\hat{R}_{k_z}^{B_\varepsilon}(x)f(z)d\mu(z)=\int_\Sigma\hat{R}_{k_z}^{B_\varepsilon}(x)f(z)d\mu(z)=\int_\Sigma k(x, z)f(z)d\mu(z)=\int k_zfd\mu\,.$$

In view of the Riesz decomposition $w^+=h-p$, $h\leq\varepsilon+p$ on B_ε. Hence $h=R_h^{B_\varepsilon}\leq\varepsilon+p$ on X. Since ε is arbitrary, $h\leq p$ and hence $h=0$, which means $w^+=0$ i.e., $w\leq 0$.

REMARK. In the same way we have

Proposition 4.1′. *Let w be an upper bounded hypoharmonic function on X and let T be a subset of Δ such that $H_{\chi_T}=0$. If* $\limsup_{\mathcal{G}_z} w\leq 0$ *for every $z\in\Delta_2\setminus T$ then $w\leq 0$.*

As an application of the above maximum principle, we have

Theorem 4.2 ([12] Théorème 23). *Let f be a numerical function on Δ such that $\underline{H}_f$ is harmonic, the defining family of $\underline{H}_f$ contains subharmonic functions, and let T be a subset of Δ satisfying $\mu(T)=0$ and $\Delta\setminus T\subset\Delta_2$. Let $\underline{\mathcal{V}}_f$ be the family of all functions w each of which is upper bounded and hypoharmonic on X and such that for every $z\in\Delta_2\setminus T$ there exists a set E_z which is not thin at z and for which* $\limsup_{\substack{x\to z\\ x\in E_z}} w(x)\leq f(z)$. *Then $\underline{H}_f(x)=\sup\{w(x);\, w\in\underline{\mathcal{V}}_f\}$.*

To prove the theorem we prepare the following lemma.

Lemma 4.3. *Let f be a numerical function on Δ. If $\underline{H}_f$ is harmonic and the defining family of $\underline{H}_f$ contains subharmonic functions, then there exists a resolutive Borel function φ on Δ satisfying $f\geq\varphi$ and $H_\varphi=\underline{H}_f$.*

Proof. We first remark that

$$\underline{H}_f(x)=\sup\{w(x);\ w \text{ is upper bounded, subharmonic, } \limsup w\leq f \text{ on } \Delta\}.$$

Let $\{a_n\}$ be a countable dense subset of X, and let w_n be an upper bounded subharmonic function on X such that $\varphi_n:=\limsup w_n\leq f$ on Δ and $\underline{H}_f(a_m)-1/n<w_n(a_m)$ for $m=1, 2, \cdots, n$. We may take $w_{n-1}\leq w_n$. Since φ_n is upper semicontinuous on Δ, we can find a decreasing sequence $\{g_k\}$ of continuous functions on Δ with $\lim_{k\to\infty} g_k=\varphi_n$. To show that φ_n is resolutive it is enough to see that $\{H_{g_k}\}$ is locally bounded from below. This is true, because $w_n\leq H_{g_k}$ for all k. Thus φ_n is resolutive and $\lim_{k\to\infty} H_{g_k}=H_{\varphi_n}$. The function $\varphi=\lim_{n\to\infty}\varphi_n$ is a resolutive Borel function and $\varphi\leq f$. Since $\underline{H}_f(a_m)-1/n<w_n(a_m)\leq H_{\varphi_n}(a_m)\leq H_\varphi(a_m)\leq\underline{H}_f(a_m)$ for $n\geq m$ we have $\underline{H}_f(a_m)=H_\varphi(a_m)$, and since both $\underline{H}_f$ and H_φ are harmonic we have $\underline{H}_f=H_\varphi$.

Now we proceed to the proof of Theorem 4.2. Clearly $\underline{H}_f\leq\sup\underline{\mathcal{V}}_f$. We prove the converse inequality. By our assumption and by the above lemma, there exists a resolutive Borel function φ on Δ satisfying $\varphi\leq f$ and $H_\varphi=\underline{H}_f$. For every $\eta>0$, we define $\Sigma_\eta=\{z\in\Delta_2;\, f(z)-\varphi(z)\geq\eta\}$. Then $\eta\underline{H}_{\chi_{\Sigma_\eta}}\leq\underline{H}_{f-\varphi}=\underline{H}_f-H_\varphi=0$ implies $\underline{H}_{\chi_{\Sigma_\eta}}=0$. Let v be lower bounded and hyperharmonic on X satisfying $\liminf v\geq\varphi+\eta$ on Δ, and let $w\in\underline{\mathcal{V}}_f$. In Proposition 4.1 if we consider $(T\cap\Delta_2)\cup\Sigma_\eta$ instead of T then the function $w-v$ satisfies the condition of Proposition 4.1. Thus $w \leq v$ and hence $w \leq H_{\varphi+\eta}$ which implies $\sup \underline{\mathcal{V}}_f \leq H_\varphi + \eta H_1 \leq H_\varphi + \eta$. Since η is arbitrary, $\sup \underline{\mathcal{V}}_f \leq H_\varphi = \underline{H}_f$.

REMARK. In the same way, we have

Theorem 4.2′. *Let f be a numerical function on Δ with $\underline{H}_f$ is harmonic, and the defining family of $\underline{H}_f$ contains subharmonic functions and let T be a subset of Δ satisfying $\mu(T)=0$ and $\Delta \backslash T \subset \Delta_2$. Let*

$$\underline{\mathcal{V}}_f^F = \left\{ w;\ \begin{array}{l} \text{upper bounded and hypoharmonic on } X, \\ \limsup_{\mathcal{G}_z} w(x) \leq f(z) \text{ for every } z \in \Delta_2 \backslash T \end{array} \right\}$$

Then $\underline{H}_f = \sup \underline{\mathcal{V}}_f^F$.

Theorem 4.4. *If f is non-negatiev and resolutive, then $\lim_{\mathcal{G}_z} H_f = f(z)$ μ-a.e. on Δ.*

The idea of the proof is due to Naïm ([13]).
Let

$$g(z) = \begin{cases} \limsup_{\mathcal{G}_z} H_f & \text{if } z \in \Delta_2 \\ 0 & \text{if } z \in \Delta \backslash \Delta_2 . \end{cases}$$

Then we shall prove that $\overline{H}_g$ is harmonic and $\overline{H}_g \leq H_f$. We set $g_n = \min(g, n)$. Since $\overline{H}_{g_n}$ is harmonic and $\lim_{n\to\infty} \overline{H}_{g_n} = \overline{H}_g$, it is sufficient to prove $\overline{H}_{g_n} \leq H_f$. By Lemma 4.3, there exists a resolutive Borel function ψ on Δ with $\psi \geq g_n$, $H_\psi = \overline{H}_{g_n}$. Let w (resp. v) be an upper (resp. a lower) bounded hypoharmonic (resp. hyperharmonic) function on X satisfying $\limsup w \leq \psi - 2\eta$ $(\eta > 0)$ (resp. $\liminf v \geq f$) on Δ, $v \geq H_f$ implies that $\limsup_{\mathcal{G}_z} v \geq g(z) > g_n(z) - \eta$ on Δ_2. Hence, if we define $E_z = \{x \in X;\ v(x) > g_n(z) - \eta\}$ for every $z \in \Delta_2$, then E_z is not thin at z and $\liminf_{\substack{x\to z \\ x \in E_z}} v(x) \geq g_n(z) - \eta$. Letting $\Sigma_\eta = \{z \in \Delta_2;\ \psi(z) - g_n(z) \geq \eta\}$, we have $\underline{H}_{\chi_{\Sigma_\eta}} = 0$ and $\limsup_{\substack{x\to z \\ x \in E_z}} [w(x) - v(x)] \leq 0$ for every $z \in \Delta_2 \backslash \Sigma_\eta$. Thus, by Proposition 4.1, $w - v \leq 0$ and hence $H_\psi - 2\eta \leq H_{\psi - 2\eta} \leq H_f$, which implies $\overline{H}_{g_n} = H_\psi \leq H_f$ since η is arbitrary.

Analogously, for

$$g'(z) = \begin{cases} \liminf_{\mathcal{G}_z} H_f & \text{if } z \in \Delta_2 \\ 0 & \text{if } z \in \Delta \backslash \Delta_2 \end{cases}$$

we have $\underline{H}_{g'}$ is harmonic and $H_f \leq \underline{H}_{g'}$. The inequalities

$$H_f \leq \underline{H}_{g'} \leq \overline{H}_{g'}, \quad \underline{H}_g \leq \overline{H}_g \leq H_f$$

mean that both g and g' are resolutive and $H_g=H_{g'}=H_f$. And we have $g=g'=f\ d\lambda_x$-a.e. on Δ for every $x\in X$ and finally $g=g'=f$ μ-a.e. on Δ (§1, 2).

5. Fatou-Doob-Naïm theorem

The aim of this section is to establish the Fatou-Doob-Naïm theorem for compactifications of Martin type, that is, every superharmonic function has a finite fine limit (a limit along the fine filters $\mathcal{G}_z$) at μ-almost every boundary point.

Let h be a non-negative harmonic function on X. h is called quasi-bounded (resp. singular) if there exists an increasing sequence h_n in $HB^+(X)$ with $\lim_{n\to\infty} h_n=h$ (resp. $\inf(h, \alpha H_1)$ is a potential for every $\alpha>0$), i.e., h is quasi-bounded if and only if $h\in MHB^+(X)$.

Lemma 5.1. *Let h be a non-negative harmonic function on X. h is quasi-bounded if and only if $h=\sup_n(h\wedge nh_0)$, where $h_0=H_1$.*

It is enough to prove the "only if" part. Let $\{h_n\}$ be an increasing sequence in $HB^+(X)$ with $\lim_{n\to\infty} h_n=h$. We may suppose that $h_n(x)=H_{f_n}(x)$ for an increasing sequence $\{f_n\}$ in $L^1(d\mu)$. The function $f=\lim_{n\to\infty} f_n$ is resolutive and $H_f=\lim_{n\to\infty} H_{f_n}=h$. We set $g_n=\min(f, n)$. Then $h\wedge nH_1=H_{\min(f,n)}=H_{g_n}$ and $\sup_n(h\wedge nh_0)=H_{\sup g_n}=H_f=h$.

For $h\in H^+(X)$ we define

$$h^q=\sup_n(h\wedge nh_0)\,.$$

Then as in [15], we can prove

Lemma 5.2. *Let h be a non-negative harmonic function.*

(1) *h is quasi-bounded if and only if $h^q=h$,*

(2) *h is singular if and only if $h^q=0$,*

(3) *h is decomposed into the sum of quasi-bounded and singular harmonic functions.*

Proposition 5.3. *Every potential has a fine limit 0 for μ-almost every point of Δ.*

Proof. Let p be a potential, $\varepsilon>0$ and $E_\varepsilon=\{x\in X; p(x)>\varepsilon h_0(x)\}$, where $h_0=H_1$. Since $R_{h_0}^{E_\varepsilon}\leq p/\varepsilon$, $\hat{R}_{h_0}^{E_\varepsilon}$ is a potential. Letting $\Sigma_\varepsilon=\{z\in\Delta_1;\ E_\varepsilon \textit{ is not thin at } z\}$, we have $\mu(\Sigma_\varepsilon)=0$. For, $\hat{R}_{h_0}^{E_\varepsilon}(x)=\int \hat{R}_{k_z}^{E_\varepsilon}(x)d\mu(z)=\int_{\Delta\setminus\Sigma_0}\hat{R}_{k_z}^{E_\varepsilon}(x)d\mu(z)+\int_{\Sigma_\varepsilon} k_z(x)d\mu(z)$ implies that $\int_{\Sigma_\varepsilon} k_z(x)d\mu(z)=0$ for every $x\in X$, i.e., Σ_ε is negligible, thus, $\mu(\Sigma_\varepsilon)=0$. For every $z\in\Delta_2\setminus\Sigma_\varepsilon$, if $E\in\mathcal{G}_z$ then $E\cap(X\setminus E_\varepsilon)=E\setminus E_\varepsilon\in\mathcal{G}_z$;

therefore $\limsup_{\mathcal{G}_z} p = \inf_{E\in\mathcal{G}_z}[\sup_{x\in E\setminus E_\varepsilon} p(x)] \leq \inf_{E\in\mathcal{G}_z}[\sup_{x\in E\setminus E_\varepsilon} h_0(x)] = \varepsilon \limsup_{\mathcal{G}_z} h = \varepsilon$ μ-a.e. on Δ_2.

Corollary 5.4. *Let $h\in H^+(X)$ be singular. Then* $\lim_{\mathcal{G}_z} h=0$ μ-a.e. *on* Δ.

In fact, we set $E_\varepsilon=\{x\in X; h(x)>\varepsilon h_0(x)\}$ for $\varepsilon>0$. By assumption, $p_\varepsilon=\min(h, \varepsilon h_0)$ is a potential. As before E_ε is thin μ-a.e. on Δ, and we conclude $\limsup_{\mathcal{G}_z} h\leq\varepsilon$ for μ-almost every z.

Finally we have

Theorem 5.5 (Fatou-Doob-Naïm). *Every non-negative superharmonic function on X has a finite fine limit for μ-almost every point of Δ.*

Proof. Let s be non-negative and superharmonic. We decompose s as $s=p+h^q+h^s$, where p is a potential and h is the greatest harmonic minorant of s ($h=h^q+h^s$). By the result of §1.3, $h^q=H_f$ for a resolutive function $f\geq 0$. By Proposition 5.3 and Corollary 5.4, p and h^s has a fine limit 0 for μ-almost all points. On the other hand, by Theorem 4.4, h^q has a fine limit f for μ-almost all points and since f is $d\lambda_x$-integrable for every $x\in X$, f is finite μ-almost everywhere.

6. The space $\mathcal{H}^\Phi$

Let X^* be of Martin type in which μ is a dilation of a normalized reference measure τ, i.e., τ is a probability measure on X such that the smallest absorbent set containing the support of τ is X and $\mu=\int\lambda_x d\tau(x)$, and let positive constant be harmonic.

As in [9], [14], we consider a function Φ defined on $\boldsymbol{R}^+$ which is strictly increasing, convex, and satisfying $\Phi(0)=0$ and $\lim_{t\to\infty}\Phi(t)/t=\infty$.

We define, in terms of Φ, the following spaces:

$\mathcal{H}^\Phi=\{u\in H(X);\ \Phi(|u|)$ *has a τ-integrable harmonic majorant*$\}$,

$\mathcal{H}^\Phi_m=\{u\in H(X);\ au\in\mathcal{H}^\Phi$ *for some* $a>0\}$,

$L^\Phi_m(d\mu)=\{f;$ *μ-measurable*, $\Phi(a|f|)\in L^1(d\mu)$ *for some* $a>0\}$.

It is known that $\mathcal{H}^\Phi_m$ (*resp.* $L^\Phi_m(d\mu)$) is a Banach space with the norm $\|u\|_\Phi=\inf\{1/k;\ k>0, \int LHM\,\Phi(k|u|)d\tau\leq 1\}$ (*resp.* $\|f\|_\Phi=\inf\{1/k;\ k>0, \int\Phi(k|f|)d\mu\leq 1\}$), where LHM denotes the least harmonic majorant.

Following the idea of Janssen [9], we can obtain the following results for which we shall give proofs for completeness.

Lemma 6.1 ([9], Proposition 4.9). *Every function of $\mathcal{H}^\Phi$ is the difference of two non-negative functions of $\mathcal{H}^\Phi$.*

Proof. Let $\{X_n\}$ be a compact exhaustion of X, i.e., X_n is a relatively compact open set, $\bar{X}_n \subset X_{n+1}$ and $\bigcup_{n=1}^{\infty} X_n = X$, and let $\lambda_x^{X_n}$ be the harmonic measure of x with respect to $\bar{X}_n$. Consider $u \in \mathcal{H}^\Phi$ and suppose that v is a τ-integrable harmonic majorant of $\Phi(|u|)$. By assumption, there is a number $c_0>0$ such that $t \leq \Phi(t)+c_0$ for every $t \in \boldsymbol{R}^+$. Thus

$$u^+ = \max(u, 0) \leq |u| \leq \Phi(|u|)) + c_0 \leq v + c_0 .$$

In view of the subharmonicity of u^+ and the harmonicity of $v+c_0$, we see that $\int u^+ d\lambda_x^{X_n}$ is increasing and is bounded above by $v(x)+c_0$, which induces that $u_1(x) = \lim_{n\to\infty} \int u^+ d\lambda_x^{X_n} \in H^+(X)$. Similarly we have $u_2(x) = \lim_{n\to\infty} \int \max(-u, 0) d\lambda_x^{X_n} \in H^+(X)$, and since

$$\int \max(u, 0) d\lambda_x^{X_n} - \int \max(-u, 0) d\lambda_x^{X_n} = \int u d\lambda_x^{X_n} = u(x) ,$$

we have $u = u_1 - u_2$. $u_1 \in \mathcal{H}^\Phi$ is derived from

$$\Phi(\int u^+ d\lambda_x^{X_n}) \leq \int \Phi(u^+) d\lambda_n^{X_n} \leq \int \Phi(|u|) d\lambda_x^{X_n} \leq v(x) ,$$

since $\int d\lambda_x^{X_n} = 1$.

Lemma 6.2. *Every non-negative function of $\mathcal{H}^\Phi$ is quasi-bounded.*

Proof. Let $u \in \mathcal{H}^\Phi$, $u \geq 0$. First of all, we show that for every $\varepsilon>0$ there is $a_0>0$ such that $\sup_n \int_{\bar{X}_n} [\int_{[u\geq a]} u d\lambda_x^{X_{n+1}}] d\tau(x) \leq \varepsilon$ whenever $a>a_0$. For,

$$\int_{\bar{X}_n} [\int \Phi(u) d\lambda_x^{X_{n+1}}] d\tau \leq \int_{\bar{X}_n} v(x) d\tau(x) \leq \int_X v d\tau$$

implies that $M = \sup_n \int_{\bar{X}_n} [\int \Phi(u) d\lambda_x^{X_{n+1}}] d\tau < \infty$. Given $\varepsilon>0$ we find a_0 such that $\Phi(t)/t \geq M/\varepsilon$ for every $t \geq a_0$. This means that

$$\int_{\bar{X}_n} [\int_{[u\geq a_0]} u d\lambda_x^{X_{n+1}}] d\tau(x) \leq (\varepsilon/M) \int_{\bar{X}_n} [\int_{[u\geq a_0]} \Phi(u) d\lambda_x^{X_{n+1}}] d\tau \leq \varepsilon$$

for every n.

Next, we see that for every $\varepsilon>0$ there is $w \in HB^+(X)$ such that $w \leq u$ and $\int (u-w) d\tau < \varepsilon$. In fact, let $a>0$ be a number satisfying

$$\sup_n \int_{\bar{X}_n} [\int_{[u\geq a]} u d\lambda_x^{X_{n+1}}] d\tau(x) \leq \varepsilon .$$

$u_a=\max(-u, -a)$ is subharmonic and

$$0\leq\int u_a d\lambda_x^{X_{n+1}}+u(x)\leq\int_{[u\geq a]}(u_a+u)\,d\lambda_x^{X_{n+1}}\leq\int_{[u\geq a]}ud\lambda_x^{X_{n+1}}\,.$$

Now, we define $w=-LHM\,u_a$. Since $-u\leq u_a\leq 0$, $0\leq w\leq -u_a\leq u$. For every compact subset K of X we have

$$\begin{aligned}\int_K(u-w)d\tau &= \int_K(u+LHM\,u_a)d\tau=\int_K\lim_{n\to\infty}[u(x)+\int u_a d\lambda_x^{X_{n+1}}]\,d\tau\\ &=\lim_{n\to\infty}\int_K[u(x)+\int u_a d\lambda_x^{X_{n+1}}]d\tau\leq\overline{\lim_{n\to\infty}}\int_{\overline{X}_n}[u(x)+\int u_a d\lambda_x^{X_{n+1}}]\,d\tau\\ &\leq\overline{\lim_{n\to\infty}}\int_{\overline{X}_n}[\int_{[u\geq a]}ud\lambda_x^{X_{n+1}}]d\tau\leq\varepsilon\,.\end{aligned}$$

Therefore $\int(u-w)d\tau\leq\varepsilon$.

Finally, letting u_n the greatest harmonic minorant of $\min(u, n)$ and defining $h=\lim_{n\to\infty}u_n$ we are going to show $u=h$. By the above argument we have $w_n\in HB^+(X)$ such that $w_n\leq\min(u, a_n)$ for some $a_n>0$ and $\int(u-w_n)d\tau\leq 1/n$. Then $w_n\leq h\leq u$ so that $\int(u-h)\,d\tau\leq\int(u-w_n)d\tau\leq 1/n$, which implies $\int(u-h)d\tau=0$. Since the set $\{x\in X;\ u(x)-h(x)=0\}$ is an absorbent set containing the support of τ, $u=h$ on X.

Proposition 6.3. *For every $u\in\mathcal{H}^\Phi$ there exist $f_i\in L^1(d\mu)$ such that $f_i\geq 0$, $H_{f_i}\in\mathcal{H}^\Phi$ $(i=1, 2)$ and $u=H_{f_1}-H_{f_2}$.*

The only thing to prove is that every non-negative $u\in\mathcal{H}^\Phi$ is the Dirichlet solution H_f with $f\in L^1(d\mu)$. This is seen from $u=H_f$ for resolutive $f\geq 0$ (§ 1,3) and $\int fd\mu=\int\lambda_x(f)d\tau(x)=\int ud\tau<\infty$.

Theorem 6.4 ([9], Theorem 5.5). *There is a linear bijection of $L_m^\Phi(d\mu)$ and $\mathcal{H}_m^\Phi$ which is isometric.*

Proof. Let $f\in L_m^\Phi(d\mu)$. Then $f\in L^1(d\mu)$ since $a|f|\leq\Phi(a|f|)+c_0$ for some $c_0>0$ and $a>0$ with $\Phi(a|f|)\in L^1(d\mu)$. By the result of § 1,3, $u(x):=H_f(x)=\int fk^x d\mu\in H(X)$. We assert that $u\in\mathcal{H}_m^\Phi$; in fact, since $\int k^x d\mu=H_1(x)=1$,

$$\Phi(a|u|)(x)=\Phi(|\int afk^x d\mu|)\leq\int\Phi(a|f|)k^x d\mu=H_{\Phi(a|f|)}(x)$$

and

$$\int H_{\Phi(a|f|)}(x)d\tau(x)=\int[\int\Phi(a|f|)k^x d\mu]\,d\tau(x)=\int\Phi(a|f|)d\mu<\infty\,.$$

Thus, we can define a mapping $\mathcal{J}$: $L_m^\Phi(d\mu)\to\mathcal{H}_m^\Phi$ by $\mathcal{J}(f)=H_f$.

Now, we shall prove that $\mathcal{J}$ is a surjection. To this end it is sufficient to show that for every $u\in\mathcal{H}^\Phi\cap H^+(X)$ there is a function f such that $\Phi(|f|)\in L^1(d\mu)$ and $u=H_f$. For, letting $u\in\mathcal{H}_m^\Phi$ we may find $a>0$ with $au\in\mathcal{H}^\Phi$ and, by Lemma 6.1, there are $u_1, u_2\in\mathcal{H}^\Phi\cap H^+(X)$ such that $au=u_1-u_2$. If $u_i=H_{f_i}$ with $\Phi(f_i)\in L^1(d\mu)$ $(i=1, 2)$, then $u=H_{(f_1-f_2)/a}$ and $\Phi((a/2)\cdot(|f_1-f_2|/a))\leq\Phi((|f_1|+|f_2|)/2)\leq(\Phi(|f_1|)+\Phi(|f_2|))/2$, which implies $(f_1-f_2)/a\in L_m^\Phi(d\mu)$.

Let $u\in\mathcal{H}^\Phi\cap H^+(X)$. $u=H_f$ and let $\Phi(u)=h^q+h^s-p$, where p is a potential and h^q(*resp.* h^s) is a quasi-bounded (*resp.* singular) part of $LHM\ \Phi(u)$. By Fatou-Doob-Naïm theorem

$$\lim_{\mathcal{G}_z} h^q = \lim_{\mathcal{G}_z}\Phi(u) = \Phi(\lim_{\mathcal{G}_z} u) = \Phi(f) \qquad \mu\text{-a.e. on } \Delta\,,$$

that is, $\Phi(f)$ is resolutive and $h^q=H_{\Phi(f)}$. The inequality $H_{\Phi(f)}\leq LHM\ \Phi(u)$ and the τ-integrability of $LHM\ \Phi(u)$ implies $\Phi(f)\in L^1(d\mu)$, since $\int\Phi(f)d\mu=\int[\int\Phi(f)d\lambda_x]d\tau(x)=\int H_{\Phi(f)}(x)d\tau(x)<\infty$.

Here, we remark that $LHM\ \Phi(|H_f|)=H_{\Phi(|f|)}$. In fact,

$$\Phi(|H_f|)(x) = \Phi(|\int fk^x d\mu|)\leq\Phi(\int|f|k^x d\mu)\leq\int\Phi(|f|)k^x d\mu = H_{\Phi(|f|)}(x).$$

Thus, $LHM\ \Phi(|H_f|)\leq H_{\Phi(|f|)}$. On the other hand, we know, by Fatou-Doob-Naïm theorem, $H_{\Phi(|f|)}$ is the quasi-bounded part of $LHM\ \Phi(|H_f|)$. Therefore $H_{\Phi(|f|)}\leq LHM\ \Phi(|H_f|)$.

We can immediately prove the isometry from above remark:

$$\begin{aligned}||H_f||_\Phi &= \inf\{1/k;\ k>0,\ \int LHM\ \Phi(k|H_f|)d\tau\leq 1\}\\ &= \inf\{1/k;\ k>0,\ \int H_{\Phi(k|f|)}d\tau\leq 1\}\\ &= \inf\{1/k;\ k>0,\ \int\Phi(k|f|)d\mu\leq 1\}\\ &= ||f||_\Phi\,.\end{aligned}$$

Corollary 6.5. *Let $p>1$. Banach spaces*

$$\mathcal{H}^p = \{u\in H(X);\ \int LHM|u|^p d\tau<\infty\}$$

and

$$L^p(d\mu) = \{f;\ \int|f|^p d\mu<\infty\}$$

are isometric and the mapping $\mathcal{J}_p$: $L^p(d\mu)\to\mathcal{H}^p$ defined by $\mathcal{J}_p(f)=H_f$ gives an isometry.

For, it is easily checked that, for $\Phi(t)=t^p$, $\mathcal{H}_m^\Phi=\mathcal{H}^p$, $L_m^\Phi(d\mu)=L^p(d\mu)$, $||u||_\Phi^p=\int LHM|u|^p d\tau$ and $||f||_\Phi^p=\int |f|^p d\mu$.

REMARK. We give a remark of some importance. In Corollary 6.5 we do not need the assumption that 1 is harmonic while we are assuming its superharmonicity. For general Φ, it is inevitably necessary to ensure the inequalities $\Phi(\int u^+ d\lambda_x^{X_n})\leq\int \Phi(u^+)d\lambda_x^{X_n}$ and $\Phi(\int |f|k^x d\mu)\leq\int\Phi(|f|)k^x d\mu$. However, in the case where $\Phi(t)=t^p$ ($p>1$) we can use Hölder's inequality instead.

Now we shall consider the space $\mathcal{H}^2$. By Corollary 6.5, $\mathcal{H}^2$ is a Hilbert space and is isometric to $L^2(d\mu)$ and the inner product is $(u, v)=\int fg d\mu$, where $u=H_f$ and $v=H_g$ with $f, g\in L^2(d\mu)$.

Let σ be a probability measure on X and

$$\mathcal{M} = \{u\in\mathcal{H}^2; \int u d\sigma = 1\}.$$

We are going to discuss the minimizing problem under the assumption that
1) *μ is a dilation of a normalized reference measure τ,*
2) *1 is harmonic,*
3) *every element of $\mathcal{H}^2$ is σ-integrable,*
4) *$\mathcal{M}$ is a closed subset of $\mathcal{H}^2$.*

The measure σ in the following examples satisfy our assumptions 3) and 4).

EXAMPLE. 1. $\sigma=\varepsilon_x$ ($x\in X$). For, $u_n\in\mathcal{M}$, $u_n\to u$ means that if $u_n=H_{f_n}$ and $u=H_f$ then $\int|f_n-f|^2 d\mu\to 0$ and $\sigma(u_n)=u_n(x)=H_{f_n}(x)=1$. Thus, $|H_{f_n}(x)-H_f(x)|=|\int(f_n-f)k^x d\mu|\leq||f_n-f||(\int[k(x, z)]^2 d\mu(z))^{1/2}=||f_n-f||\times[K(x, x)]^{1/2}\to 0$, where $K(x, x)=H_{k^x}(x)$. Therefore $H_f(x)=1$, which implies that $u(x)=\sigma(u)=1$, i.e., $u\in\mathcal{M}$.

EXAMPLE 2. $\sigma=\tau$. For, if $u_n\in\mathcal{M}$, $u_n\to u$ then, as in Example 1, $|\int(u_n-u)d\sigma|=|\int[\int(f_n-f)d\lambda_x]d\tau(x)|=|\int(f_n-f)d\mu|\to 0$, which means that $\int u d\sigma=1$, i.e., $u\in\mathcal{M}$.

Theorem 6.6. *There exists a unique $u_0\in\mathcal{M}$ such that*

$$||u_0|| = \min\{||u||; u\in\mathcal{M}\}.$$

This u_0 satisfies $\int u d\sigma=(u, u_0/||u_0||^2)$ for every $u\in\mathcal{H}^2$.

Proof. Let $m_0=\inf\{||u||; u\in\mathcal{M}\}$. Then, by the usual argument and by the assumption 4), there is only one $u_0 \in \mathcal{M}$ such that $||u_0||=m_0$. For every $u \in \mathcal{H}^2$ the function $v=u-\sigma(u)\cdot 1 \in \mathcal{H}^2$ and $\int v d\sigma=0$, thus $u_0+\eta v \in \mathcal{M}$ for every η. This implies that $(u_0, v)=0$, since $m_0^2 \leq ||u_0+\eta v||^2=||u_0||^2+2\eta(u_0, v)+\eta^2||v||^2$ for every η. $(u, u_0)=(v+\sigma(u)\cdot 1, u_0)=(v, u_0)+\sigma(u)(1, u_0)=\sigma(u)\tau(u_0)$, for, if $u_0=H_{f_0}$, $(1, u_0)=\int 1\cdot f_0 d\mu=\int [\int f_0 d\lambda_x] d\tau(x)=\int H_{f_0}(x)d\tau(x)=\tau(u_0)$. In particular, $||u_0||^2 =\sigma(u_0)\tau(u_0)=\tau(u_0)$ and $(u, u_0)=\sigma(u)||u_0||^2$; that is, $\sigma(u)=(u, u_0/||u_0||^2)$.

Corollary 6.7. *Let μ be a dilation of a normalized reference measure and* 1 *be harmonic. Then,*

(1) *if $H_{f_0}=u_0$ is the solution of the minimizing problem for $\sigma=\varepsilon_x$ in Theorem* 6.6 *then* $k^x=f_0/||f_0||^2$ μ-a.e.,

(2) $u_0=1$ *is the solution of the minimizing problem for $\sigma=\tau$.*

Indeed, in case (1), $u(x)=H_f(x)=\int fk^x d\mu=(u, K_x)=(u, u_0/||u_0||^2)$ for every $u \in \mathcal{H}^2$, where $K_x(y)=H_{k^x}(y)$. This means that $H_{k^x}(y)=K_x(y)=u_0(y)/||u_0||^2= H_{f_0}(y)/||f_0||^2$ and hence $k^x=f_0/||f_0||^2$. In the second case $(u, u_0)=\tau(u)\tau(u_0)$ implies $||u_0||=\tau(u_0)=1$. On the other hand, $\tau(K_x)=\int K_x d\tau=\int [\int k^x d\lambda_y] d\tau(y)= \int k^x d\mu=H_1(x)=1$ implies $K_x \in \mathcal{M}$ for every $x \in X$ and the result is derived from $u_0(x)=(K_x, u_0)=\int K_x d\tau=1$.

Remark. The kernel function k^x is proportional to the solution of the minimizing problem: $\inf\{||f||; f \in L^2(d\mu), \lambda_x(f)=1\}$, and the function $K(x, y) =\int k(x, z)k(y, z)d\mu(z)$ is the reproducing kernel of the Hilbert space $\mathcal{H}^2$; for $(K_x, u)=\int fk^x d\mu=u(x)$ if $u(y)=H_f(y)=\int fk^y d\mu$.

7. Poles

Let $\hat{X}$ be an arbitrary metrizable and resolutive compactification. Dirichlet solutions considered in $\hat{X}$ are denoted by $H^{\hat{X}}$.

We define, for $u \in H^+(X)$ and $A' \subset \Delta' := \hat{X} \setminus X$

$$u_{\Delta'}(x) = \inf\{R_u^{U' \cap X}(x);\ U' \text{ is open in } \hat{X},\ A' \dot{\subset} U'\}.$$

If $u=k_z$ with $z \in \Delta_1$, then $u_{A'}$ is either 0 or k_z. By the compactness of Δ' there exists at least one point $\zeta \in \Delta'$ such that $(k_z)_{\{\zeta\}}=k_z$. The point $\zeta \in \Delta'$ is termed the *unique pole* of k_z $(z \in \Delta_1)$ if

$$(k_z)_{\{\zeta'\}} = \begin{cases} k_z & \text{if } \zeta'=\zeta \\ 0 & \text{if } \zeta' \in \Delta' \setminus \{\zeta\}. \end{cases}$$

In the sequel, we use the following notations:

$$\Delta_3 = \{z \in \Delta_1;\ k_z \text{ has a unique pole}\},$$
$$\Psi\colon \Delta_3 \to \Delta',\ \Psi(z) \text{ is the unique pole of } k_z.$$

The notion of poles was introduced by M. Brelot and developed by L. Naïm [12].

Let $h_0 = H_1^{\hat{X}}$, then $h_0(x) = H_1(x) = \int k^x d\mu$. We have, as in Lemma 3.6,

Proposition 7.1. $(h_0)_{A'} = \bar{H}_{\chi_{A'}}^{\hat{X}}$, *for every* $A' \subset \Delta'$.

Proposition 7.2. $[(h_0)_{A'}]_{B'}(x) = \int [(k_z)_{A'}]_{B'}(x) d\mu(z)$ *for every compact* A', $B' \subset \Delta'$.

This is proved from the fact that

$$(h_0)_{A'}(x) = \int (k_z)_{A'}(x) d\mu(z) \text{ for every compact } A' \subset \Delta'.$$

We have also

Proposition 7.3. *If* A' *and* B' *are disjoint compact subsets of* Δ' *then* $[(h_0)_{A'}]_{B'} = 0$.

Finally, we have

Proposition 7.4. $\mu(\Delta \backslash \Delta_3) = 0$. *In other words,* $\mathcal{G}_z$ *converges to a single point* $\Psi(z)$ *for* μ*-almost every* z.

Denoting by $\mathcal{B}(\Delta)$ (*resp.* $\mathcal{B}(\Delta')$) the Borel family on Δ (*resp.* Δ') and by $\overline{\mathcal{B}(\Delta)}^{\mu}$ the completion of $\mathcal{B}(\Delta)$ by the Borel measure μ, i.e., the σ-algebra of μ-measurable subsets of Δ, we have

Proposition 7.5. Ψ *is* $\overline{\mathcal{B}(\Delta)}^{\mu} - \mathcal{B}(\Delta')$*-measurable.*

In fact, it is just a consequence of

$$\{z \in \Delta_3;\ \Psi(z) \in A'\} = \bigcap_{n=1}^{\infty} \{z \in \Delta_3;\ G'_n \cap X \text{ is not thin at } z\}, \tag{7.1}$$

where $A' \subset \Delta'$ is compact and $\{G'_n\}$ is a descending sequence of open sets in $\hat{X}$ with $\bigcap_{n=1}^{\infty} G'_n = A'$. We note that each set of the right-hand side of (7.1) is the intersection of Δ_3 with a G_δ-set.

We define a Borel measure $\Psi\mu$ on $\mathcal{B}(\Delta')$ by

$$\Psi\mu(M') := \mu(\Psi^{-1}(M')) \qquad M' \in \mathcal{B}(\Delta').$$

Following [12], we shall establish the relation between Dirichlet solutions which are considered in X^* and $\hat{X}$. To this purpose, we require some lemmas. To simplify the notation we abbreviate in the following χ_A to A; for example, $\underline{H}_{\chi_A}$ is abbreviated to $\underline{H}_A$.

Lemma 7.6. *Let A be an arbitrary subset of Δ. Then $\underline{H}_A=0$ if and only if $H_K=0$ for every compact $K\subset A$.*

It is enough to show the "if" part. Suppose, on the contrary, $\underline{H}_A(x)>0$ for some $x\in X$. Then there is a non-negative hypoharmonic function w such that $\varphi=\overline{\lim}\, w\leq \chi_A$ and $w(x)>0$. φ is upper semi-continuous on Δ and $\lambda_x(\varphi)>0$. We define $K_n=\{z\in\Delta;\ \varphi(z)\geq 1/n\}$. $\{K_n\}$ is an ascending sequence of compact sets with $K_n\subset A$. Since $\min(\chi_{K_n},\varphi)\uparrow\varphi$ as $n\to\infty$, there is an n such that $0<\int\min(\chi_{K_n},\varphi)d\lambda_x\leq\int\chi_{K_n}d\lambda_x=H_{K_n}(x)$.

Lemma 7.7. *Let A' be an arbitrary subset of Δ'. Then $\underline{H}^{\hat{X}}_{A'}=0$ implies $\underline{H}_{\Psi^{-1}(A')}=0$.*

By Lemma 7.6, it is sufficient to show that $H_K=0$ for every compact $K\subset\Psi^{-1}(A')$. Since Ψ is $\overline{\mathscr{B}(\Delta)}^{\mu}-\mathscr{B}(\Delta')$-measurable, by the well-known theorem of Lusin in the measure theory, for every $\varepsilon>0$ there is a compact set $K_1\subset K$ with $\mu(K\backslash K_1)<\varepsilon$ and the restriction of Ψ on K_1 is continuous. $\Psi(K_1)\subset A'$ implies $H^{\hat{X}}_{\Psi(K_1)}=0$, and $H_{K_1}\leq H_{\Psi^{-1}(\Psi(K_1))}\leq H^{\hat{X}}_{\Psi(K_1)}$ means that $\mu(K_1)=0$. Here we used Theorem 4.2′ and the fact that $\liminf_{\mathcal{G}_z} v\geq\liminf_{x\to\Psi(z)} v(x)$ for every $z\in\Delta_3$. Thus $\mu(K)\leq\mu(K_1)+\mu(K\backslash K_1)<\varepsilon$ and, ε being arbitrary, $\mu(K)=0$, i.e., $H_K=0$.

Theorem 7.8. *Let $f'\colon\Delta'\to\overline{\boldsymbol{R}}$ be bounded above and let*

$$f=\begin{cases} f'\circ\Psi & \text{on } \Delta_3 \\ 0 & \text{on } \Delta\backslash\Delta_3\,. \end{cases}$$

If $\underline{H}^{\hat{X}}_{f'}$ and $\underline{H}_f$ are harmonic and their defining families contain subharmonic functions then

$$\underline{H}^{\hat{X}}_{f'}=\underline{H}_f\,.$$

Proof. Let $z\in\Delta_3$ and $\Psi(z)=\zeta$. Since $\mathscr{F}=\{U(\zeta)\cap X;\ U(\zeta)$ is a neighborhood of ζ in $\hat{X}\}\subset\mathcal{G}_z$, $\limsup_{\mathcal{G}_z} w\leq\limsup_{x\to\zeta} w(x)$. By Theorem 4.2′, $\underline{H}_f=\sup\{w;$ upper bounded, hypoharmonic, $\limsup_{\mathcal{G}_z} w\leq f(z)$ for every $z\in\Delta_3\}\geq\sup\{w;$ upper bounded, hypoharmonic, $\limsup_{x\to\zeta} w(x)\leq f'(\zeta)$ for every $\zeta\in\Delta'\}=\underline{H}^{\hat{X}}_{f'}$, i.e., $\underline{H}^{\hat{X}}_{f'}\leq\underline{H}_f$. To prove the converse inequality, let ψ' be a resolutive Borel function on Δ' such that $\psi'\leq f'$ and $\underline{H}^{\hat{X}}_{f'}=H^{\hat{X}}_{\psi'}$. The function ψ' is constructed in the same way as in Lemma 4.3. Let, as before,

$$\psi = \begin{cases} \psi' \circ \Psi & \text{on } \Delta_3 \\ 0 & \text{on } \Delta \backslash \Delta_3 . \end{cases}$$

The function ψ is $\overline{\mathcal{B}(\Delta)}^{\mu}$-measurable and $\psi \leq f$. We proceed as in the proof of Theorem 4.2. We define for $\eta > 0$, $T'_\eta := [f' - \psi' \geq \eta]$. By Lemma 7.7, $\underline{H}_{\Psi^{-1}(T'_\eta)} = 0$ since $\underline{H}^{\hat{X}}_{T'_\eta} = 0$. Letting w be an upper bounded, hypoharmonic function with $\lim\sup_{\mathcal{G}_z} w \leq f(z)$ for every $z \in \Delta_2$ and v be a lower bounded, hyperharmonic function with $\liminf_{x \to z} v(x) \geq \psi(z) + \eta$ for every $z \in \Delta$, we consider $w - v$. This is upper bounded, hypoharmonic and on Δ_2,

$$\begin{aligned} \lim\sup_{\mathcal{G}_z}(w-v) &\leq \lim\sup_{\mathcal{G}_z} w - \lim\inf_{\mathcal{G}_z} v \\ &\leq \lim\sup_{\mathcal{G}_z} w - \liminf_{x \to z} v(x) \\ &\leq f(z) - (\psi(z) + \eta) \end{aligned}$$

Therefore $\lim\sup_{\mathcal{G}_z}(w-v) \leq 0$ on $(\Delta_2 \backslash \Psi^{-1}(T'_\eta)) \cap \Delta_3$. By Proposition 4.1′, we have $w \leq v$ and hence $\underline{H}_f \leq \bar{H}_\psi + \eta$ by Theorem 4.2′. Since $\eta > 0$ is arbitrary, $\underline{H}_f \geq \bar{H}_\psi$. By assumption, $\underline{H}_f$ is harmonic, and since ψ is upper bounded the defining family of $\bar{H}_\psi$ contains superharmonic function, from what we have proved at the begining of our proof, we have $\bar{H}_\psi \leq \bar{H}^{\hat{X}}_{\psi'}$ and $\bar{H}^{\hat{X}}_{\psi'} = H^{\hat{X}}_{\psi'} = H^{\hat{X}}_{f'}$, i.e., $\underline{H}_f \leq H^{\hat{X}}_{f'}$.

Corollary 7.9. *If μ is a dilation of a normalized reference measure τ then $\Psi\mu = \int \lambda'_x d\tau(x)$, where λ'_x is the harmonic measure for $\hat{X}$.*

Indeed, since $f' \circ \psi$ is resolutive for every $f' \in \boldsymbol{C}(\Delta')$

$$\begin{aligned} \Psi\mu(f') = \mu(f' \circ \Psi) &= \int \lambda_x(f' \circ \Psi) d\tau(x) = \int H_{f' \circ \Psi}(x) d\tau(x) \\ &= \int H^{\hat{X}}_{f'}(x) d\tau(x) = \int \lambda'_x(f') d\tau(x) . \end{aligned}$$

We define

$$\mathcal{M}' = \{A' \subset \Delta' ; \Psi^{-1}(A') \in \overline{\mathcal{B}(\Delta)}^{\mu}\}$$

and

$$\mu'(A') = \mu(\Psi^{-1}(A')) \qquad \text{for} \quad A' \in \mathcal{M}' .$$

$\mathcal{M}'$ is a σ-algebra and μ' is a measure on $\mathcal{M}'$ and it coincides with $\Psi\mu$ on $\mathcal{B}(\Delta')$.

Now, we shall prove the following theorem:

Theorem 7.10. *If every bounded harmonic function is the Dirichlet solution on $\hat{X}$, i.e., $HB(X) \subset \{H^{\hat{X}}_{f'} ; f'$ is resolutive on $\Delta'\}$, then there are T and T' with*

$\Delta\backslash T\subset\Gamma$, $\Delta'\backslash T'\subset\Gamma'$ and $\mu(T)=\mu'(T')=0$, such that Ψ is a bijection between $\Delta\backslash T$ and $\Delta'\backslash T'$, where Γ and Γ' are the harmonic boundaris of X^* and $\hat{X}$ respectively.

Before proving the theorem, we prepare two lemmas:

Lemma 7.11 ([6], Lemma 3). *Under the condition of Theorem 7.10, for every $A\in\overline{\mathcal{B}(\Delta)}^{\mu}$ there is $A'\in\mathcal{M}'$ such that*

1) $H_A=H^{\hat{X}}_{A'}$,
2) $A'\subset\Psi(A\cap\Delta_3)$,
3) $\mu(A\backslash\Psi^{-1}(A'))=0$.

By assumption, there is a resolutive function f' on Δ' with $H_A=H^{\hat{X}}_{f'}$. We have then $0=H_A\wedge H_{\Delta\backslash A}=H^{\hat{X}}_{\min(f',1-f')}$, which implies that $f'=1$ or 0 λ'_x-a.e. on Δ' for every $x\in X$. Let $A'_1=\{\zeta\in\Delta';f'(\zeta)=1\}$. Then $A'=A'_1\cap\Psi(A\cap\Delta_3)$ fulfils the conditions of the lemma. For, $\bar{H}^{\hat{X}}_{A'_1\backslash\Psi(A\cap\Delta_3)}\leq H^{\hat{X}}_{A'_1}=H^{\hat{X}}_{f}=H_A$ and, by Theorem 7.8, $\bar{H}^{X}_{A'_1\backslash\Psi(A\cap\Delta_3)}\leq\bar{H}^{\hat{X}}_{\Delta'\backslash\Psi(A\cap\Delta_3)}=\bar{H}_{\Delta\backslash\Psi^{-1}(\Psi(A\cap\Delta_3))}\leq H_{\Delta\backslash A}$, which implies $\bar{H}^{\hat{X}}_{A'_1\backslash\Psi(A\cap\Delta_3)}\leq H_A\wedge H_{\Delta\backslash A}=0$. Thus, $A'=A'_1\backslash(A'_1\backslash\Psi(A\cap\Delta_3))\in\mathcal{M}'$ and $H^{\hat{X}}_{A'_1}=H^{\hat{X}}_{A'}$. Hence A' satisfies conditions 1) and 2). Next, since A' is the symmetric difference of a Borel set A'_0 and a μ'-null set N', $\Psi^{-1}(A')$ is the symmetric difference of $\Psi^{-1}(A'_0)\in\overline{\mathcal{B}(\Delta)}^{\mu}$ and a μ-null set $\Psi^{-1}(N')$. Hence $\Psi^{-1}(A')\in\overline{\mathcal{B}(\Delta)}^{\mu}$ and $A\backslash\Psi^{-1}(A')\in\overline{\mathcal{B}(\Delta)}^{\mu}$. From what we have proved above, there is a resolutive set $B'\in\mathcal{M}'$ such that $H_{A\backslash\Psi^{-1}(A')}=H^{\hat{X}}_{B'}$ and $B'\subset\Psi(\Delta_3\cap(A\backslash\Psi^{-1}(A')))$. Thus, $H_{A\backslash\Psi^{-1}(A')}=0$, since $H_{A\backslash\Psi^{-1}(A')}\leq H^{\hat{X}}_{A'}\wedge H^{\hat{X}}_{B'}$ and $A'\cap B'=\emptyset$. This proves 3).

Lemma 7.12. *Under the same assumption of Theorem 7.10, $A'\subset\Delta'$ is negligible if and only if $A'\in\mathcal{M}'$ and $\mu'(A')=0$.*

For, if $\bar{H}^{\hat{X}}_{A'}=0$, then since $\bar{H}^{\hat{X}}_{A'}=\bar{H}_{\Psi^{-1}(A')}$ we have $\mu(\Psi^{-1}(A'))=\mu'(A')=0$. Conversely, if $\mu'(A')=0$ then $\lambda_x(\Psi^{-1}(A'))=0$ for every $x\in X$ and $\bar{H}^{X}_{A'}=\bar{H}_{\Psi^{-1}(A')}$ $=0$.

Now we proceed to the proof of Theorem 7.10.

Let $\{A_n\}$ be a countable base of open subsets of Δ, and, for $z\in\Delta$, let $\mathcal{W}(z)=\{A_j;\ z\in A_j\}$. By Lemma 7.11, we can form a family $\{A'_n\}\subset\mathcal{M}'$ with the following properties: (1) $H^{\hat{X}}_{A'_n}=H_{A_n}$, (2) $A'_n\subset\Psi(A_n\cap\Delta_3)$, (3) $\mu(A_n\backslash\Psi^{-1}(A'_n))$ $=0$. If $A_m\cap A_n=\emptyset$ then $A'_m\cap A'_n$ is negligible and, by Lemma 7.12, it is a μ'-null set. We define

$$\Sigma:=\{z\in\Delta_3;\ \Psi(z)\in\cap\{A'_j;\ A_j\in\mathcal{W}(z)\}\}\cap\Gamma\,.$$

Then $\mu(\Delta\backslash\Sigma)=0$, for $\Delta\backslash\Sigma\subset(\Delta\backslash\Gamma)\cup(\Delta\backslash\Delta_3)\cup\bigcup_{j=1}^{\infty}(A_j\backslash\Psi^{-1}(A'_j))$. Also, $\mu'(\Delta'\backslash\Psi(\Sigma))=0$. We shall show that

$$T:=(\Delta\backslash\Sigma)\cup\Psi^{-1}(\Sigma') \quad\text{and}\quad T':=(\Delta'\backslash\Psi(\Sigma))\cup\Sigma'$$

fulfil the requirement of the theorem, where

$$\Sigma'=\cup\{A'_n\cap A'_m;\, A_n\cap A_m=\emptyset\}\cup(\Delta'\backslash\Gamma').$$

It is easily seen that $\Delta\backslash T\subset\Gamma$, $\Delta'\backslash T'\subset\Gamma'$ and $\mu(T)=\mu'(T')=0$. We note that $\Delta\backslash T=\Sigma\backslash\Psi^{-1}(\Sigma')$ and $\Delta'\backslash T'=\Psi(\Sigma\backslash\Sigma)'$. Hence $\Psi(\Delta\backslash T)=\Delta'\backslash T'$. To show that the mapping Ψ is one-to-one on $\Sigma\backslash\Psi^{-1}(\Sigma')$, suppose, on the contrary, that $\Psi(z_1)=\Psi(z_2)=\zeta$ for distinct two points $z_1, z_2\in\Sigma\backslash\Psi^{-1}(\Sigma')$. Then, $\zeta\in[\cap\{A'_{1j};\, A_{1j}\in\mathcal{W}(z_1)\}]\cap[\cap\{A'_{2j};\, A_{2j}\in\mathcal{W}(z_2)\}]$. However, there are $A_{1j_1}\in\mathcal{W}(z_1)$ and $A_{2j_2}\in\mathcal{W}(z_2)$ with $A_{1j_1}\cap A_{2j_2}=\emptyset$, and we are led to the contradiction that $A'_{ij_1}\cap A'_{2j_2}\in\Sigma'$, q.e.d..

8. Remarks on the structure of Martin type compactifications

As an application of the previous consideration, we give informations on the kernel functions.

Let X^* and $\hat{X}$ be compactifications of Martin type with μ and μ' are dilations of the same normalized reference measure τ, and we suppose that 1 is harmonic.

Then, by Theorem 7.10, there exist boundary sets T, T' with $\mu(T)=\mu'(T')=0$ and a bijection Ψ between $\Delta\backslash T$ and $\Delta'\backslash T'$. The mapping Ψ sends every $z\in\Delta_3$ to its pole.

Lemma 8.1. *$\mu(\Psi^{-1}(A'))=\mu'(A')$ for every $A'\in\mathcal{B}(\Delta')$.*

For since $\Psi^{-1}(A')\in\overline{\mathcal{B}(\Delta)}^{\mu}$, $\mu(\Psi^{-1}(A'))=\int\lambda_x(\Psi^{-1}(A'))d\tau(x)=\int H_{\Psi^{-1}(A')}(x)d\tau(x)=\int H^{\hat{X}}_{A'}(x)d\tau(x)=\int\lambda'_x(A')d\tau(x)=\mu'(A')$.

Lemma 8.2. *$k'^x\circ\Psi=k^x$ μ-almost everywhere for every $x\in X$.*

In fact, for every $x\in X$ there is a function $u_x\in\mathcal{H}^2$ satisfying $u_x(x)=1$ and $\|u_x\|=m=\inf\{\|u\|;\, u\in\mathcal{H}^2,\, u(x)=1\}$; and, by Corollary 6.7, $u_x=H_{m^2k^x}=H^{\hat{X}}_{m^2k'^x}$. On the other hand, by Theorem 7.8, $H^{\hat{X}}_{m^2k'^x}=H_{m^2k'^x\circ\Phi}$, therefore $k^x=k'^x\circ\Psi$ μ-almost everywhere.

Now we give a theorem which reveals the structure of the Martin type compactifications.

Theorem 8.3. *Under the assumption that 1 is harmonic, let $(X^*, k(x, z), \Delta_1, \mu)$ and $(\hat{X}, k'(x, z'), \Delta'_1, \mu')$ be compactifications of Martin type and suppose that*

(1) *μ and μ' are dilations of a normalized reference measure τ,*

(2) *k^x and k'^x separates points of harmonic boundaries Γ and Γ', respectively, i.e., for example, if $z_1, z_2 \in \Gamma$, $z_1 \neq z_2$ then $k^x(z_1) \neq k^x(z_2)$ for some $x \in X$.*

Then there is a homoemorphism $\tilde{\Psi}$ between Γ and Γ' such that $k(x, z) = k'(x, \tilde{\Psi}(z))$ for every $x \in X$ and $z \in \Gamma$.

Proof. By Theorem 7.10, there exist two sets T, T' and a mapping Ψ satisfying $\Delta \backslash T \subset \Gamma$, $\Delta' \backslash T' \subset \Gamma'$, $\mu(T) = \mu'(T') = 0$ and Ψ is a bijection between $\Delta \backslash T$ and $\Delta' \backslash T'$. Let $\{a_n\}$ be a countable dense subset of X and let

$$\Theta = \bigcup_{n=1}^{\infty} \{z \in \Delta_3;\ k(a_n, z) \neq k'(a_n, \Psi(z))\} \cup (\Delta \backslash \Delta_3) \cup T\,.$$

Then, by Lemma 8.2, $\mu(\Theta) = 0$, which means that $\Delta \backslash \Theta$ is dense in Γ. We assert that $\{\Psi(z_n)\}$ is convergent for every convergent sequence $\{z_k\}$ in $\Delta \backslash \Theta$. In fact, let $z_k \to z$ and let $\{\zeta_k\}$ be a subsequence of $\{z_k\}$ with $\Psi(\zeta_k) \to z'$, Then, $k(a_n, z) = \lim_{k \to \infty} k(a_n, \zeta_k) = \lim_{k \to \infty} k'(a_n, \Psi(\zeta_k)) = k'(a_n, z')$ for every n, and, by assumption, this implies that $\{\Psi(z_k)\}$ converges to a point of Γ'. Hence the restriction of Ψ to $\Delta \backslash \Theta$ is uniformly continuous, that is, denoting by ρ and ρ' the metric of X^* and $\hat{X}$ respectively, we have for every $\varepsilon > 0$ there is $\eta > 0$ such that $z_1, z_2 \in \Delta \backslash \Theta$ and $\rho(z_1, z_2) < \eta$ implies $\rho'(\Psi(z_1), \Psi(z_2)) < \varepsilon$. Therefore if we define, for every $z \in \Gamma$, $\tilde{\Psi}(z) = \lim_{\substack{z_n \to z \\ z_n \in \Delta \backslash \Theta}} \Psi(z_n)$, then $\tilde{\Psi}$ is a well-defined continuous mapping of Γ and $\tilde{\Psi}(z) \in \Gamma'$. We show that $\tilde{\Psi}$ is a bijection. Since $k(a_n, z) = k'(a_n, \Psi(z))$ for every n and $z \in \Delta \backslash \Theta$, $k(a_n, z) = k'(a_n, \tilde{\Psi}(z))$ for every n and $z \in \Gamma$, which implies $k(x, z) = k'(x, \tilde{\Psi}(z))$ for every $x \in X$ and $z \in \Gamma$. Hence if $\tilde{\Psi}(z_1) = \tilde{\Psi}(z_2)$ then $k_{z_1} = k_{z_2}$, and hence $z_1 = z_2$ by assumption. The proof is completed if we show $\tilde{\Psi}(\Gamma) = \Gamma'$. The relation $\Delta \backslash \Psi^{-1}(\Psi(\Delta \backslash \Theta)) \subset \Theta$ implies $0 = \mu(\Delta \backslash \Psi^{-1}(\Psi(\Delta \backslash \Theta))) = \mu(\Psi^{-1}(\Delta' \backslash \Psi(\Delta \backslash \Theta))) = \mu'(\Delta' \backslash \Psi(\Delta \backslash \Theta))$ by Corollary 7.9, and this means that $\overline{\Psi(\Delta \backslash \Theta)} = \Gamma'$. Thus, for every $z' \in \Gamma'$ there is a sequence $\{z_k\}$ in $\Delta \backslash \Theta$ with $\Psi(z_k) \to z'$. If $z_k \to z \in \Gamma$ then $\tilde{\Psi}(z) = z'$, q.e.d..

REMARK. We know form Theorem 8.3 that in the case of Example 6 in §2, the family of kernel functions described there is just all that is possible.

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