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<td>Kutami, Mamoru; Oshiro, Kiyoichi</td>
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Osaka University
AN EXAMPLE OF A RING WHOSE PROJECTIVE MODULES HAVE THE EXCHANGE PROPERTY

Dedicated to Professor Goro Azumaya on his 60th birthday

MAMORU KUTAMI AND KIYOICHI OSHIRO

(Received July 9, 1979)

Harada and Ishii [2] and Yamagata [5] have shown the following result: If $R$ is a right perfect ring, then every projective right $R$-module has the exchange property, and in the case when $R$ is a direct sum of indecomposable right ideals, the converse also holds. In this paper, we show that this converse statement does not hold in general by showing the following theorem: If $R$ is a Boolean ring whose socle is a maximal ideal, then every projective $R$-module has the exchange property. It should be noted that Nicholson [3] has also shown that every projective right module over a right perfect ring has the finite exchange property, and he then asked whether the converse hold or not. Our theorem answers this question in the negative.

Throughout this paper we assume that a ring $R$ has identity 1 and all $R$-modules are unitary.

A right $R$-module $M$ has the exchange property if for any right $R$-module $X$ and any two decompositions

$$X = M' \oplus N = \sum_{\lambda \in \Lambda} A_\lambda$$

where $M' \approx M$, there exist submodules $A'_\lambda \subseteq A_\lambda$ such that

$$X = M' \oplus \left( \sum_{\lambda \in \Lambda} A'_\lambda \right).$$

$M$ has the finite exchange property if this holds whenever the index set $\Lambda$ is finite.

For a given projective right $R$-module $P$, the following condition (N) due to Nicholson [3] seems to be useful for the study of the exchange property:

(N) If $P = \sum_{\lambda \in \Lambda} P_\lambda$, where $P_\lambda$ are submodules, there exists a decomposition $P = \sum_{\lambda \in \Lambda} \oplus P'_\lambda$ with $P'_\lambda \subseteq P_\lambda$ for each $\lambda \in \Lambda$.

Lemma 1. If $R$ is a right hereditary ring whose projective right modules satisfy the condition (N), then every projective $R$-module has the exchange property.
Proof. Let $P$ be a projective right $R$-module and consider decompositions of a right $R$-module $X$:

$$X = \sum_{\lambda \in \Lambda} \oplus X_\lambda = P' \oplus Q$$

where $P' \approx P$. In order to show that there exists a submodule $X'_\lambda \subseteq X_\lambda$ for each $\lambda$ such that

$$X = P' \oplus (\sum_{\lambda \in \Lambda} \oplus X'_\lambda)$$

we can assume, by [1, Theorem 8.2], each $X_\lambda$ to be isomorphic to a submodule of $P$. Since $R$ is a right hereditary ring and $P$ is projective, we see that $X = \sum_{\lambda \in \Lambda} \oplus X_\lambda$ is also projective; whence so is $Q$. Now $\psi$ denotes the projection: $X \to Q$. Inasmuch as $Q$ is projective and satisfies the condition (N), there exists a direct summand $X'_\lambda \oplus X_\lambda$ for each $\lambda$ such that

$$Q = \sum_{\lambda \in \Lambda} \oplus \psi(X'_\lambda) \quad \text{and} \quad \sum_{\lambda \in \Lambda} \oplus X'_\lambda \approx \sum_{\lambda \in \Lambda} \oplus \psi(X'_\lambda).$$

Then it is easy to verify that

$$X = \sum_{\lambda \in \Lambda} \oplus X_\lambda = P' \oplus (\sum_{\lambda \in \Lambda} \oplus X'_\lambda).$$

Lemma 2. Let $R$ be a ring and $\{e_1, \ldots, e_n\}$ be a set of central idempotents in $R$. Then there are orthogonal central idempotents $f_1, \ldots, f_m$ in $R$ such that

$$\sum_{i=1}^{n} e_i R = \sum_{i=1}^{m} f_i R,$$

$$e_i f_j = \begin{cases} 0 & \text{or} \\ f_j & \end{cases}$$

for any $i, j$.

Proof. We show this by induction on the number $n$ of $\{e_1, \ldots, e_n\}$. In the case of $n=1$, this is obvious. We assume that $n>1$ and the assertion is true on $n=k-1$. Let $n=k$. Then there exist orthogonal central idempotents $g_1, \ldots, g_m$ in $R$ such that

$$\sum_{i=1}^{k-1} e_i R = \sum_{i=1}^{m} g_i R \quad \text{and} \quad e_i g_j = \begin{cases} 0 & \text{or} \\ g_j & \end{cases}$$

for any $i=1, \ldots, n-1$ and $j=1, \ldots, m$. Put $L = \{e_d(1-\sum_{i=1}^{m} g_i), (1-e_n)g_i, \ldots, (1-e_n)g_m, e_n g_1, \ldots, e_n g_m\}$. Then $L$ is a set of orthogonal central idempotents and it holds that
Let $R$ be a commutative (Von Neumann) regular ring, and let $x$ be a prime ideal of $R$. For a given $R$-module $M$, we denote its Pierce stalk for $x$ by $M_x$, i.e., $M_x = M/Mx$. Then, note that $M_x = M \otimes x R$ and $R_x$ is a flat $R$-module; hence, for a submodule $N$ of $M$, we have $N_x \subseteq M_x$. For an element $a$ in $M$, $a_x$ denotes the image of $a$ under the canonical map: $M \rightarrow M_x$. If \{a_1, \ldots, a_n, b_1, \ldots, b_s\} is a finite subset of $M$ and $(a_i)_x = (b_i)_x$ for all $i$, then, as is well known (see [4]), there exists $e$ in $R - x$ such that $a_i e = b_i e$ for all $i$.

**Lemma 3.** Let $R$ be a Boolean ring and let $x$ be a prime ideal of $R$. If $F$ is a free $R$-module with a base \{m_i | i \in I\} and $F = \bigoplus_{\lambda \in \Lambda} A_\lambda$ for submodules $A_\lambda$, then there exists $e_i$ in $R - x$ for each $i \in I$ and a submodule $A^*_\lambda \subseteq A_\lambda$ for each $\lambda \in \Lambda$ such that \{\{A^*_\lambda | \lambda \in \Lambda\} is an independent set and

$$\sum_{i \in I} \oplus m_i e_i R \subseteq \sum_{\lambda \in \Lambda} \oplus A^*_\lambda.$$

Proof. By $\psi'$ and $\psi'_i$, we denote the projections: $F = \sum_{i \in I} \oplus m_i R \rightarrow m_i R$ and $F_x = \sum_{i \in I} \oplus (m_i)_x R_x \rightarrow (m_i)_x R_x$, respectively. Note that $(\psi'(a))_x = \psi'_i(a_i)$ for any $a \in F$ and $i \in I$.

Firstly we show the lemma in the case of the cardinal $|\Lambda| = 2$, and let $F = A + B$. Since $F_x = A_x + B_x$ is a vector space over $R_x = GF(2)$, we can take submodules $A' \subseteq A$ and $B' \subseteq B$ such that \{A'_i, B'_i\} is independent and $F_x = A'_i + B'_i$. In this we express each $(m_i)_x$ as

$$\left(1\right) \quad (m_i)_x = (a_i)_x + (b_i)_x$$

where $a_i \in A'$ and $b_i \in B'$. Since $R_x = GF(2)$, we see that

$$\left(2\right) \quad \psi'_i ((a_i)_x) = \begin{cases} 0_x & \text{or} \\ (m_x)_x & \text{for any } i, k. \end{cases}$$

Hence from these (1) and (2) we can take $e_i \in R - x$ for each $i$ such that $m_i e_i = a_i e_i + b_i e_i$ and

$$\left(3\right) \quad \psi'_i (a_i e_i) = \begin{cases} 0 & \text{if } \psi'_i ((a_i)_x) = 0_x \\ m_x e_i & \text{if } \psi'_i ((a_i)_x) = (m_x)_x \end{cases}$$
for any $k$. Here put $A^* = \sum_{i \in I} a_i e_i R$ and $B^* = \sum_{i \in I} b_i e_i R$, and claim that these are required submodules.

In fact, clearly $\sum_{i \in I} m_i e_i R \subseteq A^* + B^*$. Assume that $\{A^*, B^*\}$ is not independent and take $0 \neq c \in A^* \cap B^*$ and express it as

$$c = \sum_{i \in V} a_i e_i f_i = \sum_{j \in W} b_j e_j g_j$$

where $V$ and $W$ are finite subsets of $I$ and $\{f_i | i \in V\} \cup \{g_j | j \in W\} \subseteq R$. Put $S = \{i \in I | i \in V\} \cup \{j \in W \} \cup \{f_i | i \in V\} \cup \{g_j | j \in W\}$. Applying Lemma 2 for $S$, we can take orthogonal elements $\{h_1, \ldots, h_l\} \subseteq R$ for which

$$\sum_{s \in S} sR = h_1 R \oplus \cdots \oplus h_l R,$$

or

$$hs = \begin{cases} h, \\ 0 \end{cases}$$

for any $s \in S$ and $h \in \{h_1, \ldots, h_l\}$. Since $u = u(h_1 + \cdots + h_l)$ for any $u \in h_1 R + \cdots + h_l R = \sum_{s \in S} sR$, we see that $c = c(h_1 + \cdots + h_l)$, so that $ch_0 \neq 0$ for some $h_0 \in \{h_1, \ldots, h_l\}$. Set $V' = \{i \in V | e_i f_i h_0 = h_0\}$ and $W' = \{j \in W | b_j e_j h_0 = h_0\}$. Note that $V' \neq \emptyset$ and $W' \neq \emptyset$, since $ch_0 \neq 0$. Now, for these $V'$ and $W'$, we see that

$$\sum_{i \in V'} a_i e_i f_i h_0 = \sum_{j \in W'} b_j e_j g_j h_0$$

and by (3) and (4),

$$\psi^k(a_i e_i f_i h_0) = \begin{cases} m_i h_0, & \text{if } \psi^k((a_i)_z) = (m_i)_z \\ 0, & \text{if } \psi^k((a_i)_z) = 0 \end{cases}$$

and

$$\psi^k(b_j e_j g_j h_0) = \begin{cases} m_j h_0, & \text{if } \psi^k((b_j)_z) = (m_j)_z \\ 0, & \text{if } \psi^k((b_j)_z) = 0 \end{cases}$$

for any $i \in V'$, $j \in W'$ and $k$. Thus, noting that the characteristic of $R$ is 2, we see from (5), (6) and (7) that

$$\sum_{i \in V'} (a_i)_z = \sum_{j \in W'} (b_j)_z$$

and furthermore $0 \neq ch_0$ shows that $\psi^k(ch_0) = m_i h_0$ for some $k$; then $\psi^k(\sum_{i \in V'} (a_i)_z) = (m_i)_z$ and hence $0 \neq \sum_{i \in V'} (a_i)_z = \sum_{j \in W'} (b_j)_z$, a contradiction. Therefore $\{A^*, B^*\}$ must be independent. In view of the argument above, the proof works on
LEMMA 4. Let $R$ be a Boolean ring whose socle is a maximal ideal. Then every projective $R$-module satisfies the condition (N).

Proof. $\text{Soc}(\ )$ denotes the socle of ( ). We may show that every free $R$-module satisfies the condition (N).

Let $F$ be a free $R$-module with a base $\{m_i|i\in I\}$, and let $F=\sum_{\lambda\in \Lambda} A_\lambda$ where $A_\lambda$ are submodules of $F$. We show that $\text{Soc}(F)=\sum_{\lambda\in \Lambda} \text{Soc}(A_\lambda)$. Consider $a\in F$ such that $aR$ is a simple module. Since $F=\sum_{\lambda\in \Lambda} A_\lambda$, there exist $\lambda_1, \cdots, \lambda_n \in \Lambda$ and $a_1 \in A_{\lambda_1}, \cdots, a_n \in A_{\lambda_n}$ such that $a=a_1+\cdots+a_n$. Express $a_1, \cdots, a_n$ as

\[
\begin{aligned}
a_1 &= m_k r^1_{i_1} + \cdots + m_k r^1_{i_t} \\
a_2 &= m_k r^2_{i_1} + \cdots + m_k r^2_{i_t} \\
&\quad \vdots \\
a_n &= m_k r^n_{i_1} + \cdots + m_k r^n_{i_t}
\end{aligned}
\]

where $r^i_{j}, i=1, \cdots, n, j=1, \cdots, t$. Using Lemma 2 for $L=\{r^i_{j}|i=1, \cdots, n, j=1, \cdots, t\}$ we can take orthogonal elements $h_1, \cdots, h_t$ such that

\[
\sum_{s\in S} sR = h_1 R + \cdots + h_t R
\]

\[
sh_i = \begin{cases} h_i & \text{or for any } h_i \text{ and } s\in S. \\
0 & \text{otherwise} \end{cases}
\]

Then $a=a(h_1+\cdots+h_t)$; hence $aR=ah_1R\oplus \cdots \oplus ah_t R$. Inasmuch as $aR$ is simple, this implies that $aR=ah_0R$ for some $h_0\in \{h_1, \cdots, h_t\}$. By $\psi^k$, we denotes the projection: $F\rightarrow m_k R$. Then we see that $\psi^k(ah_0)=0$ or $m_k h_0$ for any $k$, and, for each $i$, $\psi^k(a_0 h_0)=0$ or $m_k h_0$ for any $k$. Therefore we see that if $a_0 h_0 R\neq 0$, then $ah_0 R$ is canonically isomorphic to $a_0 h_0 R$. As a result we see that $aR=ah_0R\subseteq \text{Soc}(A_{\lambda_1}) + \cdots + \text{Soc}(A_{\lambda_n})$. Consequently we get $\text{Soc}(F)=\sum_{\lambda\in \Lambda} \text{Soc}(A_\lambda)$.

Now, by Lemma 3, there exists $e_i\in R-\text{Soc}(R)$ for each $i\in I$ and a submodule $A'_\lambda \subseteq A_\lambda$ for each $\lambda \in \Lambda$ such that $\{A'_\lambda|\lambda\in \Lambda\}$ is independent and

\[
\sum_{i\in I}+m_i e_i R \subseteq \sum_{\lambda\in \Lambda} A'_\lambda.
\]

Since $1-e_i\in \text{Soc}(R)$, we see that

\[
\sum_{i\in I}+m_i(1-e_i) R \subseteq \text{Soc}(F) = \sum_{\lambda\in \Lambda} \text{Soc}(A_\lambda).
\]

Consequently,
Then we can easily take a submodule $A'_{\lambda} \subseteq \text{Soc}(A_\lambda)$ for each $\lambda \in \Lambda$ such that 
\[
\{ \bigoplus_{\lambda \in \Lambda} A'_{\lambda} \} \cup \{ A''_{\lambda} | \lambda \in \Lambda \} \text{ is independent and}
\]
\[
F = \left( \bigoplus_{\lambda \in \Lambda} A'_{\lambda} \right) \oplus \left( \bigoplus_{\lambda \in \Lambda} A''_{\lambda} \right).
\]
Putting $A^*_k = A'_{\lambda} \oplus A''_{\lambda}$, $A^*_k$ is a submodule of $A_\lambda$ and $F = \bigoplus_{\lambda \in \Lambda} A^*_k$.

**Lemma 5.** If $R$ is a Boolean ring such that its socle is a maximal ideal, then $R$ is a hereditary ring.

**Proof.** Let $I$ be an ideal of $R$. If $I \subseteq \text{Soc}(R)$, then $I$ is surely a projective $R$-module. If $I \not\subseteq \text{Soc}(R)$, then $I + \text{Soc}(R) = R$ since $\text{Soc}(R)$ is a maximal ideal. Let $1 = e + f$ where $e \in I$ and $f \in \text{Soc}(R)$. Then $I = eR + (\text{Soc}(R) \cap I)$; whence it follows that $I = eR \oplus J$ for some ideal $J \subseteq \text{Soc}(R) \cap I$. Hence $I$ is projective.

From Lemmas 1, 4 and 5 we have our main theorem:

**Theorem 6.** If $R$ is a Boolean ring whose socle is a maximal ideal, then every projective $R$-module has the exchange property.

**Example.** Let $X$ be any infinite set, and let $R$ denotes the Boolean ring consisting of all elements $a$ in $2^X$ such that $a$ or $X - a$ is finite. Then $R$ is a Boolean ring whose socle is a maximal ideal.

**References**


