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AN EXAMPLE OF A RING WHOSE PROJECTIVE MODULES HAVE THE EXCHANGE PROPERTY

Dedicated to Professor Goro Azumaya on his 60th birthday

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Harada and Ishii [2] and Yamagata [5] have shown the following result: If R is a right perfect ring, then every projective right R-module has the exchange property, and in the case when R is a direct sum of indecomposable right ideals, the converse also holds. In this paper, we show that this converse statement does not hold in general by showing the following theorem: If R is a Boolean ring whose socle is a maximal ideal, then every projective R-module has the exchange property. It should be noted that Nicholson [3] has also shown that every projective right module over a right perfect ring has the finite exchange property, and he then asked whether the converse hold or not. Our theorem answers this question in the negative.

Throughout this paper we assume that a ring R has identity 1 and all R-modules are unitary.

A right R-module M has the *exchange property* if for any right R-module X and any two decompositions

$$X = M' \oplus N = \sum_{\lambda \in \Lambda} \oplus A_\lambda$$

where $M' \approx M$, there exist submodules $A'_\lambda \subseteq A_\lambda$ such that

$$X = M' \oplus (\sum_{\lambda \in \Lambda} \oplus A'_\lambda).$$

M has the *finite exchange property* if this holds whenever the index set Λ is finite.

For a given projective right R-module P, the following condition (N) due to Nicholson [3] seems to be useful for the study of the exchange property:

(N)　If $P = \sum_{\lambda \in \Lambda} P_\lambda$, where P_λ are submodules, there exists a decomposition

$P = \sum_{\lambda \in \Lambda} \oplus P'_\lambda$ with $P'_\lambda \subseteq P_\lambda$ for each $\lambda \in \Lambda$.

Lemma 1. *If R is a right hereditary ring whose projective right modules satisfy the condition* (N), *then every projective R-module has the exchange property.*

Proof. Let P be a projective right R-module and consider decompositions of a right R-module X:

$$X = \sum_{\lambda \in \Lambda} \oplus X_\lambda = P' \oplus Q$$

where $P' \approx P$. In order to show that there exists a submodule $X'_\lambda \subseteq X_\lambda$ for each λ such that

$$X = P' \oplus (\sum_{\lambda \in \Lambda} \oplus X'_\lambda)$$

we can assume, by [1, Theorem 8.2], each X_λ to be isomorphic to a submodule of P. Since R is a right hereditary ring and P is projective, we see that $X = \sum_{\lambda \in \Lambda} \oplus X_\lambda$ is also projective; whence so is Q. Now ψ denotes the projection: $X \to Q$. Inasumch as Q is projective and satisfies the condition (N), there exists a direct summand $X'_\lambda < \oplus X_\lambda$ for each λ such that

$$Q = \sum_{\lambda \in \Lambda} \oplus \psi(X'_\lambda) \quad \text{and} \quad \sum_{\lambda \in \Lambda} \oplus X'_\lambda \overset{\psi}{\approx} \sum_{\lambda \in \Lambda} \oplus \psi(X'_\lambda).$$

Then it is easy to verify that

$$X = \sum_{\lambda \in \Lambda} \oplus X_\lambda = P' \oplus (\sum_{\lambda \in \Lambda} \oplus X'_\lambda).$$

Lemma 2. *Let R be a ring and $\{e_1, \cdots, e_n\}$ be a set of central idempotents in R. Then there are orthogonal central idempotents $f_1, \cdots, f_m$ in R such that*

$$\sum_{i=1}^{n} e_i R = \sum_{i=1}^{m} \oplus f_i R,$$

$$e_i f_j = \begin{cases} 0 \\ or \\ f_j \end{cases}$$

for any i, j.

Proof. We show this by induction on the number n of $\{e_1, \cdots, e_n\}$. In the case of $n=1$, this is obvious. We assume that $n>1$ and the assertion is true on $n=k-1$. Let $n=k$. Then there exist orthogonal central idempotents $g_1, \cdots, g_m$ in R such that

$$\sum_{i=1}^{n-1} e_i R = \sum_{i=1}^{m} \oplus g_i R \quad \text{and} \quad e_i g_j = \begin{cases} 0 \\ or \\ g_j \end{cases}$$

for any $i=1, \cdots, n-1$ and $j=1, \cdots, m$. Put $L = \{e_n(1 - \sum_{i=1}^{m} g_i), (1-e_n)g_1, \cdots, (1-e_n)g_m, e_n g_1, \cdots, e_n g_m\}$. Then L is a set of orthogonal central idempotents and it holds that

$$\sum_{i=1}^{n} e_i R = (\sum_{i=1}^{m} \oplus g_i R) + e_n R = \sum_{f \in L} \oplus f R ,$$

$$fe_i = \begin{cases} f \\ \text{or} \\ 0 \end{cases}$$

for any $f \in L$ and $i = 1, \cdots, n$.

Let R be a commutative (Von Neumann) regular ring, and let x be a prime ideal of R. For a given R-module M, we denote its Pierce stalk for x by M_x, i.e., $M_x = M/Mx$. Then, note that $M_x = M \otimes_R Rx$ and R_x is a flat R-module; hence, for a submodule N of M, we have $N_x \subseteq M_x$. For an element a in M, a_x denotes the image of a under the canonical map: $M \to M_x$. If $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$ is a finite subset of M and $(a_i)_x = (b_i)_x$ for all i, then, as is well known (see [4]), there exists e in $R - x$ such that $a_i e = b_i e$ for all i.

Lemma 3. *Let R be a Boolean ring and let x be a prime ideal of R. If F is a free R-module with a base $\{m_i \mid i \in I\}$ and $F = \sum_{\lambda \in \Lambda} A_\lambda$ for submodules A_λ, then there exists e_i in $R - x$ for each $i \in I$ and a submodule $A_\lambda^* \subseteq A_\lambda$ for each $\lambda \in \Lambda$ such that $\{A_\lambda^* \mid \lambda \in \Lambda\}$ is an independent set and*

$$\sum_{i \in I} \oplus m_i e_i R \subseteq \sum_{\lambda \in \Lambda} \oplus A_\lambda^* .$$

Proof. By ψ^i and ψ_x^i, we denote the projections: $F = \sum_{i \in I} \oplus m_i R \to m_i R$ and $F_x = \sum_{i \in I} \oplus (m_i)_x R_x \to (m_i)_x R_x$, respectively. Note that $(\psi^i(a))_x = \psi_x^i(a_x)$ for any $a \in F$ and $i \in I$.

Firstly we show the lemma in the case of the cardinal $|\Lambda| = 2$, and let $F = A + B$. Since $F_x = A_x + B_x$ is a vector space over $R_x = GF(2)$, we can take submodules $A' \subseteq A$ and $B' \subseteq B$ such that $\{A'_x, B'_x\}$ is independent and $F_x = A'_x \oplus B'_x$. In this we express each $(m_i)_x$ as

(1) $(m_i)_x = (a_i)_x + (b_i)_x$

where $a_i \in A'$ and $b_i \in B'$. Since $R_x = GF(2)$, we see that

(2) $$\psi_x^k((a_i)_x) = \begin{cases} 0_x \\ \text{or} \\ (m_k)_x \end{cases}$$

for any i, k. Hence from these (1) and (2) we can take $e_i \in R - x$ for each i such that $m_i e_i = a_i e_i + b_i e_i$ and

(3) $$\psi^k(a_i e_i) = \begin{cases} 0 & \text{if } \psi_x^k((a_i)_x) = 0_x \\ m_k e_i & \text{if } \psi_x^k((a_i)_x) = (m_k)_x \end{cases}$$

$$(4) \qquad \psi^k(b_i e_i) = \begin{cases} 0 & \text{if } \psi_x^k((b_i)_x) = 0_x \\ m_k e_i & \text{if } \psi_x^k((b_i)_x) = (m_k)_x \end{cases}$$

for any k.   Here put $A^* = \sum_{i \in I} a_i e_i R$ and $B^* = \sum_{i \in I} b_i e_i R$, and claim that these are required submodules.

In fact, clearly $\sum_{i \in I} m_i e_i R \subseteq A^* + B^*$.   Assume that $\{A^*, B^*\}$ is not independent and take $0 \neq c \in A^* \cap B^*$ and express it as

$$c = \sum_{i \in V} a_i e_i f_i = \sum_{j \in W} b_j e_j g_j$$

where V and W are finite subsets of I and $\{f_i \,|\, i \in V\} \cup \{g_j \,|\, j \in W\} \subseteq R$.   Put $S = \{e_i \,|\, i \in V\} \cup \{e_j \,|\, j \in W\} \cup \{f_i \,|\, i \in V\} \cup \{g_j \,|\, j \in W\}$.   Applying Lemma 2 for S, we can take orthogonal elements $\{h_1, \cdots, h_l\} \subseteq R$ for which

$$\sum_{s \in S} sR = h_1 R \oplus \cdots \oplus h_l R \,,$$

$$hs = \begin{cases} h \\ \text{or} \\ 0 \end{cases}$$

for any $s \in S$ and $h \in \{h_1, \cdots, h_l\}$.   Since $u = u(h_1 + \cdots + h_l)$ for any $u \in h_1 R + \cdots + h_l R = \sum_{s \in S} sR$, we see that $c = c(h_1 + \cdots + h_l)$, so that $ch_0 \neq 0$ for some $h_0 \in \{h_1, \cdots, h_l\}$.   Set $V' = \{i \in V \,|\, e_i f_i h_0 = h_0\}$ and $W' = \{j \in W \,|\, b_j e_j h_0 = h_0\}$.   Note that $V' \neq \phi$ and $W' \neq \phi$, since $ch_0 \neq 0$.   Now, for these V' and W', we see that

$$(5) \qquad \sum_{i \in V'} a_i e_i f_i h_0 = \sum_{j \in W'} b_j e_j g_j h_0$$

and by (3) and (4),

$$(6) \qquad \psi^k(a_i e_i f_i h_0) = \begin{cases} m_k h_0 & \text{if } \psi_x^k((a_i)_x) = (m_k)_x \\ 0 & \text{if } \psi_x^k((a_i)_x) = 0_x \end{cases}$$

$$(7) \qquad \psi^k(b_j e_j g_j h_0) = \begin{cases} m_k h_0 & \text{if } \psi_x^k((b_j)_x) = (m_k)_x \\ 0 & \text{if } \psi_x^k((b_j)_x) = 0_x \end{cases}$$

for any $i \in V'$, $j \in W'$ and k.   Thus, noting that the characteristic of R is 2, we see from (5), (6) and (7) that

$$\sum_{i \in V'} (a_i)_x = \sum_{j \in W'} (b_i)_x$$

and furthermore $0 \neq ch_0$ shows that $\psi^k(ch_0) = m_k h_0$ for some k; then $\psi_x^k(\sum_{i \in V} (a_i)_x) = (m_k)_x$ and hence $0_x \neq \sum_{i \in V} (a_i)_x = \sum_{j \in W} (b_j)_x$ a contradiction.   Therefore $\{A^*, B^*\}$ must be independent.   In view of the argument above, the proof works on

arbitrary index set Λ, so the proof follows.

Lemma 4. *Let R be a Boolean ring whose socle is a maximal ideal. Then every projective R-module satisfies the condition* (N).

Proof. *Soc*( ) denotes the socle of ( ). We may show that every free R-module satisfies the condition (N).

Let F be a free R-module with a base $\{m_i \,|\, i \in I\}$, and let $F = \sum_{\lambda \in \Lambda} A_\lambda$ where A_λ are submodules of F. We show that $Soc(F) = \sum_{\lambda \in \Lambda} Soc(A_\lambda)$. Consider $a \in F$ such that aR is a simple module. Since $F = \sum_{\lambda \in \Lambda} A_\lambda$, there exist $\lambda_1, \cdots, \lambda_n \in \Lambda$ and $a_1 \in A_{\lambda_1}, \cdots, a_n \in A_{\lambda_n}$ such that $a = a_1 + \cdots + a_n$. Express $a_1, \cdots, a_n$ as

$$a_1 = m_{k_1} r_{k_1}^1 + \cdots + m_{k_t} r_{k_t}^1$$
$$a_2 = m_{k_1} r_{k_1}^2 + \cdots + m_{k_t} r_{k_t}^2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_n = m_{k_1} r_{k_1}^n + \cdots + m_{k_t} r_{k_t}^n$$

where $r_{k_j}^i \in R$, $i = 1, \cdots, n$, $j = 1, \cdots, t$. Using Lemma 2 for $L = \{r_{k_j}^i \,|\, i = 1, \cdots, n, \ j = 1, \cdots, t\}$ we can take orthogonal elements $h_1, \cdots, h_l$ such that

$$\sum_{s \in L} sR = h_1 R + \cdots + h_l R$$

$$sh_i = \begin{cases} h_i \\ \text{or} \\ 0 \end{cases} \quad \text{for any } h_l \text{ and } s \in S.$$

Then $a = a(h_1 + \cdots + h_l)$; hence $aR = ah_i R \oplus \cdots \oplus ah_l R$. Inasmuch as aR is simple, this implies that $aR = ah_0 R$ for some $h_0 \in \{h_1, \cdots, h_l\}$. By ψ^k, we denotes the projection: $F \to m_k R$. Then we see that $\psi^k(ah_0) = 0$ or $m_k h_0$ for any k, and, for each i, $\psi^k(a_i h_0) = 0$ or $m_k h_0$ for any k. Therefore we see that if $a_i h_0 R \neq 0$, then $ah_0 R$ is canonically isomorphic to $a_i h_0 R$. As a result we see that $aR = ah_0 R \subseteq Soc(A_{\lambda_1}) + \cdots + Soc(A_{\lambda_n})$. Consequently we get $Soc(F) = \sum_{\lambda \in \Lambda} Soc(A_\lambda)$.

Now, by Lemma 3, there exists $e_i \in R - Soc(R)$ for each $i \in I$ and a submodule $A_\lambda' \subseteq A_\lambda$ for each $\lambda \in \Lambda$ such that $\{A_\lambda' \,|\, \lambda \in \Lambda\}$ is independent and

$$\sum_{I \in i} \oplus m_i e_i R \subseteq \sum_{\lambda \in \Lambda} \oplus A_\lambda'.$$

Since $1 - e_i \in Soc(R)$, we see that

$$\sum_{i \in I} \oplus m_i (1 - e_i) R \subseteq Soc(F) = \sum_{\lambda \in \Lambda} Soc(A_\lambda).$$

Consequently,

$$F = \sum_{i \in I} m_i e_i R + \sum_{i \in I} m_i (1 - e_i) R$$
$$= (\sum_{\lambda \in \Lambda} \oplus A'_\lambda) + (\sum_{\lambda \in \Lambda} Soc(A_\lambda)).$$

Then we can easily take a submodule $A''_\lambda \subseteq Soc(A_\lambda)$ for each $\lambda \in \Lambda$ such that $\{\sum_{\lambda \in \Lambda} \oplus A'_\lambda\} \cup \{A''_\lambda \mid \lambda \in \Lambda\}$ is independent and

$$F = (\sum_{\lambda \in \Lambda} \oplus A'_\lambda) \oplus (\sum_{\lambda \in \Lambda} \oplus A''_\lambda).$$

Putting $A^*_\lambda = A'_\lambda \oplus A''_\lambda$, A^*_λ is a submodule of A_λ and $F = \sum_{\lambda \in \Lambda} \oplus A^*_\lambda$.

Lemma 5. *If R is a Boolean ring such that its socle is a maximal ideal, then R is a hereditary ring.*

Proof.   Let I be an ideal of R.   If $I \subseteq Soc(R)$, then I is surely a projective R-module.   If $I \nsubseteq Soc(R)$, then $I + Soc(R) = R$ since $Soc(R)$ is a maximal ideal. Let $1 = e + f$ where $e \in I$ and $f \in Soc(R)$.   Then $I = eR + (Soc(R) \cap I)$; whence it follows that $I = eR \oplus J$ for some ideal $J \subseteq Soc(R) \cap I$.   Hence I is projective.

From Lemmas 1, 4 and 5 we have our main theorem:

Theorem 6. *If R is a Boolean ring whose socle is a maximal ideal, then every projective R-module has the exchange property.*

EXAMPLE.   Let X be any infinite set, and let R denotes the Boolean ring consisting of all elements a in 2^X such that a or $X - a$ is finite.   Then R is a Boolean ring whose socle is a maximal ideal.

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